Zbornik radova 9(17)

FOUR TOPICS IN MATHEMATICS

Matematički institut SANU

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Editor:
Bogoljub Stanković

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Издавач: Математички институт САНУ, Београд, Кнеза Михаила 35
серија: Зборник радова, књига 9(17)
За издавача: Богољуб Станковић, главни уредник
Уредник: Богољуб Станковић
Технички уредник: Драган Благојевић
Штампа: "KUM Commerce", Земун
Штампање завршено јуна 2000.
ISBN 86-80593-31-1

CIP – Каталогизација у публикацији
Народна библиотека Србије, Београд

517.982
FOUR topics in Mathematics / editor Bogoljub Stanković.
– Beograd : Matematički institut SANU, 2000 (Zemun : KUM Commerce).
– 274 str. : graf. prikazi ; 24 cm.
– (Zbornik radova / Matematički institut SANU ; 9(17))

Bibliografija uz svaki rad.
ISBN 86–80593–31–1
1. Станковић, Богољуб

512.53 514.75 519.21
a) Функционална анализа
b) Полугрупе (математика)
c) Диференцијална геометрија
d) Теорија вероватноће
ID=84379660

PREFACE

The aim of *Zbornik radova* is to foster further growth of pure and applied mathematics, publishing papers which contain new ideas and scopes in the mathematics. The papers have to be prepared in such a manner that they can inform readers in a favourable way, introducing them in a narrower field of mathematical theories pointing at research possibilities. It can be for the individual use or for discussions in College or University seminars.

We are open for contacts and cooperations.

Bogoljub Stanković
Editor-in-Chief

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Typeset by $\mathcal{AMS}$-TEX

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AMS Subject Classification (1991): 16S70, 16U80, 20M25, 20M10

Supported by Grant 04M03B of RFNS through Math. Inst. SANU.

Introduction

The main aim of this paper is to give a survey of the most important structural properties of uniformly π-regular rings and semigroups. It is well-known that there are many similarities between certain types of semigroups and related rings. For example, we will see in Theorem 2.1 that the regularity of a semigroup can be characterized by means of the properties of its left and right ideals, and in the same way, the regularity of a ring can be characterized through its ring left and right ideals. On the other hand, there are many significant differences between the properties of certain types of semigroups and the properties of related rings. For example, many concepts such as the left, right and complete regularity and other, are different in Theory of semigroups, but they coincide in Theory of rings. One of the main goals of this paper is exactly to underline both the similarities and differences between related types of rings and semigroups. For that purpose many interesting results of Theory of rings or Theory of semigroups will be omitted here if they are not similar or essentially different than the corresponding result of another theory.

There are two central places in the paper. The first one is Theorem 5.11 which asserts that a π-regular ring is uniformly π-regular if and only if it is an ideal extension of a nil-ring by a Clifford ring. This theorem makes possible to represent such rings by the Everett's sums of nil-rings and Clifford rings. This has shown oneself to be very useful in many situations. For example, using Theorem 5.11, a lot of known results concerning uniformly π-regular semigroups can be very successfully applied in Theory of rings.

Another crucial result is Theorem 5.44. This theorem describes rings whose multiplicative semigroups are nil-extensions of unions of groups and it asserts that such rings are exactly the direct sums of nil-rings and Clifford rings. We present numerous known methods for decomposition of semigroups into a nil-extension of a union of groups and we show that these methods have very significant applications in Theory of rings, in decompositions of rings into the direct sum of a nil-ring and a Clifford ring.

The purpose of this paper is twofold. At first, we intend to present the known results concerning uniformly π-regular semigroups and applications of these results in Theory of rings. On the other hand, we want also to interest ring-theoretists and semigroup-theoretists for more intensive investigations in the considered area.

The paper is divided into six sections. In the first section we introduce the necessary notions and notations and we present the main results concerning ideal extensions of rings and their representation by the known Everett's sums of rings. In Sections 2 and 3 we introduce the notions of a regular, π-regular, completely

π-regular and periodic ring and semigroup, and of a completely Archimedean semigroup and we describe their basic properties. Structural characterizations of completely regular semigroups and rings are given in Section 4. The main tools that we use there, are certain decomposition methods: semilattice decompositions, in the case of semigroups, and subdirect sum decompositions, in the case of rings.

The main part of the whole paper is Section 5. In this section we first give structural descriptions of uniformly π-regular semigroups and rings. After that we present various characterizations of semigroups decomposable into a nil-extension of a union of groups, and using these results we characterize the rings decomposable into the direct sum of a nil-ring and Clifford ring.

Finally, in Section 6 we present certain applications of the results given in the previous section. Here we study various types of semigroup identities satisfied on the various classes of semigroups and rings. The classes of all identities satisfied on the classes of the semilattices of Archimedean semigroups, the nil-extensions of unions of groups, the bands of π-regular semigroups are described. The main result in the part about the rings satisfying certain semigroup identities is the characterization of all rings satisfying a semigroup identity of the form $x_1x_2\cdots x_n = w(x_1, x_2, \dots, x_n)$, where $|w| \geq n+1$, given in the Theorem 6.29.

1. Preliminaries

In this section we introduce necessary notions and notations.

1.1. Basic notions and notations. Throughout this paper $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}^0$ the set of all non-negative integers, and $\mathbb{Z}$ will denote the ring of integers. By $\mathbb{Z}\langle x, y\rangle$ we will denote the ring of all polynomials with the variables x and y and the coefficients in $\mathbb{Z}$.

For a semigroup (ring) S, $E(S)$ will denote the set of all idempotents of S, and for $A \subseteq S$, $\sqrt{A}$ will denote the subset of S defined by $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{N})\, x^n \in A\}$. For a ring R, $\mathcal{M}R$ will denote the multiplicative semigroup of R. A subset A of a semigroup (ring) S is called *completely semiprime* if for $x \in S$, $x^2 \in A$ implies $x \in A$, *completely prime* if for $x, y \in S$, $xy \in A$ implies that either $x \in A$ or $y \in A$, *left consistent* if for $x, y \in S$, $xy \in A$ implies $x \in A$, *right consistent* if for $x, y \in S$, $xy \in A$ implies $y \in A$, and it is *consistent* if it is both left and right consistent.

The expression $S = S^0$ means that S is a semigroup with the zero 0. Let S be a semigroup (ring) with the zero 0. An element $a \in S$ is called a *nilpotent element* (or a *nilpotent*) if there exists $n \in \mathbb{N}$ such that $a^n = 0$, and the smallest number $n \in \mathbb{N}$ having this property is called the *index of nilpotency* of a. The set of all nilpotents of S is denoted by $\mathrm{Nil}(S)$, and also $N_2(S) = \{a \in S \mid a^2 = 0\}$. A semigroup (ring) whose any element is nilpotent is called a *nil-semigroup* (*nil-ring*). For $n \in \mathbb{N}$, $n \geq 2$, a semigroup (ring) S is called *n-nilpotent* if $S^n = 0$, and is called *nilpotent* if it is n-nilpotent, for some $n \in \mathbb{N}$, $n \geq 2$. A 2-nilpotent semigroup (ring) is called a *null-semigroup* (*null-ring*).

For a semigroup S we say that is an *ideal extension* of a semigroup T by a semigroup Q if T is an ideal of S and the factor semigroup S/T is isomorphic to Q. An ideal extension of a semigroup S by a nil-semigroup (resp. n-nilpotent semigroup, nilpotent semigroup, null-semigroup) is called a *nil-extension* (resp. *n-nilpotent extension*, *nilpotent extension*, *null-extension*) T. A subsemigroup T of a semigroup S is called a *retract* of S if there exists a homomorphism φ of S onto T such that $a\varphi = a$, for any $a \in T$, and then φ is called a *retraction* of S onto T. An ideal extension S of a semigroup T is called a *retractive extension* of T if T is a retract of S.

By A^+ we denote the *free semigroup* over an alphabet A and by A^* we denote the *free monoid* over A. For $n \in \mathbb{N}$, $n \geq 4$, $A_n = \{x_1, x_2, \dots, x_n\}$, $A_3 = \{x, y, z\}$ and $A_2 = \{x, y\}$. For a word $w \in A^+$, w^+ will denote the set $w^+ = \{w^n \mid n \in \mathbb{N}\}$. By $|w|$ we denote the *length* of a word $w \in A^+$ and by $|x|_w$ we denote the number of appearances of the letter $x \in A$ in the word $w \in A^+$. A word $v \in A^+$ is a *left* (*right*) *cut* of a word $w \in A^+$ if $w = vu$ ($w = uv$), for some $u \in A^*$, and v is a *subword* of w if $w = u'vu''$, for some $u', u'' \in A^*$. For $w \in A^+$ such that $|w| \geq 2$, by $h^{(2)}(w)$ ($t^{(2)}(w)$) we denote the left (right) cut of w of the length 2. By $h(w)$ ($t(w)$) we denote the first (last) letter of a word $w \in A^+$, called the *head* (*tail*) of w, and by $c(w)$ we denote the set of all letters which appear in w, called the *content* of w [**246**]. An expression $w(x_1, \dots, x_n)$ will mean that w is a word with $c(w) = \{x_1, \dots, x_n\}$. If $w \in A^+$ and $i \in \mathbb{N}$, $i \leq |w|$, then $l_i(w)$ ($r_i(w)$) will denote the left (right) cut of w of the length i, $c_i(w)$ will denote the i-th letter of w and for $i, j \in \mathbb{N}$, $i, j \leq |w|$, $i \leq j$, $m_i^j(w)$ will denote the subword w determined by: $w = l_{i-1}(w)m_i^j(w)r_{|w|-j}(w)$. For $n \in \mathbb{N}$, Π_n will denote the word $x_1x_2\dots x_n \in A_n^+$. If $w \in A^+$ and $x \in A$, then $x \underset{l}{\parallel} w$ ($x \underset{r}{\parallel} w$) if $w = xv$ ($w = vx$), $v \in A^+$ and $x \notin c(v)$. Otherwise we write $x \underset{l}{\not\parallel} w$ ($x \underset{r}{\not\parallel} w$).

Let $n \in \mathbb{N}$, $w \in A_n^+$ and let S be a semigroup. By the *value* of the word w in S, in a *valuation* $a = (a_1, a_2, \dots, a_n)$, $a_i \in S$, $i \in \{1, 2, \dots, n\}$, in notation $w(a)$ or $w(a_1, a_2, \dots, a_n)$, we mean the element $w\varphi \in S$, where $\varphi : A_n^+ \to S$ is the homomorphism determined by $x_i\varphi = a_i$, $i \in \{1, 2, \dots, n\}$. Also, we then say that for $i \in \{1, 2, \dots, n\}$, the letter x_i *assumes the value* a_i in S, in notation $x_i := a_i$. For two words $u, v \in A_n^+$, the formal expression $u = v$ we call an *identity* (or a *semigroup identity*) over the alphabet A_n, and for a semigroup S we say that it *satisfies the identity* $u = v$, in notation $S \models u = v$, if $u\varphi = v\varphi$, for any homomorphism φ from A_n^+ into S, i.e. if u and v have the same value for any valuation in S. The class of all semigroups satisfying the identity $u = v$ is denoted by $[u = v]$, and is called the *variety* determined by the identity $u = v$. Identities $u = v$ and $u' = v'$ over an alphabet A_n^+ are *p-equivalent* if $u' = v'$ can be obtained from $u = v$ by some permutation of letters. It is clear that p-equivalent identities determine the same variety.

Let φ be a homomorphism of a free semigroup A^+ into a semigroup S. For an identity over A, which is treated as a pair of words from A^+, we say that it is a *solution of the equation* $u\varphi = v\varphi$ if it is contained in the kernel of φ. Any trivial identity over A, i.e. an identity of the form $w = w$, is clearly a solution of the equation $u\varphi = v\varphi$, called the *trivial solution* of $u\varphi = v\varphi$. All other solutions of $u\varphi = v\varphi$, if they exist, are called *non-trivial solutions* of $u\varphi = v\varphi$.

Let Σ be a set of non-trivial identities over an alphabet A. i.e. a subset of $A^+ \times A^+$ having the empty intersection with the equality relation on A^+. For a semigroup S we say that it *satisfies variabily* the set Σ of identities, or that it *satisfies the variable identity* Σ, in notation $S \models_v \Sigma$, if for any homomorphism φ from A^+ to S, the equation $u\varphi = v\varphi$ has a solution in Σ (clearly, such solutions are non-trivial). The class of all semigroups which satisfy the variable identity Σ is denoted by $[\Sigma]_v$ and is called a *variable variety*.

A semigroup S is called a *band* (resp. *left zero band*, *right zero band*, *rectangular band*, *left regular band*, *right regular band*, *semilattice*) if it belongs to the variety $[x = x^2]$ (resp. $[xy = x]$, $[xy = y]$, $[x = x^2,\ xyx = x]$, $[x = x^2,\ xyz = xzy]$, $[x = x^2,\ xyz = yxz]$, $[x = x^2,\ xy = yx]$). If B is a band, we say that a semigroup S is a *band B of semigroups* if B is a homomorphic image of S. When B is semilattice (resp. left zero band, right zero band, rectangular band), then we say that S is a *semilattice* (resp. *left zero band*, *right zero band*, *matrix*) *of semigroups*.

In this paper we will use several semigroups given by the following presentations:

$$
\begin{aligned}
\mathbb{B}_2 &= \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b \rangle \\
\mathbb{A}_2 &= \langle a, e \mid a^2 = 0,\ e^2 = e,\ aea = a,\ eae = e \rangle \\
\mathbb{N}_m &= \langle a \mid a^{m+1} = a^{m+2},\ a^m \neq a^{m+1} \rangle \\
\mathbb{L}_{3,1} &= \langle a, f \mid a^2 = a^3,\ f^2 = f,\ a^2 f = a^2,\ fa = f \rangle \\
\mathbb{C}_{1,1} &= \langle a, e \mid a^2 = a^3,\ e^2 = e,\ ae = a,\ ea = a \rangle \\
\mathbb{C}_{1,2} &= \langle a, e \mid a^2 = a^3,\ e^2 = e,\ ae = a,\ ea = a^2 \rangle
\end{aligned}
$$

where $m \in \mathbb{N}$, and $\mathbb{R}_{3,1}$ (resp. $\mathbb{C}_{2,1}$) will denote the dual semigroup of $\mathbb{L}_{3,1}$ (resp. $\mathbb{C}_{1,2}$). By $\mathbb{L}_2$ (resp. $\mathbb{R}_2$) we denote the two-element left zero (resp. right zero) semigroup. Let A_N^+ be the free semigroup over an alphabet $A_N = \{x_k \mid k \in \mathbb{N}\}$ and let $I = \{u \in A_N^+ \mid (\exists x_i \in A_N)\, |x_i|_u \geq 2\}$. Then I is an ideal of A_N^+. By $\mathbb{D}_N$ we will denote the factor semigroup $(A_N^+)/I$. It is clear that D_N is isomorphic to the semigroup

$$(\{u \in A_N^+ | \Pi(u) = c(u)\} \cup \{0\}, \cdot),$$

where the multiplication "$\cdot$" is defined by

$$u \cdot v = \begin{cases} uv & \text{if } u, v \neq 0 \text{ and } c(u) \cap c(v) = \varnothing \\ 0 & \text{otherwise} \end{cases}.$$

$\mathbb{D}_N$ is a nil-semigroup and it is not nilpotent.

The *principal twosided* (resp. *left*, *right*) *ideal* of a semigroup (ring) S generated by an element $a \in S$ will be denoted by (a) (resp. $(a)_L$, $(a)_R$). The *Green's relations*

$\mathcal{J}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a semigroup S are defined by

$$a\mathcal{L}b \;\Leftrightarrow\; (a)_L = (b)_L; \qquad a\mathcal{R}b \;\Leftrightarrow\; (a)_R = (b)_R;$$
$$a\mathcal{J}b \;\Leftrightarrow\; (a) = (b); \quad \mathcal{H} = \mathcal{L}\cap\mathcal{R}, \quad \mathcal{D} = \mathcal{L}\mathcal{R},$$

where $a, b \in S$. The *division relations* $\mid$, $\underset{l}{\mid}$, $\underset{r}{\mid}$ and $\underset{t}{\mid}$ on a semigroup S are defined by

$$a \underset{l}{\mid} b \;\Leftrightarrow\; b \in (a)_L, \qquad a \underset{r}{\mid} b \;\Leftrightarrow\; b \in (a)_R,$$
$$a \mid b \;\Leftrightarrow\; b \in (a), \qquad \underset{t}{\mid} = \underset{l}{\mid} \cap \underset{r}{\mid},$$

and the relations $\xrightarrow{l}$, $\xrightarrow{r}$, $\xrightarrow{t}$ and $\longrightarrow$ on S are defined by

$$a \xrightarrow{l} b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \underset{l}{\mid} b, \qquad a \xrightarrow{r} b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \underset{r}{\mid} b,$$
$$a \xrightarrow{t} b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \underset{t}{\mid} b, \qquad a \longrightarrow b \;\Leftrightarrow\; (\exists n \in \mathbb{N})\; a^n \mid b,$$

for $a, b \in S$.

If a semigroup T is a homomorphic image of a subsemigroup T' of a semigroup S, then we say that T *divides* S *through* T'. If the intersection of all ideals of a semigroup S is non-empty, then it is an ideal of S called the *kernel* of S. With respect to set-theoretical union and intersection, the set of all left ideals of a semigroup S, with the empty set included, is a lattice and it is denoted by $\mathcal{LI}d(S)$. By a *discrete partially ordered set* we mean a partially ordered set in which any two elements are incomparable. An element of a semigroup (ring) S is called *central* if it commutes with any element of S, and the set of all central elements of S is called the *center* of S. A ring without non-zero nilpotent elements is called a *reduced ring*.

For undefined notions and notations we refer to the books **[36]**, **[48]**, **[105]**, **[106]**, **[128]**, **[144]**, **[147]**, **[153]**, **[195]**, **[210]**, **[241]**, **[243]**, **[245]**, **[246]**, **[247]**, **[270]**, **[291]**, **[292]**, **[301]** and **[313]**.

1.2. Everett's sums of rings. In this section we talk about the general problem of ideal extensions of rings. This problem is formulated in the following way: Given rings A and B, construct all ideal extensions of a ring A by a ring B, i.e. construct all rings R having the property that A is an ideal of R and the factor ring R/A is isomorphic to B. A solution of this problem was given by Everett in **[113]**, 1942, and is referred here as the Everett's theorem.

The original version of the Everett's theorem can be found in the book of Rédei **[270]**, 1961. The version which will be given here due to Müller and Petrich **[217]**, 1971. The Everett's construction, given in such a version, is a combination of the well-known Schreier's construction of all extensions of a group by another, and the construction of all ideal extensions of a semigroup by a semigroup with zero, due to Yoshida [**348**], 1965. Namely, as in the group case, one chooses a system of representatives of the cosets of A in R, and as in the semigroup case, one makes a bitranslation of A by any of these representatives. Moreover, because the representatives are chosen in different cosets, two "factor systems", one for addition and one for multiplication, have to be introduced. For more information concerning Schreier's extensions of groups we refer to Hall [**131**] and Rédei [**270**], 1961, and for more information about ideal extensions of semigroups we refer to the survey article written by Petrich [**240**], 1970, and the book of the same author [**241**], 1973.

To present the Everett's construction we need the notion of a translational hull of a ring. The translational hull occurs naturally when one is concerned with a construction of ideal extensions of semigroups, and seeing that ring extensions can be treated as their particular case, it appears also in ring theory, with the necessary modification that all the functions in the definition be additive.

Let R be a ring. An endomorphism λ (ϱ) of the additive group of R, written on the left (right), is a *left* (*right*) *translation* of R if $\lambda(xy) = (\lambda x)y$ ($(xy)\varrho = x(y\varrho)$), for all $x, y \in R$. A left translation λ and a right translation ϱ of R are *linked* if $x(\lambda y) = (x\varrho)y$, for all $x, y \in R$, and in such a case the pair (λ, ϱ) is called a *bitranslation* of R. It is sometimes convenient to consider a bitranslation (λ, ϱ) as a bioperator denoted by a single letter, say π, which acts as λ, if it is written on the left, and as ϱ, if it is written on the right, i.e. $\pi x = \lambda x$ and $x\pi = x\varrho$, for $x \in R$. For any $a \in R$, the *inner left* (*right*) *translation* induced by a is the mapping λ_a (ϱ_a) of R into itself defined by $\lambda_a x = ax$ ($x\varrho_a = xa$), for $x \in R$, and the pair $\pi_a = (\lambda_a, \varrho_a)$ is called the *inner bitranslation* of R induced by a.

A left translation λ and a right translation ϱ of a ring R are *permutable* if $(\lambda x)\varrho = \lambda(x\varrho)$, for all $x \in R$, and a set T of bitranslations of R is *permutable* if for all $(\lambda, \varrho), (\lambda', \varrho') \in T$, λ and ϱ' are permutable.

The set $\Lambda(R)$ ($\mathrm{P}(R)$) of all left (right) translations of a ring R is a ring under the addition and the multiplication defined by:

$$(\lambda + \lambda')x = \lambda x + \lambda' x \qquad (x(\varrho + \varrho') = x\varrho + x\varrho'),$$
$$(\lambda\lambda')x = \lambda(\lambda' x) \qquad (x(\varrho\varrho') = (x\varrho)\varrho'),$$

for $\lambda, \lambda' \in \Lambda(R)$ ($\varrho, \varrho' \in \mathrm{P}(R)$) and $x \in R$. The subring $\Omega(R)$ of the direct sum of rings $\Lambda(R)$ and $\mathrm{P}(R)$, consisting of all bitranslations of R, is called the *translational hull* of R. More information about translational hulls of rings and semigroups can be found in [**240**] and [**241**].

Theorem 1.1. (Everett's theorem) *Let A and B be disjoint rings. Let θ be a function of B onto a set of permutable bitranslations of A, in notation $\theta : a \mapsto \theta^a \in \Omega(A)$, $a \in B$, and let $[,]$, $\langle,\rangle : B \times B \to A$ be functions such that for all $a, b, c \in B$ the following conditions hold:*

(E1) $\theta^a + \theta^b - \theta^{a+b} = \pi_{[a,b]}$;
(E2) $\theta^a \cdot \theta^b - \theta^{ab} = \pi_{\langle a,b\rangle}$;
(E3) $\langle ab, c\rangle + \langle a, b\rangle\theta^c = \langle a, bc\rangle + \theta^a\langle b, c\rangle$;

(E4) $[0,0] = 0$;
(E5) $[a,b] = [b,a]$;
(E6) $[a,b] + [a+b,c] = [a,b+c] + [b,c]$;
(E7) $[a,b]\theta^c + \langle a+b,c\rangle = [ac,bc] + \langle a,c\rangle + \langle b,c\rangle$;
(E8) $\theta^a[b,c] + \langle a,b+c\rangle = [ab,ac] + \langle a,b\rangle + \langle a,c\rangle$.

Define an addition and a multiplication on $R = A \times B$ *by:*

(E9) $(\alpha,a) + (\beta,b) = (\alpha+\beta+[a,b], a+b)$;
(E10) $(\alpha,a)\cdot(\beta,b) = (\alpha\beta + \langle a,b\rangle + \theta^a\beta + \alpha\theta^b, ab)$,

$\alpha,\beta \in A$, $a,b \in B$. *Then* $(R,+,\cdot)$ *is a ring isomorphic to an ideal extension of* A *by* B.

Conversely, every ideal extension of A *by* B *can be so constructed.*

A ring constructed as in the Everett's theorem we call an *Everett's sum* of rings A and B *by a triplet* $(\theta;[,];\langle,\rangle)$ of functions and we denote it by $E(A,B;\theta;[,];\langle,\rangle)$. The representation of a ring R as an Everett's sum of some rings we call an *Everett's representation* of R.

More information about the Everett's theorem can be found in **[240]** and **[270]**. There we can see that an Everett's representation $E(A,B;\theta;[,];\langle,\rangle)$ of some ring R is determined by the choice of a set of representatives of the cosets of A in R. Namely, if for every coset $a \in B$ we choose a representative, in notation a', then the set $\{a' \mid a \in B\}$ determines the triplet $(\theta;[,];\langle,\rangle)$ in the following way:

(E11) $\alpha\theta^a = \alpha\cdot a', \theta^a\alpha = a'\cdot\alpha, \quad \alpha \in A,\ a \in B$;
(E12) $[a,b] = a' + b' - (a+b)', \quad a,b \in B$;
(E13) $\langle a,b\rangle = a'\cdot b' - (a\cdot b)', \quad a,b \in B$.

Although an Everett's representation of a ring is determined by the choice of representatives of the related cosets, for any such choice we obtain equivalent Everett's sums. The precise conditions under which two Everett's sums are equivalent were given by Müller and Petrich in **[217]**, 1971, by the following theorem:

Theorem 1.2. *Two Everett's sums* $E(A,B;\theta;[,];\langle,\rangle)$ *and* $E(A,B;\theta';[,]';\langle,\rangle')$ *of rings* A *and* B *are equivalent if and only if there exists a mapping* $\xi : B \to A$ *such that* $0\xi = 0$ *and for all* $a,b \in B$ *the following conditions hold:*

(a) $(\theta')^b = \theta^b + \pi_{b\xi}$;
(b) $[a,b]' = [a,b] + a\xi + b\xi - (a+b)\xi$;
(c) $\langle a,b\rangle' = \langle a,b\rangle + \theta^a(b\xi) + (a\xi)\theta^b + (a\xi)(b\xi) - (ab)\xi$.

Let $n \in \mathbb{N}$ and let $w \in A_n^+$. If $X_1, X_2, \dots, X_n$ are sets, then we will denote by $w(X_1,X_2,\dots,X_n)$ the set obtained by replacement of letters $x_1,x_2,\dots,x_n$ in w by sets $X_1,X_2,\dots,X_n$, respectively, considering the Cartesian multiplication of sets instead of the juxtapositions in w. Let R be a ring, let P be a set of permutable bitranslations of R and let μ be an element of the Cartesian n-th power of $R \cup P$. If at least one projection of μ is in R, then $w(\mu)$ will denote the element of R obtained by replacement of any letter x_i, $i \in \{1,2,\dots,n\}$, by the i-th projection of μ, considering the multiplications in $\mathcal{M}R$ and $\mathcal{M}\Omega(R)$ and acting of bitranslations from P on elements of R, instead of the juxtapositions in w. Otherwise, if all the projections of μ are in P, then $w(\mu)$ will denote the value of w in the semigroup $\mathcal{M}\Omega(R)$, for the valuation μ.

The following theorem, given by Ćirić, Bogdanović and Petković in [94], 1995, describes more complicated multiplications in Everett's sums of rings.

Theorem 1.3. *Let $R = E(A, B; \theta; [,]; \langle,\rangle)$, let $n \in \mathbb{N}$, $n \geq 2$, and assume that $w = w(x_1, \dots, x_n) \in A_n^+$, $|w| = k$, $a = (a_1, \dots, a_n) \in B^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in A^n$, $\xi_i = (\alpha_i, a_i)$, $i \in \{1, \dots, n\}$, $\xi = (\xi_1, \dots, \xi_n)$, and $\theta^a = (\theta^{a_1}, \dots, \theta^{a_n})$. Then for*

$$\beta = \sum_{j=1}^{k-2} \langle (l_j(w))(a), (c_{j+1}(w))(a) \rangle (r_{k-j-1}(w))(\theta^a) + \langle (l_{k-1}(w))(a), (t(w))(a) \rangle,$$

the following statements hold:

$$\text{(i)}\ \ w(\theta^a) = \theta^{w(a)} + \pi_\beta, \qquad \text{and} \qquad \text{(ii)}\ \ w(\xi) = \left(\sum_{\mu \in M_w} \Pi_k(\mu) + \beta, w(a) \right),$$

where $M_w = w(X_1, \dots, X_n) - \{\theta^a\}$, $X_i = \{\alpha_i, a_i\}$, $i \in \{1, \dots, n\}$.

Furthermore, if $\theta^b A \theta^c = 0$, for all $b, c \in B$ and if $k \geq 3$, then

$$\beta = \langle (h(w))(a), (m_2^{k-1}(w))(a) \rangle \theta^{(t(w))(a)} + \langle (l_{k-1}(w))(a), (t(w))(a) \rangle.$$

There are many known constructions in Theory of rings which are special cases of Everett's sums. For example, the well known *split extension* of rings is in fact an Everett's sum of rings in which the functions $[,]$ and $\langle,\rangle$ are zero functions, i.e. $[a, b] = \langle a, b \rangle = 0$, for all a, b. In such a way we obtain also the well-known *Dorroh extension* of a ring by a ring of integers, which realizes an embedding of a ring into a ring with unity.

An interesting specialization of Everett's sums was given by Ćirić and Bogdanović in [80], 1990. An Everett's sum $E(A, B; \theta; [,]; \langle,\rangle)$ was called by them a *strong Everett's sum* if θ is a zero homomorphism of B into $\Omega(A)$, i.e. if $\theta^a = \pi_0$, for any $a \in B$. Such an Everett's sum is denoted by $E(A, B; [,]; \langle,\rangle)$, and a representation of a ring R by such an Everett's sum is called a *strong Everett's representation* of R. A ring R is called a *strong extension* of a ring A by a ring B if there exists a strong Everett's representation $R = E(A, B; [,]; \langle,\rangle)$.

Using the concept of strong extensions of rings, Ćirić and Bogdanović in [80], 1990, gave the following construction of nilpotent rings:

Theorem 1.4. *Let $n \in \mathbb{N}$, $n \geq 2$. A ring R is an $(n+1)$-nilpotent ring if and only if it is a strong extension of a null-ring by an n-nilpotent ring.*

Recall that by a null-ring we mean a 2-nilpotent ring.

The same authors investigated also some other strong extensions of rings, and some of the obtained results will be presented in the next sections. Here we will give only some general properties of strong extensions.

Theorem 1.5. *Any strong extension of a ring by a ring with identity is isomorphic to their direct sum.*

The previous result was obtained by Ćirić and Bogdanović in [**80**], 1990, who also stated the following problem: Is any strong extension of two rings isomorphic to their direct sum?

An example of an Everett's sum of two rings which is not equivalent to a strong Everett's sum of these rings is the following: Let $n \in \mathbb{N}$, $n \geq 2$, and let R be the ring of all $n \times n$ upper triangular matrices over a field F. The set N of nilpotents of R is the set of all matrices (a_{ij}) from R for which $a_{ij} = 0$, whenever $i \geq j$, and we have that N is an ideal of R, the factor ring R/N is isomorphic to the ring F^n, and by the previous theorem, R cannot be a strong extension of N by F^n.

Note that the previous theorem is similar to the following well-known result:

Theorem 1.6. *Let A be a ring with an identity. Then a ring R is an ideal extension of A if and only if A is a direct summand of R.*

This theorem is in fact an immediate consequence of the result given by Ćirić and Bogdanović in [**80**], 1990, concerning retractive extensions of rings. A subring A of a ring R is called a *retract* of R if there exists a homomorphism φ of R onto A such that $a\varphi = a$, for any $a \in A$. Such a homomorphism is called a *retraction* of R onto A. If R is an ideal extension of A and there exists a retraction of R onto A, we say that R is a *retractive extension* of A and that A is a *retractive ideal* of R.

Theorem 1.7. *A ring R is a retractive ideal of a ring R if and only if A is a direct summand of R.*

Note that any ideal A with an identity of a ring R is a retract of R. Namely, a retraction φ of R onto A is given by $x\varphi = xe$, where $x \in R$ and e is an identity of A.

More information concerning retractions of semigroups will be given in Section 5.

2. On π-regular semigroups and rings

In this section we present the main properties of regular and π-regular semigroups and rings.

2.1. The regularity in semigroups and rings. The regularity was first defined in Ring theory by von Neumann in [**224**], 1936, and after that this definition was naturally transmitted in Semigroup theory. By this definition, an element a of a ring (semigroup) R is a *regular element* if there exists $x \in R$ such that $a = axa$, and a ring (semigroup) is defined to be a *regular ring* (*regular semigroup*) if all its elements are regular. Thierrin, who first investigated some general properties of regular semigroups in [**322**], 1951, called them *inversive semigroups* (*demi-groupes* *inversifs*). The set of all regular elements of a semigroup (ring) S we call the *regular part* of S and we denote it by $\mathrm{Reg}(S)$.

Many very important kinds of rings are regular. For example, such a property have division rings, the full matrix ring over a division ring, the ring of linear transformations of a vector space over a division ring, and many other rings. This also holds for many significant concrete semigroups. For example, the full transformation semigroup of an arbitrary finite set is regular, and the statement that the full transformation semigroup of a set X is regular for any set X is equivalent to the famous Axiom of Choice. For more information about general properties of regular rings and semigroups we refer to the books: Goodearl [**128**], Steinfeld [**301**], Petrich [**245**] and others. Here we give only some their properties which we need in the further work.

Theorem 2.1. *The following conditions on a semigroup (ring) S are equivalent:*

(i) *S is regular;*
(ii) *$A \cap B = BA$, for any left ideal A and any right ideal B of S;*
(iii) *any one-sided ideal of S is globally idempotent and BA is a quasi-ideal of S, for any left ideal A and any right ideal B of S;*
(iv) *any principal left (right) ideal of S has an idempotent generator.*

The equivalence of conditions (i) and (ii) was established by Iséki in [**145**], 1956, for semigroups, and Kovács in [**160**], 1956, for rings. Similar characterizations of regular elements by principal one-sided ideals, and related characterizations of regular semigroups and rings, were given by Lajos in [**164**], 1961, for semigroups, and Szász in [**308**], 1961, for rings. For many information on other interesting properties of two-sided, one-sided, quasi- and bi-ideals of regular semigroups and rings we refer to the book of Steinfeld [**301**], 1978.

The equivalence of conditions (i) and (iii) was proved by Calais in [**67**], 1961, for semigroups, and by Steinfeld in [**301**], 1978, for rings. Finally, (i) $\Leftrightarrow$ (iv) was proved by von Neuman in [**224**], 1936 (see also Clifford and Preston [**105**], 1961).

If a is a regular element of a semigroup (ring) S, then the element x, whose existence was postulated by the definition of the regularity, can be chosen such that $a = axa$ and $x = xax$, and any element x satisfying this condition, which is not necessary unique, is called an *inverse* of a. This property of regular elements was first observed by Thierrin in [**323**], 1952. A regular semigroup (ring) whose any element has a unique inverse is called an *inverse semigroup* (*inverse ring*). Inverse semigroups were first defined and investigated by Vagner in [**335**], 1952, and [**337**], 1953, who called them *generalized groups*, and independently by Preston in [**252**], [**253**], [**254**], 1954. The most significant example of inverse semigroups is the semigroup of partial one-to-one mappings of a set X into itself, and is called the *symmetric inverse semigroup* on X. Just as any group can be embedded in a symmetric group, by the Cayley theorem, and any semigroup can be embedded in a full transformation semigroup, so every inverse semigroup can be embedded into a symmetric inverse semigroup. This result is due to Vagner [**332**], 1952, and Preston [**254**], 1954, and is known as the Vagner-Preston Representation Theorem.

For more information on inverse semigroups we refer to the books of Howie [**144**, Chapter V], 1976, and Petrich [**247**], 1984. Here we quote only some characterizations of these semigroups that we need in the further work.

Theorem 2.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is inverse;*
(ii) *S is regular and the idempotents of S commute;*
(iii) *any principal one-sided ideal of S has a unique idempotent generator.*

The implication (ii) $\Rightarrow$ (iii) was proved by Vagner in [**335**], 1952, and independently by Preston in [**252**], 1954, (i) $\Rightarrow$ (ii) was proved by Liber in [**197**], 1954, whereas the equivalence of all three conditions was proved by Munn and Penrose in [**219**], 1955.

A natural generalization of inverse semigroups was given by Venkatesan in [**338**], 1974, who defined a regular semigroup (ring) to be a *left inverse* (resp. *right inverse*) *semigroup* (*ring*) if for all $a, x, y \in S$, $a = axa = aya$ implies $ax = ay$ (resp. $a = axa = aya$ implies $xa = ya$). Left inverse semigroups are characterized by the following theorem:

Theorem 2.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is left inverse;*
(ii) *S is regular and $E(S)$ is a left regular band;*
(iii) *any principal left ideal of S has a unique idempotent generator.*

Another important kind of the regularity was introduced by Clifford in [**99**], 1941, who studied elements a of a semigroup S having the property that there exists $x \in S$ such that $a = axa$ and $ax = xa$, which we call now *completely regular elements*, and semigroups whose any element is completely regular, called *completely regular semigroups*. The complete regularity was also investigated by Croisot in [**107**], 1953, who also studied elements a of a semigroup S for which $a \in Sa^2S$ (resp. $a \in Sa^2$, $a \in a^2S$), called *intra-regular* (resp. *left regular*, *right regular*) *elements*, and semigroups whose every element is intra-regular (resp. left regular, right regular), called *intra-regular* (resp. *left regular*, *right regular*) *semigroups*. Analogously we define intra-, left, right and completely regular rings and elements of rings. As we will see in Section 4, the concepts of the left, right and completely regular rings coincide, and in Ring theory such rings are known under the names *strongly regular* and *Abelian regular rings*. The results of A. H. Clifford and R. Croisot from the above mentioned papers concerning intra-, left, right and completely regular semigroups will be also presented in Section 4. Here we give only some their results which characterizes completely regular elements of a semigroup:

Theorem 2.4. *The following conditions for an element a of a semigroup S are equivalent:*

(i) *a is completely regular;*

(ii) *a has an inverse which commutes with a;*
(iii) *a is contained in a subgroup of S;*
(iv) *a is left regular and right regular.*

In view of the previous theorem, completely regular elements are often called *group elements*, and the set of all completely regular elements of a semigroup (ring) S is denoted by $\mathrm{Gr}(S)$ and is called the *group part* of S. For any idempotent e of a semigroup S, $G_e = \{a \in S \mid a \in eS \cap Se,\ e \in aS \cap Sa\}$ is the *maximal subgroup* of S having e as its identity, and $\mathrm{Gr}(S)$ is a disjoint union of all maximal subgroups of S. The existence of maximal subgroups was established by Schwarz in [**278**], 1943, for periodic semigroups, and by Wallace in [**340**], 1953, and Kimura in [**155**], 1954, for an arbitrary semigroup. The sets of all left, right and intra-regular elements of a semigroup (ring) S are called the *left regular*, *right regular* and *intra-regular part* of S, and are denoted by $\mathrm{LReg}(S)$, $\mathrm{RReg}(S)$ and $\mathrm{Intra}(S)$, respectively.

For any pair $m, n \in \mathbb{N}^0$, $m + n > 1$, Croisot in [**107**], 1953, also defined an element a of a semigroup S to be (m, n)-*regular* if $a \in a^m S a^n$, where a^0 denotes the identity adjoined to S. He proved that for all $m, n \geq 2$, the $(m, 0)$-regularity is equivalent to the right regularity and the $(0, n)$-regularity is equivalent to the left regularity, and for all $m, n \in \mathbb{N}$ for which $m + n \geq 3$, the (m, n)-regularity of a semigroup is equivalent to the complete regularity. As we see, the intra-regularity is not included in this Croisot's concept. But, by Lajos and Szász in [**192**], 1975, for $p, q, r \in \mathbb{N}^0$, an element a of a semigroup S was defined to be (p, q, r)-*regular* if $a \in a^p S a^q S a^r$, and a semigroup S was defined to be a (p, q, r)-*regular semigroup* if any its element is (p, q, r)-regular. This definition obviously includes the intra-regularity and many other interesting concepts. For example, this definition includes the concept of quasi-regularity introduced by Calais in [**67**], 1961, as a generalization of the ordinary regularity, seeing that by Theorem 2.1, in a regular semigroup (ring) any its one-sided ideal is globally idempotent. Namely, J. Calais defined a semigroup (ring) to be *left quasi-regular* (resp. *right quasi-regular* if any its left ideal (resp. right ideal) is globally idempotent, and to be *quasi-regular* if it is both left and right quasi-regular. The corresponding definitions can be given for elements: an element a of a semigroup (ring) S is called *left quasi-regular* (resp. *right quasi-regular*) if the principal left ideal $(a)_L$ (resp. the principal right ideal $(a)_R$) generated by a is globally idempotent, and is called *quasi-regular* if it is both left and right quasi-regular. It is easy to see that a semigroup (ring) is (left, right) quasi-regular if and only if any its element is (left, right) quasi-regular. As Lajos and Szász proved in [**192**], 1975, the left quasi-regular and the right quasi-regular elements of a semigroup S are exactly the $(0, 1, 1)$-regular and the $(1, 1, 0)$-regular elements of S, respectively.

Note that this concept of quasi-regularity differs to the well-known concept of quasi-regularity of elements of rings which is used in the definition of the Jacobson radical of a ring.

2.2. The π-regularity in semigroups and rings. In order to give a generalization both of regular rings and of algebraic algebras and rings with minimum conditions on left or right ideals, Arens and Kaplansky in [**11**], 1948, and Kaplansky in [**150**], 1950, defined π-regular rings. Following their terminology, an element a of a semigroup (ring) S is called *π-regular* (resp. *left π-regular*, *right π-regular*, *completely π-regular*, *intra-π-regular*) if some its power is regular (resp. left regular, right regular, completely regular, intra-regular), and S is called a *π-regular* (resp. *left π-regular*, *right π-regular*, *completely π-regular*, *intra-π-regular*) if any its element is π-regular (resp. left π-regular, right π-regular, completely π-regular, intra-π-regular). In some origins several other names were used. For example, Putcha in [**255**], 1973, Galbiati and Veronesi in [**121**]–[**125**], Shum, Ren and Guo in [**289**], [**290**], [**272**] and [**273**], and others called π-regular semigroups *quasi regular*, whereas Edwards in [**112**], 1993, called them *eventually regular*. Completely π-regular semigroups were sometimes called *quasi-completely regular* or *group-bound*, and Shevrin in [**285**] and [**296**], 1994, called them *epigroups*. In theory of rings, completely π-regular rings are known as *strongly π-regular rings*, as they were called by Azumaya in [**14**], 1954. In order to unify the terminology used in this paper, we use the name completely π-regular both for semigroups and rings.

Some variations of the π-regularity were also investigated by Fuchs and Rangaswamy in [**119**], 1968. For a positive integer m, they called an element a of a semigroup (ring) S *m-regular* if the power a^m is regular, and *$\overline{m}$-regular*, if a^n is regular for any $n \geq m$, and S is called an *m-regular* (resp. *$\overline{m}$-regular*) *semigroup* (*ring*) if any its element is m-regular (resp. $\overline{m}$-regular). Clearly, an element a is π-regular if and only if it is m-regular for some $m \in \mathbb{N}$. If for an element a of a semigroup (ring) S there exists $m \in \mathbb{N}$ such that a is $\overline{m}$-regular, we then say that a is *$\overline{\pi}$-regular*, and a semigroup (ring) whose any element is $\overline{\pi}$-regular is called a *$\overline{\pi}$-regular semigroup* (*ring*). If a is an element of a semigroup (ring) S and a^m is left (resp. right, completely) regular for some $m \in \mathbb{N}$, then a^n is left (resp. right, completely) regular for any $n \geq m$.

Some relationships between the π-regularity, left π-regularity, right π-regularity, complete π-regularity and intra-π-regularity were investigated by many authors. We give here the most important results concerning these relationships. The first theorem that we give was proved by Bogdanović and Ćirić in [**55**], 1996:

Theorem 2.5. *A semigroup S is left π-regular if and only if it is intra-π-regular and* $\mathrm{Intra}(S) = \mathrm{LReg}(S)$.

By this theorem we obtain the following interesting result:

Theorem 2.6. *If S is a completely π-regular semigroup, then*

$$\mathrm{Gr}(S) = \mathrm{LReg}(S) = \mathrm{RReg}(S) = \mathrm{Intra}(S) \subseteq \mathrm{Reg}(S).$$

Note that there exists a completely π-regular semigroup S in which $\mathrm{Gr}(S)$ is a proper subset of $\mathrm{Reg}(S)$. Completely π-regular semigroups whose regular part coincide with the group part will be considered in Section 5.

Another theorem gives some connections between the complete π-regularity, π-regularity and left (or right) π-regularity:

Theorem 2.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely π-regular;*

(ii) *S is left and right π-regular;*

(iii) *S is π-regular and left (or right) π-regular;*

(iv) *for any $a \in S$ there exists $n \in \mathbb{N}$ such that a^n is regular and left (or right) regular.*

The equivalence of conditions (i) and (iv) was proved by Hongan in [**143**], 1986, and of (i) and (iii) by Bogdanović and Ćirić in [**44**], 1992.

For rings a more rigorous theorem holds:

Theorem 2.8. *The following conditions on a ring R are equivalent:*

(i) *R is left π-regular;*

(ii) *R is right π-regular;*

(iii) *R is completely π-regular.*

This very important theorem was proved by Dischinger in [**108**], 1976, and another proof was given by Hirano in [**139**], 1978.

Clearly, any completely π-regular ring is π-regular. Various conditions under which a π-regular ring is completely regular were investigated by many authors. The best known results from this area are the results obtained by Azumaya in [**14**], 1954. He investigated rings in which the indices of nilpotency of all nilpotent elements are bounded, called the *rings of bounded index* and he proved the following two theorems:

Theorem 2.9. *If R is a ring of bounded index, then*

$$\mathrm{RReg}(R) = \mathrm{LReg}(R) = \mathrm{Gr}(R).$$

Theorem 2.10. *Let R be a ring of bounded index. Then R is π-regular if and only if it is completely π-regular.*

In connection with the π-regularity, rings of bounded index were also investigated by Tominaga in [**329**], 1955, and Hirano in [**140**], 1990.

As known, Moore in [**215**], 1936, Penrose in [**234**], 1955, and Rado in [**264**], 1956, introduced the notion of a generalized inverse of a matrix. Namely, by a result obtained by Moore, but stated in a more convenient form by Penrose, for any square complex matrix a there exists a unique complex matrix x such that $axa = a$, $xax = x$ and both ax and xa are hermitian. Such a matrix x is called the *generalized inverse*, or the *Moore-Penrose inverse*, of a. In order to give a further generalization of generalized inverses, Drazin introduced in [**110**], 1958, the following notion: Given a semigroup (ring) S and an element $a \in S$. An element $x \in S$ is called the *pseudo-inverse*, or the *Drazin inverse*, of a, if $ax = xa$, $x^2a = x$ and there exists $m \in \mathbb{N}$ such that $a^m = a^{m+1}x$. An element having a pseudo-inverse is called *pseudo-invertible*, and also, a semigroup (ring) whose any element is pseudo-invertible is called a *pseudo-invertible semigroup* (*ring*). As was shown by Drazin, a pseudo-inverse of an element a, if it exists, is unique. He also proved the following:

Theorem 2.11. *An element a of a semigroup (ring) S is pseudo-invertible if and only if it is completely π-regular.*

Let us note that an element a of a semigroup S is completely π-regular if and only if there exists $n \in \mathbb{N}$ such that the power a^n lies in some subgroup of S (see Theorem 2.4). The next theorem, proved by Drazin in [**110**], 1958, and in a slightly simplified form by Munn in [**218**], 1961, and known in Theory of semigroups as the *Munn's lemma*, gives an interesting property of such elements:

Theorem 2.12. *Let a be an element of a semigroup S such that for some $n \in \mathbb{N}$, a^n belongs to some subgroup G of S, and let e be the identity of this group. Then $ea = ae \in G_e$ and $a^m \in G_e$, for each integer $m \geq n$.*

Using the previous two theorems, pseudo-inverses can be represented in another way. Namely, if a is a pseudo-invertible, or equivalently, a completely π-regular element of a semigroup S, then $a^n \in G_e$, for some $n \in \mathbb{N}$ and $ae \in G_e$, and then the pseudo-inverse x of a is given by $x = (ae)^{-1}$, i.e. x is the ***group inverse*** of the element ae in the group G_e. If a is an element of a completely π-regular semigroup S and $a^n \in G_e$, for some $n \in \mathbb{N}$ and $e \in E(S)$, then a^0 denotes the identity of G_e, i.e. $a^0 = e$.

An interesting characterization of completely π-regular rings was given by Ôhori in [**229**], 1985. Before we exhibit this result, we must introduce some new notions. These notions were introduced by Hirano, Tominaga and Yaqub in [**142**], 1988, but they are given here in a slightly modified form. Let A and B be two subsets of a ring R. We say that R is *(A, B)-representable* if for any $x \in R$ there exist $a \in A$ and $b \in B$ such that $x = a + b$, and that it is *uniquely (A, B)-representable* if for any $x \in X$ there exist unique $a \in A$ and $b \in B$ such that $x = a + b$. Similarly, we say that R is *$[A, B]$-representable* if for any $x \in R$ there exist $a \in A$ and $b \in B$ such that $x = a + b$ and $ab = ba$, and that it is *uniquely $[A, B]$-representable* if for any $x \in R$ there exist unique $a \in A$ and $b \in B$ such that $x = a + b$ and $ab = ba$. Clearly, any uniquely (A, B)-representable ring is (A, B)-representable, any uniquely $[A, B]$-representable ring is $[A, B]$-representable, and all these rings are (A, B)-representable.

The characterization of completely π-regular rings given by Ôhori in [**229**], 1985, is the following:

Theorem 2.13. *A ring R is completely π-regular if and only if it is $[\mathrm{Nil}(R), \mathrm{Gr}(R)]$-representable.*

In order to generalize the concept of an inverse semigroup, Galbiati and Veronesi defined in [**120**], 1980, a semigroup (and also ring) to be ***π-inverse*** if it is π-regular and any its regular element has a unique inverse. A further generalization of these concept was given by Bogdanović in [**35**], 1984, who defined a semigroup (or ring) S to be *left* (resp. *right*) *π-inverse* if it is π-regular and for all $a, x, y \in S$, $a = axa = aya$ implies $ax = xa$ (resp. $a = axa = aya$ implies $xa = ya$).

Similarly, a semigroup (ring) S is called *completely π-inverse* (resp. *left completely π-inverse*, *right completely π-inverse*) if it is completely π-regular and π-inverse (resp. left π-inverse, right π-inverse).

The following theorem, which characterizes left π-inverse semigroups, was proved by Bogdanović in [**35**], 1984:

Theorem 2.14. *The following conditions on a semigroup S are equivalent:*

(i) *S is left π-inverse;*
(ii) *S is π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (ef)^n e$;*
(iii) *S is π-regular and for any pair $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n \mathcal{L} (fe)^n$;*
(iv) *for any $a \in S$ there exists $n \in \mathbb{N}$ such that $(a^n)_L$ has a unique idempotent generator.*

A consequence of the previous theorem and its dual is the following result obtained by Galbiati and Veronesi in [**120**], 1980, and Bogdanović in [**33**], 1982, and [**35**], 1984.

Theorem 2.15. *The following conditions on a semigroup S are equivalent:*

(i) *S is π-inverse;*
(ii) *S is left and right π-inverse;*
(iii) *S is π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (fe)^n$.*
(iv) *S is π-regular and for any $a \in S$ there exists $n \in \mathbb{N}$ such that $(a^n)_L$ and $(a^n)_R$ have unique idempotent generators.*

Left completely π-inverse semigroups were studied by Bogdanović and Ćirić in [**44**], 1992, where the following result was obtained:

Theorem 2.16. *A semigroup S is left completely π-inverse if and only if it is π-regular and for all $a \in S$, $e \in E(S)$, there exists $n \in \mathbb{N}$ such that $(ea)^n = (ea)^n e$.*

Finally, completely π-inverse semigroups are characterized by the following theorem, due to Galbiati and Veronesi [**124**], 1984.

Theorem 2.17. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely π-inverse;*
(ii) *S is left and right completely π-inverse;*
(iii) *S is π-regular and for all $a \in S$, $e \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ea)^n = (ae)^n$.*

2.3. Periodic semigroups and rings. Periodic semigroups and rings are among the most important special types of completely π-regular semigroups and rings. They are defined as semigroups (rings) in which for any element a there exist different $m, n \in \mathbb{N}$ such that $a^m = a^n$, or equivalently, as semigroups (rings) in which for any element a, some power of a is an idempotent.

Periodic semigroups and rings have many very interesting properties. For example, the property "being periodic" is a hereditary property, both for semigroups and rings, and many subclasses of the class of periodic semigroups (rings) can be characterized in terms of variable identities, as we will see in Section 5. Clearly, the whole class of periodic semigroups is definable by a variable identity $\{x^m = x^n \mid m, n \in \mathbb{N}, m \neq n\}$ over the one-element alphabet. Also, all finite semigroups and rings are periodic, and the periodicity was often investigated as a generalization of the finiteness.

An element a of a semigroup (ring) S having the property that $a^m = a^n$, for some different $m, n \in \mathbb{N}$, will be called a *periodic element*. An interesting type of periodic elements of a semigroup (ring) are potent elements defined as follows: an element a of a semigroup (ring) S is *potent* if $a = a^n$, for some $n \in \mathbb{N}$, $n \geq 2$. The set of all potent elements of S is denoted by $P(S)$ and called the *potent part* of S.

Periodic rings have especially interesting properties. The next theorem, which is due to Chacron [**68**], 1969, gives a criterion of periodicity of rings, known as the *Chacron's criterion of the periodicity.*

Theorem 2.18. *A ring R is periodic if and only if for any $a \in R$ there exists $n \in \mathbb{N}$ and a polynomial $p(x)$ with integer coefficients such that $a^n = a^{n+1}p(a)$.*

Another proof of this theorem can be found in Bell [**19**], 1980.

The following properties of periodic rings were found by Bell in [**18**], 1977.

Theorem 2.19. *Let R be a periodic ring. Then the following conditions hold:*

(a) *for any $a \in R$ there exists $n \in \mathbb{N}$ such that $a - a^n \in \mathrm{Nil}(R)$;*
(b) *R is $(\mathrm{Nil}(R), P(R))$-representable;*
(c) *if I is an ideal of R and $a+I$ is a non-zero nilpotent of R/I, then R contains a nilpotent element u such that $a \equiv u \pmod I$.*

By Grosen, Tominaga and Yaqub in [**129**], 1990, rings satisfying the condition (b) of the above theorem were called *weakly periodic rings*. Therefore, the Bell's theorem asserts that any periodic ring is weakly periodic. The converse does not hold, but Ôhori in [**229**], 1985, found the conditions under which a weakly periodic rings is periodic, and this result is given here as the following theorem:

Theorem 2.20. *A ring R is periodic if and only if it is $[P(R), \mathrm{Nil}(R)]$-representable.*

3. On completely Archimedean semigroups

The topic of this paper are uniformly π-regular semigroups and rings, i.e. semigroups and rings decomposable into a semilattice of completely Archimedean semigroups, or equivalently, into a semilattice of nil-extensions of completely simple semigroups, so we must present the main properties of completely Archimedean and completely simple semigroups.

3.1. Completely simple semigroups. As known, a semigroup S having no an ideal different than the whole S is called a *simple semigroup*, and similarly, a semigroup S having no a left (resp. right) ideal different than the whole S is called a *left simple* (resp. *right simple*) *semigroup*. In other words, a semigroup S is simple (resp. left simple, right simple) if and only if $a \mid b$ (resp. $a \underset{l}{\mid} b$, $a \underset{r}{\mid} b$), for all $a, b \in S$. The first papers from Theory of semigroups were devoted exactly to these semigroups, because they are the closest generalization of groups. Namely, a semigroup is a group if and only if it is both left and right simple. By Sushkevich in [**304**], 1928, and [**305**], 1937, and Rees in [**271**], 1940, finite simple semigroups and other significant special types of simple semigroups were investigated. In this section we talk about the most important special types of these semigroups.

Semigroups which are both simple and left (resp. right) regular were called by Bogdanović and Ćirić in [**55**], 1996, *left* (resp. *right*) *completely simple*. Some characterizations of these semigroups are given by the following theorem:

Theorem 3.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is left completely simple;*
(ii) *S is simple and left π-regular;*
(iii) *S is simple and has a minimal left ideal;*
(iv) *S is a union of its minimal left ideals;*
(v) *S is a disjoint union of its principal left ideals;*
(vi) *any principal left ideal of S is a left simple subsemigroup of S;*
(vii) *any left ideal of S is right consistent;*
(viii) *S is a matrix of left simple semigroups;*
(ix) *S is a right zero band of left simple semigroups;*
(x) *$\underset{l}{\mid}$ is a symmetric relation on S;*
(xi) *$S/\mathcal{L}$ is a discrete partially ordered set;*
(xii) *$\mathcal{LId}(S)$ is a Boolean algebra;*
(xiii) $(\forall a, b \in S)\ a \in Sba$.

The equivalence of the conditions (iii), (iv), (vii) and (xiii) was proved by Croisot in [**107**], 1953, of (vi), (ix) and (xiii) by Bogdanović in [**33**], 1982, and of (i), (ii), (viii), (ix), (x), (xi) and (xiii) by Bogdanović and Ćirić in [**55**], 1996. The equivalence of the conditions (vii), (ix) and (xii) is an immediate consequence of the results of Bogdanović and Ćirić from [**53**], 1995, concerning so-called right sum decomposition of semigroups with zero. In the book of Clifford and Preston [**106**], 1967, semigroups satisfying the condition (xiii) of the above theorem were called *left stratified semigroups*.

Another important type of simple semigroups are simple semigroups having a primitive idempotent, called *completely simple semigroups*. Recall that an idempotent e of a semigroup S is called *primitive* if it is minimal in the partially ordered set of idempotents on S, i.e. if for $f \in E(S)$, $ef = fe = f$ implies $e = f$. Completely simple semigroups were first studied also by Sushkevich in [**304**], 1928, and [**305**], 1937, and Rees in [**271**], 1940, who gave the following fundamental representation theorem for these semigroups:

Theorem 3.2. *Let G be a group, let I and Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G. Define a multiplication on $S = G \times I \times \Lambda$ by:*

$$(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu).$$

Then S with so defined multiplication is a completely simple semigroup.

Conversely, any completely simple semigroup is isomorphic to some semigroup constructed in this way.

The semigroup constructed in accordance with this recipe is called the *Rees matrix semigroup of type $\Lambda \times I$ over a group G with the sandwich matrix P*, and is denoted by $M(G; I, \Lambda, P)$. The previous theorem is usually called the *Rees-Sushkevich theorem.*

Some other characterizations of completely simple semigroups are given by the following theorem:

Theorem 3.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely simple;*
(ii) *S is simple and completely π-regular;*
(iii) *S is simple and completely regular;*
(iv) *S is simple and has a minimal left ideal and a minimal right ideal;*
(v) *S is simple and has a minimal quasi-ideal;*
(vi) *S is a union of its minimal quasi-ideals;*
(vii) *S is left and right completely simple;*
(viii) *S is left (or right) completely simple and has an idempotent;*
(ix) *S is regular and all its idempotents are primitive;*
(x) *S is regular and $a = axa$ implies $x = xax$;*
(xi) *S is regular and weakly cancellative;*
(xii) $(\forall a, b \in S)\, a \in aSba$;
(xii') $(\forall a, b \in S)\, a \in abSa$;
(xiii) $\underset{t}{|}$ *is a symmetric relation on S;*
(xiv) *$S/\mathcal{H}$ is a discrete partially ordered set.*

The equivalence of conditions (i) and (iv) is from Clifford [**100**], 1948. The assertion (i) $\Leftrightarrow$ (ii) was proved by Munn in [**218**], 1961, and is known as the *Munn theorem*. For periodic semigroups this assertion was proved by Rees in [**271**], 1940. The equivalence of the conditions (iv) and (v) is a result of Schwarz from [**279**], 1951, and the equivalence of the conditions (v) and (vi) is derived from the results of Steinfeld from [**296**], 1956 (see also his book [**301**]). For the proof of the equivalence of conditions (i), (ix), (x) and (xi) we refer to the book of Petrich [**241**], 1973. The equivalence of the conditions (vii), (xii), (xii'), (xiii) and (xiv) is an immediate consequence of Theorem 3.1 and its dual.

Special types of completely simple semigroups are left, right and rectangular groups. A semigroup S is called a *rectangular group* if it is a direct product of a rectangular band and a group, and is called a *left group* (resp. ***right group***) if it is a direct product of a left zero band (resp. right zero band) and a group. Rectangular groups and left groups are characterized by the following two theorems:

Theorem 3.4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a rectangular group;*
(ii) *S is completely simple and $E(S)$ is a subsemigroup of S;*
(iii) *S is regular and $E(S)$ is a rectangular band;*
(iv) *$S \cong M(G; I, \Lambda, P)$ with $p_{\lambda i}^{-1} p_{\lambda j} = p_{\mu i}^{-1} p_{\mu j}$, for all $i, j \in I$, $\lambda, \mu \in \Lambda$.*

For the proof of this theorem we refer to the book of Petrich [**241**], 1973.

Theorem 3.5. *The following conditions on a semigroup S are equivalent:*

(i) *S is a left group;*
(ii) *S is left simple and right cancellative;*
(iii) *S is left simple and has an idempotent;*
(iv) *S has a right identity e and $e \in Sa$, for any $a \in S$;*
(v) *S is regular and right cancellative;*
(vi) *S is regular and $E(S)$ is a right zero band;*
(vii) *for all $a, b \in S$, the equation $xa = b$ has a unique solution in S;*
(viii) *for any $a \in S$, the equation $xa^2 = a$ has a unique solution in S;*
(ix) *S is a left zero band of groups;*
(x) *$(\forall a, b \in S)\, a \in aSb$;*
(xi) *$S \cong M(G; I, \Lambda, P)$ with $|I| = 1$.*

The equivalence of conditions (i), (ii) and (iii) was proved by Sushkevich in [**304**], 1928, for finite semigroups, and in [**305**], 1937, in the general case, and it was also formulated (without proofs) by Clifford in [**98**], 1933. The assertion (i) $\Leftrightarrow$ (iv) was proved by Clifford in [**98**], 1933, (i) $\Leftrightarrow$ (v) is an unpublished result of Munn, and (i) $\Leftrightarrow$ (x) was proved by Bogdanović and Stamenković in [**66**], 1988.

Now, in terms of left groups, right groups and groups, completely simple semigroups can be characterized as follows:

Theorem 3.6. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely simple;*
(ii) *S is a left zero band of right groups;*
(iii) *S is a right zero band of left groups;*
(iv) *S is a matrix of groups.*

The above theorem is an immediate consequence of the Rees-Sushkevich representation theorem for completely simple semigroups, and also, of Theorem 3.1, its dual and Theorem 3.3.

Note finally that the multiplicative semigroup of a non-trivial ring may not be simple, since a semigroup with zero is simple only if it is trivial. But, simple semigroups can appear in Theory of rings as subsemigroups of multiplicative semigroups of rings, as we will see later. On the other hand, in investigations of semigroups with zero one introduces other more suitable concepts. For example, one defines a semigroup $S = S^0$ to be a *0-simple semigroup* if $S^2 \neq 0$ and it has no an ideal different than 0 and the whole S. Similarly, *completely 0-simple semigroups* one defines as 0-simple semigroups having a *0-primitive idempotent*, by which we mean a minimal element in the partially ordered set of all non-zero idempotents of S. It is interesting to note that these semigroups have also a representation theorem of the Rees-Sushkevich type, through so-called Rees matrix semigroups over a group with zero adjoined. More information on completely 0-simple semigroups can be found in the books: Clifford and Preston [**105**], 1961, and [**106**], 1967, Howie [**144**], 1976, Steinfeld [**301**], 1978, Bogdanović and Ćirić [**48**], 1993, and others.

In theory of rings, a ring R having no an ideal different than 0 and the whole ring R is called a *simple ring*. More information about them and on so-called *Rees matrix rings over a division ring* can be found in the Petrich's book [**243**], 1974.

3.2. Completely Archimedean semigroups. By a natural generalization of semigroups considered in the previous section, the following semigroups one obtains: A semigroup S is called an *Archimedean semigroup* if $a \longrightarrow b$, for all $a, b \in S$, and similarly, S is called a *left Archimedean* (resp. *right Archimedean*) *semigroup* if $a \xrightarrow{l} b$ (resp. $a \xrightarrow{r} b$), for all $a, b \in S$. A semigroup which is both left and right Archimedean is called *two-sided Archimedean*, or shortly, a *t-Archimedean semigroup*.

The structure of Archimedean semigroups is quite complicated, but when an Archimedean semigroup is supplied by some additional property, such as the π-regularity, intra-, left, right or complete π-regularity, then its structure can be described more precisely, as we will see in the further text.

First we present the following two theorems, due mostly to Putcha [**255**], 1973.

Theorem 3.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a simple semigroup;*
(ii) *S is Archimedean and intra-π-regular;*
(iii) *S is Archimedean and has an intra-regular element;*
(iv) *S is Archimedean and has a kernel;*
(v) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in Sb^{2n}S$.

Theorem 3.8. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a left simple semigroup;*
(ii) *S is left Archimedean and intra-π-regular;*
(iii) *S is left Archimedean and left π-regular;*
(iv) *S is left Archimedean and has an intra-regular element;*
(v) *S is left Archimedean and has a left regular element;*
(vi) *S is left Archimedean and has a kernel;*
(vii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in Sb^{n+1}$.

By Theorem 3.7 it follows that a semigroup S is Archimedean and π-regular if and only if it is a nil-extension of a regular simple semigroup.

Left (resp. right) π-regular Archimedean semigroups were studied under the name *left* (resp. *right*) *completely Archimedean semigroups* by Bogdanović and Ćirić in [**59**], where the following theorem was proved:

Theorem 3.9. *The following conditions on a semigroup S are equivalent:*

(i) *S is left completely Archimedean;*
(ii) *S is a nil-extension of a left completely simple semigroup;*
(iii) *S is Archimedean and has a minimal left ideal;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in Sba^n$.

In analogy with completely simple semigroups, Archimedean semigroups having a primitive idempotents was called by Bogdanović in [**36**], 1985, *completely Archimedean semigroups*. The structure of these semigroups is described by the following theorem:

Theorem 3.10. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely Archimedean;*
(ii) *S is a nil-extension of a completely simple semigroup;*
(iii) *S is Archimedean and completely π-regular;*
(iv) *S is Archimedean and has a minimal left ideal and a minimal right ideal;*
(v) *S is Archimedean and has a minimal quasi-ideal;*
(vi) *S is left and right completely Archimedean;*
(vii) *S is left (or right) completely Archimedean and has an idempotent;*
(viii) *S is π-regular and all its idempotents are primitive;*
(ix) $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in a^nSba^n$;
(ix') $(\forall a, b \in S)(\exists n \in \mathbb{N})\, a^n \in a^nbSa^n$.

The equivalence of the conditions (ii), (viii), (ix) and (ix') was proved by Bogdanović and Milić in [**64**], 1984, the assertion (i) $\Leftrightarrow$ (iii) due to Galbiati and Veronesi [**123**], 1984, while (i) $\Leftrightarrow$ (ii) is an immediate consequence of Theorems 3.7 and 3.3.

A representation theorem of the Rees-Sushkevich type for completely Archimedean semigroups was given by Shum and Ren in [**289**], 1995.

Before we give a theorem which characterizes nil-extensions of rectangular groups, we must introduce the following notion: Let S and T be semigroups and let a semigroup H be a common homomorphic image of S and T, with respect to homomorphisms φ and ψ, respectively. Then

$$P = \{(a, b) \in S \times T \mid a\varphi = b\psi\},$$

is a subsemigroup of the direct product $S \times T$ of semigroups S and T, and is called a *spined product* of S and T with respect to H. It is known that P is a subdirect product of S and T. In Universal algebra this notion is known as a *pullback product*. It was introduced by Fuchs in [**117**], 1952, and since studied by Fleischer in [**116**], 1955, and Wenzel in [**343**], 1968. In Theory of semigroups these products have been intensively studied by Kimura, Yamada, Ćirić and Bogdanović and others, and the name "spined product" was introduced by Kimura in [**156**], 1958.

Theorem 3.11. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a rectangular group;*
(ii) *S is completely Archimedean and $E(S)$ is a subsemigroup of S;*
(iii) *S is π-regular and $E(S)$ is a rectangular band;*
(iv) *S is π-regular and Archimedean and for any $e \in E(S)$, the mapping φ_e : $x \mapsto exe$ is a homomorphism of S onto eSe;*
(v) *S is a subdirect product of a group and a nil-extension of a rectangular band;*
(vi) *S is a subdirect product of a group, a nil-extension of a left zero band and a nil-extension of a right zero band;*
(vii) *S is a spined product of a nil-extension of a left group and a nil-extension of a right group with respect to a nil-extension of a group.*

The equivalence of the conditions (i), (v) and (vi) was established by Putcha in [**255**], 1973, and of (i), (iii), (iv) and (vii) by Ren, Shum and Guo in [**273**]. Ren, Shum and Guo also gave a representation theorem of the Rees-Sushkevich type for these semigroups.

The next theorem, which characterizes nil-extensions of left groups, is mostly due to Bogdanović and Milić [**64**], 1984.

Theorem 3.12. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a left group;*
(ii) *S is left Archimedean and π-regular;*
(iii) *S is left Archimedean and right π-regular;*
(iv) *S is left Archimedean and completely π-regular;*
(v) *S is left Archimedean and has an idempotent;*
(vi) *S is π-regular and $E(S)$ is a left zero band;*
(vii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ a^n \in a^n S a^n b$.

A Rees-Sushkevich type representation theorem for nil-extensions of left groups was given by Shum, Ren and Guo in [**290**].

The previous theorem and its dual give the following:

Theorem 3.13. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a group;*
(ii) *S is π-regular and has a unique idempotent;*
(iii) *S is Archimedean and has a unique idempotent;*
(iv) *S is t-Archimedean and intra-π-regular;*
(v) *S is t-Archimedean and π-regular;*
(vi) *S is t-Archimedean and has an intra-regular element;*
(vii) *S is t-Archimedean and has an idempotent.*

The equivalence of the conditions (i) and (iii) was established by Tamura in [**318**], 1982.

Note finally that a semigroup with zero may be Archimedean if and only if it is a nil-semigroup, so Ćirić and Bogdanović introduced in [**89**], 1996, a concept more convenient for semigroups with zero, which generalizes both 0-simple and Archimedean semigroups. Namely, they defined a semigroup $S = S^0$ to be a *0-Archimedean semigroup* if $a \longrightarrow b$, for all $a, b \in S - 0$. These semigroups and some their special types were also studied by Ćirić and Bogdanović in [**86**], 1996, and Ćirić, Bogdanović and Bogdanović in [**97**].

4. Completely regular semigroups and rings

Although in Section 2 we have already discussed intra-, left, right and completely regular semigroups and rings, here we present their precise structure.

4.1. Completely regular semigroups. We start with intra-regular semigroups.

Theorem 4.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is intra-regular;*
(ii) *S is a union of simple semigroups;*
(iii) *any $\mathcal{J}$-class of S is a subsemigroup;*
(iv) *S is a semilattice of simple semigroups;*
(v) *any ideal of S is completely semiprime;*
(vi) $(\forall a, b \in S)\, (a) \cap (b) = (ab)$;
(vii) *$A \cap B \subseteq AB$, for any left ideal A and any right ideal B of S.*

The equivalence of conditions (i) and (vii) was proved by Lajos and Szász in [**192**], 1975. The rest of the theorem due to Croisot [**107**], 1953, and Anderson [**7**], 1952.

Combining the previous theorem with Theorem 2.1, the following theorem was obtained:

Theorem 4.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is regular and intra-regular;*
(ii) *S is a semilattice of regular simple semigroups;*
(iii) *$A \cap B = AB \cap BA$, for any left ideal A and any right ideal B of S;*
(iv) *$A \cap B \subseteq AB$, for all bi-ideals (or quasi-ideals) A and B of S;*
(v) *any quasi-ideal of S is globally idempotent.*

The equivalence of conditions (i) and (v) was established by Lajos in [**177**], 1972, and of (i) and (iv) by Lajos and Szász in [**192**], 1975. By Lajos in [**187**], 1991, the proof of (i) $\Leftrightarrow$ (iii) was attributed to Pondelíček. Finally, (i) $\Leftrightarrow$ (ii) is an immediate consequence of Theorem 4.1.

Structure of left regular semigroups was described by Croisot, 1953, and Bogdanović and Ćirić, 1996, who proved the following:

Theorem 4.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is left regular;*
(ii) *S is intra-regular and left π-regular;*
(iii) *S is a union of left simple semigroups;*
(iv) *any $\mathcal{L}$-class of S is a subsemigroup;*
(v) *S is a semilattice of left completely simple semigroups;*
(vi) *any left ideal of S is completely semiprime.*

The equivalence of conditions (i), (ii) and (v) was proved by Bogdanović and Ćirić in [**107**], 1996, and the rest is from Croisot [**55**], 1953.

For an element a of a semigroup (ring) S we say that it is *left duo* (*right duo*) if the principal left (right) ideal generated by a is a two-sided ideal, and that a is *duo* if it is both left and right duo. Similarly, a semigroup (ring) S is called *left duo* (*right duo*) if any left (right) ideal of S is a two-sided ideal, and is called *duo* if it is both left and right duo. The notion of a duo ring (semigroup) was introduced by Feller in [**114**], 1958, and Thierrin in [**325**], 1960, the corresponding definition for elements was given first by Steinfeld in [**300**], 1973, and left and right duo semigroups, rings and elements were first defined and studied by Lajos in [**181**] and [**182**], 1974. Between these notions the following relationship holds:

Theorem 4.4. *A semigroup (ring) is duo (resp. left duo, right duo) if and only if any its element is duo (resp. left duo, right duo).*

The previous theorem was proved by Kertész and Steinfeld in [**154**], 1974, and Steinfeld in [**300**], 1973, for the case of duo semigroups and rings.

Note also that the following holds:

Theorem 4.5. *An element a of a semigroup (ring) S is duo (resp. left duo, right duo) if and only if $(a)_L = (a)_R$ (resp. $(a)_R \subseteq (a)_L$, $(a)_L \subseteq (a)_R$).*

Recall that $(a)_L$ and $(a)_R$ denote the principal left and the principal right ideal of S generated by a, respectively.

Now we are ready to give the following characterization of semilattices of left simple semigroups.

Theorem 4.6. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left simple semigroups;*
(ii) *S is left (or intra-) regular and left duo;*
(iii) *S is left quasi-regular and left duo;*
(iv) *$A \cap B = AB$, for all left ideals A and B of S.*

The equivalence of conditions (i) and (ii) was proved by Petrich in [**236**], 1964, the proof of (i) $\Leftrightarrow$ (iv) was given by Saitô in [**274**], 1973, and the equivalence of (ii) and (iii) is an immediate consequence of Theorem 4.3 and Theorem 1 from the paper of Lajos and Szász [**192**], 1975.

Now we go to the completely regular semigroups. Various characterizations of these semigroups are collected in the following theorem:

Theorem 4.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is completely regular;*
(ii) *S is regular and left (or right) regular;*
(iii) *S is a union of groups;*
(iv) *any $\mathcal{H}$-class of S is a subsemigroup;*
(v) *S is a semilattice of completely simple semigroups;*
(vi) *any one-sided ideal of S is completely semiprime;*
(vii) *any left (or right, bi-) ideal of S is a regular semigroup;*
(viii) *any principal bi-ideal of S has an idempotent generator.*

The equivalence of conditions (i), (iii) and (v) was established by Clifford in [**99**], 1941, of (i), (ii) and (vi) by Croisot in [**107**], 1953, of (i) and (vii) by Lajos in [**184**], 1983. As was noted by Lajos in [**187**], 1991, (i) $\Leftrightarrow$ (viii) was proved in his paper from 1976. Note that the analogue of the condition (vii) for two-sided ideals is valid in any regular semigroup and ring (see Kaplansky [**151**], 1969, or Steinfeld [**301**], 1978).

For various constructions of completely regular semigroups we refer to Lallement [**194**], 1967, Petrich [**244**], 1974, and [**245**], 1977, Clifford [**104**], 1976, Warne [**341**], 1973, and Yamada [**346**], 1971.

Next we present the structure descriptions of the most important special types of completely regular semigroups.

Theorem 4.8. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of rectangular groups;*
(ii) *S is regular and $a = axa$ implies $a = ax^2a^2$;*
(iii) *S is completely regular and $E(S)$ is a subsemigroup.*
(iv) *S is completely regular and any inverse of any idempotent of S is an idempotent.*

The equivalence of conditions (iii) and (iv) is an immediate consequence of the result of Reilly and Scheiblich from [**269**], 1967, by which in any regular semigroup S, the idempotents of S form a subsemigroup if and only if any inverse of any idempotent of S is an idempotent. For the proof of the rest of the theorem we refer to Petrich [**241**], 1973.

Theorem 4.9. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left groups;*
(ii) *S is regular and $a = axa$ implies $ax = ax^2a$;*
(iii) *S is completely regular and $E(S)$ is a left regular band;*
(iv) *S is regular (or right regular) and left duo;*
(v) *S is quasi-regular (or right quasi-regular) and left duo;*
(vi) *$A \cap B = BAB$, for any left ideal A and any right ideal B of S;*
(vii) *$A \cap B = AB$, for any bi-ideal A and any right ideal B of S;*
(viii) *$A \cap B = AB$, for any bi-ideal A and any two-sided ideal B of S;*
(ix) *$A \cap B = BA$, for any left ideal A and any quasi-ideal B of S.*

The equivalence of conditions (i) and (iv) was established by Lajos in [**178**], 1972, and [**182**], 1974, of (i) and (viii) by Lajos in [**177**], 1972, and (i) $\Leftrightarrow$ (v) is a consequence of Theorem 4.6 and Theorem 1 from the paper of Lajos and Szász [**192**], 1975. The proofs of (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) can be found in Petrich [**241**], 1973. Finally, the conditions (vi), (vii) and (ix) are assumed from the survey paper of Lajos [**187**], 1991.

As we said before, completely regular semigroups were first investigated by Clifford in [**99**], 1941, and in some origins these semigroups were called the Clifford semigroups. But, some other authors, for example Howie in [**144**], 1976, used this name for another class of semigroups, studied first also by Clifford in [**99**], 1941, and following the terminology of these authors, in this paper by a *Clifford semigroup* (*ring*) we mean a regular semigroup (ring) whose all idempotents are central. These semigroups are characterized by the following theorem:

Theorem 4.10. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of groups;*
(ii) *S is a strong semilattice of groups;*
(iii) *S is a Clifford semigroup;*
(iv) *S is regular and $a = axa$ implies $ax = xa$;*
(v) *S is completely regular and $E(S)$ is a semilattice;*
(vi) *S is completely regular and inverse;*
(vii) *S is regular (or left, right, intra regular) and duo;*
(viii) *S is quasi-regular (or left, right quasi-regular) and duo;*
(ix) *$A \cap B = AB$, for any left ideal A and any right ideal B of S;*
(x) *$A \cap B = AB$, for all bi-ideals A and B of S;*
(xi) *$A \cap B = AB$, for all quasi-ideals A and B of S;*
(xii) *S is regular and a subdirect product of groups with a zero possibly adjoined.*

By Clifford in [**99**], 1941, the equivalence of conditions (i), (ii) and (iii) was proved, the equivalence of conditions (iv), (v) and (vi) is an immediate consequence of Theorem 2.2, (i) $\Leftrightarrow$ (vii) was proved by Petrich in [**236**], 1964, and (i) $\Leftrightarrow$ (xii) by the same author in [**242**], 1973. The equivalence of the condition (i) or (vii) and the conditions (ix), (x) and (xi) was established by Lajos in [**167**] and [**168**], 1969, [**170**] and [**171**], 1970, and [**174**] and [**175**], 1971.

The previous theorem, applied to commutative semigroups, gives the following their characterizations:

Theorem 4.11. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of Abelian groups;*
(ii) *S is a strong semilattice of Abelian groups;*
(iii) *S is regular and commutative;*
(iv) *S is quasi-regular and commutative;*
(v) *S is regular and a subdirect product of Abelian groups with a zero possibly adjoined.*

Various other characterizations of the semigroups considered here in terms of two-sided, one-sided, bi- and quasi-ideals we refer to the book of Steinfeld [**301**], 1978, the survey paper of Lajos [**187**], 1991, and other their papers given in the list of references.

4.2. Completely regular rings. In this section we will see that many of the concepts from Theory of semigroups considered in the previous section coincide in Theory of rings and are equivalent to the complete regularity. But, in Theory of rings we have many interesting special types of completely regular rings, such as Jacobson rings, p-rings, Boolean rings etc, whose main properties will be presented here.

The first theorem that we quote here gives various equivalents of the complete regularity of rings.

Theorem 4.12. *The following conditions on a ring R are equivalent:*

(i) *R is completely regular;*
(ii) *R is left (right) regular;*
(iii) *R is regular and intra-regular;*
(iv) *R is inverse;*
(v) *R is a Clifford ring;*
(vi) *R is regular and has no non-zero nilpotents;*
(vii) *R is a regular (left, right) duo ring;*
(viii) *R is an intra-regular (left, right) duo ring;*
(ix) *R is a (left, right) quasi-regular (left, right) duo ring;*
(x) *R is regular and a subdirect sum of division rings;*
(xi) *any left (right, bi-) ideal of R is a regular ring;*
(xii) *$A \cap B = AB$, for any left ideal A and any right ideal B of R;*
(xiii) *$A \cap B = AB$, for all left (right) ideals A and B of R;*
(xiv) *$A \cap B = AB$, for all quasi-ideals A and B of R;*
(xv) *any quasi-ideal of R is globally idempotent.*

The equivalence of conditions (v), (vi) and (vii) was proved by Schein in [**276**], 1966, although (vi) $\Rightarrow$ (vii) was first stated by Calais in [**67**], 1961. The equivalence of (vi) and (xvi) is due to Kovács [**160**], 1956, of (vii), (xii) and (xiii) is due to Lajos [**166**], 1969, and [**169**], 1970, while (i) $\Leftrightarrow$ (xiii) is due to Andrunakievich [**8**], 1964, (vii) $\Leftrightarrow$ (xiv) was proved by Lajos in [**175**], 1971, and Steinfeld in [**299**], 1971, (vi) $\Leftrightarrow$ (x) by Forsythe and Mc Coy in [**117**], 1946, (ii) $\Leftrightarrow$ (xi) by Lajos in [**184**], 1983, (ii) $\Leftrightarrow$ (v) is from Lajos and Szász [**189**] and [**190**], 1970. The proof of (iii) $\Leftrightarrow$ (xv) can be found in Steinfeld [**301**], 1978. Finally, the equivalence of (i) and (ii) is an immediate consequence of the result of Azumaya given in Section 2 as Theorem 2.9.

Let us hold our attention on the equivalence of the conditions (vi) and (x) of the above theorem. This result can be viewed as a consequence of a more general result obtained by Andrunakievich and Ryabuhin in [**9**], 1968, given by the following theorem:

Theorem 4.13. *A ring R has no non-zero nilpotent elements if and only if it is a subdirect sum of rings without zero divisors.*

A proof of this theorem can be found also in their book [**10**], 1979 (see also Thierrin [**327**], 1967). In the commutative case this theorem was proved by Krull in [**161**], 1929, and [**162**], 1950.

An analogue of the previous theorem holds in Theory of semigroups. It was proved by Park, Kim and Sohn in [**233**], 1988, and it follows directly from the theorem that asserts that any completely semiprime ideal of a semigroup is an intersection of some family of their completely prime ideals. The proofs of this theorem given by Petrich in [**241**], 1973, and Park, Kim and Sohn in [**233**], 1988, include an essential use of the Zorn lemma, but Ćirić and Bogdanović showed in [**87**] and [**91**], 1996, that its proof can be derived from the general theory of semilattice decompositions of semigroups, without recourse to transfinite methods.

Let us also note that direct sums of division rings were characterized by Gerchikov in [**126**], 1940, by the following theorem:

Theorem 4.14. *A ring R is a direct sum of division rings if and only if it has no non-zero nilpotent elements and it satisfies minimum conditions on left (or right) ideals.*

In the case of commutative rings we have

Theorem 4.15. *The following conditions on a ring R are equivalent:*

(i) *R is regular and commutative;*
(ii) *R is quasi-regular and commutative;*
(iii) *R is regular and a subdirect sum of fields.*

As we said before, several special types of completely regular rings are of the great importance in Theory of rings. The first of these types are Jacobson rings, which one defines in the following way: A ring R is called a *Jacobson ring* if for any $a \in R$ there exists $n \in \mathbb{N}$, $n \geq 2$ such that $a^n = a$. This condition is known as the *Jacobson's $a^n = a$ condition*. This condition has appeared in investigations of algebraic algebras without nilpotent elements over a finite field, carried out by Jacobson in [**148**], 1945. In this paper Jacobson proved that such algebras are commutative and as a consequence he obtained the following very important result:

Theorem 4.16. (Jacobson's $a^n = a$ theorem) *Any Jacobson ring is commutative.*

This theorem can be viewed as a generalization of the celebrated Wedderburn's theorem from [**342**], 1905, which asserts that any finite division ring must be a field.

A complete characterization of Jacobson rings, in few ways, is given by the next theorem, which is an immediate consequence of the Jacobson's $a^n = a$ theorem and Theorem 4.12.

Theorem 4.17. *The following conditions on a ring R are equivalent:*

(i) *R is a Jacobson ring;*

(ii) *R is commutative, regular and periodic;*
(iii) *R is completely regular and periodic;*
(iv) *R is regular and a subdirect sum of periodic fields;*
(v) *$\mathcal{M}R$ is a semilattice of periodic groups;*
(vi) *$\mathcal{M}R$ is a semilattice of periodic Abelian groups.*

A special case of Jacobson rings are the rings satisfying the semigroup identity of the form $x^n = x$, where $n \geq 2$ is an integer. Such rings were studied by Ayoub and Ayoub in [**13**], 1965, Luh in [**203**] and [**204**], 1967, and others. Luh characterized in [**203**], 1967, these rings in terms of p^k-rings, which are introduced by Mc Coy and Montgomery in [**211**], 1937, in the following way: A ring R is called a *p^k-ring* if there exists a prime p and a positive integer k such that R has the characteristic p and it satisfies the identity $x^{p^k} = x$. Rings defined in such a way with $k = 1$ are known as *p-rings*. The theorem proved by Luh in [**203**], 1967, is the following:

Theorem 4.18. *The following conditions on a ring R are equivalent:*

(i) *R satisfies the identity $x^n = x$, for some integer $n \geq 2$;*
(ii) *R satisfies the identity $x^p = x$, for some prime p;*
(iii) *R is a direct sum of finitely many p^k-rings.*

Particularly, p-rings are characterized by the following theorem:

Theorem 4.19. *Let p be a prime. A ring R is a p-ring if and only if it is a subdirect sum of fields of integers modulo p.*

Let us emphasize that p-rings, and consequently p^k-rings, trace one's origin to the famous *Boolean rings*, defined as rings whose any element is an idempotent. The following theorem characterizes these rings:

Theorem 4.20. *The following conditions on a ring R are equivalent:*

(i) *R is a Boolean ring;*
(ii) *R is a 2-ring;*
(iii) *R is a subdirect sum of fields of integers modulo 2;*
(iv) *$\mathcal{M}R$ is a band;*
(v) *$\mathcal{M}R$ is a semilattice.*

For more information on Boolean rings, and especially on their connections with Boolean algebras, we refer to the book of Abian [**1**], 1976, the paper of Stone [**302**], 1936, and others.

Various subdirect and direct sums whose summands are division rings or integral domains were studied by Kovácz in [**160**], 1956, Sussman in [**306**], 1958, Abian in [**2**], 1970, Chacron in [**69**], 1971, Wong in [**345**], 1976 and others.

5. Uniformly π-regular semigroups and rings

In Section 2 we seen that the left regular, right regular, intra-regular and group part of a completely π-regular semigroup (ring) coincide, but in the general case, they form a proper subset of the regular part of S. This motivates as to give the following definition: a π-regular semigroup (ring) S is called ***uniformly π-regular*** if every its regular element is completely regular. Similarly, a π-regular semigroup (ring) whose any regular element is left (resp. right) regular will be called *left* (resp. *right*) *uniformly π-regular*. We will see later that all of these notions coincide.

The subject of this section are some general structural properties of uniformly π-regular semigroups and rings. We will also consider *uniformly π-inverse* (resp. *left uniformly π-inverse*, *right uniformly π-inverse*) *semigroups* (*rings*), defined as uniformly π-regular semigroups (rings) which are also π-inverse (resp. left π-inverse, right π-inverse), and *uniformly periodic semigroups* (*rings*), defined as semigroups (rings) which are both uniformly π-regular and periodic.

5.1. Uniformly π-regular semigroups. One of the celebrated results in Theory of semigroups is the theorem of Tamura from [**314**], 1956, which asserts that any semigroup has a greatest semilattice decomposition, whose components are semilattice indecomposable semigroups. The smallest semilattice congruence on a semigroup, which corresponds to this decomposition, has various characterizations, but two of these characterization, given by Tamura in [**317**], 1972, and Putcha in [**257**], 1974, are especially interesting. Namely, T. Tamura proved that the transitive closure of the relation $\longrightarrow$ on a semigroup S is a quasi-order on S whose symmetric opening, i.e. its natural equivalence, equals the smallest semilattice congruence on S. On the other hand, M. S. Putcha started from the relation $—$ on S, defined as the symmetric opening of $\longrightarrow$, i.e. $— = \longrightarrow \cap (\longrightarrow)^{-1}$, and he proved that the smallest semilattice congruence on S equals the transitive closure of $—$. In the special case when the relations $\longrightarrow$ and $—$ are transitive, we obtain exactly semigroups having a decomposition into a semilattice of Archimedean semigroups, as demonstrated by the following theorem:

Theorem 5.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of Archimedean semigroups;*
(ii) $(\forall a,b \in S)\ a \longrightarrow b \ \Rightarrow\ a^2 \longrightarrow b$;
(iii) $(\forall a,b,c \in S)\ a \longrightarrow b \ \&\ b \longrightarrow c \ \Rightarrow\ a \longrightarrow c$;
(iv) $(\forall a,b,c \in S)\ a \longrightarrow c \ \&\ b \longrightarrow c \ \Rightarrow\ ab \longrightarrow c$;
(v) $(\forall a,b \in S)\ a — b \ \Rightarrow\ a^2 — b$;
(vi) $(\forall a,b,c \in S)\ a — b \ \&\ b — c \ \Rightarrow\ a — c$;
(vii) $(\forall a,b,c \in S)\ a — c \ \&\ b — c \ \Rightarrow\ ab — c$;
(viii) $(\forall a,b \in S)\ a^2 \longrightarrow ab$;
(viii)' $(\forall a,b \in S)\ b^2 \longrightarrow ab$;
(ix) *$\sqrt{A}$ is an ideal (or left ideal, right ideal) of S, for any ideal A of S;*
(x) *$\sqrt{SabS} = \sqrt{SaS} \cap \sqrt{SbS}$, for all $a,b \in S$.*

The first characterization of semilattices of Archimedean semigroups was given by Putcha in [**255**], 1973, who proved the equivalence of conditions (i) and (ii) of the above theorem. The equivalence of the conditions (ii), (iii) and (iv) was proved by Tamura in [**316**], 1972. The condition (ii) is known as the *power property*, (iii) is the transitivity and (iv) is known as the *common multiple property*, or shortly *cm-property* of a relation. The equivalence of the conditions (i), (v), (vi) and (vii) was established by Bogdanović, Ćirić and Popović in [**62**], Ćirić and Bogdanović in [**83**], 1993, showed the equivalence of the conditions (i), (viii), (viii)' and (ix), while Kmeť in [**157**], 1988, proved (i) $\Leftrightarrow$ (ix). The condition (x) is obtained from a more general result given by Ćirić and Bogdanović in [**87**], 1996.

Semilattices of left Archimedean semigroups are characterized by the following theorem:

Theorem 5.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ a \longrightarrow b \ \Rightarrow\ a \xrightarrow{l} b$;
(iii) $(\forall a, b \in S)\ a \xrightarrow{l} ab$;
(iv) *$\sqrt{L}$ is an ideal (or right ideal) of S, for any left ideal L of S;*
(v) $\sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb}$, *for all $a, b \in S$.*

The equivalence (i) $\Leftrightarrow$ (ii) was proved by Putcha in [**258**], 1981, (i) $\Leftrightarrow$ (iii) by Bogdanović in [**34**], 1984, and the equivalence of (i), (iv) and (v) was established by Bogdanović and Ćirić in [**43**], 1992.

By the previous theorem and its dual one obtains:

Theorem 5.3. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of t-Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ a \longrightarrow b \ \Rightarrow\ a \xrightarrow{t} b$;
(iii) $(\forall a, b \in S)\ a \xrightarrow{l} ab \ \&\ b \xrightarrow{r} ab$;
(iv) *$\sqrt{B}$ is an ideal of S, for any bi-ideal B of S.*

Supplying the above considered semigroups with the intra-π-regularity we obtain the following two theorems:

Theorem 5.4. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of simple semigroups;*
(ii) *S is a semilattice Archimedean semigroups and it is intra π-regular;*
(iii) *S is intra-π-regular and any $\mathcal{J}$-class of S containing an intra-regular element is a subsemigroup;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in S(ba)^n(ab)^nS$.

The equivalence of conditions (i), (ii) and (iii) was given by Putcha in [**255**], 1973.

Theorem 5.5. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left simple semigroups;*
(ii) *S is a semilattice left Archimedean semigroups and it is intra-π-regular;*
(iii) *S is a semilattice left Archimedean semigroups and it is left π-regular;*
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in S(ab)^na$.

The next theorem, which characterizes semilattices of left completely Archimedean semigroups, was proved by Bogdanović and Ćirić in [**59**].

Theorem 5.6. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of left completely Archimedean semigroups;*
(ii) *S is a semilattice of Archimedean semigroups and it is left π-regular;*
(iii) *S is left π-regular and each $\mathcal{L}$-class of S containing a left regular element is a subsemigroup;*
(iv) *S is left π-regular and each $\mathcal{J}$-class of S containing a left regular element is a subsemigroup;*
(v) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in Sa(ab)^n$.

Finally, we go to the uniformly π-regular semigroups. These semigroups are characterized by the following theorem:

Theorem 5.7. *The following conditions on a semigroup S are equivalent:*

(i) *S is uniformly π-regular;*
(ii) *S is left (or right) uniformly π-regular;*
(iii) *S is a semilattice of completely Archimedean semigroups;*
(iv) *S is a semilattice of Archimedean semigroups and it is completely π-regular;*
(v) *S is a semilattice of left completely Archimedean semigroups and it is right π-regular;*
(vi) *S is a semilattice of left completely Archimedean semigroups and it is π-regular;*
(vii) *S is π-regular and any $\mathcal{L}$-class of S containing an idempotent is a subsemigroup;*
(viii) *S is completely π-regular and any $\mathcal{J}$-class of S containing an idempotent is a subsemigroup;*
(ix) *S is completely π-regular and any $\mathcal{D}$-class of S containing a regular element is a subsemigroup;*
(x) *S is completely π-regular and $\mathbb{A}_2$ and $\mathbb{B}_2$ don't divide S through completely π-regular subsemigroups of S;*
(xi) $(\forall a, b \in S)(\exists n \in \mathbb{N})(ab)^n \in (ab)^n bS(ab)^n$;

The first characterization of semilattices of completely Archimedean semigroups was given by Putcha in [**255**], 1973, who proved that the conditions (iii), (iv) and (viii) are equivalent. The equivalence of the conditions (i), (iii), (ix) and (x) was stated without proofs by Shevrin in [**282**], 1977, and [**284**], 1981, and it was proved in [**285**], 1994. Some of these conditions, and also some other conditions equivalent to the uniform π-regularity of semigroups, were independently found by Veronesi in [**339**], 1984. The conditions (ii), (v), (vi) and (vii) were given by Bogdanović and Ćirić in [**59**], while (xi) is from Bogdanović [**38**], 1987.

Next we give the results obtained by Bogdanović in [**34**], 1984, Bogdanović and Ćirić in [**54**], 1995, and Shevrin in [**286**], 1994.

Theorem 5.8. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of rectangular groups;*

(ii) *S is π-regular and for all $a, x \in S$, $a = axa$ implies $a = ax^2a^2$;*
(iii) *S is uniformly π-regular and any inverse of any idempotent of S is an idempotent;*
(iv) *S is uniformly π-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (ef)^{n+1}$;*
(v) *S is completely π-regular and $(ab)^0 = (ab)^0(ba)^0(ab)^0$, for all $a, b \in S$.*

The results collected in the next theorem were also obtained by Bogdanović in [**34**], 1984, Bogdanović and Ćirić in [**54**], 1995, and Shevrin in [**286**], 1994.

Theorem 5.9. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of left groups;*
(ii) *S is a semilattice of left Archimedean semigroups and it is π-regular (or right π-regular, completely π-regular);*
(iii) *S is left uniformly π-inverse;*
(iv) *S is π-regular and for all $a, x \in S$, $a = axa$ implies $ax = xa^2x$;*
(v) *S is completely π-regular and $(ab)^0 = (ab)^0(ba)^0$, for all $a, b \in S$.*
(vi) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in (ab)^nS(ba)^n$.

Finally, semilattices of nil-extensions of groups are characterized by the following theorem:

Theorem 5.10. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of groups;*
(ii) *S is a semilattice of t-Archimedean semigroups and it is π-regular (or intra-π-regular, left π-regular, right π-regular, completely π-regular);*
(iii) *S is π-regular and for all $a, x \in S$, $a = axa$ implies $ax = xa$;*
(iv) *S is uniformly π-inverse;*
(v) *S is completely π-regular and $(ab)^0 = (ba)^0$, for all $a, b \in S$.*
(vi) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in (ba)^nS(ba)^n$.

The equivalence of the conditions (i) and (iv) was established by Veronesi in [**339**], 1984, of (i), (ii), (iii) and (vi) by Bogdanović in [**34**], 1984, and of (i) $\Leftrightarrow$ (v) was proved by Shevrin in [**286**], 1994, and Bogdanović and Ćirić in [**54**], 1995.

5.2. Uniformly π-regular rings. Uniformly π-regular rings have also a very interesting structure characterization, given by the following theorem:

Theorem 5.11. *The following conditions on a ring R are equivalent:*

(i) *R is uniformly π-regular;*
(ii) *R is π-regular and $\mathrm{Nil}(R)$ is an ideal od $\mathcal{M}R$;*
(iii) *R is π-regular and $\mathrm{Nil}(R)$ is an ideal od R;*
(iv) *R is π-regular and an ideal extension of a nil-ring by a Clifford ring;*
(v) *$\mathcal{M}R$ is a semilattice of completely Archimedean semigroups;*
(vi) *$\mathcal{M}R$ is a semilattice of left (or right) completely Archimedean semigroups;*
(vii) *$\mathcal{M}R$ is a semilattice of Archimedean semigroups and R is π-regular.*

The equivalence of the conditions (v), (ii) and (iii) was proved by Putcha in [**258**], 1981, for completely π-regular rings, and the same proof was translated to π-regular rings by Ćirić and Bogdanović in [**81**], 1992, where also it was proved that (i), i.e. (v), is equivalent to (iv) and (vii). The equivalence of the conditions (v) and (vi) follows by Theorems 2.8 and 5.7.

Note that an analogue of the equivalence (v) $\Leftrightarrow$ (vii) is not valid in Theory of semigroups. For example, bicyclic semigroups are regular and simple, but these are no uniformly π-regular.

As we will see later, the condition (iv) has a great importance, since it gives a possibility to represent uniformly π-regular rings by Everett's sums.

Problem. Can the equivalence of the conditions (i), (ii), and (iii) of the previous theorem be proved if in (ii) and (iii) we omit the assumption that R is π-regular?

Some special cases of uniformly π-regular rings are also interesting. First we give

Theorem 5.12. *The following conditions on a ring R are equivalent:*

(i) *R is uniformly π-regular and for all $e, f \in E(R)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (ef)^{n+1}$;*
(ii) *R is uniformly π-regular and $(ef)^2 = (ef)^3$, for all $e, f \in E(R)$;*
(iii) *$\mathcal{M}R$ is a semilattice of nil-extensions of rectangular groups.*

Rings whose multiplicative semigroups can be decomposed into a semilattice of nil-extensions of left groups were investigated by Bogdanović and Ćirić in [**44**], 1992, who proved the following theorem:

Theorem 5.13. *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a semilattice of nil-extensions of left groups;*
(ii) *R is left π-inverse;*
(iii) *R is left completely π-inverse;*
(iv) *R is π-regular and $ea = eae$, for any $a \in R$ and $e \in E(R)$;*
(v) *R is π-regular and $E(R)$ is a left regular band.*

The previous theorem and its dual yield the next theorem, proved also by Bogdanović and Ćirić in [**44**], 1992.

Theorem 5.14. *The following conditions on a ring R are equivalent:*

(i) *$\mathcal{M}R$ is a semilattice of nil-extensions of groups;*
(ii) *R is π-inverse;*
(iii) *R is completely π-inverse;*
(iv) *R is uniformly π-inverse;*
(v) *R is π-regular and the idempotents of R are central;*
(vi) *R is π-regular and $E(R)$ is a semilattice.*

In the case of completely π-regular rings with an identity, Putcha in [**258**], 1981, showed that the above considered concepts coincide. Namely, he proved the following:

Theorem 5.15. *The following conditions on a completely π-regular ring R with the identity are equivalent:*

(i) $\mathcal{M}R$ *is a semilattice of left Archimedean semigroups;*
(ii) $\mathcal{M}R$ *is a semilattice of right Archimedean semigroups;*
(iii) $\mathcal{M}R$ *is a semilattice of t-Archimedean semigroups;*
(iv) *the idempotents of R are central.*

In the mentioned paper, M. S. Putcha gave an example of a ring that is a semilattice of right Archimedean semigroups, but it is not a semilattice of left Archimedean semigroups.

5.3. Uniformly periodic semigroups and rings. There are examples that the property "being a semilattice of Archimedean semigroups" is not a hereditary property. Semigroups on which this property is hereditary were investigated by Bogdanović, Ćirić and Mitrović in [**60**], 1995, where the following theorem was given:

Theorem 5.16. *The following conditions on a semigroup S are equivalent:*

(i) *any subsemigroup of S is a semilattice of Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ ab \uparrow a^2$;
(ii)' $(\forall a, b \in S)\ ab \uparrow b^2$;
(iii) *S satisfies one of the following variable identities over A_2:*
  (a) $\{(xy)^n = w \mid w \in A_2^* x^2 A_2^* \cup A_2^* x,\ n \in \mathbb{N}\}$;
  (b) $\{(xy)^n = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^*,\ n \in \mathbb{N}\}$;
  (c) $\{(xy)^n x = w \mid w \in A_2^* x^2 A_2^*,\ n \in \mathbb{N}\}$;
  (d) $\{(xy)^n x = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^* \cup A_2^* y,\ n \in \mathbb{N}\}$.

Semigroups in which the property "being uniformly π-regular" is hereditary are exactly the uniformly periodic semigroups. This is demonstrated by the next theorem, proved in the same paper of S. Bogdanović, M. Ćirić and M. Mitrović.

Theorem 5.17. *The following conditions on a semigroup S are equivalent:*

(i) *S is uniformly periodic;*
(ii) *S is a semilattice of nil-extensions of periodic completely simple semigroups;*
(iii) *S is periodic and a semilattice of Archimedean semigroups;*
(iv) *any subsemigroup of S is uniformly π-regular;*
(v) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n = (ab)^n ((ba)^n (ab)^n)^n$;
(v)' $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n = ((ab)^n (ba)^n (ab)^n)^n$;
(vi) *S satisfies one of the following variable identities over A_2:*
  (a) $\{(xy)^n = w \mid w \in A_2^* x^2 A_2^* \cup A_2^* x,\ |w| \neq 2n,\ n \in \mathbb{N}\}$;
  (b) $\{(xy)^n = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^*,\ |w| \neq 2n,\ n \in \mathbb{N}\}$;
  (c) $\{(xy)^n x = w \mid w \in A_2^* x^2 A_2^*,\ |w| \neq 2n+1,\ n \in \mathbb{N}\}$;
  (d) $\{(xy)^n x = w \mid w \in A_2^* y^2 A_2^* \cup y A_2^* \cup A_2^* y,\ |w| \neq 2n+1,\ n \in \mathbb{N}\}$.

The next theorem, which describes the structure of uniformly periodic rings, is due to the authors.

Theorem 5.18. *The following conditions on a ring R are equivalent:*

(i) *R is uniformly periodic;*
(ii) *R is an ideal extension of a nil-ring by a Jacobson's ring;*
(iii) *any subring of R is uniformly π-regular;*
(iv) *any subsemigroup of $\mathcal{M}R$ is uniformly π-regular;*
(v) *$\mathcal{M}R$ is a semilattice of nil-extensions of periodic completely simple semigroups.*

A special type of the above considered rings, namely the rings which are ideal extensions of a nil-ring by a Boolean ring, were studied by Hirano, Tominaga and Yaqub in [**142**], 1988, where the following theorem was proved:

Theorem 5.19. *The following conditions on a ring R are equivalent:*

(i) *R is an ideal extension of a nil-ring by a Boolean ring;*
(ii) $(\forall a \in R)\ a - a^2 \in \mathrm{Nil}(R)$;
(iii) *R is $[E(R), \mathrm{Nil}(R)]$-representable;*
(iv) *R is uniquely $[E(R), \mathrm{Nil}(R)]$-representable.*

In the same paper, Y. Hirano, H. Tominaga and A. Yaqub also considered the condition of the form

$$(\#)_n \qquad\qquad (\forall a \in R)\ x - x^n \in \mathrm{Nil}(R),$$

where $n \in \mathbb{N}$, $n \geq 2$. By the above theorem, $\mathrm{Nil}(R)$ form an ideal of R, whenever a ring R satisfies $(\#)_2$, but this does not holds for all $n \in \mathbb{N}$. Necessary and sufficient conditions for n, under which $\mathrm{Nil}(R)$ is an ideal of R, for any ring R satisfying $(\#)_n$, are determined by the following theorem, proved also in the above mentioned paper.

Theorem 5.20. *Let $n \in \mathbb{N}$, $n \geq 2$. Then the following conditions are equivalent:*

(i) *$\mathrm{Nil}(R)$ is an ideal of R, for any ring R which satisfies $(\#)_n$;*
(ii) *$n \not\equiv 1 \pmod 3$ and $n \not\equiv 1 \pmod 8$;*
(iii) *for each prime p, $n \not\equiv 1 \pmod{p^2 - 1}$;*
(iv) *for each prime p, $M_2(GF(p))$ fails to satisfy $(\#)_n$.*

5.4. Nil-extensions of unions of groups. The subject of this section are semigroups decomposable into a nil-extension of a union of groups. In other words, these are π-regular semigroups in which the group part form an ideal, and they are one of the most significant special cases of uniformly π-regular semigroups.

Except the mentioned semigroups, here we also consider certain their special types, such as retractive, nilpotent and retractive nilpotent extensions of unions of groups. Recall that we say that a semigroup S is a *retractive extension* of a semigroup K if S is an ideal extension of K and there exists a retraction of S onto K. These are extensions which can be more easily constructed than many other kinds of extensions, and this make important their investigation.

For $n \in \mathbb{N}$, a retractive $(n+1)$-nilpotent extension S of a semigroup K was called by Bogdanović and Milić in [**65**], 1987, an *n-inflation* of K. These authors also gave a general construction for such extensions. It is important to note that 1-inflations are called simply *inflations*, while 2-inflations are also known as *strong inflations*. Inflations of semigroups were first defined and studied by Clifford in [**102**], 1950, and strong inflations by Petrich in [**238**], 1967.

The first theorem which we quote here was proved by Bogdanović and Ćirić in [**41**], 1991, and it describes nil-extensions of regular semigroups.

Theorem 5.21. *A semigroup S is a nil-extension of a regular semigroup if and only if for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^ny \in xa^nySxa^ny$.*

An immediate consequence of the previous theorem is the following:

Theorem 5.22. *A semigroup S is a nil-extension of a union of groups if and only if for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^ny \in xa^nyxSxa^ny$.*

Nil-extensions of semilattices of left groups are characterized similarly:

Theorem 5.23. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of left groups;*
(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^ny \in xa^nySya^nx;$
(iii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^ny \in xSx$.*

The equivalences (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv) are from Bogdanović and Ćirić [**41**], 1991, and [**46**], 1992, respectively.

When we deal with retractive nil-extensions of regular semigroups, the following theorem has a crucial role:

Theorem 5.24. *A semigroup S is a retractive nil-extension of a regular semigroup if and only if it is a subdirect product of a nil-semigroup and a regular semigroup.*

The above theorem was proved by Bogdanović and Ćirić in [**45**], 1992. The same authors in another paper [**46**], 1992, proved the following:

Theorem 5.25. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a union of groups;*
(ii) *S is a subdirect product of a nil-semigroup and a union of groups;*
(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^ny \in x^2Sy^2.$

A very interesting property of retractive nil-extensions of unions of groups was found by Bogdanović and Ćirić in [**41**], 1991, who gave:

Theorem 5.26. *Let a semigroup S be a nil-extension of a union of groups K. Then an arbitrary retraction φ of S onto K has the following representation:*

$$a\varphi = ea \quad \text{if } a \in \sqrt{G_e},\ \text{for } e \in E(S) \qquad (a \in S).$$

In view of the Munn's lemma (Theorem 2.12), if the above condition is fulfilled, then $ae = ea \in G_e$, so also $a\varphi = ae$.

The next theorem was proved by Bogdanović and Ćirić in [**46**], 1992:

Theorem 5.27. *A semigroup S is a retractive nil-extension of a semilattice of left groups if and only if it is π-regular and the following condition holds:*

$$(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y \in x^2 Sx.$$

Nil-extensions of Clifford semigroups (semilattices of groups) one considers in the following theorem:

Theorem 5.28. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of groups;*
(ii) *S is a retractive nil-extension of a semilattice of groups;*
(iii) *$(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y \in xa^n ySya^n x \cap ya^n xSxa^n y$;*
(iv) *S is π-regular and for all $x, a, y \in S$ there exists $n \in \mathbb{N}$ such that $xa^n y \in xSx \cap ySy$.*

The equivalence (i) $\Leftrightarrow$ (ii) was proved by Bogdanović and Ćirić in [**41**], 1991. The remaining conditions are derived from Theorem 5.23 and its dual.

Now we pass from nil-extensions to the nilpotent ones. For an arbitrary $n \in \mathbb{N}$, a semigroup S which is an $(n+1)$-nilpotent extension of a regular semigroup (resp. union of groups) can be characterized by a simple condition $S^{n+1} \subseteq \mathrm{Reg}(S)$ (resp. $S^{n+1} \subseteq \mathrm{Gr}(S)$). But, $(n+1)$-nilpotent extensions of semilattices of left groups have a more interesting characterization, given by the following theorem:

Theorem 5.29. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of left groups;*
(ii) *S is π-regular (or right π-regular) and $xS^n = xS^n x$, for any $x \in S$;*
(iii) *$(\forall x_1, x_2, \dots, x_{n+1} \in S)\ x_1 x_2 \cdots x_{n+1} \in x_1 x_2 \cdots x_{n+1} Sx_1$.*

The equivalence of the conditions (i) and (iii) was proved by Bogdanović and Stamenković in [**66**], 1988.

As was proved by Bogdanović and Ćirić in [**45**], 1992, retractive nilpotent extensions of regular semigroups can be also characterized in terms of subdirect products:

Theorem 5.30. *Let $n \in \mathbb{N}$. A semigroup S is an n-inflation of a regular semigroup K if and only if it is a subdirect product of K and an $(n+1)$-nilpotent semigroup.*

Applying this theorem to n-inflations of unions of groups we obtain the following:

Theorem 5.31. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an n-inflation of a union of groups;*
(ii) *S is a subdirect product of a union of groups and an $(n+1)$-nilpotent semigroup;*
(iii) $(\forall x, y \in S)\ xS^{n-1}y = x^2S^ny^2$.

The equivalence of conditions (i) and (ii) was proved by Bogdanović and Milić in [**65**], 1987. In the case $n = 1$ this was shown by Bogdanović in [**37**], 1985.
Next we quote

Theorem 5.32. *Let $n \in \mathbb{N}$. A semigroup S is an n-inflation of a semilattice of left groups if and only if the following condition holds:*

$$(\forall x \in S)xS^n = x^2S^nx.$$

This theorem was proved by Bogdanović and Stamenković in [**66**], 1988 (see also Bogdanović and Ćirić [**46**], 1992), and by Bogdanović in [**38**], 1987, in the case $n = 1$.
This section we finish giving the following theorem:

Theorem 5.33. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of groups;*
(ii) *S is an n-inflation of a semilattice of groups;*
(iii) $(\forall x, y \in S)\ xa^ny \in y^2S^nx$.

These results are due to Bogdanović and Milić [**65**], 1987. Inflations of semilattices of groups were described in a similar way by Bogdanović in [**37**], 1985.

5.5. Nil-extensions of unions of periodic groups. The class of semigroups which are nil-extensions of unions of groups, and certain its subclasses, have very nice characterizations in terms of variable identities, which will be presented in this section. Except the results whose origins we quote explicitly, all remaining results are unpublished results of the authors.

For a given $n \in \mathbb{N}$, $n \geq 3$, at the start of the section we deal with semigroup identities over A_n of the form

$$x_1u(x_2, \dots, x_{n-1})x_n = w(x_1, x_2, \dots, x_n), \tag{1}$$

and the following conditions concerning them:

(A1) for a fixed $i \in \{1, \dots, n\}$, x_i appears once on one side of (1) and at most twice on another side;
(B1) $|w| \neq |u| + 2$;

(C1.1) $x_1 \underset{l}{\nmid} w$; (C1.2) $h^{(2)}(w) = x_1^2$; (C1.3) $h(w) \neq x_1$;

(D1.1) $x_n \underset{r}{\nmid} w$; (D1.2) $t^{(2)}(w) = x_n^2$; (D1.3) $t(w) \neq x_n$;

and we also deal with identities of the form

$$x_1 u(x_2, \dots, x_n) = v(x_1, \dots, x_{n-1}) x_n, \tag{2}$$

and the following conditions concerning them:

(A2) for a fixed $i \in \{1, \dots, n\}$, x_i appears once on one side of (2) and at most twice on another side;

(B2) $|u| \neq |v|$;

(C2.2) $h^{(2)}(v) = x_1^2$; (C2.3) $h(v) \neq x_1$;

(D2.2) $t^{(2)}(u) = x_n^2$; (D2.3) $t(u) \neq x_n$.

Let us observe that the following implications hold: (C1.2) $\Rightarrow$ (C1.1) & (A.1), (C1.3) $\Rightarrow$ (C1.1), (D1.2) $\Rightarrow$ (D1.1) & (A.1), (D1.3) $\Rightarrow$ (D1.1), (C2.2) $\Rightarrow$ (A2) and (D2.2) $\Rightarrow$ (A2).

The next five theorems are due to the authors:

Theorem 5.34. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a union of periodic groups;*

(ii) $(\forall x, a, y \in S)(\exists m, n \in \mathbb{N})\ xa^n y = (xa^n y)^{m+1}$;

(iii) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (A1), (B1), (C1.1) *and* (D1.1).

Theorem 5.35. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a union of periodic groups;*

(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = x^{n+1} a^n y^{n+1}$;

(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y^{n+1} = x^{n+1} a^n y$;

(iv) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (B1), (C1.2) *and* (D1.2);

(v) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (2) *having the properties* (B.2), (C2.2) *and* (D2.2).

Theorem 5.36. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a semilattice of periodic left groups;*

(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = xa^n yx^n$;

(iii) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (A1), (B1), (C1.1) *and* (D1.3)

Theorem 5.37. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a semilattice of periodic left groups;*

(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = x^{n+1} a^n yx^n$;

(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n ya^n = x^{n+1} a^n y$;

(iv) *for an integer $n \geq 3$, S satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (B1), (C1.2) *and* (D1.3);

(v) *for an integer* $n \geq 3$, S *satisfies the variable identity consisting of all identities of the form* (2) *having the properties* (B2), (C2.2) *and* (D2.3).

Theorem 5.38. *The following conditions on a semigroup* S *are equivalent:*

(i) S *is a nil-extension of a semilattice of periodic groups;*

(ii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y = y^n x a^n y x^n$;

(iii) $(\forall x, a, y \in S)(\exists n \in \mathbb{N})\ xa^n y^{n+1} a^n = a^n x^{n+1} a^n y$;

(iv) *for an integer* $n \geq 3$, S *satisfies the variable identity consisting of all identities of the form* (1) *having the properties* (A1), (B1), (C1.3) *and* (D1.3);

(v) *for an integer* $n \geq 3$, S *satisfies the variable identity consisting of all identities of the form* (2) *having the properties* (A2), (B2), (C2.3) *and* (D2.3).

For $n \in \mathbb{N}$, now we deal with semigroup identities over A_{n+1} of the form

$$x_1 x_2 \cdots x_{n+1} = w(x_1, x_2, \ldots, x_{n+1}), \tag{3}$$

and the following conditions concerning them:

(A3) for a fixed $i \in \{1, \ldots, n+1\}$, x_i appears once on one side of (3) and at most twice on another side;

(B3) $|w| \geq n+2$;

(C3.1) $x_1 \not|_l\, w$; (C3.2) $h^{(2)}(w) = x_1^2$; (C3.3) $h(w) \neq x_1$;

(D3.1) $x_{n+1} \not|_r\, w$; (D1.2) $t^{(2)}(w) = x_{n+1}^2$; (D1.3) $t(w) \neq x_{n+1}$;

The next theorems characterize various types of nilpotent extensions of unions of groups.

Theorem 5.39. *Let* $n \in \mathbb{N}$. *Then the following conditions on a semigroup* S *are equivalent:*

(i) S *is an* $(n+1)$*-nilpotent extension of a union of periodic groups;*

(ii) $(\forall x_1, x_2, \ldots, x_n \in S)(\exists m \in \mathbb{N})\ x_1 x_2 \cdots x_{n+1} = (x_1 x_2 \cdots x_{n+1})^{m+1}$;

(iii) S *satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (A3), (B3), (C3.1) *and* (D3.1).

The equivalence (i) $\Leftrightarrow$ (ii) was proved by Bogdanović and Milić in [**65**], 1987, and for $n = 1$ by Bogdanović in [**37**], 1985. A condition similar to (iii) was given by Putcha and Weissglass in [**263**], 1972 (they required that the condition (A3) holds for all $i \in \{1, 2, \ldots, n+1\}$).

Theorem 5.40. *Let* $n \in \mathbb{N}$. *Then the following conditions on a semigroup* S *are equivalent:*

(i) S *is an* n*-inflation of a union of periodic groups;*

(ii) $(\forall x_1, x_2, \ldots, x_n \in S)(\exists m \in \mathbb{N})\ x_1 x_2 \cdots x_{n+1} = x_1^{m+1} x_2 \cdots x_n x_{n+1}^{m+1}$;

(iii) S *satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (B3), (C3.2) *and* (D3.2).

The equivalence of the conditions (i) and (ii) was established by Bogdanović and Milić in [**65**], 1987, whereas in the case $n = 1$ this was shown by Bogdanović in [**37**], 1985.

Theorem 5.41. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of periodic left groups;*
(ii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1x_2\cdots x_{n+1} = x_1x_2\cdots x_{n+1}x_1^m$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (A3), (B3), (C3.1) *and* (D3.3).

A condition equivalent to (i), and similar to (ii), was given by Bogdanović and Stamenković in [**66**], 1988, and by Bogdanović in [**38**], 1987, for the case $n = 1$. These remarks hold also for the next theorem.

Theorem 5.42. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an n-inflation of a semilattice of periodic left groups;*
(ii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1x_2\cdots x_{n+1} = x_1^{m+1}x_2\cdots x_{n+1}x_1^m$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (B3), (C3.2) *and* (D3.3).

Theorem 5.43. *Let $n \in \mathbb{N}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is an $(n+1)$-nilpotent extension of a semilattice of periodic groups;*
(ii) *S is an n-inflation of a semilattice of periodic groups;*
(iii) $(\forall x_1, x_2, \dots, x_n \in S)(\exists m \in \mathbb{N})\ x_1x_2\cdots x_{n+1} = x_{n+1}^m x_1\cdots x_{n+1}x_1^m$;
(iii) *S satisfies the variable identity consisting of all identities of the form* (3) *having the properties* (A3), (B3), (C3.3) *and* (D3.3).

The equivalence (i) $\Leftrightarrow$ (iii) was proved by Bogdanović and Milić in [**65**], 1987, while (i) $\Leftrightarrow$ (iv) was shown by Putcha and Weissglass in [**263**], 1972. The related results concerning the case $n = 1$ were given by Bogdanović in [**37**], 1985, and Putcha and Weissglass in [**262**], 1971. The equivalence of the conditions (i) and (ii) was obtained as a consequence of Theorem 5.28.

The theorems characterizing nilpotent and nil-extensions of bands, left regular bands and semilattices, and their retractive analogues, are very similar to the previous ones, so they will be omitted. We only note that the variable identities describing these semigroups consist of the corresponding identities from the above theorems, having an additional property:

(A1-3)* for a fixed $i \in \{1, \dots, x_n\}$ (resp. $i \in \{1, \dots, x_n\}$, $i \in \{1, \dots, x_{n+1}\}$), x_i appears once on one side of (1) (resp. (2), (3)), and exactly twice on another side.

This condition forces all subgroups of a semigroup to be one-element.

5.6. Direct sums of nil-rings and Clifford rings. In Section 5.2 we have seen that the set of nilpotents of a π-regular ring is a ring ideal if and only if it is a semigroup ideal. Here we show that this property also holds for the group part of such a ring, i.e. that the group part of a π-regular ring is a ring ideal if and only if it is a semigroup ideal. In this case we get a decomposition of this ring into a direct sum of a nil-ring and a Clifford ring, as it is demonstrated by the following theorem:

Theorem 5.44. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a nil-ring and a Clifford ring;*
(ii) *R is a subdirect sum of a nil-ring and a Clifford ring;*
(iii) *R is a strong extension of a nil-ring by a Clifford ring;*
(iv) *R is uniquely $(\mathrm{Gr}(R), \mathrm{Nil}(R))$-representable;*
(v) *R is π-regular and uniquely $(\mathrm{LReg}(R), \mathrm{Nil}(R))$-representable;*
(vi) *R is π-regular and $E(R)$ is contained in a reduced ideal of R;*
(vii) *$\mathcal{M}R$ is a nil-extension of a completely regular (or a Clifford) semigroup;*
(viii) *$\mathcal{M}R$ is a retractive nil-extension of a completely regular (or a Clifford) semigroup;*
(ix) *$\mathcal{M}R$ is a subdirect product of a nil-semigroup and a completely regular (or a Clifford) semigroup;*
(x) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a completely regular (or a Clifford) semigroup.*

The equivalence of conditions (i), (v) and (vi) was proved by Hirano and Tominaga in [**141**], 1985, and of (i) and (ii) by Bell and Tominaga in [**23**], 1986, Tominaga [**332**]. For some related results see also Tominaga [**331**]. Ćirić and Bogdanović in [**80**], 1990, showed that the conditions (iii), (vii) and (viii) are equivalent, and in [**90**], 1996, they established the equivalence of the conditions (i), (vii) and (x). The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (ix) are obvious, while (ix) $\Rightarrow$ (viii) is an immediate consequence of Theorem 5.24.

For some related results see also Bell and Yaqub [**24**], 1987, and Abu-Khuzam and Yaqub [4], 1985. Certain more general decompositions can be found in Hirano and Tominaga [**141**], 1985, and Bell and Tominaga [**23**], 1986.

On the other hand, a special case of the above decompositions are decompositions into a direct sum of a nil-ring and a Jacobson ring. The results concerning these decompositions are collected in the following theorem:

Theorem 5.45. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a nil-ring and a Jacobson ring;*
(ii) *R is a subdirect sum of a nil-ring and a Jacobson ring;*
(iii) *R is uniquely $(P(R), \mathrm{Nil}(R))$-representable;*
(iv) *$E(R) \cdot N_2(R) = N_2(R) \cdot E(R) = 0$ and R is $[P(R), \mathrm{Nil}(R)]$-representable;*
(v) *R is periodic and $E(R)$ is contained in a reduced ideal of R;*
(vi) *$\mathcal{M}R$ is a nil-extension of a union (or a semilattice) of periodic groups;*
(vii) *$\mathcal{M}R$ is a retractive nil-extension of a union (or a semilattice) of periodic groups;*
(viii) *$\mathcal{M}R$ is a subdirect product of a nil-semigroup and a union (or a semilattice) of periodic groups;*
(ix) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a union (or a semilattice) of periodic groups.*

The conditions (i), (iii) and (iv) are equivalent by theorems proved by Bell and Tominaga in [**23**], 1986, and Hirano, Tominaga and Yaqub in [**142**], 1988, although (i) $\Leftrightarrow$ (iii) was proved by Bell in [**20**], 1985, and Hirano and Tominaga in [**141**], 1985, under the assumption that R is periodic. The condition (v) is assumed from Hirano and Tominaga [**141**], 1985. The equivalence of the conditions (i), (vii), (viii) and (ix) was established by Ćirić and Bogdanović in [**90**], 1996.

By the next theorem we describe direct sums of nil-rings and Boolean rings.

Theorem 5.46. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a nil-ring and a Boolean ring;*
(ii) *R is a subdirect sum of a nil-ring and a Boolean ring;*
(iii) *R is a strong extension of a nil-ring by a Boolean ring;*
(iv) *$E(R) \cdot N_2(R) = N_2(R) \cdot E(R) = 0$ and R satisfies one of the conditions of Theorem 5.19;*
(v) *$E(R) \cdot N_2(R) = N_2(R) \cdot E(R) = 0$ and R is $(E(R), \mathrm{Nil}(R))$-representable;*
(vi) *$E(R) \cdot N_2(R) = N_2(R) \cdot E(R) = 0$ and R is uniquely $(E(R), \mathrm{Nil}(R))$-representable;*
(vii) *$\mathcal{M}R$ is a nil-extension of a band* (*or a semilattice*)*;*
(viii) *$\mathcal{M}R$ is a retractive nil-extension of a band* (*or a semilattice*)*;*
(ix) *$\mathcal{M}R$ is a subdirect product of a nil-semigroup and a band* (*or a semilattice*)*;*
(x) *$\mathcal{M}R$ is a direct product of a nil-semigroup and a band* (*or a semilattice*)*.*

Hirano, Tominaga and Yaqub in [**142**], 1988, proved that (i), (iv), (v) and (vi) are equivalent. The remaining conditions were given by Ćirić and Bogdanović in [**80**], 1990, and [**90**], 1996.

In the rest of the section we present the results characterizing direct sums of nilpotent rings and of Clifford, Jacobson and Boolean rings.

Theorem 5.47. *Let $n \in \mathbb{N}$. Then the following conditions on a ring R are equivalent:*

(i) *R is a direct sum of an $(n+1)$-nilpotent ring and a Clifford ring;*
(ii) *R is a subdirect sum of an $(n+1)$-nilpotent ring and a Clifford ring;*
(iii) $(\forall a \in R)\ aR^n = aR^na$*;*
(iv) $(\forall a \in R)\ aR^n \subseteq R^na^2$*;*
(v) $(\forall a \in R)\ aR^n = a^2R^n\ \&\ R^na = R^na^2$*;*
(vi) $R^{n+1} \subseteq \mathrm{LReg}(R)$ (*or* $R^{n+1} \subseteq \mathrm{RReg}(R)$)*;*
(vii) *$\mathcal{M}R$ is an $(n+1)$-nilpotent extension of a completely regular* (*or a Clifford*) *semigroup;*
(viii) *$\mathcal{M}R$ is an n-inflation of a completely regular* (*or a Clifford*) *semigroup;*
(ix) *$\mathcal{M}R$ is a subdirect product of an $(n+1)$-nilpotent semigroup and a completely regular* (*or a Clifford*) *semigroup;*
(x) *$\mathcal{M}R$ is a direct product of an $(n+1)$-nilpotent semigroup and a completely regular* (*or a Clifford*) *semigroup.*

The conditions (iii) *and* (iv) *are equivalent to their left-right analogues.*

The equivalence of the conditions (i), (iii), (iv) and (v) was proved by Chiba and Tominaga in [**75**], 1976, the condition (vi) is assumed from Komatsu and Tominaga [**138**], 1989, and the remaining conditions are from Ćirić and Bogdanović [**80**], 1990, and [**90**], 1996. Note that the assertion (i) $\Leftrightarrow$ (iii) is a consequence of Theorems 2.8 and 5.29.

Theorem 5.48. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a null-ring and a Clifford ring;*
(ii) *R is a subdirect sum of a null-ring and a Clifford ring;*
(iii) $(\forall a \in R)\ aR = aRa;$
(iv) $(\forall a \in R)\ aR \subseteq Ra^2;$
(v) $(\forall a \in R)\ aR = a^2R\ \&\ Ra = Ra^2;$
(vi) *$\mathcal{M}R$ is a null-extension of a completely regular (or a Clifford) semigroup;*
(viii) *$\mathcal{M}R$ is an inflation of a completely regular (or a Clifford) semigroup;*
(ix) *$\mathcal{M}R$ is a subdirect product of a null-semigroup and a completely regular (or a Clifford) semigroup;*
(x) *$\mathcal{M}R$ is a direct product of a null-semigroup and a completely regular (or a Clifford) semigroup.*

The conditions (ii) *and* (iv) *are equivalent to their left-right analogues.*

Rings satisfying (iii) were first studied by Szász in [**309**], 1972, and they are known as *P_1-rings*. The equivalence of the conditions (i) and (iii) was established by Ligh and Utumi in [**301**], 1974, and of (i), (iv) and (v) by Chiba and Tominaga in [**74**], 1975.

Theorem 5.49. *Let $n \in \mathbb{N}$. Then the following conditions on a ring R are equivalent:*

(i) *R is a direct sum of an $(n+1)$-nilpotent ring and a Jacobson ring;*
(ii) *R is a subdirect sum of an $(n+1)$-nilpotent ring and a Jacobson ring;*
(iii) $R^{n+1} \subseteq P(R);$
(iv) *$\mathcal{M}R$ satisfies a variable identity consisting of all identities of the form:*

$$x_1x_2\cdots x_{n+1} = (x_1x_2\cdots x_{n+1})^2u,$$

with $u \in A^+_{n+1}$;
(v) *$\mathcal{M}R$ is an $(n+1)$-nilpotent extension of a union (or a semilattice) of periodic groups;*
(viii) *$\mathcal{M}R$ is an n-inflation of a union (or a semilattice) of periodic groups;*
(ix) *$\mathcal{M}R$ is a subdirect product of an $(n+1)$-nilpotent semigroup and a union (or a semilattice) of periodic groups;*
(x) *$\mathcal{M}R$ is a direct product of an $(n+1)$-nilpotent semigroup and a union (or a semilattice) of periodic groups.*

Characterizations of direct sums of $(n+1)$-nilpotent rings and Jacobson rings through the conditions (iii) and (iv) were given by H. Komatsu and H. Tominaga, while the remaining conditions are due to the first two authors of this paper.

Theorem 5.50. *The following conditions on a ring R are equivalent:*

(i) *R is a direct sum of a null-ring and a Jacobson ring;*
(ii) *R is a subdirect sum of a null-ring and a Jacobson ring;*
(iii) $(\forall a, b \in R)(\exists p(x,y) \in \mathbb{Z}\langle x,y\rangle)\ ab = (ab)^2 p(a,b)$;
(iv) $(\forall a, b \in R)(\exists p(x,y) \in \mathbb{Z}\langle x,y\rangle)\ ab = (ba)^2 p(a,b)$;
(v) *$\mathcal{M}R$ satisfies a variable identity consisting of all identities of the form $xy = (xy)^{m+1}$, with $m \in \mathbb{N}$;*
(vi) *$\mathcal{M}R$ is a null-extension of a union (or a semilattice) of periodic groups;*
(vii) *$\mathcal{M}R$ is an inflation of a union (or a semilattice) of periodic groups;*
(viii) *$\mathcal{M}R$ is a subdirect product of a null-semigroup and a union (or a semilattice) of periodic groups;*
(ix) *$\mathcal{M}R$ is a direct product of a null-semigroup and a union (or a semilattice) of periodic groups.*

The equivalence (i) $\Leftrightarrow$ (v) was proved by Ligh and Luh in [**200**], 1989, and (iii) and (iv) are assumed from Bell and Ligh [**22**], 1989.

Note that the above considered rings are commutative. An elementary proof of the commutativity of rings satisfying the condition (v) was given by Ó Searcóid and Mac Hale in [**232**], 1986.

Note that all direct sums of nil-, nilpotent and null-rings and Jacobson rings considered above can be characterized in terms of variable identities, using the semigroup-theoretical results presented in the previous section. The next three theorems, which were proved by Bell in [**18**], 1977, follow immediately from such obtained characterizations.

Theorem 5.51. *Let R be a ring satisfying one of the following variable identities over A_2:*

(a) $\{xy = w \mid |x|_w \geq 2,\ |y|_w \geq 2\}$;
(b) $\{xy = w \mid w = yx^n,\ n \in \mathbb{N},\ n \geq 2\}$;
(c) $\{xy = w \mid w = y^n x,\ n \in \mathbb{N},\ n \geq 2\}$;
(d) $\{xy = w \mid |y|_w = 0,\ |w| \geq 3\}$;
(e) $\{xy = w \mid |x|_w = 0,\ |w| \geq 3\}$;
(f) $\{xy = w \mid w = x^m y x^n,\ m, n \in \mathbb{N}\}$;
(g) $\{xy = w \mid w = y^m x y^n,\ m, n \in \mathbb{N}\}$.

Then R is commutative.

Theorem 5.52. *If a periodic ring R satisfies a variable identity $xy = w(x,y)$, with $w = yx$, or $h(w) = y$ and $|x|_w \geq 2$, then R is commutative.*

Theorem 5.53. *If a ring R satisfies a variable identity $xy = w(x,y)$, with $h(w) = y$ and $|x|_w \geq 2$, then R is commutative.*

6. Semigroups and rings satisfying certain semigroup identities

There are many semigroup identities for which it was observed that they induce certain structural properties on semigroups on which they are satisfied. But, the general problem of finding all semigroup identities inducing a given structural property was first stated by Clarke in [**78**], 1981, and in a more general form in the Ph. D. thesis of Ćirić [**79**], 1991, and in the paper of Ćirić and Bogdanović [**83**], 1993. This problem was formulated in the following way:

(P1) for a given class $\mathcal{X}$ of semigroups, find all semigroup identities $u = v$ having the property $[u = v] \subseteq \mathcal{X}$.

It was also stated one similar problem:

(P2) for given classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of semigroups, find all semigroup identities $u = v$ having the property $[u = v] \cap \mathcal{X}_1 \subseteq \mathcal{X}_2$.

Identities having the property $[u = v] \subseteq \mathcal{X}$ are called *$\mathcal{X}$-identities*, and identities having the property $[u = v] \cap \mathcal{X}_1 \subseteq \mathcal{X}_2$ are called *$\mathcal{X}_1 \triangleright \mathcal{X}_2$-identities*.

In other words, (P1) is the problem of finding all identities having the property that every semigroup satisfying them must be in $\mathcal{X}$, and (P2) is the problem of finding all identities having the property that every semigroup from $\mathcal{X}_1$ satisfying them must be in $\mathcal{X}_2$. Problems of this type were treated only in the mentioned papers of Clarke, Ćirić and Bogdanović, and also by Ćirić and Bogdanović in [**84**], 1994, and [**88**], 1996. The results obtained in these papers, which characterize all identities that induce decompositions of semigroups into a semilattice of Archimedean semigroups and nil-extensions into a union of groups, will be presented in Sections 1 and 2. In Section 3 we show how these results can be applied in Theory of rings.

As was proved by Chrislock in [**77**], 1969, any semigroup which satisfies a heterotype identity is a nil-extensions of a periodic completely simple semigroup, and hence, any ring satisfying a heterotype semigroup identity is a nil-ring. Therefore, studying of heterotype semigroup identities is not so interesting, and in this section we aim our attention only to homotype semigroup identities. Our topic under question will be identities of the form

$$u(x_1, x_2, \dots, x_n) = v(x_1, x_2, \dots, x_n), \tag{1}$$

where $u, v \in A_n^+$ and $c(u) = c(v) = A_n$, $n \in \mathbb{N}$, $n \geq 2$. We will also treat a particular case

$$u(x, y) = v(x, y), \tag{2}$$

where $u, v \in A_2^+$ and $c(u) = c(v) = A_2$.

Before we give the results promised above, we introduce the following notations:

Notation	Class of semigroups	Notation	Class of semigroups
$\mathcal{A}$	*Archimedean*	$\mathcal{CA}$	*completely Archimedean*
$\mathcal{LA}$	*left Archimedean*	$\mathcal{LG}$	*left groups*
$\mathcal{TA}$	*t-Archimedean*	$\mathcal{G}$	*groups*
$\mathcal{S}$	*semilattices*	$\mathcal{N}$	*nil-semigroups*
$\pi\mathcal{R}$	*π-regular*	$\mathcal{N}_k$	*$k+1$-nilpotent*
$\mathcal{CS}$	*completely simple*	$\mathcal{UG}$	*unions of groups*
$\mathcal{M} \times \mathcal{G}$	*rectangular groups*		

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be classes of semigroups. By $\mathcal{X}_1 \circ \mathcal{X}_2$ we denote the *Maljcev's product* of classes $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ρ such that S/ρ is in $\mathcal{X}_2$ and every ρ-class which is a subsemigroup is in $\mathcal{X}_1$. This product was introduced by Mal'cev in [**206**], 1967. The related decomposition is called an $\mathcal{X}_1 \circ \mathcal{X}_2$*-decomposition*. It is clear that $\mathcal{X} \circ \mathcal{S}$ is the class of all semilattices of semigroups from the class $\mathcal{X}$. If $\mathcal{X}_2$ is a subclass of the class $\mathcal{N}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is a class of all semigroups which are ideal extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$. Also, in such a case, by $\mathcal{X}_1 \circledast \mathcal{X}_2$ we denote a class of all semigroups which are retract extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

6.1. On $\mathcal{A} \circ \mathcal{S}$-identities.

Various types of $\mathcal{A} \circ \mathcal{S}$-identities have been investigated by many authors. The commutativity identity $xy = yx$ is an identity for which it has been first proved that it is an $\mathcal{A} \circ \mathcal{S}$-identity. This was done by Tamura and Kimura in [**319**], 1954. After that, the same property was established by Chrislock in [**76**], 1969, for the *medial identity*: $x_1x_2x_3x_4 = x_1x_3x_2x_4$, by Tamura and Shafer in [**321**], 1972, Tamura and Nordahl in [**320**], 1972, and Nordahl in [**225**], 1974, for the *exponential identity*: $(xy)^n = x^ny^n$, $n \in \mathbb{N}$, $n \geq 2$, by Schutzenberger in [**277**], 1976, for the identity $(xy)^n = \big((xy)^n(yx)^n(xy)^n\big)^n$, $n \in \mathbb{N}$, by Sapir and Suhanov in [**275**], 1985, for the identity $(xy)^m = \big((xy)^m(yx)^m\big)^n(xy)^m$, $m, n \in \mathbb{N}$, and identities of the form $x_1x_2\cdots x_{n+1} = w(x_1, x_2, \dots, x_{n+1})$, $n \in \mathbb{N}$, etc. But, the first general characterization of all $\mathcal{A} \circ \mathcal{S}$-identities was given by Ćirić and Bogdanović in [**83**], 1993, who proved the following theorem:

Theorem 6.1. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is an $\mathcal{A} \circ \mathcal{S}$-identity;*
(ii) (1) *is not satisfied on the semigroup $\mathbb{B}_2$;*
(iii) *there exists a homomorphism $\varphi : A_n^+ \to A_2^+$ and a permutation π of a set $\{u, v\}$ such that one of the following conditions hold:*
 $(A1)$ $(u\pi)\varphi \in (xy)^+$ *and* $(v\pi)\varphi \notin (xy)^+$*;*
 $(A2)$ $(u\pi)\varphi \in (xy)^+x$ *and* $(v\pi)\varphi \notin (xy)^+x$*;*
(iv) *there exists $k \in \mathbb{N}$ and $w \in A_2^*x^2A_2^* \cup A_2^*y^2A_2^*$ such that*

$$[u = v] \subseteq [(xy)^k = w].$$

One description of all identities which are satisfied on the semigroup $\mathbb{B}_2$ was given by Mashevitskiĭ in [**208**], 1979, but it is quite complicated.

Using the above theorem, for many other significant semigroup identities it can be proved that they are $\mathcal{A} \circ \mathcal{S}$-identities. For example, this can be proved for *permutation identities*, by which we mean identities of the form $x_1x_2\cdots x_n = x_{1\sigma}x_{2\sigma}\cdots x_{n\sigma}$, where σ is a non-identical permutation of the set $\{1, 2, \dots, n\}$, for *quasi-permutation identities*, which have the form

$$x_1\cdots x_{k-1}yx_k\cdots x_n = x_{1\sigma}\cdots x_{(l-1)\sigma}y^2x_{l\sigma}\cdots x_{n\sigma},$$

for some permutation σ of the set $\{1,2,\dots,n\}$ and some $k,l \in \{2,\dots,n\}$, and other.

In the mentioned paper of Ćirić and Bogdanović [**83**], from 1993, the authors also investigated some special types of $\mathcal{A}\circ\mathcal{S}$-identities, and all theorems from 6.4 to 6.9 were proved in this paper.

The next theorem says that the set of all $\mathcal{A}\circ\mathcal{S}$-identities coincides with the set of all identities which forces all π-regular semigroups to be uniformly π-regular.

Theorem 6.2. *The identity* (1) *is a* $\pi\mathcal{R}\triangleright\mathcal{CA}\circ\mathcal{S}$*-identity if and only if* (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity.*

The following theorem, which is a consequence of the previous two theorems, give an answer to one problem stated by Shevrin and Suhanov in [**288**], 1989, concerning semigroup varieties consisting of semilattices of Archimedean semigroups.

Theorem 6.3. *Let* $\mathcal{X}$ *be a variety of semigroups. Then the following conditions are equivalent:*

(i) $\mathcal{X}\subseteq\mathcal{A}\circ\mathcal{S}$;
(ii) $\mathcal{X}$ *does not contain the semigroup* $\mathbb{B}_2$;
(iii) *any regular semigroup from* $\mathcal{X}$ *is completely regular;*
(iv) *any completely 0-simple semigroup from* $\mathcal{X}$ *has no zero divisors;*
(v) *in any semigroup with zero from* $\mathcal{X}$ *the set of all nilpotents is a subsemigroup;*
(vi) *in any semigroup with zero from* $\mathcal{X}$ *the set of all nilpotents is an ideal.*

More information on semigroup varieties contained in $\mathcal{A}\circ\mathcal{S}$ can be found in Schutzenberger [**277**], 1976, Sapir and Suhanov [**275**], 1985, Shevrin and Volkov [**287**], 1985, and Shevrin and Suhanov [**288**], 1989.

The next two theorems characterize identities which induce decompositions of π-regular semigroups into a semilattice of left Archimedean semigroups and into a semilattice of t-Archimedean semigroups.

Theorem 6.4. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{LG}\circ\mathcal{N})\circ\mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$ *and* $\mathbb{R}_2$;
(iii) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity and* $t(u)\neq t(v)$.

Theorem 6.5. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R}\triangleright(\mathcal{G}\circ\mathcal{N})\circ\mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{R}_2$ *and* $\mathbb{L}_2$;
(iii) (1) *is an* $\mathcal{A}\circ\mathcal{S}$*-identity,* $h(u)\neq h(v)$ *and* $t(u)\neq t(v)$.

Using the previous theorems, it can be proved that the identities of the form $x_1\cdots x_n x_{m+1}\cdots x_{m+n}=x_{m+1}\cdots x_{m+n}x_1\cdots x_n$, called the (m,n)*-commutativity identities*, are $\mathcal{TA}\circ\mathcal{S}$-identities. These identities were intensively studied by Babcsanyi in [**15**], 1991, Babcsanyi and Nagy in [**16**], 1993, Lajos in [**185**], 1990, and

[**186**], 1991, and by Nagy in [**221**], 1992, and [**222**], [**223**], 1993. The assertion that these identities are $\mathcal{TA} \circ \mathcal{S}$-identities was proved by Lajos in [**185**], 1990.

The next two theorems were also given by Ćirić and Bogdanović in [**83**], 1993:

Theorem 6.6. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{CS} \circledast \mathcal{N}) \circ \mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_{3,1}$*;*
(iii) (1) *is an* $\mathcal{A} \circ \mathcal{S}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t^{(2)}(u) \neq t^{(2)}(v)$.

Theorem 6.7. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{LG} \circledast \mathcal{N}) \circ \mathcal{S}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_2$*;*
(iii) (1) *is an* $\mathcal{A} \circ \mathcal{S}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t(u) \neq t(v)$.

Identities over the two-element alphabet were systematically investigated by Ćirić and Bogdanović in [**88**], 1996. In this paper it was shown that $\mathcal{A} \circ \mathcal{S}$-identities over the two-element alphabet have a more simple characterization, given by the following theorem:

Theorem 6.8. *The identity* (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity if and only if it is p-equivalent to one of the following identities:*

(B1) $xy = w(x,y)$, *where* $w \neq xy$*;*
(B2) $(xy)^k = w(x,y)$, *where* $k \in \mathbb{N}$, $k \geq 2$ *and* $w \notin (xy)^+$*;*
(B3) $(xy)^k x = w(x,y)$, *where* $k \in \mathbb{N}$ *and* $w \notin (xy)^+x$*;*
(B4) $xy^k = w(x,y)$, *where* $k \in \mathbb{N}$, $k \geq 2$ *and* $w \notin xy^+$*;*
(B5) $x^k y = w(x,y)$, *where* $k \in \mathbb{N}$, $k \geq 2$ *and* $w \notin x^+y$.

In the same paper the authors proved the following two theorems:

Theorem 6.9. *The following conditions for the identity* (2) *are equivalent:*

(i) (2) *is a* $\mathcal{LA} \circ \mathcal{S}$*-identity;*
(ii) (2) *is not satisfied on semigroups* $\mathbb{B}_2$ *and* $\mathbb{R}_2$*;*
(iii) (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity and* $t(u) \neq t(v)$.

Theorem 6.10. *The following conditions for the identity* (2) *are equivalent:*

(i) (2) *is a* $\mathcal{TA} \circ \mathcal{S}$*-identity;*
(ii) (2) *is not satisfied on semigroups* $\mathbb{B}_2$, $\mathbb{R}_2$ *and* $\mathbb{L}_2$*;*
(iii) (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity,* $t(u) \neq t(v)$ *and* $h(u) \neq h(v)$.

Note that there are not any characterizations of $\mathcal{LA} \circ \mathcal{S}$-identities and $\mathcal{TA} \circ \mathcal{S}$-identities over the alphabet with more than two letters.

The next two theorems were also proved in [**88**]:

Theorem 6.11. *The identity* (2) *is a* $\mathcal{CS} \triangleright \mathcal{M} \times \mathcal{G}$*-identity if and only if one of the following conditions holds:*

(C1) $h(u) \neq h(v)$ *or* $t(u) \neq t(v)$*;*

(C2) (1) *is p-equivalent to some identity of the form*

$$x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots x^{m_h}y^{n_h} = x^{k_1}y^{l_1}x^{k_2}y^{l_2}\cdots x^{k_s}y^{l_s}$$

$m_i, n_i, k_j, l_j \in \mathbb{N}$, *with* $\gcd(p_x, p_y, h-s) = 1$, *where* $p_x = \sum_{i=1}^{h} m_i - \sum_{j=1}^{s} k_j$ *and* $p_y = \sum_{i=1}^{h} n_i - \sum_{j=1}^{s} l_j$.

(C3) (1) *is p-equivalent to some identity of the form*

$$x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots x^{m_h}y^{n_h}x^{m_{h+1}} = x^{k_1}y^{l_1}x^{k_2}y^{l_2}\cdots x^{k_s}y^{l_s}x^{k_{s+1}}$$

$m_i, n_i, k_j, l_j \in \mathbb{N}$, *with* $\gcd(p_x, p_y, h-s) = 1$, *where* $p_x = \sum_{i=1}^{h+1} m_i - \sum_{j=1}^{s+1} k_j$ *and* $p_y = \sum_{i=1}^{h} n_i - \sum_{j=1}^{s} l_j$.

Theorem 6.12. *The identity* (2) *is a* $\pi\mathcal{R} \triangleright (\mathcal{M} \times \mathcal{G} \circ \mathcal{N}) \circ \mathcal{S}$*-identity if and only if* (2) *is a* $\mathcal{A} \circ \mathcal{S}$*-identity and a* $\mathcal{CS} \triangleright \mathcal{M} \times \mathcal{G}$*-identity.*

6.2. On $\mathcal{UG} \circ \mathcal{N}$-identities. There are many papers in which some types of $\mathcal{UG} \circ \mathcal{N}$-identities have been investigated. The identity $xy = y^m x^m$, for $m, n \in \mathbb{N}$, $m + n \geq 3$, was studied by Tully in [**334**], the identity $xy = (xy)^m$, $m \in \mathbb{N}$, $m \geq 2$, by Gerhard in [**127**], 1977, the *distributive identities* $xyz = xyxz$ and $xyz = xzyz$ by Petrich in [**239**], 1969, etc. Various $\mathcal{UG} \circ \mathcal{N}$-identities of the form $x_1 x_2 \cdots x_{n+1} = w(x_1, x_2, \dots, x_{n+1})$ were investigated by Bogdanović and Stamenković in [**66**], 1988, Ćirić and Bogdanović in [**80**], 1990, Tishchenko in [**328**], 1991, and others. Tamura in [**310**], 1969, stated the general problem of describing structure of semigroups satisfying an identity of the form $xy = w(x, y)$, where $|w| \geq 3$, known as *Tamura's problem.* Various cases appearing in this problem were treated in the mentioned paper of Tamura, and also by Lee in [**196**], 1973, Clarke in [**78**], 1981, and Bogdanović in [**38**], 1987. Complete solutions of all possible cases of the Tamura's problem were given by Ćirić and Bogdanović in [**88**], 1996. More information on problems of Tamura's type can be found in another survey paper of Bogdanović and Ćirić [**49**], 1993.

A complete description of all $\mathcal{UG} \circ \mathcal{N}$-identities was given by Ćirić and Bogdanović in [**84**], 1994, by the following theorem:

Theorem 6.13. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity;*

(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$ *and* $\mathbb{C}_{2,1}$;

(iii) $\Pi(u) \neq \Pi(v)$ *and* (1) *is p-equivalent to some identity of one of the following forms:*

$$(D1) \qquad x_1 u'(x_2, \dots, x_n) = v'(x_1, \dots, x_{n-1})x_n,$$

where $x_1 \underset{l}{\nmid\nmid} v'$ *and* $x_n \underset{r}{\nmid\nmid} u'$;

$$(D2) \qquad x_1 u' x_n = v',$$

where $x_1, x_n \nmid u'$, $x_1 \underset{l}{\nmid\nmid} v'$ *and* $x_n \underset{r}{\nmid\nmid} v'$;

$$(D3) \qquad x_1 u'(x_2, \dots, x_n) = v'(x_2, \dots, x_n)x_1.$$

In the same paper the next two theorems were obtained:

Theorem 6.14. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $(\mathcal{LG} \circ \mathcal{S}) \circ \mathcal{N}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$ *and* $\mathbb{R}_2$*;*
(iii) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity and* $t(u) \neq t(v)$.

Theorem 6.15. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $(\mathcal{G} \circ \mathcal{S}) \circ \mathcal{N}$*-identity;*
(ii) (1) *is a* $(\mathcal{G} \circ \mathcal{S}) \circledast \mathcal{N}$*-identity;*
(iii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{R}_2$ *and* $\mathbb{L}_2$*;*
(iv) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $t(u) \neq t(v)$ *and* $h(u) \neq h(v)$.

Identities which induce retractive nil-extensions of a union of groups were characterized in the following way:

Theorem 6.16. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circledast \mathcal{N}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_{3,1}$*;*
(iii) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t^{(2)}(u) \neq t^{(2)}(v)$.

Theorem 6.17. *The following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $(\mathcal{LG} \circ \mathcal{S}) \circledast \mathcal{N}$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{L}_{3,1}$ *and* $\mathbb{R}_2$*;*
(iii) (1) *is a* $\mathcal{UG} \circ \mathcal{N}$*-identity,* $h^{(2)}(u) \neq h^{(2)}(v)$ *and* $t(u) \neq t(v)$.

Further we consider identities which induce nilpotent and retractive nilpotent extensions of a union of groups. These identities are described by the next two theorems which are also due to Ćirić and Bogdanović [**84**], 1994.

Theorem 6.18. *Let* $k \in \mathbb{N}$. *Then the following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circ \mathcal{N}_k$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{D}_N$ *and* $\mathbb{N}_{k+1}$*;*
(iii) $n \leq k+1$ *and* (1) *is p-equivalent to some identity of the form*

$$x_1 x_2 \dots x_n = w,$$

where $|w| \geq n+1$, $x_1 \underset{l}{\nmid} w$ *and* $x_n \underset{l}{\nmid} w$.

Theorem 6.19. *Let* $k \in \mathbb{N}$. *Then the following conditions for an identity* (1) *are equivalent:*

(i) (1) *is a* $\mathcal{UG} \circledast \mathcal{N}_k$*-identity;*
(ii) (1) *is not satisfied on semigroups* $\mathbb{C}_{1,1}$, $\mathbb{C}_{1,2}$, $\mathbb{C}_{2,1}$, $\mathbb{L}_{3,1}$, $\mathbb{R}_{3,1}$, $\mathbb{D}_N$ *and* $\mathbb{N}_{k+1}$*;*
(iii) (1) *is p-equivalent to some identity of the form*

$$x_1 x_2 \dots x_n = w,$$

where $|w| \geq n+1$, $h^{(2)}(u) \neq x_1 x_2$ *and* $t^{(2)}(v) \neq x_{n-1} x_n$.

Applying the above results to the case of identities over the two-element alphabet, Ćirić and Bogdanović obtained in [**88**], 1996, the following two theorems:

Theorem 6.20. *The identity (2) is a $\mathcal{UG} \circ \mathcal{N}$-identity if and only if it is p-equivalent to an identity of one of the following forms:*

(F1) $xy = w(x,y)$, *where* $w \neq yx$, $w \notin xy^+$ *and* $w \notin x^+y$;
(F2) $xy^m = x^ny$, *where* $m,n \in \mathbb{N}$, $m,n \geq 2$.

Theorem 6.21. *The identity (2) is a $\mathcal{UG} \circledast \mathcal{N}$-identity if and only if it is p-equivalent to an identity of one of the following forms:*

(G1) $xy = w$, *where* $w \in A_2^+$, $|w| \geq 3$, $h^{(2)}(w) \neq xy$ *and* $t^{(2)}(w) \neq xy$;
(G2) $xy^m = x^ny$, *where* $m,n \in \mathbb{N}$, $m,n \geq 2$.

Finally, a consequence of the previous theorem is the following theorem proved by Clarke in [**78**], 1981:

Theorem 6.22. *A semigroup identity determines a variety of inflations of unions of groups if and only if this identity has one of the following forms:*

(i) $x = w$, *where* $w \neq x$;
(ii) $xy = w$, *where* $w \neq yx$ *is a word which neither begins nor ends with* xy.

6.3. Rings satisfying certain semigroup identities. The results presented in the previous two sections, together with the results given in Section 5, make a possibility to give very nice descriptions of the structure of rings satisfying certain semigroup identities. These descriptions will be presented in this section. But, we first introduce some necessary notions.

For a semigroup identity $u = v$ over the alphabet A_n, $n \in \mathbb{N}$, $n \geq 2$, and for $i \in \{1,2,\dots,n\}$, let $p_i = \big||x_i|_u - |x_i|_v\big|$. If there exists $i \in \{1,2,\dots,n\}$ such that $p_i \neq 0$, then we say that $u = v$ is a *periodic identity*, and the number $p = \gcd(p_1,p_2,\dots,p_n)$ is called the *period* of this identity. When we deal with the two-element alphabet $A_2 = \{x,y\}$, then $p_x = \big||x|_u - |x|_v\big|$, $p_y = \big||y|_u - |y|_v\big|$ and $p = \gcd(p_x,p_y)$. Otherwise, if $p_i = 0$, for any $i \in \{1,2,\dots,n\}$, then we say that the identity $u = v$ is *aperiodic*. In some origins periodic identities were called *unbalanced*, and aperiodic identities were called *balanced*. But, our terminology is justified by the following theorem:

Theorem 6.23. *The following conditions for a semigroup identity $u = v$ are equivalent:*

(i) $[u = v]$ *consists of π-regular semigroups;*
(ii) $[u = v]$ *consists of completely π-regular semigroups;*
(iii) $[u = v]$ *consists of periodic semigroups;*
(iv) $u = v$ *is a periodic identity.*

As was noted by Ćirić and Bogdanović in [**90**], 1996, any group satisfying a semigroup identity of the period p satisfies also the identity $x = x^{p+1}$, and any commutative semigroup satisfying the identity $x = x^{p+1}$, satisfies also any identity of the period p. Using these properties and Theorems 5.44 and 6.13, in the mentioned paper Ćirić and Bogdanović proved the following

Theorem 6.24. *A ring R satisfies an $\mathcal{UG} \circ \mathcal{N}$-identity of the period p if and only if R is a direct sum of a nil-ring that satisfies the same identity and a nil-ring that satisfies the identity $x = x^{p+1}$.*

As a consequence of this result, the same authors also obtained

Theorem 6.25. *Any ring which satisfies the identity $xy = w(x, y)$, with $w \notin xy^+ \cup x^+y$, is commutative.*

In the same paper the authors gave some examples which justify that the previous assertion does not hold for identities of the form $xy = xy^n$ and $xy = x^ny$, $n \in \mathbb{N}$.

Many well-known results in Theory of rings are consequences of the above quoted theorems. Here we present the results obtained by Abian and Mc Worter in [**3**], 1964, and Lee in [**196**], 1973.

Let p be a prime. A ring R is called a *pre p-ring* if it is a commutative ring of the characteristic p and it satisfies an identity $xy^p = x^py$. The structure of these rings was described by Abian and Mc Worter in [**3**], 1964, in the following way:

Theorem 6.26. *Let p be a prime. A ring R is a pre-p-ring if and only if it is a direct sum of a p-ring and a pre-p-nil-ring.*

On the other hand, Lee investigated in [**196**], 1973, rings satisfying a system of identities $(xy)^n = xy = x^ny^n$. He proved the following two theorems:

Theorem 6.27. *Let $n \in \mathbb{N}$, $n \geq 2$. A ring R satisfies a system of identities $(x + y)^n = x^n + y^n$, $(xy)^n = xy = x^ny^n$, if and only if it is a direct sum of a ring satisfying the identity $x = x^n$ and a null-ring.*

Theorem 6.28. *A ring R satisfies a system of identities $(xy)^2 = xy = x^2y^2$ if and only if it is a direct sum of a Boolean ring a null-ring.*

Except for the rings satisfying a $\mathcal{UG} \circ \mathcal{N}$-identity, very nice structural descriptions can be given for rings satisfying certain other $\mathcal{A} \circ \mathcal{S}$-identities, especially the periodic ones. The main tool used in these descriptions are Theorems 5.11 and 6.1, and the Everett's representations of rings which follow by these theorems.

Here we present results concerning the structure of rings satisfying a semigroup identity of the form

$$x_1 \cdots x_n = w(x_1, \dots, x_n), \tag{3}$$

where $n \in \mathbb{N}$, $n \geq 2$, $c(w) = A_n$ and $|w| \geq n + 1$. Identities of this form have been investigated by many authors, and the general result characterizing rings satisfying an arbitrary semigroup identity of this form was given by Ćirić, Bogdanović and Petković in [**94**], 1995.

The identity (3) is a periodic $\mathcal{A} \circ \mathcal{S}$-identity. Let p be its period and let

$$h = \max\Big(\{i \mid w = x_1 \cdots x_i u(x_{i+1}, \dots, x_n)\} \cup 0\Big),$$
$$t = \max\Big(\{l \mid w = u'(x_1, \dots, x_{n-l}) x_{n-l+1} \cdots x_n\} \cup 0\Big).$$

The quadruplet (n, p, h, t) was called the *characteristic quadruplet* of the identity (3) [**94**]. Clearly, $h + t \leq n - 1$, and the following conditions hold:

$$x_1 \cdots x_n = x_1 \cdots x_h u(x_{h+1}, \cdots, x_n),$$

with $x_{h+1} \underset{l}{\nparallel} u$, if $h \geq 1$,

$$x_1 \cdots x_n = u'(x_1, \dots, x_{n-t}) x_{n-t+1} \dots x_n,$$

with $x_{n-t} \underset{r}{\nparallel} u'$, if $t \geq 1$, and

$$x_1 \cdots x_n = x_1 \cdots x_h v(x_{h+1}, \cdots, x_{n-t}) x_{n-t+1} \cdots x_n,$$

with $x_{h+1} \underset{l}{\nparallel} v$, $x_{n-t} \underset{r}{\nparallel} v$, if $h \geq 1$ and $t \geq 1$.

Using the above notion, M. Ćirić, S. Bogdanović and T. Petković proved the following theorem:

Theorem 6.29. *Let* (3) *be an identity with the characteristic quadruplet* (n, p, h, t)*. Then the following conditions for a ring* R *are equivalent:*

(i) R *satisfies* (3)*;*

(ii) R *is an ideal extension of an* n*-nilpotent ring* N *by a ring satisfying the identity* $x = x^{p+1}$ *and*

$$N^{h+1} \cdot E(R) = E(R) \cdot N^{t+1} = E(R) \cdot N \cdot E(R) = 0;$$

(iii) R *is an ideal extension of an* n*-nilpotent ring* N *by a ring satisfying the identity* $x = x^{p+1}$ *and*

$$N^{h+1} \cdot \mathrm{Reg}(R) = \mathrm{Reg}(R) \cdot N^{t+1} = \mathrm{Reg}(R) \cdot N \cdot \mathrm{Reg}(R) = 0;$$

(iv) $R = E(N, Q; \theta; [,]; \langle , \rangle)$*, where* N *is an* n*-nilpotent ring,* Q *is a ring satisfying the identity* $x = x^{p+1}$*, and*

$$\theta^b N \theta^c = 0, \qquad \text{for all } b, c \in Q.$$
$$N^{h+1}\theta^b = \theta^b N^{t+1} = 0, \qquad \text{for each } b \in Q.$$

In the particular case when the characteristic quadruplet of (3) has the form $(n, p, 0, 0)$, the same authors obtained the following:

Theorem 6.30. *Let* (3) *be an identity with the characteristic quadruplet* $(n, p, 0, 0)$. *Then a ring* R *satisfies the identity* (3) *if and only if* R *is a direct sum of an* n*-nilpotent ring and a ring satisfying the identity* $x = x^{p+1}$.

The previous theorem is a consequence both of Theorems 6.24 and 6.29.

M. Ćirić, S. Bogdanović and T. Petković gave also a consequent classification of semigroup identities over the two-element and the three-element alphabet.

Theorem 6.31. *For the identity*

$$xy = w(x, y), \tag{4}$$

with $w \in A_2^+$, $|w| \geq 3$, *there are exactly three possibilities:*

(i) (4) *has the characteristic quadruplet* $(2, p, 0, 0)$, *and then a ring satisfies* (4) *if and only if it is a direct sum of a ring satisfying* $x = x^{p+1}$ *and a null-ring, and consequently these rings are commutative.*

(ii) (4) *has the characteristic quadruplet* $(2, p, 1, 0)$, *and this holds if and only if it is of the form* $xy = xy^{p+1}$.

(iii) (4) *has the characteristic quadruplet* $(2, p, 0, 1)$, *and this holds if and only if it is of the form* $xy = x^{p+1}y$.

Theorem 6.32. *For the identity*

$$xyz = w(x, y, z), \tag{5}$$

with $w \in A_3^+$, $|w| \geq 4$, *there are exactly six possibilities:*

(i) (5) *has the characteristic quadruplet* $(3, p, 0, 0)$, *and then a ring satisfies* (5) *if and only if it is a direct sum of a ring satisfying* $x = x^{p+1}$ *and a 3-nilpotent ring.*

(ii) (5) *has the characteristic quadruplet* $(3, p, 1, 0)$, *and this holds if and only if it is of the form* $xyz = xu(y, z)$, $|u| \geq 3$.

(iii) (5) *has the characteristic quadruplet* $(3, p, 0, 1)$, *and this holds if and only if it is of the form* $xyz = v(x, y)z$, $|v| \geq 3$.

(iv) (5) *has the characteristic quadruplet* $(3, p, 2, 0)$, *and this holds if and only if it is of the form* $xyz = xyz^{p+1}$.

(v) (5) *has the characteristic quadruplet* $(3, p, 0, 2)$, *and this holds if and only if it is of the form* $xyz = x^{p+1}yz$.

(vi) (5) *has the characteristic quadruplet* $(3, p, 1, 1)$, *and this holds if and only if it is of the form* $xyz = xy^{p+1}z$.

Important particular types of the identities of the form (3) are the identities $xyz = xyxz$ and $xyz = xzyz$. Rings satisfying the first one are known as *left distributive* (or *left self distributive*) *rings*, and the rings satisfying another identity are *right distributive* (or *right self distributive*). These rings have an important role when we study rings whose any additive endomorphism is also multiplicative (see Birkenmeier and Heatherly [**27**], 1990). Left distributive rings were investigated by Birkenmaier, Heatherly and Kepka in [**29**], 1992. Using Theorem 6.29, these rings can be characterized as follows:

Theorem 6.33. *A ring R is left distributive if and only if it is an ideal extension of a 3-nilpotent ring N by a Boolean ring, and the following conditions hold:*

$$E(R) \cdot N \cdot E(R) = N \cdot E(R) = E(R) \cdot N^2 = 0.$$

Rings which are both left and right distributive are known as ***distributive rings***. These rings are characterized by the following theorem proved by Petrich in [**239**], 1969.

Theorem 6.34. *A ring R is distributive if and only if it is a direct sum of a Boolean ring and a 3-nilpotent ring.*

One generalization of distributive rings was introduced by Ćirić and Bogdanović in [**80**], 1990, who defined a ring R to be ***n-distributive***, where $n \in \mathbb{N}$, $n \geq 2$, if it satisfies the system of identities

$$x_1x_2 \cdots x_{n+1} = (x_1x_2)(x_1x_3) \cdots (x_1x_{n+1}),$$
$$x_1x_2 \cdots x_{n+1} = (x_1x_{n+1})(x_2x_{n+1}) \cdots (x_nx_{n+1}).$$

These rings can be characterized as follows:

Theorem 6.35. *A ring R is n-distributive if and only if it is a direct sum of a ring satisfying the identity $x = x^n$ and a $(n+1)$-nilpotent ring.*

Note finally that rings satisfying identities of the form

$$x_1x_2 \cdots x_n = w(x_1, x_2, \ldots, x_n)$$

(without the assumption $|w| \geq n+1$) were studied by Putcha and Yaqub in [**260**], 1972. They proved that in such a ring R, the commutator ideal $C(R)$ is a nilpotent ideal, and there exists $m \in \mathbb{N}$ such that $R^mC(R)R^m = 0$. Rings satisfying permutation identities were studied by Birkenmeier and Heatherly in [**26**] and [**28**].

References

[1] A. Abian, *Boolean rings*, Branded Press, Boston, 1976.

[2] A. Abian, *Direct product decomposition of commutative semigroups and rings*, Proc. Amer. Math. Soc. **24** (1970), 502–507.

[3] A. Abian and W. A. McWorter, *On the structure of pre p-rings*, Amer. Math. Monthly **71** (1964), 155–157.

[4] H. Abu-Khuzam and A. Yaqub, *Structure of certain periodic rings*, Canad. Math. Bull. **28** (1985), 120–123.

[5] A. Ya. Aĭzenshtat, *On permutable identities*, Sovr. Algebra, L. **3** (1975), 3–12.

[6] D. D. Anderson, *Generalizations of Boolean rings, Boolean-like rings and von Neumann regular rings*, Comment. Math. Univ. St. Paul. **35** (1986), 69–76.

[7] O. Anderson, *Ein Bericht uber Structur abstracter Halbgruppen*, Thesis, Hamburg, 1952.

[8] V. A. Andrunakievich, *Strongly regular rings*, Izv. Akad. Nauk. Moldavskoĭ SSR **11** (1963), 75–77. (in Russian)

[9] V. A. Andrunakievich and Yu. M. Ryabuhin, *Rings without nilpotent elements and completely simple ideals*, Dokl. Akad. Nauk SSSR **180** (1968), 9–11. (in Russian)

[10] V. A. Andrunakievich and Yu. M. Ryabuhin, *Radicals of a algebras and the structure theory*, Nauka, Moskva, 1979. (in Russian)

[11] R. F. Arens and I. Kaplansky, *Topological representation of algebras*, Trans. Amer. Math. Soc. **63** (1948), 457–481.

[12] A. G. Athanassiadis, *A note on V-rings*, Bull. Greek Math. Soc. (N.S.) **2** (1972), 91–95.

[13] R. Ayoub and C. Ayoub, *On the commutativity of rings*, Amer. Math. Monthly **71** (1965), 267–271.

[14] G. Azumaya, *Strongly π-regular rings*, J. Fac. Sci. Hokkaido Univ., Ser 1 **13** (1954), 34–39.

[15] I. Babcsány, *On (m,n)-commutative semigroups*, PU. M. A. Ser. A **2** (1991), no. 3-4, 175–180.

[16] I. Babcsány and A. Nagy, *On a problem of $n_{(2)}$-permutable semigroups*, Semigroup Forum **46** (1996), 398–400.

[17] H. E. Bell, *Some commutativity results for periodic rings*, Acta Math. Acad. Sci. Hungar. **28** (1976), 279–283.

[18] H. E. Bell, *A commutativity study for periodic rings*, Pacific J. Math. **70** (1977), 29–36.

[19] H. E. Bell, *On commutativity of periodic rings and near-rings*, Acta Math. Acad. Sci. Hungar. **36** (1980), 293–302.

[20] H. E. Bell, *On commutativity and structure of periodic rings*, Math. J. Okayama Univ. **27** (1985), 1–3.

[21] H. E. Bell and A. A. Klein, *On finiteness, commutativity and periodicity in rings*, Math. J. Okayama Univ. **35** (1993), 181–188.

[22] H. E. Bell and S. Ligh, *Some periodic theorems for periodic rings and near-rings*, Math. J. Okayama Univ. **31** (1989), 93–99.

[23] H. E. Bell and H. minaga, *On periodic rings and related rings*, Math. J. Okayama Univ. **28** (1986), 101–103.

[24] H. E. Bell and A. Yaqub, *Some periodicity conditions for rings*, Math. J. Okayama Univ. **29** (1987), 179–183.

[25] B. Belluce, S. K. Jain and I. N. Herstein, *Generalized commutative rings*, Nagoya Math. J. **27** (1966), 1–5.

[26] G. F. Birkenmeier and H. Heatherly, *Medial rings and an associated radical*, Czech. Math. J. **40 (115)** (1990), 258–283.

[27] G. F. Birkenmeier and H. Heatherly, *Rings whose additive endomorphisms are ring endomorphisms*, Bull. Austral. Math. Soc. **42** (1990), 145–152.

[28] G. F. Birkenmeier and H. Heatherly, *Permutation identity rings and the medial radical*, Springer-Verlag Lect. Notes in Math. 1448, Noncommutative Ring Theory, 1990, pp. 125–138.

[29] G. F. Birkenmeier, H. Heatherly and T. Kepka, *Rings with left self distributive multiplication*, Acta Math. Hung. **60** (1992), no. 1–2, 107–114.

[30] A. R. Blass and Č. V. Stanojević, *On certain classes of associative rings*, Amer. Math. Monthly **75** (1968), 52–53.

[31] A. R. Blass and Č. V. Stanojević, *On certain classes of associative rings*, Math. Balkanica **1** (1971), 19–21.

[32] S. Bogdanović, *O slabo komutativnoj polugrupi*, Mat. Vesnik **5 (18) (33)** (1981), 145–148.

[33] S. Bogdanović, *Power regular semigroups*, Zbornik radova PMF Novi Sad **12** (1982), 418–428.

[34] S. Bogdanović, *Semigroups of Galbiati-Veronesi*, Algebra and Logic, Zagreb, Inst. of Math., Novi Sad, 1984, pp. 9–20.

[35] S. Bogdanović, *Right π-inverse semigroups*, Zbornik radova PMF, Novi Sad, Ser. Mat. **14** (1984), 187–195.

[36] S. Bogdanović, *Semigroups with a system of subsemigroups*, Inst. of Math., Novi Sad, 1985.

[37] S. Bogdanović, *Inflation of a union of groups*, Mat. Vesnik **37** (1985), 351–355.

[38] S. Bogdanović, *Semigroups of Galbiati-Veronesi* II, Facta Univ. Niš, Ser. Math. Inform. **2** (1987), 61–66.

[39] S. Bogdanović, *Nil-extensions of a completely regular semigroup*, Algebra and Logic, Sarajevo, 1987, Inst. Of Math., Novi Sad, 1989, pp. 7–15.

[40] S. Bogdanović and M. Ćirić, *Semigroups of Galbiati-Veronesi* III (*Semilattice of nil-extensions of left and right groups*), Facta Univ. Niš, Ser. Math. Inform. **4** (1989), 1–14.

[41] S. Bogdanović and M. Ćirić, *A nil-extension of a regular semigroup*, Glasnik matematički **25 (2)** (1991), 3–23.

[42] S. Bogdanović and M. Ćirić, *Semigroups of Galbiati-Veronesi* IV (*Bands of nil-extensions of groups*), Facta Univ. (Niš), Ser. Math. Inform. **7** (1992), 23–35.

[43] S. Bogdanović and M. Ćirić, *Semigroups in which the radical of every ideal is a subsemigroup*, Zbornik radova Fil. fak. Niš **6** (1992), 129–135.

[44] S. Bogdanović and M. Ćirić, *Right π-inverse semigroups and rings*, Zb. rad. Fil. fak. (Niš), Ser. Mat. **6** (1992), 137–140.

[45] S. Bogdanović and M. Ćirić, *Retractive nil-extensions of regular semigroups* I, Proc. Japan Acad. **68** (1992), 115–117.

[46] S. Bogdanović and M. Ćirić, *Retractive nil-extensions of regular semigroups* II, Proc. Japan Acad. **68** (1992), 126–130.

[47] S. Bogdanović and M. Ćirić, *Retractive nil-extensions of bands of groups*, Facta Univ. (Niš), Ser. Math. Inform. **8** (1993), 11–20.

[48] S. Bogdanović and M. Ćirić, *Semigroups*, Prosveta, Niš, 1993. (in Serbian)

[49] S. Bogdanović and M. Ćirić, *Semilattices of Archimedean semigroups and (completely) π-regular semigroups* I (*A survey*), Filomat (Niš) **7** (1993), 1–40.

[50] S. Bogdanović and M. Ćirić, *A new approach to some greatest decompositions of semigroups* (*A survey*), Southeast Asian Bulletin of Math. **18 (3)** (1994), 27–42.

[51] S. Bogdanović and M. Ćirić, *Chains of Archimedean semigroups* (*Semiprimary semigroups*), Indian J. Pure Appl. Math. **25 (3)** (1994), 331–336.

[52] S. Bogdanović and M. Ćirić, *Orthogonal sums of semigroups*, Israel J. Math. **90** (1995), 423–428.

[53] S. Bogdanović and M. Ćirić, *Decompositions of semigroups with zero*, Publ. Inst. Math. Belgrade **57 (71)** (1995), 111–123.

[54] S. Bogdanović and M. Ćirić, *Semilattices of nil-extensions of rectangular groups*, Publ. Math. Debrecen **47/3–4** (1995), 229–235.

[55] S. Bogdanović and M. Ćirić, *A note on left regular semigroups*, Publ. Math. Debrecen **48** (1996), no. 3-4, 285–291.

[56] S. Bogdanović and M. Ćirić, *Semilattices of weakly left Archimedean semigroups*, Filomat (Niš) **9:3** (1995), 603–610.

[57] S. Bogdanović and M. Ćirić, *Radicals of Green's relations*, Czechoslov. Math. J. (to appear).

[58] S. Bogdanović and M. Ćirić, *A note on radicals of Green's relations*, Pure Math. Appl. Ser A **7** (1996), no. 3–4, 215–219.

[59] S. Bogdanović and M. Ćirić, *Semilattices of left completely Archimedean semigroups*, Math. Moravica **1** (1997), 11–16.

[60] S. Bogdanović, M. Ćirić and M. Mitrović, *Semilattices of hereditary Archimedean semigroups*, Filomat (Niš) **9:3** (1995), 611–617.

[61] S. Bogdanović, M. Ćirić and B. Novikov, *Bands of left Archimedean semigroups*, Publ. Math. Debrecen **52 / 1–2** (1998), 85–101.

[62] S. Bogdanović, M. Ćirić and Ž. Popović, *Semilattice decompositions of semigroups revisited*, Semigroup Forum (to appear).

[63] S. Bogdanović and T. Malinović, *(m,n)-two-sided pure semigroups*, Comment. Math. Univ. St. Pauli **35 (2)** (1986), 219–225.

[64] S. Bogdanović and S. Milić, *A nil-extension of a completely simple semigroup*, Publ. Inst. Math. **36 (50)** (1984), 45–50.

[65] S. Bogdanović and S. Milić, *Inflations of semigroups*, Publ. Inst. Math. **41 (55)** (1987), 63–73.

[66] S. Bogdanović and B. Stamenković, *Semigroups in which S^{n+1} is a semilattice of right groups (Inflations of a semilattice of right groups)*, Note di matematica **8** (1988), 155–172.

[67] J. Calais, *Demi-groupes quasi-inversifs*, Compt. Rend. Acad. Sci Paris **252** (1961), 2357–2359.

[68] M. Chacron, *On a theorem of Herstein*, Canad. J. Math. **21** (1969), 1348–1353.

[69] M. Chacron, *Direct products of division rings and a paper of Abian*, Proc. Amer. Math. Soc. **29** (1971), 259–262.

[70] M. Chacron, I. N. Herstein and S. Montgomery, *Structure of certain class of rings with involution*, Canad. J. Math. **27** (1975), 1114–1126.

[71] M. Chacron and G. Thierrin, *σ-reflexive semigroups and rings*, Canad. Math. Bull. **15 (2)** (1972), 185–188.

[72] A. Cherubini and A. Varisco, *A further generalization of a theorem of Outcalt and Yaqub*, Math. Japonica **28 (6)** (1983), 701–703.

[73] K. Chiba and H. Tominaga, *On strongly regular rings*, Proc. Japan Acad. **49** (1973), 435–437.

[74] K. Chiba and H. Tominaga, *Note on strongly regular rings and P_1-rings*, Proc. Japan Acad. **51** (1975), 259–261.

[75] K. Chiba and H. Tominaga, *Generalizations of P_1-rings and gsr-rings*, Math. J. Okayama Univ. **18** (1976), 149–152.

[76] J. L. Chrislock, *On medial semigroups*, Journal of Algebra **12** (1969), 1–9.

[77] J. L. Chrislock, *A certain class of identities on semigroups*, Proc. Amer. Math. Soc. **21** (1969), 189–190.

[78] G. Clarke, *Semigroup varieties of inflations of union of groups*, Semigroup Forum **23** (1981), 311–319.

[79] M. Ćirić, *Decompositions of semigroups and identities*, Ph. D. thesis, University of Belgrade, 1991. (in Serbian)

[80] M. Ćirić and S. Bogdanović, *Rings whose multiplicative semigroups are nil-extensions of a union of groups*, Pure Math. Appl. Ser. A **1** (1990), no. 3-4, 217–234.

[81] M. Ćirić and S. Bogdanović, *A note on π-regular rings*, Pure Math. Appl. Ser. A **3** (1992), no. 1–2, 39–42.

[82] M. Ćirić and S. Bogdanović, *Spined products of some semigroups*, Proc. Japan. Acad., Ser. A **69 (9)** (1993), 357–362.

[83] M. Ćirić and S. Bogdanović, *Decompositions of semigroups induced by identities*, Semigroup Forum **46** (1993), 329–346.

[84] M. Ćirić and S. Bogdanović, *Nil-extensions of unions of groups induced by identities*, Semigroup Forum **48** (1994), 303–311.

[85] M. Ćirić and S. Bogdanović, *Theory of greatest decompositions of semigroups (A survey)*, Filomat (Niš) **9:3** (1995), 385–426.

[86] M. Ćirić and S. Bogdanović, *Orthogonal sums of 0-σ-simple semigroups*, Acta Math. Hung. **70 (3)** (1996), 199–205.

[87] M. Ćirić and S. Bogdanović, *Semilattice decompositions of semigroups*, Semigroup Forum **52** (1996), 119–132.

[88] M. Ćirić and S. Bogdanović, *Identities over the twoelement alphabet*, Semigroup Forum **52** (1996), 365–379.

[89] M. Ćirić and S. Bogdanović, *0-Archimedean semigroups*, Indian J. Pure Appl. Math. **27 (5)** (1996), 463–468.

[90] M. Ćirić and S. Bogdanović, *Direct sums of nil-rings and of rings with Clifford's multiplicative semigroups*, Math. Balkanica **10** (1996), 65–71.

[91] M. Ćirić and S. Bogdanović, *The lattice of positive quasi-orders on a semigroup*, Israel J. Math. **98** (1997), 157–166.

[92] M. Ćirić and S. Bogdanović, *A five-element Brandt semigroup as a forbidden divisor*, Semigroup Forum (to appear).

[93] M. Ćirić, S. Bogdanović and T. Petković, *Semigroups satisfying some variable identities* (to appear).

[94] M. Ćirić, S. Bogdanović and T. Petković, *Rings satisfying some semigroup identities*, Acta Sci. Math. Szeged **61** (1995), 123–137.

[95] M. Ćirić, S. Bogdanović and T. Petković, *Semigroup identities and their recognition by automata* (to appear).

[96] M. Ćirić, S. Bogdanović and T. Petković, *Sums and limits of generalized direct families of algebras* (to appear).

[97] M. Ćirić, S. Bogdanović and M. Bogdanović, *0-Archimedean semigroups* II (to appear).

[98] A. H. Clifford, *A system arising from a weakened set of group postulates*, Annals of Math. **34** (1933), 865–871.

[99] A. H. Clifford, *Semigroups admitting relative inverses*, Annals of Math. **42** (1941), no. 2, 1037–1049.

[100] A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math. **70** (1948), 521–526.

[101] A. H. Clifford, *Semigroups without nilpotent ideals*, Amer. J. Math. **71** (1949), 834–844.

[102] A. H. Clifford, *Extensions of semigroups*, Trans. Amer. Math. Soc. **68** (1950), 165–173.

[103] A. H. Clifford, *Bands of semigroups*, Proc. Amer. Math. Soc. **5** (1954), no. 2, 499–504.

[104] A. H. Clifford, *The fundamental representation of completely regular semigroups*, Semigroup Forum **12** (1976), 341–346.

[105] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups* **I**, Amer. Math. Soc., 1961.

[106] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups* **II**, Amer. Math. Soc., 1967.

[107] R. Croisot, *Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples*, Ann. Sci. Ecole Norm. Sup. **70** (1953), no. 3, 361–379.

[108] F. Dischinger, *Sur les anneaux fortement π-réguliers*, C. R. Acad. Sci. Paris **283 A** (1976), 571–573.

[109] J. L. Dorroh, *Concerning adjunktions to algebras*, Bull. Amer. Math. Soc. **38** (1932), 85–88.

[110] M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly **65** (1958), 506–514.

[111] M. P. Drazin, *Rings with central idempotent or nilpotent elements*, Proc. Edinburgh Math. Soc. **(2) 9** (1958), 157–165.

[112] P. Edwards, *Eventually regular semigroups*, Bull. Austral. Math. Soc. **28** (1983), 23–28.

[113] C. J. Everett, *An extension theory for rings*, Amer. J. Math. **64** (1942), 363–370.

[114] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Mat. Soc. **89** (1958), 79–91.

[115] J. W. Fisher and R. L. Snider, *On the von Neumann regularity of rings with regular prime factor rings*, Pacific J. Math. **54** (1974), 135–144.

[116] I. Fleischer, *A note on subdirect products*, Acta. Math. Acad. Sci. Hungar. **6** (1955), 463–465.

[117] A. Forsythe and N. H. Mc Coy, *On the commutativity of certain rings*, Bull. Amer. Math. Soc. **52** (1946), 523–526.

[118] L. Fuchs, *On subdirect unions* I, Acta. Math. Acad. Sci. Hungar. **3** (1952), 103–120.

[119] L. Fuchs and K. M. Rangaswamy, *On generalized regular rings*, Math. Zeitschr. **107** (1968), 71–81.

[120] J. L. Galbiati e M. L. Veronesi, *Sui semigruppi che sono un band di t-semigruppi*, Istituto Lombardo (Rend. Sc.) **A 114** (1980), 217–234.

[121] J. L. Galbiati e M. L. Veronesi, *Sui semigruppi quasi regolari*, Rend. Ist. Lombardo, Cl. Sc. **A 116** (1982), 1–11.

[122] J. L. Galbiati and M. L. Veronesi, *Semigruppi quasi regolari*, Atti del convegno: Teoria dei semigruppi, (ed. F. Migliorini), Siena, 1982.

[123] J. L. Galbiati and M. L. Veronesi, *On quasi completely regular semigroups*, Semigroup Forum **29** (1984), 271–275.

[124] J. L. Galbiati e M. L. Veronesi, *Sui semigruppi quasi completamente inversi*, Istituto Lombardo (Rend. Sc.) (1984).

[125] J. L. Galbiati e M. L. Veronesi, *Bande di semigruppi quasi-bisemplici*, Sci. Math. Vitae Pensiero (1994), 157–172.

[126] A. I. Gerchikov, *Rings decomposable into a direct sum of division rings*, Mat. Sbornik **7** (1940), 591–597. (in Russian)

[127] J. Gerhard, *Semigroups with idempotent power* II, Semigroup Forum **14** (1977), 375–388.

[128] K. R. Goodearl, *Von Neumann regular rings*, Pitman, London, 1979.

[129] J. Grosen, H. Tominaga and A. Yaqub, *On weakly periodic rings, periodic rings and commutativity theorems*, Math. J. Okayama Univ. **32** (1990), 77–81.

[130] V. Gupta, *Weakly π-regular rings and group rings*, Math. J. Okayama Univ. **19** (1977), 123–127.

[131] M. Hall, *The theory of groups*, Mac Millan, New York, 1959.

[132] I. N. Herstein, *A generalization of a theorem of Jacobson* I, Amer. J. Math. **73** (1951), 756–762.

[133] I. N. Herstein, *A generalization of a theorem of Jacobson* III, Amer. J. Math. **75** (1953), 105–111.

[134] I. N. Herstein, *The structure of a certain class of rings*, Amer. J. Math. **75** (1953), 866–871.

[135] I. N. Herstein, *A note on rings with central nilpotents elements*, Proc. Amer. Math. Soc. **5** (1954), 620.

[136] I. N. Herstein, *An elementary proof of a theorem of Jacobson*, Duke Math. J. **21** (1954), 45–48.

[137] I. N. Herstein, *A conditions for the commutativity of rings*, Canad. J. Math. **9** (1957), 583–586.

[138] I. N. Herstein, *Wedderburn's theorem and a theorem of Jacobson*, Amer. Math. Monthly **68** (1961), 249–251.

[139] Y. Hirano, *Some studies on strongly π-regular rings*, Math. J. Okayama Univ. **20** (1978), 115–118.

[140] Y. Hirano, *Some characterizations of π-regular rings of bounded index*, Math. J. Okayama Univ. **32** (1990), 97–101.

[141] Y. Hirano and H. Tominaga, *Rings decomposed into direct sums of nil rings and certain reduced rings*, Math. J. Okayama Univ. **27** (1985), 35–38.

[142] Y. Hirano, H. Tominaga and A. Yaqub, *On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element*, Math. J. Okayama Univ. **30** (1988), 33–40.

[143] M. Hongan, *Note on strongly regular near-rings*, Proc. Edinburgh. Math. Soc. **29** (1986), 379–381.

[144] J. M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.

[145] K. Iséki, *A characterization of regular semigroups*, Proc. Japan Acad. **32** (1956), 676–677.

[146] J. Ivan, *On the decomposition of a simple semigroup into a direct product*, Mat. Fyz. Časopis **4** (1954), 181–202. (in Slovak)

[147] N. Jacobson, *Structure of rings*, Amer. Math. Soc., Providence, Rhode Island, 1956.

[148] N. Jacobson, *Structure theory for algebraic algebras of bounded degree*, Ann. of Math. **46** (1945), 695–707.

[149] T. Kandô, *Strong regularity of arbitrary rings*, Nagoya Math. J. **4** (1952), 51–53.

[150] I. Kaplansky, *Topological representation of algebras* II, Trans. Amer. Math. Soc. **68** (1950), 62–75.

[151] I. Kaplansky, *Fields and rings*, Univ. of Chicago Press, Chicago and London, 1969.

[152] A. Kertész, *Vorlesungen über artinische Ringe*, Akadémiai Kiadó, Budapest, 1968.

[153] A. Kertész, *Lectures on Artinian rings*, Akadémiai Kiadó, Budapest, 1987.

[154] A. Kertész and O. Steinfeld, *Über reguläre Z-ringe*, Beiträge zur Algebra und Geometrie **3** (1974), 7–10.

[155] N. Kimura, *Maximal subgroups of a semigroup*, Kodai Math. Sem. Rep. (1954), 85–88.

[156] N. Kimura, *The structure of idempotent semigroups* I, Pacific J. Math. **8** (1958), 257–275.

[157] F. Kmeť, *Radicals and their left ideal analogues in a semigroup*, Math. Slovaca **38** (1988), 139–145.

[158] H. Komatsu and H. Tominaga, *Some decompositions theorems for rings*, Math. J. Okayama Univ. **31** (1989), 121–124.

[159] H. Komatsu and H. Tominaga, *Commutativity theorems for algebras and rings*, Math. J. Okayama Univ. **33** (1991), 71–95.

[160] L. Kovácz, *A note on regular rings*, Publ. Math. Debrecen **4** (1956), 465–468.

[161] W. Krull, *Idealtheorie in Ringe ohne Endlichkeitsbedungung*, Math. Ann. **101** (1929), 729–744.

[162] W. Krull, *Subdirecte Summendarstellungen von Integritätsbereichen*, Math. Z. **52** (1950), 810–826.

[163] S. Lajos, *Generalized ideals in semigroups*, Acta Sci. Math. (Szeged) **22** (1961), 217–222.

[164] S. Lajos, *A remark on regular semigroups*, Proc. Japan Acad. **37** (1961), 29–30.

[165] S. Lajos, *Note on semigroups which are semilattices of groups*, Proc. Japan Acad. **44** (1968), 805–806.

[166] S. Lajos, *On regular duo rings*, Proc. Japan Acad. **45** (1969), 157–158.

[167] S. Lajos, *On semilattices of groups*, Proc. Japan Acad. **45** (1969), 383–384.

[168] S. Lajos, *On semigroups which are semilattices of groups*, Acta Sci. Math. **30** (1969), 133–135.

[169] S. Lajos, *A characterization of regular duo rings*, Annales Univ. Budapest, Sect. Math. **13** (1970), 71–72.

[170] S. Lajos, *A characterization of regular duo semigroups*, Mat. Vesnik **7 (22)** (1970), 401–402.

[171] S. Lajos, *Notes on semilattices of groups*, Proc. Japan Acad. **46** (1970), 151–152.

[172] S. Lajos, *Notes on regular semigroups*, Proc. Japan Acad. **46** (1970), 253–254.

[173] S. Lajos, *On semilattices of groups* II, Proc. Japan Acad. **47** (1971), 36–37.

[174] S. Lajos, *A new characterization of regular duo semigroups*, Proc. Japan Acad. **47** (1971), 394–395.

[175] S. Lajos, *On semigroups that are semilattices of groups*, Depth. of Math., Karl Marx Univ. of Econ., Budapest (1971), 1–14.

[176] S. Lajos, *A remark on semigroups that are semilattices of right groups*, Mat. Vesnik **8 (23)** (1971), 315–316.

[177] S. Lajos, *A characterization of semilattices of left groups*, Mat. Vesnik **9 (24)** (1972), 363–364.

[178] S. Lajos, *On regular right duo semigroups*, Elemente der Math. **27** (1972), 86–87.

[179] S. Lajos, *Notes on regular semigroups* V, Math. Balkanica **3** (1973), 308–309.

[180] S. Lajos, *Characterizations of semilattices of groups*, Math. Balkanica **3** (1973), 310–311.

[181] S. Lajos, *Semilattice decompositions of semigroups*, Abstracts of 5th Balkan Math. Congress, Beograd, 1974.

[182] S. Lajos, *Theorems on* $(1,1)$*-ideals in semigroups* II, Depth. of Math., Karl Marx Univ. of Econ., Budapest (1974), 1–17.

[183] S. Lajos, *Characterizations of completely regular elements in semigroups*, Acta. Sci. Math. (Szeged) **40** (1978), 297–300.

[184] S. Lajos, *A remark on Abelian regular rings*, Notes on semigroups IX, Depth. of Math., Karl Marx Univ. of Econ., Budapest (1983), no. 4, 1–6.

[185] S. Lajos, *Fibonacci characterizations and* (m,n)*-commutativity in semigroup theory*, Pure Math. Appl. Ser. A **1** (1990), 59–65.

[186] S. Lajos, *Notes on externally commutative semigroups*, Pure Math. Appl. Ser. A **2** (1991), no. 1-2, 67–72.

[187] S. Lajos, *Bi-ideals in semigroups,* I; *A survey*, Pure Math. Appl. Ser. A **2** (1991), no. 3–4, 215–237.

[188] S. Lajos and F. Szász, *Some characterizations of two-sided regular rings*, Acta. Sci. Math. Szeged **31** (1970), 223–228.

[189] S. Lajos and F. Szász, *Characterizations of strongly regular rings* I, Proc. Japan Acad. **46** (1970), 38–40.

[190] S. Lajos and F. Szász, *Characterizations of strongly regular rings* II, Proc. Japan Acad. **46** (1970), 287–289.

[191] S. Lajos and F. Szász, *Bi-ideals in associative rings*, Acta. Sci. Math. (Szeged) **32** (1971), 185–193.

[192] S. Lajos and G. Szász, *Generalized regularity in semigroups*, Depth. of Math., Karl Marx Univ. of Econ., Budapest (1975), no. 7, 1–23.

[193] S. Lajos and G. Szász, *On regular rings*, Math. Vesnik **13** **(28)** (1976), 163–165.

[194] G. Lallement, *Demi-groupes réguliers*, Ann. Math. Pure Appl. **(4)** **77** (1967), 47–129.

[195] G. Lallement, *Semigroups and combinatorial applications*, Willey Interscience, New York, 1979.

[196] S.-M. Lee, *Rings and semigroups which satisfy the identity* $(xy)^n = xy = x^ny^n$, Nanta Math. **6** (1973), no. 1, 21–28.

[197] A. E. Liber, *On theory of generalized groups*, Dokl. Akad. Nauk SSSR **97** (1954), 25–28. (in Russian)

[198] L. Li and B. M. Schein, *Strongly regular rings*, Semigroup Forum **32** (1985), 145–161.

[199] S. Ligh, *A generalization of a theorem of Wedderburn*, Bull. Austral. Math. Soc. **8** (1973), 181–185.

[200] S. Ligh and J. Luh, *Direct sum of J-rings and zero rings*, Amer. Math. Monthly **96** (1989), 40–41.

[201] S. Ligh and Y. Utumi, *Direct sums of strongly regular rings and zero rings*, Proc. Japan Acad. **50** (1974), 589–592.

[202] J. Luh, *A characterization of regular rings*, Proc. Japan Acad. **39** (1963), 741–742.

[203] J. Luh, *On the structure of J-rings*, Amer. Math. Monthly **74** (1967), 164–166.

[204] J. Luh, *On the commutativity of J-rings*, Canad. J. Math. **19** (1967), 1289–1292.

[205] J. Luh, *On the structure of pre-J-rings*, Hung-ching Chow Sixty-fifth Anniversary Volume, Math. Res. Center Nat. Taiwan Univ., Taipei, 1967, pp. 47–52.

[206] A. I. Mal'cev, *On products of classes of algebraic systems*, Sib. Mat. Žurn **2** (1967), no. 8, 346–365. (in Russian)

[207] A. I. Mal'cev, *Algebraic systems*, Nauka, Moskva, 1970. (in Russian)

[208] G. I. Mashevickiĭ, *Identities on Brandt semigroups*, Polugruppov. mnogoobraziya i polugruppy endomorfizmov, L. (1979), 126–137.

[209] N. H. Mc Coy, *Generalized regular rings*, Bull. Amer. Math. Soc. **45** (1939), 175–177.

[210] N. H. Mc Coy, *The theory of rings*, Macmillan, New York–London, 1964.

[211] N. H. Mc Coy and D. Montgomery, *A representation of generalized Boolean rings*, Duke Math. J. **3** (1937), 455–459.

[212] S. Mc Lane, *Extensions and obstructions for rings*, Illinois J. Math. **2** (1958), 316–345.

[213] D. G. Mead and T. Tamura, *Semigroups satisfying* $xy^m = yx^m = (xy^m)^n$, Proc. Japan Acad. **44** (1968), 779–781.

[214] I. Mogami, *On certain periodic rings*, Math. J. Okayama Univ. **27** (1985), 5–6.

[215] E. H. Moore, *General analysis*, Mem. Amer. Philos. Soc. Vol 1, Philadelphia, 1936.

[216] U. Müller and M. Petrich, *Erweiterungen eines Ringes durch eine direkte Summe zyklischer Ringe*, J. Reine Angew. Math. **248** (1971), 47–74.

[217] U. Müller and M. Petrich, *Translationshülle and wesentlichen Erweiterungen eines zyklischen Ringes*, J. Reine Angew. Math. **249** (1971), 34–52.

[218] W. D. Munn, *Pseudoinverses in semigroups*, Proc. Camb. Phil. Soc. **57** (1961), 247–250.

[219] W. D. Munn and R. Penrose, *A note on inverse semigroups*, Proc. Camb. Phil. Soc. **51** (1955), 396–399.

[220] T. Nagahara and H. Tominaga, *Elementary proofs of a theorem of Wedderburn and a theorem of Jacobson*, Abh. Math. Sem. Univ. Hamburg. **41** (1974), 72–74.

[221] A. Nagy, *On the structure of (m,n)-commutative semigroups*, Semigroup Forum **45** (1992), 183–190.

[222] A. Nagy, *Semilattice decomposition of $n_{(2)}$-permutable semigroups*, Semigroup Forum **46** (1993), 16–20.

[223] A. Nagy, *Subdirectly irreducible right commutative semigroups*, Semigroup Forum **46** (1993), 187–198.

[224] J. von Neumann, *On regular rings*, Proc. Nat. Acad. Sci. **22** (1936), 707–713.

[225] T. Nordahl, *Semigroup satisfying $(xy)^m = x^m y^m$*, Semigroup Forum **8** (1974), 332–346.

[226] W. K. Nicholson, *Rings whose elements are quasi-regular or regular*, Aequationes Math. **9** (1973), 64–70.

[227] C. Niţă, *Anneaux N-réguliers*, Rev. Roum. Math. Pures Appl. **20** (1975), 793–801.

[228] M. Ôhori, *On non-commutative generalized p.p. rings*, Math. J. Okayama Univ. **26** (1984), 157–167.

[229] M. Ôhori, *On strongly π-regular rings and periodic rings*, Math. J. Okayama Univ. **27** (1985), 49–52.

[230] M. Ôhori, *Some studies on generalized p.p. rings and hereditary rings*, Math. J. Okayama Univ. **27** (1985), 53–70.

[231] M. Ó Searcóid, *A structure theorem for generalized J-rings*, Proc. Roy. Irish Acad. **87A** (1987), 117–120.

[232] M. Ó Searcóid and D. MacHale, *Two elementary generalizations of Boolean rings*, Amer. Math. Monthly. **93** (1986), 121–122.

[233] Y. S. Park, J. P. Kim and M. G. Sohn, *Semiprime ideals in semigroups*, Math. Japonica **33** (1988), 269–273.

[234] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Phil. Soc. **51** (1955), 406–413.

[235] R. Penrose, *On best approximate solutions of matrix equations*, Proc. Cambridge Phil. Soc. **52** (1956), 17–19.

[236] M. Petrich, *The maximal semilattice decomposition of a semigroup*, Math. Zeitschr. **85** (1964), 68–82.

[237] M. Petrich, *On extensions of semigroups determined by partial homomorphisms*, Nederl. Akad. Wetensch. Indag. Math. **28** (1966), 49–51.

[238] M. Petrich, *Sur certaines classes de demi-groupes* III, Acad. Roy. Belg. Cl. Sci. **53** (1967), 60–73.

[239] M. Petrich, *Structures des demi-groupes et anneaux distributifs*, C. R. Acad. Sci. Paris, Ser A **268** (1969), 849–852.

[240] M. Petrich, *The translational hull in semigroups and rings*, Semigroup Forum **1** (1970), 283–360.

[241] M. Petrich, *Introduction to semigroups*, Merill, Ohio, 1973.

[242] M. Petrich, *Regular semigroups which are subdirect products of a band an a semilattice of groups*, Glasgow Math. J. **14** (1973), 27–49.

[243] M. Petrich, *Rings and semigroups*, Lecture Notes in Math., vol. 380, Springer-Varlag, Berlin, 1974.

[244] M. Petrich, *The structure of completely regular semigroups*, Trans. Amer. Math. Soc. **189** (1974), 211–236.

[245] M. Petrich, *Structure of regular semigroups*, Cahiers Mathématique no. 11, Montpellier, 1977.

[246] M. Petrich, *Lectures in semigroups*, Akademie-Verlag, Berlin, 1977.

[247] M. Petrich, *Inverse semigroups*, J. Wiley & Sons, New York, 1984.

[248] G. Pollák, *On the consequences of permutation identities*, Acta. Sci. Math. **34** (1973), 323–333.

[249] B. Pondělíček, *A certain relation for closure operation on a semigroup*, Czech. Math. J. **20 (95)** (1970), 220–230.

[250] B. Pondělíček, *A certain equivalence on a semigroup*, Czech. Math. J. **21 (96)** (1971), 109–117.

[251] B. Pondělíček, *A characterization of semilattices of left or right groups*, Czech. Math. J. **22 (97)** (1972), 522–524.

[252] G. B. Preston, *Inverse semi-groups*, J. London Math. Soc. **29** (1954), 396–403.

[253] G. B. Preston, *Inverse semi-groups with minimal right ideals*, J. London Math. Soc. **29** (1954), 404–411.

[254] G. B. Preston, *Representations of inverse semi-groups*, J. London Math. Soc. **29** (1954), 411–419.

[255] M. S. Putcha, *Semilattice decomposition of semigroups*, Semigroup Forum **6** (1973), 12–34.

[256] M. S. Putcha, *Bands of t-archimedean semigroups*, Semigroup Forum **6** (1973), 232–239.

[257] M. S. Putcha, *Minimal sequences in semigroups*, Trans. Amer. Math. Soc. **189** (1974), 93–106.

[258] M. S. Putcha, *Rings which are semilattices of Archimedean semigroups*, Semigroup Forum **23** (1981), 1–5.

[259] M. S. Putcha and A. Yaqub, *Semigroups satisfying permutation identities*, Semigroup Forum **3** (1971), 68–73.

[260] M. S. Putcha and A. Yaqub, *Rings satisfying monomial identities*, Proc. Amer. Math. Soc. **32** (1972), 52–56.

[261] M. S. Putcha and A. Yaqub, *Structure of rings satisfying certain polynomial identities*, J. Math. Soc. Japan **24** (1972), 123–127.

[262] M. S. Putcha and J. Weissglass, *Semigroups satisfying variable identities*, Semigroup Forum **3** (1971), 64–67.

[263] M. S. Putcha and J. Weissglass, *Semigroups satisfying variable identities II*, Trans. Amer. Math. Soc. **168** (1972), 113–119.

[264] R. Rado, *Note on generalized inverses of matrices*, Proc. Cambridge Phil. Soc. **52** (1956), 600–601.

[265] V. S. Ramamuurthi, *Weakly regular rings*, Canad. Math. Bull. **16** (1973), 317-321.

[266] V. V. Rasin, *On the varieties of cliffordian semigroups*, Semigroup Forum **23** (1981), 201–220.

[267] V. N. Saliĭ, *Equational complete varieties of semigroups*, Izv. Vysh. Uchebn. Zav. Mat. **5** (1969), 61–68. (in Russian)

[268] Y. V. Reddy and C. V. L. N. Murty, *On strongly π-regular near-rings*, Proc. Edinburgh Math. Soc. **27** (1984), 61–64.

[269] N. R. Reily and H. E. Scheiblich, *Congruences on regular semigroups*, Pacific J. Math. **23** (1967), 349–360.

[270] L. Rédei, *Algebra* I, Pergamon Press, Oxford-New York, 1967.

[271] D. Rees, *On semigroups*, Proc. Camb. Phil. Soc. **36** (1940), 387–400.

[272] X. M. Ren and K. P. Shum, *Green's star relations on completely Archimedean semigroups*, Facta Univ. (Niš), Ser. Math. Inform. (to appear).

[273] X. M. Ren, K. P. Shum and Y. Q. Guo, *On spined products of quasi-rectangular groups*, Algebra Colloquium (to appear).

[274] T. Saitô, *On semigroups which are semilattices of left simple semigroups*, Math. Japonica **18** (1973), 95–97.

[275] M. V. Sapir and E. V. Suhanov, *On varieties of periodic semigroups*, **Izv.** Vyzov. Mat. **4** (1981), 48–55. (in Russian)

[276] B. M. Schein, *O-rings and LA-rings*, Izv. Vysh. Uch. Zav. MAt. **2** **(51)** (1966), 111-122, (English translation: *Amer. Math. Soc. Transl.* **96** (1970), 137–152). (in Russian)

[277] M. Schutzenberger, *Sur le produit de concatenation non ambigu*, Semigroup Forum **13** (1976), 47–75.

[278] Š. Schwarz, *Zur Theorie der Halbgruppen*, Sbornik pràc Prirodovekej Fakulty Slov. Univ. v Bratislave (1943), no. 6.

[279] Š. Schwarz, *On the structure of simple semigroups without zero*, Czech. Math. J. **1** **(76)** (1951), 41–53.

[280] Š. Schwarz, *On semigroups having a kernel*, Czech. Math. J. **1** **(76)** **(1951)**, 259–301. (in Russian)

[281] Š. Schwarz, *A theorem on normal semigroups*, Czech. Math. J. **10** **(85)** (1960), 197–200.

[282] L. N. Shevrin, *On decompositions of quasi-periodic semigroups into a band of Archimedean semigroups*, XIV Vsesoyuzn. algebr. konf. Tezisy dokl, Novosibirsk, vol. 1, **1977**, pp. 104–105. (in Russian)

[283] L. N. Shevrin, *Quasi-periodic semigroups having a decomposition on unipotent semigroups*, XVI Vsesoyuzn. algebr. konf. Tezisy dokl, L, Vol. 1, 1981, pp. 177–178. **(in** Russian)

[284] L. N. Shevrin, *Quasi-periodic semigroups decomposable into a band of Archimedean semigroups*, XVI Vsesoyuzn. algebr. konf. Tezisy dokl, L, Vol. 1, 1981, p. 188. (in Russian)

[285] L. N. Shevrin, *Theory of epigroups*, Mat. Sbornik **185** (1994), no. 8, 129–160. (in Russian)

[286] L. N. Shevrin, *Theory of epigroups* II, Mat. Sbornik **185** (1994), no. 9, 153–176. (in Russian)

[287] L. N. Shevrin and M. V. Volkov, *Identities on semigroups*, Izv. Vysh. Uchebn. Zav. Mat. **11** (1985), 3–47. (in Russian)

[288] L. N. Shevrin and E. V. Suhanov, *Structural aspects of theory of varieties of semigroups*, Izv. Vysh. Uchebn. Zav. Mat. **6** (1989), 3–39. (in Russian)

[289] K. P. Shum and X. M. Ren, *Completely Archimedean semigroups*, Proc. of the Int. Math. Conf. at Kaoshing, Taiwan, 1994, World Scientific, 1995, pp. 193–202.

[290] K. P. Shum, X. M. Ren and Y. Q. Guo, *On quasi-left groups*, Groups-Korea '94, Walter de Gruntyre, 1995, pp. 285–288.

[291] L. A. Skornyakov, ed., *General algebra*, vol. 1, Nauka, Moskva, 1990. (in Russian)

[292] L. A. Skornyakov, ed., *General algebra*, vol. 2, Nauka, Moskva, 1991. (in Russian)

[293] Č. V. Stanojević, *A sufficient condition for commutativity of division rings with characteristic p*, Bull. Soc. Math. Phys. Macedonie **12** (1961), 25–27. (in Serbian)

[294] O. Steinfeld, *On ideal-quotients and prime ideals*, Acta Math. Acad. Sci. Hung. **4** (1953), 289–298.

[295] O. Steinfeld, *Bemerkung zu einer Arbeit von T. Szele*, Acta Math. Acad. Sci. Hung. **6** (1955), 479–484.

[296] O. Steinfeld, *Über die Quasiideale von Halbgruppen*, Publ. Math. Debrecen **4** (1956), 262–275.

[297] O. Steinfeld, *Über die Quasiideale von Ringen*, Acta Sci. Math. Szeged **17** (1956), 170–180.

[298] O. Steinfeld, *Über die Quasiideale von Halbgruppen mit eigentlichem Suschkewitsch-Kern*, Acta Sci. Math. Szeged **18** (1957), 235–242.

[299] O. Steinfeld, *Über die regulären duo-Elemente in Gruppoid-Verbänden*, Acta Sci. Math. Szeged **32** (1971), 327–331.

[300] O. Steinfeld, *Notes on regular duo elements, rings and semigroups*, Studia Sci. Math. Hung. 8 (1973), 161–164.

[301] O. Steinfeld, *Quasi-ideals in rings and semigroups*, Akadémiai Kiadó, Budapest, 1978.

[302] M. H. Stone, *The theory of representations of Boolean algebras*, Trans. Amer. Math. Soc. **40** (1936), 37–111.

[303] R. P. Sullivan, *Research problems*, Period. Math. Hungar. 8 (1977), 313–314.

[304] A. K. Sushkevich, *Über die wendlichen Gruppen ohne das Gesetz des einden figen umkehrborkeit*, Math. Ann. **99** (1928), 30–50.

[305] A. K. Sushkevich, *Theory of generalized groups*, GNTI, Kharkov–Kiev, 1937.

[306] I. Sussman, *A generalization of Boolean rings*, Math. Ann. **136** (1958), 326–338.

[307] F. Szász, *Über ringe mit Minimalbedingung für Hauptrechtsideale* I, Publ. Math. Debrecen **7** (1960), 54-64.

[308] F. Szász, *Über ringe mit Minimalbedingung für Hauptrechtsideale* II, Acta Math. Acad. Sci. Hung. **12** (1961), 417–439.

[309] F. Szász, *Some generalizations of strongly regular rings* I, Math. Japonicae **17** (1972), 115–118.

[310] F. Szász, *Some generalizations of strongly regular rings* II, Math. Japonicae **18** (1973), 87–90.

[311] F. Szász, *Some generalizations of strongly regular rings* III, Math. Japonicae **18** (1973), 91–94.

[312] F. Szász, *On generalized Subcommutative Regular rings*, Monatshefte Math. **77** (1973), 67–71.

[313] F. Szász, *Radicals of rings*, Akadémiai Kiadó, Budapest, 1981.

[314] T. Tamura, *The theory of construction of finite semigroups* I, Osaka Math. J. **8** (1956), 243–261.

[315] T. Tamura, *Semigroups satisfying identity* $xy = f(x, y)$, Pacific J. Math. **3** (1969), 513–521.

[316] T. Tamura, *On Putcha's theorem concerning semilattice of archimedean semigroups*, Semigroup Forum **4** (1972), 83–86.

[317] T. Tamura, *Note on the greatest semilattice decomposition of semigroups*, Semigroup Forum **4** (1972), 255–261.

[318] T. Tamura, *Semilattice indecomposable semigroups with a unique idempotent*, Semigroup Forum **24** (1982), 77–82.

[319] T. Tamura and N. Kimura, *On decomposition of a commutative semigroup*, Kodai Math. Sem. Rep. **4** (1954), 109–112.

[320] T. Tamura and T. Nordahl, *On exponential semigroups* II, Proc. Japan Acad. **48** (1972), 474–478.

[321] T. Tamura and J. Shafer, *On exponential semigroups* I, Proc. Japan Acad. **48** (1972), 77–80.

[322] G. Thierrin, *Sur une condition nécessaire et suffisante pour qu'un semigroupe siot un groupe*, Compt. Rend. Acad. Sci. Paris **232** (1951), 376–378.

[323] G. Thierrin, *Sur les èlèments inversifs et les èlèments unitaires d'un demi-groupe inversif*, Compt. Rend. Acad. Sci. Paris **234** (1952), 33–34.

[324] G. Thierrin, *Contribution à la théorie des anneaux et des demi-groupes*, Comment. Math. Helv. **32** (1957), 93–112.

[325] G. Thierrin, *On duo rings*, Canad. Math. Bull. **3** (1960), 167–172.

[326] G. Thierrin, *Sur le radical corporoidal d'un anneau*, Canad. J. Math. **12** (1960), 101–106.

[327] G. Thierrin, *Quelques caractérisations du radical d'un anneau*, Canad. Math. Bull. **10** (1967), 643–647.

[328] A. V. Tishchenko, *Note on semigroup varieties of finite index*, Izv. Vysh. Uchebn. Zav. Mat. (1991), 79–83. (in Russian)

[329] H. Tominaga, *Some remarks on π-regular rings of bounded index*, Math. J. Okayama Univ. **4** (1955), 135–144.

[330] H. Tominaga, *On s-unital rings*, Math. J. Okayama Univ. **18** (1976), 117–134.

[331] H. Tominaga, *On s-unital rings* II, Math. J. Okayama Univ. **19** (1977), 171–182.

[332] H. Tominaga, *Rings decomposed into direct sums of J-rings and nil-rings*, Int. J. Math. Math. Sci. **8** (1985), 205–207.

[333] H. Tominaga and A. Yaqub, *On rings satisfying the identity $(x - x^2)^2 = 0$*, Math. J. Okayama Univ. **25** (1983), 181–184.

[334] E. J. Tully, *Semigroups satisfying an identity of the form $xy = y^m x^n$*, (via [309]).

[335] V. V. Vagner, *Generalized groups*, Dokl. Akad. Nauk SSSR **84** (1952), 1119–1122. (in Russian)

[336] V. V. Vagner, *On theory of partial mappings*, Dokl. Akad. Nauk SSSR **84** (1952), 653–656. (in Russian)

[337] V. V. Vagner, *Theory of generalized heaps and generalized groups*, Mat. Sbornik **32** (1953), 545–632. (in Russian)

[338] P. S. Venkatesan, *Right (left) inverse semigroups*, J. of Algebra **31** (1974), 209–217.

[339] M. L. Veronesi, *Sui semigruppi quasi fortemente regolari*, Riv. Mat. Univ. Parma (4) **10** (1984), 319–329.

[340] A. D. Wallace, *A note on mobs*, II, Anais Acad. Brasil Si. **25** (1953), 335–336.

[341] R. J. Warne, *On the structure of semigroups which are unions of groups*, Trans. Amer. Math. Soc. **186** (1973), 385–401, (announced in Semigroup Forum **5** (1973), 323–330).

[342] J. H. M. Wedderburn, *A theorem on finite algebras*, Trans. Amer. Math. Soc. **6** (1905), 349–352.

[343] G. H. Wenzel, *Note on a subdirect representation of universal algebras*, Acta. Math. Acad. Sci. Hungar. **18** (1967), 329–333.

[344] E. T. Wong, *Regular rings and integral extension of a regular ring*, Proc. Amer. Math. Soc. **33** (1970), 313–315.

[345] E. T. Wong, *Almost commutative rings and their integral extensions*, Math. J. Okayama Univ. **18** (1976), 105–111.

[346] M. Yamada, *Construction of inversive semigroups*, Mem. Fac. Lit. Sci. Shimane Univ., Nat. Sci. **4** (1971), 1–9.

[347] A. Yaqub, *The structure of pre-p^k-rings and generalized pre-p-rings*, Amer. Math. Monthly **71** (1964), 1010–1014.

[348] R. Yoshida, *Ideal extensions of semigroups and compound semigroups*, Mem. Res. Inst. Sci. Eng. Ritumeikan Univ. **13** (1965), 1–8.

Index

Neda Bokan

TORSION FREE CONNECTIONS, TOPOLOGY, GEOMETRY AND DIFFERENTIAL OPERATORS ON SMOOTH MANIFOLDS

Contents

INTRODUCTION

This lecture notes give a survey of basic facts related to geometry of manifolds endowed with a torsion free connection. We pay the attention especially on geometries which come from the existence on some characteristic torsion free connection closely related to some metric, in general case of an arbitrary signature. So we study in this spirit affine differential geometry, Weyl and Codazzi geometries. One can join naturally these two structures: a torsion free connection and a metric, and the corresponding groups of transformations. We present basic facts related to these groups. To study geometry of manifolds endowed with a torsion free connection, a powerful tool is, of course, the corresponding curvatures. Therefore the curvature is appeared in all five sections of this lecture notes. In Section I we give the definitions of curvatures and some of their properties. Section II is devoted to curvatures which are invariant with respect to some groups of transformations. These groups are closely related to classical groups: $GL(m,\mathbb{R}), U(m), SO(m)$ etc. We use well developed representation theory of these groups to enlight curvature from this point of view. It is the content of Section III. To prove the irreducibility of some vector space of curvatures one can use different methods. We pay the attention especially on these ones closely related to the Weyl classical invariance theory. It allows to study also some relations between topology and analysis of these manifolds with their geometry. So we develop the theory of characteristic classes in Section IV and differential operators of Laplace type in Section V. Among characteristic classes we pay the attention on Chern classes and their dependence on some groups of transformations and curvature symmetries. To fulfill our programme related to the influence of Weyl classical invariance theory into the theory of differential operators of Laplace type we study the heat equation method. Several operators of Laplace type are studied more sistematically.

Finally, there are various possibilities to present some material related to this topic. The author of this lecture notes choose this one closely related to her main interest through previous twenty years. Her interest yields in the cooperation with other colleagues a series of results which are presented too.

We omit the proofs as it is far from the framework of these notes. We rather give the advantage to the results to present the riches of this topic to motivate the readers into further investigations. Of course to go into this level we suggest to use the corresponding monographs and papers, mentioned in the convenient moment throughout this notes.

As it is usual the contribution of colleagues friends and institutions to the quality of manuscript is significant. I would like to acknowledge all of them, bur first of all to Prof. B. Stanković, who has initiated and encouraged writing this manuscript.

I. MANIFOLDS WITH A TORSION FREE CONNECTION

I.1. Definitions and basic notions

The straight lines play a very important role in the geometry of a plane. Therefore, it would be useful to have lines on surfaces with the analogous properties to these ones of straight lines. But the definition of such lines on surfaces is not so evident as straight lines have several characteristic properties and hence it is not clear which one should characterize "straight lines" on surfaces, i.e., which one can be generalized, and specially which generalizations give the same and which one give different lines. Among these properties are the followings:

(PL1) The curvature of a straight line (in a plane) vanishes.

(PL2) For any two points there exists the unique straight line which consists both of them.

(PL3) The tangent vectors on a straight line are mutually parallel

All of these properties can be generalized for lines on a surface. To obtain this one we use heavily a linear connection. Studying the same problem on a smooth manifold M^m of the dimension m we need also a linear connection. Hence we give its definition in a full generality. More details one can find in [**61**], [**82**], [**85**], etc.

Definition 1.1. Let $\mathfrak{X}$ be the modul of vector fields over the ring of smooth functions $C^\infty(M)$ on M. *A linear connection* on the manifold M is a map ∇ : $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, such that for all $x, y, z \in \mathfrak{X}(M)$, $r \in \mathbb{R}$ and $f \in C^\infty(M)$ it yields

$$\text{(i)} \qquad \nabla_x(y+z) = \nabla_x y + \nabla_x z \quad \text{and} \quad \nabla_x ry = r\nabla_x y,$$

$$\text{(ii)} \qquad \nabla_{x+y} z = \nabla_x z + \nabla_y z \quad \text{and} \quad \nabla_{fx} y = f\nabla_x y,$$

$$\text{(iii)} \qquad \nabla_x fy = (xf)y + f\nabla_x y \quad \text{(Leibnitz formula).}$$

The operator $\nabla_x : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is *the covariant derivative in the direction of a vector field x.* □

If (u, U) is a chart and $\{\partial/\partial u^i\}_p, 1 \le i \le m$ the corresponding coordinate base of tangent space T_pM, for any $p \in U$, then arbitrary vector field x can be given in the following way $\sum X^i \frac{\partial}{\partial u^i}$, $X^i \in C^\infty(M)$. A linear connection ∇ is determined by the vector fields $\sum_{\partial/\partial u^i}(\partial/\partial u^j)$. It allows to introduce the Christoffel symbols of ∇.

Definition 1.2. Let ∇ be a linear connection on the manifold M and (u, U) a chart. *Christoffel symbols of* ∇ with respect to the chart (u, U) are functions

$\Gamma_{ij}^k \in C^\infty(M)$ defined by

$$\nabla_{\partial/\partial u^i}\left(\frac{\partial}{\partial u^j}\right) = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u^k}. \qquad (\square)$$

Let $\alpha : I \to M$ be a curve. *The tangent vector field* T_α of a curve α is given by $(T_\alpha)_{\alpha(t)} = (\alpha_*)(d/dt), t \in \mathbb{R}$. Usually we use the notation T for T_α if there is no confusion. Locally we have

$$T_{\alpha(t)} = \sum \frac{d\alpha^i}{dt}(t)\left(\frac{\partial}{\partial u^i}\right)_{\alpha(t)}.$$

Let y be a vector field defined along a curve α. We say y is *parallel* along α if $\nabla_{T_\alpha} y = 0$. A curve α on a manifold M is *geodesic* (with respect to the connection ∇) if $\nabla_{T_\alpha} T_\alpha = 0$. Let (M, g) be a Riemannian manifold endowed with a linear connection ∇. The connection ∇ is *metric* if it satisfies

$$xg(y,z) = g(\nabla_x y, z) + g(y, \nabla_x z),$$

for all $x, y, z \in \mathfrak{X}(M)$. A linear connection ∇ is *symmetric* or *torsion free* if we have

$$\nabla_x y - \nabla_y x = [x, y] \qquad (1.1)$$

for all $x, y \in \mathfrak{X}(M)$. A connection ∇ is torsion free if and only if it yields $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all $1 \leq i, j, k \leq m$ in an arbitrary coordinate chart. There exists a unique metric symmetric connection ∇ on a Riemannian manifold (M, g). This connection ∇ is called *Levi-Civita connection.*

Definition 1.3. *The curvature tensor* of type (1,3) of arbitrary connection ∇ is the map $R : \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X} \to \mathfrak{X}$ defined by relation $R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. The curvature tensor of Levi-Civita connection is called *Riemann curvature tensor.* $\square$

In a local coordinate system one can find

$$R\left(\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right)\frac{\partial}{\partial u^i} = \sum R_{jki}{}^l \frac{\partial}{\partial u^l},$$

where the components $R_{jki}{}^l$ are defined by

$$R_{jki}{}^l = \Gamma_{ji,k}^l - \Gamma_{ki,j}^l + \sum_m \Gamma_{ji}^m \Gamma_{mk}^l \quad \text{and} \quad \Gamma_{ji,k}^l = \frac{\partial}{\partial u^k}(\Gamma_{ji}^l).$$

Riemann curvature tensor of type (0,4) is the map $R : \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X} \to C^\infty(M)$, given by the relation $R(x,y,z,w) = g(R(x,y)z, w)$.

The curvature tensor of type (1,3) of arbitrary connection ∇ satisfies the following relation

$$R(x,y)z = -R(y,x)z; \tag{1.2}$$

for torsion free connection we have also

$$R(x,y)z + R(z,x)y + R(y,z)x = 0, \tag{1.3}$$

(*the first Bianchi identity*), and

$$(\nabla_v R)(x,y)z + (\nabla_x R)(y,v)z + (\nabla_y R)(v,x)z = 0 \tag{1.4}$$

(*the second Bianchi identity*). Riemann curvature tensor fulfills all these relations (1.2)–(1.4) and

$$R(x,y,z,w) = -R(x,y,w,z), \tag{1.5}$$

$$R(w,z,x,y) = R(x,y,w,z). \tag{1.6}$$

The curvature tensor of a metric connection satisfies symmetry relations (1.2), (1.5) and (1.6).

Let Π be a 2-dimensional subspace of tangent space T_pM. *The sectional curvature* of Π is $K_p(\Pi) = R(x,y,y,x)(p)$, where $\{x,y\}$ is an orthonormal base of Π. If x,y are two arbitrary vectors in Π, then

$$K_p(\Pi) = \frac{R(x,y,y,x)}{||x||^2||y||^2 - g(x,y)^2},$$

where $||x||^2 = g(x,x)$. M is a space of *the constant sectional curvature* if $K_p(\Pi)$ is independent of the choice of Π in T_pM, where p is an arbitrary point of M and depends on $p \in M$. The Riemann curvature tensor of this space is given by

$$R(u,v,z,w) = K_p(g(u,z)g(v,w) - g(u,w)g(v,z)).$$

If $K_p(\Pi)$ is independent of Π in T_pM in all $p \in M$ then K_p is same everywhere on M.

Some information about the geometry of M give Ricci and scalar curvatures. These curvatures are very powerful tool in studying of Einstein spaces and other topics. Let $\Theta_p(x_p,y_p) : T_pM \to T_pM$ be the map defined by the relation

$$\Theta_p(x_p,y_p)v_p = R(v_p,x_p)y_p.$$

Then $\Theta_p(x_p,y_p)$ is linear for all $p \in M$ and $x_p, y_p \in T_pM$. Consequently, there exists the trace of $\Theta_p(x_p,y_p)$.

Definition 1.4. *Ricci curvature tensor* ρ is the correspondence between points $p \in M$ and maps $S_p : T_pM \times T_pM \to R$, given by $\rho_p(x_p, y_p) = \mathrm{trace}(\Theta_p(x_p, y_p))$. *Ricci curvature in a direction* x is $\rho_p(x, x)$. □

Definition 1.5. Let (M, g) be a Riemann manifold with Ricci curvature ρ. *The scalar curvature* τ of M in a point p is defined by $\tau = \sum_{i=1}^{n} \rho_p((x_i)_p, (x_i)_p)$, where $\{x_{1p}, \dots, x_{np}\}$ is an orthonormal base of the tangent space T_pM. □

Einstein space is Riemann space (M, g) such that $\rho_p = \frac{\tau}{m} g_p$.

In general case Ricci curvature tensor is neither symmetric nor skew-symmetric. But, ρ_p corresponding to Levi-Civita connection is symmetric. Manifolds endowed with special type connections will be studied in next sections.

A skew-symmetric Ricci tensor naturally appeared on manifolds which admit absolute parallelizability of directions (see for example [**100**, §§**49**, 89]). More precisely, it means the following. Let (M, ∇) be a differentiable manifold with a symmetric connection ∇. If a vector field v defined along a curve γ collinear with some parallel vector field w we say *the direction* of v *is parallel*. A manifold M admits *absolute parallelizability of directions* if every direction given in a $p \in M$ can be included in some field with parallel directions along every curve.

A skew-symmetric Ricci tensor is appeared also in the complete decomposition of a curvature tensor for ∇ in the spirit of the representation theory of classical groups (see Section III).

I.2. Affine differential geometry

Torsion free, Ricci symmetric connections arise naturally in affine differential geometry and motivate the discussion of the previous section. We review this geometry briefly and refer to [**10**], [**33**], [**97**], [**111**], [**124**] for further details.

Let $\mathfrak{A}$ be a real affine space which is modeled on a vector space V of dimension $m+1$. Let V^* be the dual space. If $a \in \mathfrak{A}$, we may identify $T_a\mathfrak{A} = V$ and $T_a^*\mathfrak{A} = V^*$.

Let $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ be the natural pairing between V^* and V. Let x be a smooth hypersurface immersion of M into $\mathfrak{A}$. If $p \in M$, let

$$C(M)_p = \{X \in V^* : \langle X, dx(v) \rangle = 0, \forall v \in T_pM\}$$

be the conormal space at p; we let $C(M)$ be the corresponding conormal line bundle over M. We assume $C(M)$ is trivial and choose a non vanishing conormal field X.

We say the hypersurface $x(M)$ is *regular* if and only if $\mathrm{rank}(X, dX) = m + 1$, for every point of M; we impose this condition henceforth. Then X is an immersion $X : M \to V^*$ which is transversal to $X(M)$. Define $y = y(X) : M \to V$ by the conditions $\langle X, y \rangle = 1$ and $\langle dX, y \rangle = 0$.

The triple (x, X, y) is called *a hypersurface with relative normalization*; we remark that y need not be an immersion. The relative structure equations given below contain the fundamental geometric quantities of hypersurface theory: two connections ∇, ∇^*, the relative shape (Weingarten) operator S, and two symmetric forms h and $\hat{S}$. Let $\bar{\nabla}$ be the flat affine connection on $\mathfrak{A}$.

$$\begin{aligned} \bar{\nabla}_v y &= dy(v) = -dx(S(v)), \\ \bar{\nabla}_w dx(v) &= dx(\bar{\nabla}_w v) + h(v,w)y, \\ \bar{\nabla}_w dX(v) &= dX(\nabla^*_w v) - \hat{S}(v,w)X. \end{aligned} \tag{2.1}$$

The first equation is called *the Weingarten equation*, the second two are *the Gauss equations.* Symmetric form h is called *the Blaschke metric.* Generally, it is indefinite. If h is positive definite, this means that the immersed hypersurface $x(M)$ is locally strongly convex.

The relative shape operator S is self-adjoint with respect to h and is related to the auxiliary shape operator $\hat{S}$ by the identity $\hat{S}(v,w) = h(S(v),w) = h(v,S(w))$. It is useful to define a $(1,2)$ *difference tensor* C, *a totally symmetric relative cubic form* $\hat{C}$, and *the Tchebychef form* $\hat{T}$ by:

$$C := \frac{1}{2}(\nabla - \nabla^*), \quad \hat{C}(v,w,z) := h(C(v,w),z), \quad \hat{T}(z) := m^{-1}\operatorname{Tr}_h(C(z,\cdot)).$$

Let ';' denotes multiple covariant differentiation with respect to the Levi-Civita connection $\nabla(h)$. $\hat{T}$ has the following useful symmetry property [**123**]: $\hat{T}_{i;j} = \hat{T}_{j;i}$.

We note that both ∇ (*the induced connection*) and ∇^* (*the conormal connection*) are torsion free connections on TM. They are conjugate with respect to the Levi-Civita connection; which implies $\frac{1}{2}(\nabla + \nabla^*) = \nabla(h)$. Consequently, we may express $\nabla = \nabla(h) + C$ and $\nabla^* = \nabla(h) - C$.

The curvature tensors R, R^*, $R(h)$ of ∇, ∇^*, $\nabla(h)$ respectively can be expressed by the Gauss equations

$$\begin{aligned} R(u,v)w &= h(v,w)Su - h(u,w)Sv, \\ R^*(u,v)w &= \hat{S}(v,w)u - \hat{S}(u,w)v, \\ R(h)(w,v)u &= C(C(w,u),v) - C(C(v,u),w) \\ &\quad + \frac{1}{2}\{\hat{S}(v,u)w - \hat{S}(w,u)v + h(v,u)S(w) - h(w,u)S(v)\}. \end{aligned} \tag{2.2}$$

Let R_{ij}, R^*_{ij}, $R(h)_{ij}$ be the components of Ricci tensors Ric, Ric*, Ric(h) for ∇, ∇^*, $\nabla(h)$ respectively relative to a local orthonormal frame. We use the metric to raise and lower indices and identify $\hat{S} = S$. Then:

$$R_{ij} = \delta_{ij}S_{kk} - Sij \quad \text{and} \quad R^*_{ij} = (m-1)S_{ij}.$$

We denote *the normalized mean curvature* by $H := m^{-1}S_{ii}$; the normalized traces are then equal

$$(m-1)^{-1}\operatorname{Tr}_h(\mathrm{Ric}) = (m-1)^{-1}\operatorname{Tr}_h \mathrm{Ric}^* = mH.$$

We construct the extrinsic curvature invariants of relative geometry from S and h. Let $\{\lambda^1, \dots, \lambda^m\}$ be the eigenvalues of S relative to h:

$$\det(S - \lambda h) = 0.$$

These are *the principal curvatures*. Let $\{H_1, \dots, H_m\}$ be the corresponding normed elementary symmetric functions. For example the relative mean curvature is given by $mH_1 = \lambda^1 + \dots + \lambda^m$.

We fix a volume form or determinant on $\mathfrak{A}$. Then there is, up to orientation, a unique equiaffine unimodular normalization which is invariant under the unimodular group. Admitting arbitrary volume forms, all such normalizations differ by a non-zero constant factor. The class of equiaffine normalizations can be characterized within the class of relative normalizations by the vanishing of the Tchebychev form. We call such a hypersurface with equiaffine normalization a *Blaschke hypersurface*. The vanishing of T simplifies the local invariants greatly.

We take $y = -x$ to define *the centroaffine normalization*. This geometry is invariant with respect to the subgroup of regular affine mappings which fix the origin of $\mathfrak{A}$. Let X_e be the equiaffine conormal field and let $\zeta = -\langle x, X_e\rangle$ be the equiaffine support function. We choose the orientation so that $\zeta > 0$. Then

$$\hat{S} = h, \quad H_r = 1 \quad \text{for} \quad r = 1, \dots, m, \quad \text{and} \quad \hat{T} = \frac{2+m}{2m} d \ln(\zeta).$$

Among relative normalizations significant one is Euclidean normalization. Let $x : M \to E$ be a hypersurface, let Y be a conormal and y transversal. The pair $\{Y, y\}$ is called *a Euclidean normalization* with respect to the given Euclidean structure of E if Y and y can be identified by the Riesz theorem and $\langle Y, y\rangle = 1$. We write $Y = y = \mu$.

As a consequence of this definition one can express the regularity of a hypersurface in terms of Euclidean hypersurface geometry. More precisely, let $x : M \to E$ be a hypersurface. Then the following properties are equivalent:

(i) x is non-degenerate.
(ii) The Euclidean Gauss-map is an immersion.
(iii) The Euclidean Weingarten operator is regular.
(iv) The third fundamental form III is positive definite on M.
(v) The second fundamental form II is regular.

We express the relative quantities (in the following on the left) for the Euclidean normalization in terms of quantities of Euclidean hypersurface theory (on the right).

(a) $S(E) = b$, (b) $h(E) = II$, (c) $\nabla(E) = \nabla(I)$,
(d) $\hat{S}(E) = III$, (e) $\nabla^*(E) = \nabla(III)$, (f) $\nabla(h) = \nabla(II)$,
(g) $-2\hat{C}(E) = \nabla(I)II = -\nabla(III)II$, (h) $-2C(E)(v, w) = b^{-1}((\nabla(I)_v b)(w))$,
(i) $C(E) = \frac{1}{2}(\nabla(I) - \nabla(III)) = \nabla(I) - \nabla(II) = \nabla(II) - \nabla(III)$,
(j) $\hat{T}(E) = -\frac{1}{2m} d \lg |H_m(E)|$, where b is the Weingarten operator.

Consider a pair of non-degenerate hypersurfaces $x : M \to V$ and ${}^*x : M \to V^*$ such that $\langle x, {}^*x\rangle = -1$, $\langle d^*x, x\rangle = 0$, and $\langle x, dx\rangle = 0$. Such a pair is called *a polar* *pair.* These relations are satisfied for a non degenerate hypersurface $x : M \to V$ and its centroaffine conormal map $^*x := X : M \to V^*$. This indicates the important role which centroaffine differential geometry has for the investigation of polar pairs. We recall some facts about the controlling geometry of polar pairs and refer to [OS, §7.2] for further details

$$\hat{S} = h = {}^*h = {}^*\hat{S},$$
$$\nabla = {}^*\nabla^*, \quad \nabla^* = {}^*\nabla,$$
$$C = -{}^*C, \quad \hat{T} = -{}^*\hat{T}, \quad R = {}^*R.$$

Let μ be a unit normal on an Euclidean sphere $S^m(r) \subset E$ of radius r and with center x_0. $S^m(r)$ can be characterized by the relation $r\mu = x - x_0$, or more generally by μ and $(x - x_0)$ being parallel. Studying quadrics we conclude that all quadrics with center x_0 have the property that the equiaffine normal satisfies $y(e) = -H(e)x + x_0$. One can generalizes this notion in relative geometry as follows.

Let $x : M \to \mathfrak{A}$ be a regular hypersurface with relative normalization $\{Y, y\}$. Then $\{x, Y, y\}$ is called *a proper relative sphere* with center x_0 if

$$y = \lambda(x - x_0), \quad \lambda \in C^\infty(M). \tag{2.3}$$

$\{x, Y, y\}$ is called *an improper relative sphere* if $y = \text{const} \neq 0$. A point $p \in M$ is called *a relative umbilic* if the relative principal curvatures coincide

$$k_1(p) = k_2(p) = \cdots = k_m(p).$$

A consequence of the Weingarten equation in (2.1) is that $\lambda = \text{const}$ in (2.3).

Since $y = -x$ per definition in centroaffine geometry it follows any hypersurface with centroaffine normalization is a relative sphere with respect to this normalization.

Usually relative spheres with respect to the equiaffine normalization are called *affine spheres* (instead of equiaffine spheres). Any quadric is an affine sphere. If a regular quadric has a center x_0, it is a proper affine sphere with center x_0 (examples: ellipsoids, hyperboloids are proper affine spheres; paraboloids are improper affine spheres). In the following theorems we give some characterizations of relative spheres and affine ones.

Theorem 2.1. *(a) Each of the following properties (i)–(vi) characterizes a relative sphere:*

(i) $S = \lambda \cdot \mathrm{id}$ *on* M *(where* $\lambda \in C^\infty(M)$ *and* $\lambda \neq 0$ *for proper relative spheres and* $\lambda = 0$ *for improper relative spheres).*

(ii) $\hat{S} = \lambda \cdot h$ *on* M, $\lambda \in C^\infty(M)$.

(iii) $m\nabla(h)\hat{T} = \div C$ *on* M.

(iv) $\nabla(h)\hat{C}$ *is totally symmetric on* M.

(v) $\nabla\hat{C}$ *is totally symmetric on* M.

(vi) $\mathrm{Ric} = \mathrm{Ric}^*$ *on* M.

(b) $\hat{S} = \lambda h$ *implies* $\lambda = \mathrm{const} = H$.

(c) *If* x *is a relative sphere, then for each* $p \in M$:

(i) p *is an umbilic;*

(ii) $\sum_{i<j}(k_i - k_j)^2 = m\|Hh - \hat{S}\|^2 = m[\|\hat{S}\|^2 - mH^2] = 0$. □

Theorem 2.2. *Let* x *be regular with relative normalization* $\{Y, y\}$. *Then* x *is a proper relative sphere with center* x_0 *if and only if* $\rho(x_0) = \langle Y, x_0 - x\rangle = \mathrm{const} \neq 0$. □

Theorem 2.3. *A regular hypersurface* x *with relative normalization* $\{Y, y\}$ *is an improper relative sphere if and only if* $S = 0$. □

We refer to [**10**], [**11**], [**32**], [**68**]–[**72**], [**74**]–[**76**], [**95**], [**111**], [**122**], [**134**], for many examples including classifications of subclasses of affine spheres.

The complete classification is yet unknown. One tries to classify subclasses of affine spheres. In the following theorem is given a result related to this topic.

Theorem 2.4. *A locally strongly convex affine hypersphere with constant equiaffine sectional curvature is either a quadric or equiaffinely equivalent to the hypersurface* $x^1x^2\dots x^{m+1} = 1$, *where* $x^i : \mathfrak{A} \to \mathbb{R}$ *is a coordinate function.* □

We refer to [**133**] for the proof.

In case of an indefinite metric there are classifications for $m = 2$ in [**75**], [**120**], and for $m = 3$ in [**76**]. Other classifications results one can find in [**31**], [**117**], [**134**], etc.

There is a serious of results about compact affine spheres where many of the results are related to the spectral geometry of the equiaffine Laplacian (see [**114**], [**115**], [**119**] etc). We refer also Section V of this paper.

In [**122**] was studied existence and uniqueness problem about 2-spheres. Certain types of PDE's play an important role for the local and global classification of affine spheres in the equiaffine theory. One of the first PDE which was used in the theory of affine spheres is an expression for the Laplacian of the Pick invariant (see [**10**, §76]). Simon [**121**] extends this PDE to non-degenerate hypersurface. Monge-Ampère equations are used to investigate improper affine spheres and hyperbolic affine spheres.

A characterization of quadrics and improper affine spheres in terms of symmetry properties of $\nabla\hat{C}$ and $\nabla^2\hat{C}$ is given in [**22**].

I.3. Weyl geometry

As we know the metric h of a semi-Riemannian manifold (M, h) is parallel or covariantly constant with respect to the corresponding Levi-Civita connection. The main purpose of this section is to study a torsion free connection ${}^{\mathfrak{w}}\nabla$ satisfying the recurrence condition for the metric. This connection has been introduced by H. Weyl.

Weyl [**138**] attempted a unification of gravitation and electromagnetism in a model of space-time geometry combining both structures. His particular approach failed for physical reasons but his model is still studied in mathematics (see, for example, [42], [49]–[53], [106], [130],) and in mathematical physics (see, for example [54]).

We begin our discussion by introducing some notational conventions. Let (M,h) be a semi-Riemannian manifold of dimension $m \geq 2$. Fix a torsion free connection ${}^{w}\nabla$, called *the Weyl connection*, on the tangent bundle of M. We begin the definition of a Weyl structure by assuming that there exists a one-form $\hat{\theta} = \hat{\theta}_h$ so that

$$(3.1) \qquad {}^{w}\nabla h = 2\hat{\theta}_h \otimes h.$$

Let $\mathfrak{C} = \mathfrak{C}\{\mathfrak{W}\}$ be the conformal class defined by h (for more details, see Section II, 4), and let $\mathfrak{T} = \mathfrak{T}\{\mathfrak{W}\}$ be the corresponding collection of one-forms $\hat{\theta}_h$. Here and in the following we identify metrics in $\mathfrak{C}$ which merely differ by a constant positive factor. So there is a bijective correspondence between elements of $\mathfrak{C}$ and of $\mathfrak{T}$. We will call the triple $\mathfrak{W} = ({}^{w}\nabla, \mathfrak{C}, \mathfrak{T})$ *a Weyl structure* on M and we will call $(M, \mathfrak{W})$ *a Weyl manifold.*

The compatibility condition described in equation (3.1) is invariant under so-called gauge transformations

$$(3.2) \qquad h \to {}_{\beta}h := \beta h \;\text{ and }\; \hat{\theta} \to {}_{\beta}\hat{\theta} := \hat{\theta} + d\frac{1}{2}(\ln \beta),$$

for $\beta \in C^{\infty}_{+}(M)$. We note that $C^{\infty}_{+}(M)$ acts transitively on $\mathfrak{C}$ and on $\mathfrak{T}$.

It is well known that a Weyl structure $\mathfrak{W}$ can be generated from a given pair $\{h, \hat{\theta}\}$ (where h is a semi-Riemannian metric and where $\hat{\theta}$ is a 1-form) in the following way. Let, $u, v, \ldots$ be vector fields on M and let ${}^{h}\nabla = \nabla(h)$ be the Levi-Civita connection of h. Let θ be the vector field dual to the 1-form $\hat{\theta}$, i.e., $h(w,\theta) = \hat{\theta}(w)$. We define $\alpha(u,v,w) := h(({}^{w}\nabla_u - {}^{h}\nabla_u)v, w)$. Since ${}^{w}\nabla$ and ${}^{h}\nabla$ are torsion free, $\alpha(u,v,w) = \alpha(v,u,w)$. Since ${}^{h}\nabla h = 0$ and since ${}^{w}\nabla$ satisfies equation (3.1), we have

$$(3.3) \qquad \begin{aligned} \alpha(u,v,w) + \alpha(u,w,v) + 2\hat{\theta}(u)h(v,w) &= 0, \\ \alpha(u,v,w) = -\hat{\theta}(u)h(v,w) - \hat{\theta}(v)h(u,w) &+ \hat{\theta}(w)h(u,v), \\ {}^{w}\nabla_u v = {}^{h}\nabla_u v - \hat{\theta}(u)v - \hat{\theta}(v)u &+ h(u,v)\theta. \end{aligned}$$

Conversely, if equations (3.3) are satisfied, then ${}^{w}\nabla = {}^{w}\nabla(h,\hat{\theta})$ is a torsion free connection and equation (3.1) is satisfied. One can generates a Weyl structure from an arbitrary semi-Riemannian metric h and from an arbitrary 1-form $\hat{\theta}$ by using equation (3.3) to define ${}^{w}\nabla$ and using the action of $C^{\infty}_{+}(M)$ defined in equation (3.2) to generate the classes $\mathfrak{C}$ and $\mathfrak{T}$; see [138] or [42] for further details.

We use the sign convention of [61] to define the curvature of ${}^{w}\nabla$. Hence

$${}^{w}R(u,v) := {}^{w}\nabla_u\, {}^{w}\nabla_v - {}^{w}\nabla_v\, {}^{w}\nabla_u - {}^{w}\nabla_{[u,v]}$$

is the curvature corresponding to Weyl connection ${}^{\mathfrak{w}}\nabla$. For $h \in \mathfrak{C}$, Weyl introduced the 2-form ${}^{\mathfrak{w}}F := d\hat{\theta}_h$ as a gauge invariant of a given Weyl structure. He called it *the length curvature* or *distance curvature* [**138**, p. 124]. We have that F and ${}^{\mathfrak{w}}R$ are related by the equation

$$h(z, {}^{\mathfrak{w}}R(u,v)z) = {}^{\mathfrak{w}}F(u,v)h(z,z). \tag{3.4}$$

Weyl defined *the directional curvature* ${}^{\mathfrak{w}}K$ by

$${}^{\mathfrak{w}}K(u,v)w := {}^{\mathfrak{w}}R(u,v)w - {}^{\mathfrak{w}}F(u,v)w. \tag{3.5}$$

The curvature ${}^{\mathfrak{w}}R$ of ${}^{\mathfrak{w}}\nabla$ and the Weyl directional curvature ${}^{\mathfrak{w}}K$ are also gauge invariants. Relations (3.4) and (3.5) imply the orthogonality relation $h({}^{\mathfrak{w}}K(u,v)w, w) = 0$, for any $h \in \mathfrak{C}$ and for any vector field w. Moreover ${}^{\mathfrak{w}}F$ and ${}^{\mathfrak{w}}K$ satisfy respectively symmetry and skew-symmetry relations

$$h({}^{\mathfrak{w}}F(u,v)w, z) = h({}^{\mathfrak{w}}F(u,v)z, w),$$
$$h({}^{\mathfrak{w}}K(u,v)w, z) = -h(K(u,v)z, w).$$

As a local result the following is known: if the Weyl connection ${}^{\mathfrak{w}}\nabla$ is metric, then the length curvature vanishes identically. Conversely, if $F = d\hat{\theta}_h = 0$, equation (3.2) implies that the cohomology class $[\hat{\theta}_h(\mathfrak{W})] \in H^1(M)$ of the closed form $\hat{\theta}_h(\mathfrak{W})$ does not depend on the choice of a metric in $\mathfrak{W}$. Conversely, if ${}^{\mathfrak{w}}F = d\hat{\theta}_h = 0$ equation (3.2) implies that the cohomology class $[\hat{\theta}_h(\mathfrak{W})] \in H^1(M)$ of the closed form $\hat{\theta}_h(\mathfrak{W})$ is gauge invariant and does not depend on the choice of a metric in $\mathfrak{W}$. The following is well known; see, for example, [**52**], [**106**], [**130**].

Proposition 3.1. *The following assertions are equivalent:*
(i) we have ${}^{\mathfrak{w}}F(\mathfrak{W}) = 0$ *and* $[\hat{\theta}_h(\mathfrak{W})] = 0$ *in* $H^1(M)$;
(ii) there exists $h \in \mathfrak{C}(\mathfrak{W})$ *such that* ${}^{\mathfrak{w}}\nabla h = 0$; *i.e.,* ${}^{\mathfrak{w}}\nabla$ *is the Levi-Civita connection of* h. □

II. SOME TRANSFORMATIONS OF SMOOTH MANIFOLDS

II.1. Projective transformations

The main purpose of this section is to study projective transformations of a smooth manifold (M, ∇) endowed with a torsion free connection ∇. More details one can find in [**41**], [**64**], [**109**], [**112**].

A map $f : (\tilde{M}, \tilde{\nabla}) \to (M.\nabla)$ of manifolds with torsion free connections is called *projective* if for each geodesic γ of $\tilde{\nabla}$, $f \circ \gamma$ is a reparametrization of a geodesic of ∇, i.e., there exists a strictly increasing C^∞ function h on some open interval such that $f \circ \gamma \circ h$ is a ∇-geodesic. Linear connections $\tilde{\nabla}$ and ∇ on M are *projectively equivalent* if the identity map of M is projective. A *projective transformation* of

(M,∇) is a diffeomorphism which is projective. The transformation s is projective on M, if the pull back $s^*\nabla$ of the connection is projectively related to ∇, i.e., if there exists a global 1-form $\pi = \pi(s)$ on M such that

$$s^*\nabla_u v = \nabla_u v + \pi(u)v + \pi(v)u, \tag{1.1}$$

for arbitrary smooth vector fields $u, v \in \mathfrak{X}(M)$. Having (1.1) in mind, if s and t are two projective transformations, we find $\pi(st) = \pi(s) + \hat{s}\cdot\pi(t)$, where $\hat{s}$ is the cotangent map, i.e. $[\hat{s}\cdot\pi]_{s(p)} = \hat{s}\cdot[\pi]p$.

If a transformation s of M preserves geodesics and the affine character of the parameter on each geodesic, then s is called *an affine transformation of the connection* ∇ or simply of the manifold M, and we say that s leaves the connection ∇ invariant.

It is well-known that *the Weyl projective curvature tensor* has the form

$$\begin{aligned} P(R)(u,v)w = R(u,v)w &+ \frac{1}{m^2-1}[m\rho(u,w)+\rho(w,u)]v \\ &- \frac{1}{m^2-1}[m\rho(v,w)+\rho(w,v)]u \\ &+ \frac{1}{m+1}[\rho(u,v)-\rho(v,u)]w, \end{aligned} \tag{1.2}$$

for any $m > 2$, and for $m = 2$ we have $P(R)(u,v)w = 0$, where $u, v, w, \dots \in \mathfrak{X}(M)$ (see for example [**110**], [**112**], [**136**]). $P(R)$ is a tensor that is invariant with respect to each projective transformation of M. $P(R)$ characterizes a space of constant sectional curvature in very nice way: $P(R) = 0$ if and only if M^m $(m > 2)$ is space of constant curvature (in that case R is the Riemannian curvature of M^m).

A manifold (M,∇) is said to be a *projectively flat*, if it can be related to a flat space by a projective map. We know that the curvature tensor of a flat space is equal to zero: $R(u,v) = 0$, and therefore the Ricci tensor ρ is equal to zero also. Due to this fact from (1.2) we have the Weyl projective curvature tensor $P(R)$ of a flat space vanishes. Since the tensor $P(R)$ is invariant with respect to projective transformations, we have immediately $P(R)$ of a projectively flat space vanishes. The inverse theorem is valid also. Namely, if $P(R)$ of a manifold (M,∇) vanishes then (M,∇) is a projectively flat space.

One can use (2.2) in Section I to see (M,∇^*) is a projectively flat space.

Ishihara studied in [**56**] the groups of projective and affine transformations. Among others he investigated the conditions that these groups coincide.

Theorem 1.1. *If (M,∇) is a compact manifold with a torsion free connection ∇ and the Ricci tensor of ∇ vanishes identically in M, then the group of projective transformations of M coincides with its subgroup of affine transformations.* □

If M is also irreducible then Ishihara has proved that the group of projective transformations of M coincides with its group of isometries.

Projective transformations are closely related with projective structures (see [**60**]). *A projective structure on a m-dimensional manifold* is determined if there exists an atlas on M with transition functions being projective transformations [**65**]. Projective structure can be considered also in terms of subbundles of principal fiber bundles of 2-frames which structure group satisfies certain conditions (see [**60**]).

We refer also [**83**] for the references related this topic.

II.2. Holomorphically projective transformations

Before studying holomorphically projective transformations we need to introduce an almost complex structure.

An almost complex structure J on a smooth manifold M^{2m} is an endomorphism J such that $J^2 = -I$ on TM, where I is the identity. We say ∇ is a *complex symmetric connection* if it satisfies (1.1) of Section I and the following relation

$$\nabla J = 0. \tag{2.1}$$

The curvature R of ∇ satisfies besides of (1.2)–(1.4) of Section I also *the Kähler identity* $R(u,v) \circ J = J \circ R(u,v)$, for $u, v \in \mathfrak{X}(M)$. A manifold M^{2m} endowed with an almost complex structure J is *an almost complex manifold* (M, J). An almost complex manifold (M, J) may be endowed with a complex symmetric connection ∇ if *the Nijenhuis tensor* S of M, given by

$$S(u,v) = [u,v] + J[Ju,v] + J[u,Jv] - [Ju,Jv]$$

vanishes (see [**101**], [**105**]). An almost complex manifold (M, J) such that $S = 0$ may be also endowed with *a complex atlas*, i.e., with complex coordinates. This manifold is called *a complex manifold.*

Especially, a complex symmetric connection ∇ is *a holomorphic affine connection* if $R(u,v) = -R(Ju, Jv)$, or *an affine Kähler connection* when one has $R(u,v) = R(Ju, Jv)$.

Holomorphic affine connections naturally appeared in the context of semi-Riemannian manifolds with the metric of signature (n,n) as well as in complex affine and projective differential geometry (see [**38**], [**59**], [**96**], [**98**], [**99**] for more details).

If a semi-Riemannian manifold (M, g) is endowed with an almost complex structure J satisfying (2.1) with respect to the corresponding Levi-Civita connection then (M, g, J) is *a Kähler manifold.*

Let Π_H be a 2-dimensional subspace of tangent space T_pM, spanned by vectors (u, Ju), for any unit vector $u \in T_pM$. *The holomorphic sectional curvature* of Π_H is $KH_p(\Pi_H) = R(u, Ju, Ju, u)(p)$. M is *a space of the constant holomorphic sectional curvature* if $KH_p(\Pi_H)$ is independent of the choice of Π_H in T_pM, where p is an arbitrary point of M and depends on $p \in M$. Its Riemann curvature tensor is given by

$$\begin{aligned} R(u,v,z,w) = KH_p(g(u,z)g(v,w) & \\ - g(u,w)g(v,z) + g(u,Jz)g(v,Jw) & \\ - g(u,Jw)g(v,Jz) + 2g(u,Jv)g(z,Jw)). & \end{aligned}$$

Let (M^{2m}, g, J) be a connected Kähler manifold ($m \geq 2$). If $KH_p(\Pi_H)$ is invariant by J, depends only on p, then M is *a space of constant holomorphic sectional curvature.*

A Hermitian manifold is a complex manifold endowed with a Riemannian metric g such that

$$g(Ju, Jv) = g(u, v), \tag{2.2}$$

for all $u, v \in \mathfrak{X}(M)$. More details about other types of almost complex manifolds endowed with a metric g satisfying (2.2) one can find in [**46**].

A curve γ is *the holomorphically planar curve* if its tangent vector field T belongs to the plane spanned by the vectors T and JT under parallel displacement with respect to a complex symmetric connection ∇ along the curve γ; i.e., if T satisfies the relation $\nabla_T T = \rho(t)T + \sigma(t)JT$, where $\rho(t)$ and $\sigma(t)$ are some functions of a real parameter t.

A diffeomorphism $f : (\tilde{M}, \tilde{\nabla}) \to (M, \nabla)$ of manifolds with complex symmetric connections is called *holomorphically projective* if the image of any holomorphically planar curve of $\tilde{M}$ is also holomorphically planar curve of M. More details about these diffeomorphisms and curves one can find in [**83**].

Let (M^{2m}, ∇, J) be a complex manifold, where ∇ is the corresponding complex symmetric connection. Tashiro [**129**] has studied some transformations of (M^{2m}, ∇, J). *The transformation s* is *holomorphically projective* on (M^{2m}, ∇, J) if it preserves the system of holomorphically planar curves, i.e., if the pull back $s^*\nabla$ of the complex symmetric connection ∇ is holomorphically projective related to ∇, i.e., if there exists a global 1-form $\pi = \pi(s)$ on M such that

$$s^*\nabla_u v = \nabla_u v + \pi(u)v + \pi(v)u - \pi(Ju)Jv - \pi(Jv)Ju,$$

for arbitrary smooth vector fields u, v. He has proved that *the holomorphically projective curvature tensor*

$$\begin{aligned} HP(R)(u,v)w = R(u,v), w &+ P(v,w)u - P(u,w)v - P(u,v)w + P(v,u)w \\ &- P(v,Jw)Ju + P(u,Jw)Jv + P(u,Jv)Jw - P(v,Ju)Jw, \end{aligned}$$

where

$$P(u,v) = -\frac{1}{2(m+1)}\Big[\rho(u,v) + \frac{1}{2(m-1)}(\rho(u,v) + \rho(v,u) - \rho(Ju,Jv) - \rho(Jv,Ju))\Big],$$

is invariant with respect to each holomorphically projective transformation of ∇. This tensor plays a similar role in studying of a manifold endowed with a complex symmetric connection as the Weyl projective curvature tensor in studying of manifolds with a torsion free connection. So, $HP(R)$ of *a holomorphically projective flat* space vanishes. A complex manifold (M^{2m}, J, ∇) is said to be a holomorphically projective flat, if it can be related to a flat space by a holomorphically projective map. $HP(R)$ characterizes a space of constant holomorphical sectional curvature as follows: $HP(R) = 0$ if and only if M^{2m} is a space of constant holomorphic sectional curvature. Ishihara in [**55**] has found the conditions that the group of holomorphically projective transformations coincides with its subgroup of affine transformations. More precisely, he proved the following theorem.

Theorem 2.1. *If a complex manifold M of complex dimension $m > 1$ is complete with respect to a complex symmetric connection ∇ and the Ricci tensor of M vanishes identically in M, then the group of holomorphically projective transformations of M coincides with its subgroup of affine transformations.* □

Moreover, if M is a compact Kähler manifold then Ishihara has proved that the identity component of its group of holomorphically projective transformations for the Levi-Civita connection coincides with the identity component of its group of isometries.

II.3. C-holomorphically projective transformations

As we have seen in II.2 a C^∞ differentiable manifold M^{2m} is said to have an almost complex structure if there exists on TM a field J of endomorphisms of tangent spaces such that $J^2 = -I$, I being the identity transformation. Every manifold carrying an almost complex structure must have an even dimension.

The notion of almost contact structure generalizes these structures in the case of the odd dimension. A $(2m+1)$-dimensional C^∞ manifold M is said to have *an almost contact structure* (φ, ξ, η) if it admits a field of endomorphisms φ, a vector field ξ and a 1-form η such that $\varphi^2 = -I + \eta \otimes \xi$, $4\eta(\xi) = 1$. The following relations also hold $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, $\operatorname{rank} \varphi = 2m$. We remark that any odd dimensional orientable compact manifold M has Euler characteristic equal to zero, and there exists at least one non singular vector field ξ on M. On every almost contact manifold M there exists a Riemannian metric g satisfying

$$g(x, \xi) = \eta(x), \quad g(\varphi x, \varphi y) = g(x, y) - \eta(x)\eta(y),$$

g is said to be ***compatible with the structure*** (φ, ξ, η) and (φ, ξ, η, g) is called *an almost contact metric* structure. We refer to [**7**] for more details.

Example 3.1. Let M^{2m+1} be a C^∞ orientable hypersurface of an almost Hermitian manifold $\bar{M}^{2m+2}$ with almost complex structure J and Hermitian metric G.

Then there exists a vector field C along M^{2m+1} transverse to M^{2m+1} such that JC is tangent to M^{2m+1} (otherwise an almost complex structure on M^{2m+1} would exist, which is impossible). Thus, we can find a vector field ξ on M^{2m+1} such that $C = J\xi$ is transverse to M^{2m+1}. The relation $Ju = \varphi u + \eta(u)C$ defines the tensor field φ of type (1,1) and the 1-form η on M^{2m+1} satisfying $\varphi^2 = -I + \eta \otimes \xi$ and $\eta \circ \varphi = 0$. Since $\varphi\xi = 0$ and $\eta(\xi) = 1$ also hold, (φ, ξ, η) defines an almost contact structure on M^{2m+1}. Moreover, the metric g induced by G is the metric compatible with the almost contact structure (φ, ξ, η). □

Example 3.2. Let M^{2m} be an almost complex manifold with almost complex structure J. We consider the manifold $M^{2m+1} = M^{2m} \times \mathbb{R}$, though a similar construction can be made for the product $M^{2m} \times S^1$. Denote a vector field on M^{2m+1} by $(u, f\frac{d}{dt})$ where u is tangent to M^{2m}, t is the coordinate of $\mathbb{R}$ and f is a C^∞ function on M^{2m+1}. Then $\eta = dt$, $\xi = (0, \frac{d}{dt})$ and $\varphi(u, f\frac{d}{dt}) = (Ju, 0)$ define an almost contact structure (φ, ξ, η) on M^{2m+1}. □

An odd-dimensional parallelizable manifold, especially any odd-dimensional Lie group, carries an almost contact structure.

As is well known, if $(M^{2m+1}, \varphi, \xi, \eta)$ is an almost contact manifold, the linear map J defined on the product $M^{2m+1} \times \mathbb{R}$ by the relation

$$J\Big(u, f\frac{d}{dt}\Big) = \Big(\varphi u - f\xi, \eta(u)\frac{d}{dt}\Big),$$

where f is a C^∞ real-valued function on $M^{2m+1} \times \mathbb{R}$, is an almost complex structure on $M^{2m+1} \times \mathbb{R}$; thus we have $J^2 = -I$. In particular, if J is integrable, the almost contact structure (φ, ξ, η) *is normal.*

The almost contact structure (φ, ξ, η) is said to be ***normal*** if and only if the tensors $N, N^{(1)}, N^{(2)}, N^{(3)}$ vanish on M^{2m+1}, where

$$\begin{aligned} N(u,v) &= [\varphi,\varphi](u,v) + d\eta(u,v)\xi, \quad & N^{(2)}(u,v) &= (\mathfrak{L}_v\varphi)(u), \\ N^{(1)}(u,v) &= (\mathfrak{L}_{\varphi u}\eta)(v) - (\mathfrak{L}_{\varphi v}\eta)(u), \quad & N^{(3)}(u) &= (\mathfrak{L}_\xi\eta)(u), \end{aligned} \tag{3.6}$$

$\mathfrak{L}$ denotes the *Lie differentiation* and $[\varphi, \varphi]$ is *the Nijenhuis torsion tensor* of φ.

The normal almost contact structure generalizes, in the odd dimension, the complex structure.

If an almost complex structure J is integrable then $[J, J] = 0$. As a consequence there exists a torsion free adopted connection $\bar{\nabla}$, i.e., satisfying $\bar{\nabla}J = 0$. Thus it appears of interest to construct some connection ∇ on the almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ which gives rise to an adapted connection $\bar{\nabla}$ on $M^{2m+1} \times \mathbb{R}$.

Definition 3.3. [79] A linear connection ∇ on an almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ is called *an adopted connection* if it satisfies the following system

$$\begin{aligned} (\nabla_u\varphi)v &= \eta(v)hu + \frac{1}{4}(d\eta(\varphi u, hv) - d\eta(u, \varphi v))\xi, \\ (\nabla_u\eta)(v) &= \frac{1}{4}(d\eta(u,v) + d\eta(\varphi u, \varphi v)), \\ \nabla_u\xi &= \varphi u - \frac{1}{4}d\eta(u,\xi)\xi, \end{aligned} \tag{3.1}$$

where $h = I - \xi \otimes \eta$. □

We refer to [80] for more details related to the results which follow. Notice that the system (3.1) is not the only solution to our initial problem $\nabla J = 0$. One can check that the general family of the adopted connections ∇ on the almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ is given by the equation

$$\nabla_u v = \check{\nabla}_u v + P(u,v),$$

where $\breve{\nabla}$ is an arbitrary initial connection and P is given by

$$P(u,v) = \frac{1}{2}(\breve{\nabla}_u\varphi)\varphi v - (\breve{\nabla}_u\xi)\eta(v) + \frac{1}{2}(\breve{\nabla}_u\varphi)(v)\xi + \frac{1}{2}\eta(\breve{\nabla}_u\xi)\eta(v)\xi$$
$$+ \eta(v)\varphi u - \frac{1}{4}\{d\eta(u,v) + d\eta(\varphi u, \varphi v)\}\xi + (\Phi - \Theta)Q(u,v).$$

Here Q denotes an arbitrary tensor field of type (1.2) and $\Phi = \frac{1}{2}(I \otimes I - \varphi \otimes \varphi)$, $\Theta = \frac{1}{2}(I \otimes I - h \otimes h)$.

We remark the curvature tensor as well as the Ricci tensor of an adopted connection on a normal almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ have some interesting properties which allow us to consider some transformations, in the spirit of the sections II.2.

Definition 3.4. Let ∇ be a torsion free connection adopted to the normal almost contact structure (φ, ξ, η) on M^{2m+1}. A curve γ is *C-flat* (*almost-contact flat*) with respect to ∇ if $\nabla_T T = \rho(t)T + \sigma(t)\varphi T$, where T denotes the vector tangent to γ and ρ, σ are smooth real valued functions along γ. □

We remark that in this case the subspace spanned by T and φT is not transported by parallelism along γ. Namely, $\nabla_T(\varphi T)$ does not belong to the space spanned by T and φT. However, one can show that the dimension of this subspace is constant along γ and this dimension can be 2,1 or 0.

Remark. We introduced in **[25]** the concept of C-flat paths, obtaining a C-projective tensor in normal almost contact manifolds, endowed, with a torsion free connection whose fundamental tensors φ, ξ and η are parallel.

The torsion free linear connections $\nabla, \tilde{\nabla}$ adapted to the normal almost contact structure (φ, ξ, η) are *C-projectively related* if they have the same C-flat curve. One can show that two torsion free connections $\nabla, \tilde{\nabla}$ adopted to the normal almost contact structure (φ, ξ, η) are C-projectively related if and only if

$$\tilde{\nabla}_u v = \nabla_u v + P(u,v),$$

where $P(u,v) = \alpha(u)hv + \alpha(v)hu - \beta(u)\varphi v - \beta(v)\varphi u$, with α an arbitrary 1-form satisfying $\alpha(\xi) = 0$, $\beta(u) = \alpha(\varphi u)$. Consequently, since $\nabla, \tilde{\nabla}$ fulfill the same conditions (3.1), their difference tensor P satisfies the same relations as in the case where φ, ξ, η are parallel.

Matzeu has proved in **[80]** that the tensor field $W(R)$ given by

$$\begin{aligned} W(R)(u,v)z = {} & hR(u,v)z + \{L(u,v) - L(v,u)\}hz + \{L(u,v) + \eta(u)\eta(z)\}hv \\ & - \{L(v,z) + \eta(v)\eta(z)\}hu - \{L(u,\varphi v) - L(v,\varphi u) + d\eta(u,v)\}\varphi z \\ & - \Big\{L(u,\varphi z) + \frac{1}{2}d\eta(u,z)\Big\}\varphi v + \Big\{L(v,\varphi z) + \frac{1}{2}d\eta(v,z)\Big\}\varphi u, \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} L(u,v) = {} & \frac{1}{2(m+1)}\Big\{\rho(R)(u,hv) + \frac{1}{2(m-1)}[\rho(R)(hu,v) \\ & + \rho(R)(hv,u) - \rho(R)(\varphi u,\varphi v) - \rho(R)(\varphi v,\varphi u)]\Big\} + kd\eta(u,\varphi v), \quad k = \text{const}, \end{aligned} \tag{3.3}$$

is C-projectively invariant. Moreover, if k in (3.3) is given by $k = \frac{1}{2m+2}$, all traces of $W(R)$

$$\operatorname{trace}(W(R)(u,v)), \quad \operatorname{trace}(u \to W(R)(u,v)z)$$
$$\operatorname{trace}(\varphi W(R)(u,v)), \quad \operatorname{trace}(u \to \varphi W(R)(u,v)z)$$

vanish.

We say torsion free connection adopted to the normal almost contact structure (φ, ξ, η) is *C-projectively flat* if its *C-projective curvature tensor* $W(R)$ vanishes.

We refer **[80]** for the proof of the following theorem.

Theorem 3.5. *For $m > 2$ the torsion free adopted connection ∇ is C-projectively flat if and only if it can be transformed locally by a C-projective transformation into a torsion free adopted connection $\tilde{\nabla}$ whose curvature tensor $\tilde{R}$ satisfies the condition*

$$h\tilde{R}(u,v)z = \{-\eta(u)hv + \eta(v)hu\}\eta(z) + d\eta(u,v)\varphi z + \frac{1}{2}d\eta(u,z)\varphi v - \frac{1}{2}d\eta(v,z)\varphi u. \square$$

The case $m = 1$ has been studied by Oproiu in **[103]**. It is also interesting that there does not exist a flat adopted connection from a C-projective transformation. But, in the framework of ∇ with parallelizable (φ, ξ, η) it exists.

A special class of normal almost contact metric spaces $(M^{2m+1}, \varphi, \xi, \eta, g)$ is Sasakian one satisfying the condition $\eta \wedge d\eta^m \neq 0$ ($d\eta^m$ is m-th exterior power). The Levi-Civita connection ∇ of g for Sasakian manifold is an adopted one. A Sasakian manifold is C-projectively flat if and only if it has constant φ-sectional curvature.

II.4. Conformal transformations

Let (M, g) be an m-dimensional Riemannian manifold. Locally the metric is given by $ds^2 = g_{ij}dx^i dx^j$, where the g_{ij} are the components of g with respect to the natural frames of a local coordinate system (x^i). A metric g^* on M is said to be conformally related to g if it is proportional to g, that is, if there is a function $\beta > 0$ on M such that $g^* = \beta^2 g$. We denote by $\mathfrak{C}$ a conformal class of Riemannian metrics on a smooth manifold M, of dimension $m \geq 2$. By *a conformal transformation* of M is meant a differentiable homeomorphism f of M onto itself with the property that $f^*(ds^2) = \beta^2 ds^2$, where f^* is the induced map in the bundle of frames and β is a positive function on M. The set of conformal transformations of M forms a group. Moreover, it can be shown that it is a Lie transformation group. A diffeomorphism f of M onto itself is called an isometry if it preserves the metric tensor.

Under a conformal transformation of metric, the curvature tensor $R(u,v)w$ will be transformed into

$$R^*(u,v)w = R(u,v)w - \sigma(w,u)v + \sigma(w,v)u - g(w,u)v + g(w,v)u,$$

where σ is the tensor of type (0,2) with components $\sigma_{jk} = \beta_{j;k} - \beta_j\beta_k + \frac{1}{2}g^{bc}\beta_b\beta_c g_{jk}$, and μ the corresponding type (1,1) tensor with components

$$\mu_l^i = \sigma_{kl}g^{ki}, \quad \left(\beta_j = \frac{\partial \log \beta}{\partial u^j}\right).$$

Let $m > 2$. The tensor

$$\begin{aligned} C(u,v)w = R(u,v)w &- \frac{1}{m-2}(\rho(w,u)v - \rho(w,v)u + g(w,u)S(v) - g(w,v)S(u)) \\ &+ \frac{\tau}{(m-1)(m-2)}(g(w,u)v - g(w,v)u), \end{aligned}$$

where S is the Ricci endomorphism $g(Su,v) = \rho(u,v)$, is invariant under a conformal transformation of a metric, i.e. $C^*(u,v)w = C(u,v)w$. This tensor is called *the Weyl conformal curvature tensor.* The case $m = 3$ is interesting. Indeed, by choosing an orthogonal coordinate system ($g_{ij} = 0$, $i \neq j$) at a point, it is readily shown that the Weyl conformal curvature tensor vanishes.

Consider a Riemannian manifold (M,g) and let g^* be a conformally related locally flat metric. Under these circumstances M is said to be locally conformally flat. Let

$$\begin{aligned} C(u,v,w) = &\frac{1}{m-2}((\nabla_w\rho)(u,v) - (\nabla_v\rho)(w,v)) \\ &- \frac{1}{2(m-1)(m-2)}(g(u,v)\nabla_w\tau - g(u,w)\nabla_v\tau). \end{aligned}$$

One can prove the following theorem

Theorem 4.1. *A necessary and sufficient condition that a Riemannian manifold of dimension $m > 3$ be a conformally flat is that its Weyl conformal curvature tensor vanish. For $m = 3$, it is necessary and sufficient that the tensor $C(u,v,w)$ vanishes, i.e. $C(u,v,w) = 0$.* □

There exist numerous examples of conformally flat spaces. For example, a Riemannian manifold of constant curvature is conformally flat, provided $m \geq 3$.

Any two 2-dimensional Riemannian manifolds are conformally related, as the quadratic form ds^2 for $m = 2$ is reducible to the form $\lambda[(du^1)^2 + (du^2)^2]$ (in infinitely many ways).

For more details one can use [**45**], [**64**], [**140**] etc.

II.5. Codazzi geometry

Codazzi structure is constructed from a conformal and a projective structure using the Codazzi equations. A torsion free connection $^*\nabla$ and a semi-Riemannian metric h are said to *satisfy the Codazzi equation* or to be *Codazzi compatible* if

$$(5.1) \qquad (^*\nabla_u h)(v,w) = (^*\nabla_v h)(u,w).$$

A projective class $\mathfrak{P}$ of torsion free connections and a conformal class $\mathfrak{C}$ of semi-Riemannian metrics are said to be *Codazzi compatible* if there exists ${}^*\nabla \in \mathfrak{P}$ and $h \in \mathfrak{C}$ which are Codazzi compatible. We extend the action of the gauge group to define ${}^*\nabla \to {}^*_\beta\nabla$ where ${}^*_\beta\nabla$ is defined by taking $\pi = d\ln\beta$ in (**1.1**):

$$
{}^*_\beta\nabla_u v = {}^*\nabla_u v + d\ln\beta(u)v + d\ln\beta(v)u. \tag{5.2}
$$

One can check easily the Codazzi equations are preserved by gauge equivalence. *A Codazzi structure* $\mathfrak{K}$ on M is a pair $(\mathfrak{C}, \mathfrak{P})$ where the conformal class of semi-Riemannian metrics $\mathfrak{C}$ and the projective class $\mathfrak{P}$ are Codazzi compatible. *A Codazzi manifold* $(M, \mathfrak{K})$ is a manifold endowed with the Codazzi structure.

Suppose now that $(h, {}^*\nabla)$ are Codazzi compatible. Let $C := {}^*\nabla - \nabla(h)$ be a (1,2) tensor and let $\check{C}$ be the associated cubic form. Since ${}^*\nabla$ and $\nabla(h)$ are torsion free, C is a symmetric (1,2) tensor and $\check{C}(u, v, w) = \check{C}(v, u, w)$. The relation (5.1) and this symmetry implies $\check{C}(u, v, w) = \check{C}(w, v, u)$ and consequently $\check{C}$ is totally symmetric.

Assuming that h is a semi-Riemannian metric and $\check{C}$ is a totally symmetric cubic form one can construct *a conjugate triple* $({}^*\nabla, h, \nabla)$, i.e. a triple $({}^*\nabla, h, \nabla)$ satisfying

$$
uh(v, w) = h(\nabla_u v, w) + h(v, {}^*\nabla_u w). \tag{5.3}
$$

Therefore, let ${}^*\nabla := \nabla(h) + C$, where C is the associated symmetric (1,2) tensor field. Since $\check{C}$ is symmetric, ${}^*\nabla$ is torsion free and the Codazzi equation (5.1) is satisfied. If we put $\nabla := \nabla(h) - C$ one can check the triple $({}^*\nabla, h, \nabla)$ satisfies (5.3), i.e., it is a conjugate triple.

If $\mathfrak{W}$ is a Weyl structure one can define an associated Codazzi structure $\mathfrak{K}(\mathfrak{W})$. We may recover also the Weyl structure from the associated Codazzi structure. We refer to [**20**] for more details.

III. DECOMPOSITIONS OF CURVATURE TENSORS UNDER THE ACTION OF SOME CLASSICAL GROUPS AND THEIR APPLICATIONS

The main purpose of this section is to consider a curvature for a torsion free connection from the algebraic point of view and to see why it does provide insight in some problems of differential geometry, topology etc. It is possible also to study the various curvatures which appear in differential geometry in different context (see [**73**]). Let us mention that it is possible in this spirit to study some classification of almost Hermitian manifolds [**48**], Riemannian homogeneous structure [**132**] etc. Of course all these decompositions are, in principle, consequences of general theorems of groups representations (see [**135**]).

More precisely, the proofs of theorems are based on the following facts. Let $\mathfrak{G}$ be a Lie group, V a real vector space and V^* its dual space. When ξ is a $\mathfrak{G}$-concomitant between two spaces, $\mathfrak{G}$ acting on these spaces then the image for ξ

of an invariant subspace is also invariant. Further, the image is irreducible when the first space is irreducible. Also an invariant subspace of $\otimes^r V^*$ is irreducible for the action of some group if and only if the space of its quadratic invariants is 1-dimensional.

III.1. Some historical remarks

The development of the theory of the decomposition was initiated by Singer and Thorpe [**125**]. Let (V, g) be an m-dimensional real vector space with positive definite inner product and denote by $\mathcal{R}_b(V)$ the vector space of all symmetric linear transformations of the space of 2-vectors of V. All tensors having the same symmetries as the Riemannian curvature tensor including the first Bianchi identity belong also to $\mathcal{R}_b(V)$. Singer and Thorpe gave a geometrically useful description of the splitting of $\mathcal{R}_b(V)$ under the action of $O(n)$ into four components. One of the projections gives the Weyl conformal tensor. Their considerations are as follows.

Let a tensor R of type (1,3) over V be a bilinear mapping

$$R : V \times V \to \mathrm{Hom}(V, V) : (x, y) \mapsto R(x, y).$$

We use the notation $R(x, y, z, w) = g(R(x, y)z, w)$. Let $\mathcal{R}_b(V)$ and $\mathcal{R}(V)$ be the subspaces of $\otimes^4 V^*$ consisting of all tensors having the same symmetries as the curvature tensor, the first for metric connections, the second for Levi-Civita connections. It means, $R \in \mathcal{R}_b(V)$ if it yields

(a) $R(x, y) = -R(y, x)$

(b) $R(x, y)$ is a skew-symmetric endomorphism of V, i.e.,

$$R(x, y, z, w) + R(x, y, w, z) = 0$$

and $R \in \mathcal{R}(V)$ if R satisfies (a), (b) and the first Bianchi identity

(c) $\sigma R(x, y)z = 0$, where σ denotes the cyclic sum over x, y and z.

The Ricci tensor $\rho(R)$ of type (0,2) associated with R is symmetric bilinear function on $V \times V$ defined by $\rho(R)(x, y) = \mathrm{trace}\,(z \in V \mapsto R(z, x)y \in V)$. Then the Ricci tensor $Q = Q(R)$ of type (1,1) is given by $\rho(R)(x, y) = g(Qx, y)$ and the trace of Q is called the scalar curvature $\tau = \tau(R)$ of R.

Further, let α be the standard representation of the orthogonal group $O(n)$ in V. Then there is a natural *induced representation* $\tilde{\alpha}$ of $O(n)$ in $\mathcal{R}_b(V)$ given by

$$\tilde{\alpha}(a)(R)(x, y, z, w) = R(\alpha(a^{-1})x, \alpha(a^{-1})y, \alpha(a^{-1})z, \alpha(a^{-1})w).$$

for all $x, y, z, w \in V$, $R \in \mathcal{R}_b(V)$ and $a \in O(n)$.

Theorem 1.1. $\mathcal{R}_b(V) = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$, $\mathcal{R}(V) = \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$.

(i) $\mathcal{R} \in \mathcal{R}_1$ *iff the sectional curvature is zero.*

(ii) $R \in \mathcal{R}_1 \oplus \mathcal{R}_2$ *iff the sectional curvature of* R *is constant.*

(iii) $R \in \mathcal{R}_1 \oplus \mathcal{R}_3$ *iff the Ricci tensor of* R *is zero.*

(iv) $R \in \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3$ *iff the Ricci tensor of* R *is a scalar multiple of the identity.*

(v) $R \in \mathcal{R}_1 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$ *iff the scalar curvature of* R *is zero.* □

Furthermore, the action of $O(n)$ in $\mathcal{R}_b(V)$ is irreducible on each $\mathcal{R}_i$, $i = 1, 2, 3, 4$. Since a curvature tensor R, corresponding to the Levi Civita connection ∇ of a Riemannian manifold M satisfies the first Bianchi identity, we have $R \in \mathcal{R}_1^{\perp} = \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$. Statement (iv) of Theorem 1.1 implies a very nice characterization of an Einstein space as follows: a Riemannian manifold M has curvature tensor in $\mathcal{E} = \mathcal{R}_2 \oplus \mathcal{R}_3$ at each point if and only if M is an Einstein space. The $\mathcal{R}_3$-component of a curvature tensor of M is its Weyl conformal curvature tensor.

To study the action of $SO(n)$, especially for $\dim V = 4$, Singer and Thorpe have used the star operator $*$. They studied also in **[125]** the problem of a *normal form* for the curvature tensor of a 4-dimensional oriented Einstein manifold by analyzing the critical point behavior of the sectional curvature function σ. In this case, the function σ on each 2-plane is equal to its value on the orthogonal complement. Using this characterization, they have shown that the curvature function σ is completely determined by its critical point behaviour and they have shown what the locus of critical points looks like.

The relationship between the Euler-Poincaré characteristic, the arithmetic genus $\alpha(M)$ and the decomposition of the space of curvature operators at a point of 4-dimensional compact Riemannian manifold has been studied by Gray **[47]**. Applications of the decomposition of $\mathcal{R}(V)$ involving orthogonal Radon transformations were given by Strichartz **[127]**. An algebraic interpretation of the Weyl conformal curvature tensor due to Singer and Thorpe makes possible the development of the theory of submanifolds in conformal differential geometry more up to date (see **[67]**).

The complete decomposition of $\mathcal{R}_k(V) \subset \mathcal{R}(V)$, $\dim V = 2m$, satisfying the Kähler identity, under the action of $U(V)$ was treated by Sitaramaya **[126]** (see also **[57]**, **[88]**). Tricerri and Vanhecke **[131]** have found the complete decomposition of $\mathcal{R}(V)$ under the action of $U(V)$. They have obtained new conformal invariants among components of the complete decomposition of $\mathcal{R}(V)$ on almost Hermitian manifolds.

We refer to **[24]** for more details.

III.2. The action of general linear group

The main purpose of this section is to interpret the Weyl projective curvature tensor as one of the projection operators in the decomposition of tensors having all the symmetries of curvature tensors for torsion free connections under the action of the general linear group $GL(V)$. We refer to **[127]** for some details.

In this section V denotes m-dimensional ($m \geq 2$) real vector space, V^* its dual space, and $\mathcal{V}_3^1$ the space of (1,3) tensors $T(u, v, z, w)$ with $u, v, z \in V$ and $w \in V^*$. The group $GL(V)$ acts naturally on $\mathcal{V}_3^1$ by

$$\pi(g)T(u, v, z, w) = T(g^{-1}u, g^{-1}v, g^{-1}z, (g^{-1})^T w).$$

Let $\pi(m)$ be representations of $GL(V)$, where $m = (m_1, \ldots, m_m)$ is the highest weight of the representation, with $m_1 \geq m_2 \geq \cdots \geq m_m$, all m_i integers, and for simplicity of notation we delete strings of zeroes (so that $\pi(2,-1)$ stands for $\pi(2,0,\ldots,0,-1)$. Let $\mathcal{R}(V)$ be a subspace of tensors with symmetries as the curvature of a torsion free connection. So for $R \in \mathcal{R}(V)$ we have

$$R(u,v,z,w) = -R(v,u,z,w), \tag{2.1}$$

$$R(u,v,z,w) + R(v,z,u,w) + R(z,u,v,w) = 0, \tag{2.2}$$

where $R(u,v,z,w) = \langle R(u,v)z, w\rangle$. We denote the Ricci contraction by $\rho(R) = \mathrm{con}(1,4)R$. It maps $\mathcal{R}(V)$ onto $\mathcal{V}_2$. The space $\mathcal{V}_2$ splits as $\pi(2) \oplus \pi(1,1)$, the symmetric and skew-symmetric tensors. Consequently, we have

$$\mathcal{R}(V) = \pi(2) \oplus \pi(1,1) \oplus \ker(\rho).$$

One can check easily that $ker(\rho)$ is also irreducible. We introduce two special products $\odot_1$ and $\odot_2$ to describe the corresponding projection operators. For $Q \in \mathcal{V}_1^1$ and $S \in \mathcal{V}_2$ we have

$$\begin{aligned} Q \odot S(u,v,z,w) &= Q(v,w)(S(u,z) + S(z,u) - Q(u,w)(S(v,z) + S(z,v)) \\ Q \odot_2 S(u,v,z,w) &= Q(v,w)(S(u,z) - S(z,u)) - Q(u,w)(S(v,z) - S(z,v)) \\ &\quad + 2Q(z,w)(S(u,v) - S(v,u)). \end{aligned}$$

By direct computation one can check $Q \odot_1 S$, $Q \odot_2 S \in \mathcal{R}(V)$ with $\rho(Q \odot_1 S)$ symmetric and ρ $(Q \odot_2 S)$ skew-symmetric. Henceforth we have

Theorem 2.1. *Under the action of $GL(V)$, the space $\mathcal{R}(V)$ decomposes as*

$$\pi(2) \oplus \pi(1,1) \oplus \pi(2,1,-1),$$

(when $m = 2$ the third component is deleted) with corresponding projections

$$\begin{aligned} P_{(2)}R &= \frac{1}{2(m-1)}\delta \odot_1 \rho(R), \quad P_{(1,1)}R = \frac{1}{2(m+1)}\delta \odot_2 \rho(R), \\ P_{(2,1,-1)}R &= R - \frac{1}{2(m-1)}\delta \odot_1 \rho(R) - \frac{1}{2(m+1)}\delta \odot_2 \rho(R), \end{aligned}$$

where δ is Kronecker symbol.

The $\pi(2,1,-1)$ component is the kernel of ρ, while

$$\begin{aligned} \rho(P_{(z)}R)(u,v) &= \frac{1}{2}(\rho(R)(u,v) + \rho(R)(v,u)) \\ \rho(P_{(1,1)}R)(u,v) &= \frac{1}{2}(\rho(R)(v,u) - \rho(T)(u,v)). \end{aligned}$$

The corresponding dimensions and highest weight vectors are as follows:

Components	Dimension	Highest Weight Vector
$\pi(2)$	$\frac{1}{2}m(m+1)$	$\sum_k(e_1 \wedge e_k) \otimes e_1 \otimes e_k^*$
$\pi(1,1)$	$\frac{1}{2}m(m-1)$	$\sum_k(2(e_1 \wedge e_2) \otimes e_k \otimes e_k^* + (e_1 \wedge e_k) \otimes e_2 \otimes e_k^* + (e_k \wedge e_2) \otimes e_1 \otimes e_k^*)$
$\pi(2,1,-1)$	$\frac{1}{3}m^2(m^2-4)$	$(e_1 \wedge e_2) \otimes e_1 \otimes e_m^*$
$\mathcal{R}(V)$	$\frac{1}{2}m^2(m^2-1)$	

If $V = T_pM$ then one can compare (1.2) in Section II with $P_{(2,,1,-1)}R$ to see that they coincide and consequently the Weyl projective curvature tensor is really an irreducible component in the proceeding decomposition. One can see easily that the Weyl projective curvature tensor fulfills the algebraic conditions (2.2), (2.3) and $\rho(P(R)) = 0$.

Strichartz in **[127]** has studied the complete decomposition of the vector space of the first covariant derivative of curvature tensors for torsion free connections under the action of $GL(V)$. Using these decompositions he has proved that a projectively flat affine manifold with skew-symmetric Ricci curvature must be locally affine symmetric.

III.3. The action of the group $SO(m)$

Studying projective transformations of a Riemannian manifold (M, g) we naturally combine two structures: the positive definite metric g and torsion free connections ∇, which can be, for example, projectively equivalent to the Levi-Civita connection. Therefore we are interested now in the complete decomposition of $\mathcal{R}(V)$ from the section III.2 under the action of $SO(m)$. We refer to **[23]** for more details.

Let V be an m-dimensional real vector space endowed with positive definite inner product $\langle \cdot, \cdot \rangle$. A tensor R of type (1,3) over V is a bilinear mapping

$$R : V \times V \to \mathrm{Hom}(V, V) : (x, y) \mapsto R(x, y).$$

R is called *a curvature tensor over* V if it has the following properties for all $x, y, z, w \in V$:

(i) $R(x, y) = -R(y, x)$,

(ii) *the first Bianchi identity*, i.e. $\sigma R(x, y)z = 0$, where σ denotes the cyclic sum with respect to x, y and z.

We also use the notation $R(x, y, z, w) = g(R(x, y)z, w)$. We denote by $\mathcal{R}(V)$ the vector space of all curvature tensors over V. In addition to *the Ricci tensor*

$\rho(R)$ for a curvature tensor $R \in \mathcal{R}(V)$ it makes sense to define *the second trace* $\hat{\rho}(R)$ by

$$\hat{\rho}(R)(x,y) = \sum_{i=1}^{m} R(e_i, x, e_i, y), \quad x, y \in V,$$

where $\{e_i\}$ is an arbitrary orthonormal basis of V. The traces $\rho(R)$ and $\hat{\rho}(R)$ are orthogonal. Moreover, they are neither symmetric nor skew-symmetric in general case. The scalar curvature $\tau = \tau(R)$ of R is defined as the trace of $Q = Q(R)$, given by $\rho(R)(x,y) = \langle Qx, y \rangle$. Now one can define all the components of the decomposition of $\mathcal{R}(V)$. We put

$$\begin{aligned}
\mathcal{R}^a(V) &= \{R \in \mathcal{R}(V) \mid \rho(R) \text{ and } \hat{\rho}(R) \text{ are skew-symmetric}\},\\
\mathcal{R}^s(V) &= \{R \in \mathcal{R}(V) \mid \rho(R) \text{ and } \hat{\rho}(R) \text{ are symmetric}\},\\
\mathcal{R}_p(V) &= \{R \in \mathcal{R}(V) \mid \rho(r) \text{ is zero}\},\\
\mathcal{R}_0(V) &= \mathcal{R}^a(V) \cap \mathcal{R}^s(V) = \{R \in \mathcal{R}(V) \mid \rho(R) \text{ and } \hat{\rho}(R) \text{ are zero}\},\\
W_4 &= \text{orthogonal complement of } \mathcal{R}_0(V) \text{ in } \mathcal{R}_p(V) \cap \mathcal{R}^a(V),\\
W_5 &= \text{orthogonal complement of } \mathcal{R}_0(V) \text{ in } \mathcal{R}_p(V) \cap \mathcal{R}^s(V),\\
W_3 &= \text{orthogonal complement of } \mathcal{R}_p(V) \cap \mathcal{R}^a(v) \text{ in } \mathcal{R}^a(V),\\
W_1 \oplus W_2 &= \text{orthogonal complement of } \mathcal{R}_p(V) \cap \mathcal{R}^s(V) \text{ in } \mathcal{R}^s(V),\\
W_2 &= \{R \in W_1 \oplus W_2 \mid \tau(R) \text{ is zero}\},\\
W_1 &= \text{orthogonal complement of } W_2 \text{ in } W_1 \oplus W_2,\\
W_6 &= \{R \in \mathcal{R}_0(V) \mid R(x,y,z,w) = -R(x,y,w,z);\ x,y,z,w \in V\},\\
W_7 &= \{R \in \mathcal{R}_0(V) \mid R(x,y,z,w) = R(x,y,w,z);\ x,y,z,w \in V\},\\
W_8 &= \text{orthogonal complement of } W_6 \oplus W_7 \text{ in } \mathcal{R}_0(V).
\end{aligned}$$

So we can state the decomposition theorem for $\mathcal{R}(V)$.

Theorem 3.1. *We have*

$$\mathcal{R}(V) = W_1 \oplus \cdots \oplus W_8, \tag{3.1}$$

where W_i are orthogonal invariant subspaces under the action of $SO(V)$ $(m \geq 2)$. Moreover,

(i) The decomposition (3.1) is irreducible for $m \geq 4$.
(ii) For $m = 4$ we have $W_6 = W^+ \oplus W^-$ where $W^\pm = \{R \in W_6 \mid R* = \pm R\}$, and the other factors are irreducible.
(iii) For $m = 3$ we have $W_6 = W_8 = \{0\}$ and the other factors are irreducible.
(iv) For $m = 2$ we have $W_4 = W_5 = W_6 = W_7 = W_8 = \{0\}$ and the other factors are irreducible. □

We compare the decompositions in subsections III.2 and III.3 to see

$$\pi(2) = W_1 \oplus W_2, \quad \pi(1,1) = W_3, \quad \pi(2,1,-1) = \mathcal{R}_p(V).$$

If we suppose $x, y, z, \ldots \in \mathfrak{X}(M)$, the algebra of C^∞ vector fields on (M, g) and R and ρ the curvature and the Ricci tensor respectively then the projective curvature tensor $P(R)$ associated with R is the orthogonal projection of R on $\mathcal{R}_p(\mathfrak{X}(M))$. We recall that the Weyl conformal curvature tensor belongs to the $\mathcal{R}_3$-component from Singer-Thorpe decomposition, which is irreducible under the action of the group $O(V)$. The projective component $\mathcal{R}_p(V)$ is not irreducible under the action of $O(V)$ or $SO(V)$.

The complete decomposition of $\mathcal{R}(V)$ given by Theorem **3.1.** is very useful in the study of the group of projective transformations on some manifold M^m and its subgroups. We recall that an equiaffine transformation is an affine volume preserving transformation of a manifold (M, ∇). On a manifold (M^m, ∇) there exists an equiaffine transformation if and only if the Ricci tensor $\rho(R)$, corresponding to any symmetric connection ∇, is symmetric. Let us mention some of these results.

Theorem 3.2. *Let (M, ∇, g) be a Riemannian manifold endowed with a symmetric connection ∇ such that $W_3 = 0$. Then the group of affine transformations coincides with its subgroup of equiaffine transformations.* □

Theorem 3.3. *If a Riemannian manifold (M^m, g) is compact and $W_1 = W_2 = W_3 = 0$ then the group $\mathcal{P}(M)$ of all projective transformations coincides with its subgroup $\mathcal{A}(M)$ of all affine transformations.* □

Some of components in (3.1) have other interesting geometric properties. So, Nikčević has proved in [**91**] that $\mathcal{R}_0(\mathfrak{X}(M))$ is conformally invariant.

Let us point out that for some torsion free connections the corresponding curvature tensor has some of its projections on W_i $(i = 1, \ldots, 8)$ equal to zero. Namely, if ∇ is the Levi-Civita connection then we have

$$\begin{aligned} &P(R) \in W_5 \oplus W_6, && \text{for } m > 4, \\ &P(R) \in W_5 \oplus W^+ \oplus W^-, && \text{for } m = 4, \\ &P(R) \in W_5, && \text{for } m = 3. \end{aligned}$$

We refer to [**28**] for more details.

The decomposition (3.1) is not unique. The second one is given in [**23**] which is closely related with the decomposition of curvature tensors for Weyl connections under the action of the conformal group $CO(m)$ [**53**].

III.4. The action of the group $U(m)$

The main purpose of this section is to study algebraic properties of a holomorphically projective curvature tensor on Hermitian manifold. Therefore we start with some considerations in a vector space endowed with some structures. We refer to [**81**] for some details.

Let V be a $2m$-dimensional real vector space endowed with the complex structure J, compatible with the positive definite inner product g, i.e.

$$J^2 = -I, \quad g(Jx, Jy) = g(x, y),$$

for all $x, y \in V$ and where I denotes the identity transformation of V. A tensor R of type (1,3) over V is bilinear mapping

$$R : V \times V \to \mathrm{Hom}(V, V) : (x, y) \mapsto R(x, y).$$

R is called *a curvature tensor over* V if it has the following properties for all $x, y, z, w \in V$:

(i) $R(x, y) = -R(y, x)$,
(ii) $\sigma R(x, y)z = 0$ (the first Bianchi identity)
(iii) $JR(x, y) = R(x, y)J$ (the Kähler identity).

We use also the notation $R(x, y, z, w) = g(R(x, y)z, w)$.

Let $\mathcal{R}(V)$ denotes the vector space of all curvature tensors over V. This space has a natural inner product defined with that on V:

$$\langle R, \tilde{R} \rangle = \sum_{i,j,k=1}^{2m} g(R(e_i, e_j)e_k, \tilde{R}(e_i . e_j)e_k),$$

where $R, \tilde{R} \in \mathcal{R}(V)$ and $\{e_i\}$ is an orthonormal basis of V. *A natural induced representation of* $U(m)$ *in* $\mathcal{R}(V)$ is the same as of $O(m)$ in the previous sections.

To describe a complete decomposition of $\mathcal{R}(V)$ under the action of $U(m)$ we need some basic notations. There are independent traces as follows:

$$\rho(R)(x, y) = \sum_{i=1}^{2m} R(e_i, x, y, e_i), \qquad \tau(R) = \sum_{i,j=1}^{2m} R(e_i, e_j, e_j, e_i),$$
$$\tilde{\rho}(R)(x, y) = \sum_{i=1}^{2m} R(e_i, x, e_i, y), \qquad \tau^*(R) = \sum_{i,j=1}^{2m} R(e_i, e_j, Je_j, e_i),$$

where $\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}$ is an arbitrary basis of V. The trace $\rho = \rho(R)$, as we have seen in the Section I, is called *the Ricci tensor*, and $\tau = \tau(R)$ is *the scalar curvature* of R.

In general, the traces ρ and $\hat{\rho}$ are neither symmetric nor skew-symmetric and always we have $\hat{\rho}(Jx, Jy) = \hat{\rho}(x, y)$; ρ and $\hat{\rho}$ belong to $\mathcal{V}^2 = V^* \otimes V^*$, where V^* is the dual space of V. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathcal{V}^2(V)$ given by:

$$\langle \alpha, \beta \rangle = \sum_{i,j=1}^{2n} \alpha(e_i, e_j)\beta(e_i, e_j), \quad \mathrm{for} \alpha, \beta \in \mathcal{V}^2(V).$$

Now we introduce some tensors, and operators that we need to define components in the complete decomposition of $\mathcal{R}(V)$.

$$\pi_1(x, y)z : = g(x, z)y - g(y, z)x + g(Jx, z)Jy - g(Jy, z)Jx + 2g(Jx, y)Jz,$$
$$\pi_2(x, y)z : = g(Jx, z)y - g(Jy, z)x + 2g(Jx, y)z - g(x, z)Jy + g(y, z)Jx.$$

Let ϕ be the operator defined by

$$\phi(R)(x,y,z,w) := R(Jx,Jy,z,w),\ \ R\in\mathcal{R}(V),$$

and $\mathcal{R}^+$ and $\mathcal{R}^-$ be the vector subspaces of $\mathcal{R}(V)$ given by

$$\mathcal{R}^+ : \{R\in\mathcal{R}(V)\mid\phi(R)=R\},\quad \mathcal{R}^- := \{R\in\mathcal{R}(V)\mid\phi(R)=-R\}.$$

The vector space $\mathcal{R}(V)$ consists some subspaces which one can define in terms of traces symmetry properties. So we have

$$\begin{aligned}
W_0 &:= \{R\in\mathcal{R}(V)\mid\tau(R)=\tau^*(R)=0\},\\
R_H &:= \{R\in\mathcal{R}(V)\mid\rho(R)=0\},\\
\mathcal{R}_0 &:= \{R\in\mathcal{R}(V)\mid\rho(R)=\hat\rho(R)=0\},\\
\mathcal{R}^s_\rho &:= \{R\in\mathcal{R}(V)\mid\rho(R)\neq 0 \text{ and } \rho(R)(x,y)=\rho(R)(y,x)\},\\
\mathcal{R}^a_\rho &:= \{R\in\mathcal{R}(V)\mid\rho(R)\neq 0 \text{ and } \rho(R)(x,y)=-\rho(R)(y,x)\},\\
\mathcal{R}^s_{\hat\rho} &:= \{R\in\mathcal{R}(V)\mid\hat\rho(R)(x,y)=\hat\rho(R)(y,x)\},\\
\mathcal{R}^a_{\hat\rho} &:= \{R\in\mathcal{R}(V)\mid\hat\rho(R)(x,y)=-\hat\rho(y,x)\}.
\end{aligned}$$

Now we can define all components in the compete decomposition of $\mathcal{R}(V)$.

Definition 4.1. We put

$$\begin{aligned}
W_9 &:= \{R\in\mathcal{R}^+\cap\mathcal{R}_0\mid R(x,y,z,w)=-R(x,y,w,z)\},\\
W_{10} &:= \{R\in\mathcal{R}^+\cap\mathcal{R}_0\mid R(x,y,z,w)=R(x,y,w,z)\},\\
W_{11} &:= \text{orthogonal complement of } W_9\oplus W_{10} \text{ in } \mathcal{R}^+\cap\mathcal{R}_0,\\
W_1 &:= \mathcal{R}^+\cap\mathcal{R}^s_\rho,\qquad W_5 := \mathcal{L}(\pi_1);\\
W_3 &:= \mathcal{R}^+\cap\mathcal{R}^a_\rho,\qquad W_6 := \mathcal{L}(\pi_2):\\
W_7 &:= \text{orthogonal complement of } \mathcal{R}_0 \text{ in } \mathcal{R}_H\cap\mathcal{R}^s_{\hat\rho},\\
W_8 &:= \text{orthogonal complement of } \mathcal{R}_0 \text{ in } \mathcal{R}_H\cap\mathcal{R}^a_{\hat\rho},\\
W_{12} &:= \mathcal{R}^-\cap\mathcal{R}_H,\\
W_2 &:= \mathcal{R}^-\cap\mathcal{R}^s_\rho = \text{orthogonal complement of } W_1 \text{ in } \mathcal{R}^s_\rho,\\
W_4 &:= \mathcal{R}^-\cap\mathcal{R}^a_\rho = \text{orthogonal complement of } W_3 \text{ in } \mathcal{R}^a_\rho.
\end{aligned}$$

Thus we obtain

Theorem 4.2. *If* $\dim V = 2m, m\geq 3$, *then* $\mathcal{R}(V) = W_1\oplus\cdots\oplus W_{12}$; *if* $m=2, W_{11}=W_{12}=\{0\}$ *and* $\mathcal{R}(V)=W_1\oplus\cdots\oplus W_{10}$. *These subspaces are mutually orthogonal and invariant under the action of* $U(m)$. □

Recalling that an invariant subspace is irreducible if it does not contain a nontrivial invariant subspace, we have also

Theorem 4.3. *The decomposition of $\mathcal{R}(V)$ given above is irreducible under the action of $U(m)$.* $\square$

The projections of $R \in \mathcal{R}(V)$ on W_i $(i = 1, 2, \dots, 12)$ and the dimensions of W_i have been done also in [**81**].

Let M be a $2m$-dimensional C^∞ manifold with an almost complex structure J and a Hermitian inner product g. Then, for all $u, v \in \mathfrak{X}(M)$, the Lie algebra of C^∞ vector fields on M, we have $J^2u = -u$, $g(Ju, Jv) = g(u, v)$. It is known (see [**90**], [**139**]) that the existence on M of an arbitrary torsion free connection ∇ s.t. $\nabla J = 0$ is equivalent, to the vanishing of the Nijenhuis tensor defined by

$$N_J(u, v) = [Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v], \quad u; v \in \mathfrak{X}(M).$$

For every $p \in M$, the tangent space T_pM has a Hermitian structure given by $(J_{|p}, g_{|p})$. Now let $\mathcal{R}(V)$ be the vector bundle on M with fibre $\mathcal{R}(T_pM)$; the decomposition of $\mathcal{R}(T_pM)$ gives rise to a decomposition of $\mathcal{R}(M)$ into orthogonal subbundles with respect to the fibre metric introduced by g on $\mathcal{R}(M)$. We shall still denote the components of this decomposition by W_i, $i = 1, 2, \dots, 12$. If ∇ is an arbitrary linear torsion free connection, the corresponding curvature tensor is a section of the vector bundle $\mathcal{R}(M)$ and it is not difficult to check that its $HP(R)$-component in each point $p \in M$ gives the well-known holomorphical projective curvature tensor associated with ∇ (see [**27**], [**139**]); as a consequence, every subspace W_i, $i = 7, 8, \dots, 12$ of the decomposition is holomorphically projective invariant.

If some of the W_i vanish then the corresponding manifold has special groups of transformations and we have the following theorems (we refer [**92**]) for more details).

Theorem 4.4. *Let (M, g) be a Hermitian manifold with a torsion free connection. If the homogeneous holonomy group of M has no invariant hyperplane, or if the restricted homogeneous holonomy group has no invariant covariant vector and $W_1 = \dots = W_6 = 0$ then $\mathcal{HP}(M) = \mathcal{A}(M)$.* $\square$

We denote here by $\mathcal{HP}(M)$ the group of all holomorphically projective transformations of M and denote by $A(M)$ the group of all affine transformations of M.

Theorem 4.5. *If a Hermitian manifold (M, g) endowed with a torsion free connection ∇ is complete with respect to ∇ and $W_1 = \dots = W_6 = 0$ then $\mathcal{HP}(M) = \mathcal{A}(M)$.* $\square$

III.5. The action of the group $U(m) \times 1$

In Section II.3 we have introduced C-projective transformations on a normal almost contact manifold and have found C-projective curvature tensor - invariant with respect to these transformations. The key point was the existence of a torsion free adopted connection ∇. The main purpose of this section is to study the curvature tensor R of ∇, especially its C-projective curvature tensor $W(R)$ from algebraic point of view.

We use (3.2) of Section II to check

$$W(R)(u,v)\varphi = \varphi W(R)(u,v), \quad W(R)(u,v)\xi = 0. \tag{5.1}$$

Now starting from (5.1) we shall define certain special curvature tensor fields on M^{2m+1}, which will become useful for our discussion.

Definition 5.1. Let $(M^{2m+1}, \varphi, \xi, \eta)$ be a normal almost contact manifold. We define the *difference curvature tensor field of the torsion free adopted connection* ∇ as $K(R) = hR - h\tilde{R}$, where R is the curvature tensor field of ∇ and $h\tilde{R}$ is given by
(5.2)

$$h\tilde{R}(u,v)z = \{-\eta(u)hv + \eta(v)hu\}\eta(z) + d\eta(u,v)\varphi z + \frac{1}{2}d\eta(u,z)\varphi v - \frac{1}{2}d\eta(v,z)\varphi u. \square$$

Notice that $h\tilde{R}$ can be considered as the component on the vector subbundle $H = Ker\eta$ of TM^{2m+1} of the curvature tensor field $\tilde{R}$ of a torsion free adopted connection $\tilde{\nabla}$ on M^{2m+1}. Taking into account the properties of R and $\rho(R)$ we find

$$K(R)(u,v)z = -K(R)(v,u)z, \tag{5.3}$$

$$\underset{uvz}{\sigma}\, K(R)(u,v)z = 0 \quad \text{(the first Bianchi identity)}, \tag{5.4}$$

$$K(R)(u,v)\varphi z = \varphi K(R)(u,v)z, \quad K(R)(u,v)\xi = 0, \tag{5.5}$$

$$\begin{aligned} \rho(K(R))(u,v) = \operatorname{tr}(z \to K(R)(z,u)v) = \rho(R)(u,v) - \rho(\tilde{R})(u,v), \\ \rho(K(R))(u,\xi) = 0. \end{aligned} \tag{5.6}$$

III.5.1. The vector space $\mathcal{K}(V)$. Let V be an $(2m+1)$-dimensional real vector space endowed with an almost contact structure (φ, ξ, η) and a compatible inner product g and let V^* be the dual of V. Then, the (1,1) tensor φ, the vector $\xi \in V$ and the one-form $\eta \in V^*$ satisfy the relations:

$$\begin{gathered} \varphi^2 = -I_V + \xi \otimes \eta, \quad \eta(\xi) = 1, \\ \varphi\xi = 0, \quad \eta \circ \varphi = 0 \\ g(\varphi x, \varphi y) = g(x,y) - \eta(x)\eta(y), \quad x, y \in V. \end{gathered}$$

A tensor R of type (1,3) *over V* is a bilinear mapping $R : V \times V \to \operatorname{Hom}(V,V)$, $(x,y) \mapsto R(x,y)$. We say that R is *a curvature tensor over V* if

$$R(x,y) = -R(y,x), \text{ and } \underset{x,y,z}{\sigma} R(x,y)z = 0$$

We denote by $\mathcal{R}(V)$ the vector space of all curvature tensors over V. One can consider the following inner product, induced by g:

$$\langle R, \bar{R} \rangle = \sum_{1}^{2m+1} g(R(e_i,e_j)e_k, \bar{R}(e_i,e_j)e_k), \qquad R, \bar{R} \in \mathcal{R}(V),$$

where $\{e_i\}$, $i = 1, \dots, 2m+1$ is an arbitrary orthonormal basis of V. Furthermore, the representation α of $U(m) \times 1$ in V induces a representation $\tilde{\alpha}$ of $U(m) \times 1$ in $\mathcal{R}(V)$ in the following way

$$\tilde{\alpha} : U(m) \times 1 \to \mathfrak{gl}(\mathcal{R}(V)), \quad r \mapsto \tilde{\alpha}(r),\ r \in U(m) \times 1,$$

where $\tilde{\alpha}(r)(R)(x, y, z, w) = R(\alpha(r^{-1})x,\ \alpha(r^{-1})y,\ \alpha(r^{-1})z,\ \alpha(r^{-1})w)$, for all x, y, z, $w \in V$. It follows that the mapping $R \mapsto \tilde{\alpha}(r)R$ is an isometry for $\mathcal{R}(V)$; therefore $\langle \tilde{\alpha}(r)R,\ \tilde{\alpha}(r)\bar{R}\rangle = \langle R, \bar{R}\rangle$, which implies that the orthogonal complement of an invariant subspace of $\mathcal{R}(V)$ is also invariant and the representation $\tilde{\alpha}$ is completely reducible.

Taking into account the properties (5.3)–(5.6) of a "difference curvature tensor field" we shall denote by K the curvature tensors over V such that

$$K(x, y)\varphi z = \varphi K(x, y)z \text{ and } K(x, y)\xi = 0, \tag{5.7}$$

for all $x, y, z \in V$, or equivalently, if $K(x, y, z, w) = g(K(x, y)z, w)$, we have

$$\begin{aligned} &K(x, y, z, w) = K(x, y, \varphi z, \varphi w), \\ &K(x, y, \xi, w) = 0, \quad K(x, y, z, \xi) = \eta(K(x, y)z) = 0. \end{aligned} \tag{5.8}$$

Let $\mathcal{K}(V)$ be the vector subspace of $\mathcal{R}(V)$, whose elements are all K, satisfying (5.7). This subspace of $\mathcal{R}(V)$ is invariant for $\tilde{\alpha}$.

$\mathcal{K}(V)$ may be splited into direct sum of two subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$, defined as follows

$$\begin{aligned} &\mathcal{K}_1 = \{K \in \mathcal{K}(V) \mid K(x, \xi, z, w) = 0\}, \\ &\mathcal{K}_2 = \{K \in \mathcal{K}(V) \mid K(x, y, z, w) = \eta(x)K(\xi, y, z, w) + \eta(y)K(x, \xi, z, w)\}. \end{aligned}$$

It means

$$\mathcal{K}(V) = \mathcal{K}_1 \oplus \mathcal{K}_2, \tag{5.9}$$

and moreover $\mathcal{K}_1$ and $\mathcal{K}_2$ are mutually orthogonal and invariant with respect to the action of $U(m) \times 1$.

Now let $H = \operatorname{Ker}\eta$; H is a $2m$-dimensional Hermitian vector space with $(\varphi|_H, g|_H)$ as Hermitian structure and $U(m) \times 1|_H \simeq U(m)$; further, the vector space $\mathcal{K}_1$ is naturally isomorphic to the vector space $\mathcal{K}(H)$ given by the curvature tensors over H which satisfy the Kähler identity. This isomorphism allows us to use the results of Section III.4. concerning the decomposition of $\mathcal{K}(H)$ with respect to the action of $U(m)$ to obtain the decomposition of $\mathcal{K}_1$.

To simplify our notation, in the following, we shall denote for every $x \in V$ the component on H by $\dot{x}$; that is, $\dot{x} = hx$, where $h = I_V - \eta \otimes \xi$ is the projection on

H. First, we notice that for any $K \in \mathcal{K}(V)$ there are only two possible independent traces associated with K, analogously to these ones in III.4. i.e.

$$\rho(K)(y,z) = \sum_{i=1}^{2m+1} K(e_i, y, z, e_i),$$
$$\hat{\rho}(K)(y,z) = \sum_i K(e_i, y, e_i, z), \quad y, z \in V,$$

where $\{e_i\}$, $i = 1, 2, \ldots, 2m+1$ is an arbitrary orthonormal basis of V. Further, we have two scalar curvatures

$$\tau(K) = \sum_{i,j} K(e_i, e_j, e_j, e_i),$$
$$\bar{\tau}(K) = \sum_{i,j} K(e_i, e_j, \varphi e_j, e_i).$$

One can check easily

$$\rho(K)(y,z) = \rho(K)(\dot{y}, \dot{z}) + \eta(y)\rho(K)(\xi, \dot{z}),$$

$$\hat{\rho}(K)(\varphi y, \varphi z) = \hat{\rho}(K)(y,z).$$

In general, $\rho(K)$ and $\hat{\rho}(K)$ are neither symmetric nor antisymmetric; moreover $\rho(K)(\xi, z) = 0$ for every $K \in \mathcal{K}_1$, while for $K \in \mathcal{K}_2$, $\rho(K)$ reduces to $\eta(y)\rho(K)(\xi, \dot{z})$ and $\hat{\rho}(K) = 0$.

We omit all details related to the decomposition of $\mathcal{K}_1$, because of the previous comments, and pay the attention only on the decomposition of $\mathcal{K}_2 \subset \mathcal{K}$. First of all, we note that $K(\xi, y, z, w) = K(\xi, z, y, w)$, for every $K \in \mathcal{K}_2$ Next, we introduce the endomorphism δ on $\mathcal{K}_2$ defined by

$$\begin{aligned}
\delta(K)(x,y,z,w) = -\frac{1}{2m+2}\{&\eta(x)[g(\varphi y, \varphi w)\rho(K)(\xi, \dot{z}) + g(\varphi z, \varphi w)\rho(K)(\xi, \dot{y}) \\
&- g(\varphi y, w)\rho(K)(\xi, \varphi z) - g(\varphi z, w)\rho(K)(\xi, \varphi y)] \\
&- \eta(y)[g(\varphi x, \varphi w)\rho(K)(\xi, \dot{z}) + g(\varphi z, \varphi w)\rho(K)(\xi, \dot{x}) \\
&- g(\varphi x, w)\rho(K)(\xi, \varphi z) - g(\varphi z, w)\rho(K)(\xi, \varphi x)]\},
\end{aligned}$$

for any $x, y, z, w \in V$. If we take into account $\rho(\delta(K)) = \rho(K)$ we can check easily $\delta(K) \in \mathcal{K}_2$, $\delta^2 = \delta$ and δ commutes with the action of $U(m) \times 1$.

Now we define the following subspaces of $\mathcal{K}_2$

$$W_{13} = \operatorname{Ker}\delta = \{K \in \mathcal{K}_2 \mid \rho(K) = 0\}, \quad W_{14} = \operatorname{Im}\delta,$$

and state the following theorem.

Theorem 5.2. *If* $\dim V = 2m+1$, $m \geq 2$, *then* $\mathcal{K}_2 = W_{13} \oplus W_{14}$. *The subspaces* W_{13} *and* W_{14} *are mutually orthogonal and invariant under the action of* $U(m) \times 1$. *In particular, for* $m = 1$, $W_{13} = \{0\}$ *and* $\mathcal{K}_2 = W_{14}$. □

We use now (5.8), the isomorphism of $\mathcal{K}_1$ and $\mathcal{R}(H)$, Theorem 4.2. and Theorem 5.2. to obtain the following decomposition theorem for $\mathcal{K}(V)$:

Theorem 5.3. *If* $\dim V = 2m+1$, $m \geq 3$, *then*

$$\mathcal{K}(V) = W_1 \oplus \cdots \oplus W_{14} \tag{5.10}$$

and the subspaces W_i *are* $U(m) \times 1$ - *invariant and mutually orthogonal. For* $m = 2$, $W_{11} = W_{12} = \{0\}$ *and when* $m = 1$, *the decomposition reduces to* $\mathcal{K}(V) = W_5 \oplus W_6 \oplus W_{14}$. □

III.5.2. Some geometric results. Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a normal almost contact metric manifold. For every $p \in M^{2m+1}$, the vector space T_pM^{2m+1} has an induced almost contact structure $(\varphi_p, \eta_p, \xi_p)$ with compatible inner product g_p. If we denote by $\mathcal{K}(M^{2m+1})$ the vector bundle on M^{2m+1} with fibre $\mathcal{K}(T_pM^{2m+1})$, the decomposition (5.10) gives rise to a decomposition of $\mathcal{K}(M^{2m+1})$ into orthogonal subbundles with respect to the fibre metric induced by g on $\mathcal{K}(M^{2m+1})$. We use the same notation W_i, $i = 1, \ldots, 14$ for the components of this decomposition.

Let ∇ be a torsion free adapted connection on M^{2m+1} with curvature tensor R. Then, the difference tensor field $K(R)$ is a section of $\mathcal{K}(M^{2m+1})$. Let Q_i be the projections of K on the subspaces W_i $(i = 1, \ldots, 14)$. Recalling that $K(R) = hR - h\tilde{R}$, where $h\tilde{R}$ is given by (5.2) with $W(\tilde{R}) = 0$, we can state

Proposition 5.4. *Let* $(M^{2m+1}, \varphi, \xi, \eta, g)$ *be a normal almost contact metric manifold. If* ∇ *is an adapted torsion free connection on* M^{2m+1} *with curvature tensor* R, *we have*

$$W(R) = \sum_{i=7}^{13} Q_i(K(R)), \qquad K(R) = hR - h\tilde{R}$$

and the spaces W_i, $i = 7, 8, \ldots, 13$ *of the decomposition (5.9) are* C*-projectively invariant.* □

If $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Sasakian manifold, then $d\eta(u,v) = 2g(\varphi u, v)$, where $u, v, z, \cdots \in \mathfrak{X}(M^{2m+1})$.

As we know, the Levi-Civita connection ∇ on M^{2m+1} is one of adapted connections, and the system (3.1) in Section II is reduced to the simpler one

$$(\nabla_u\varphi)v = \eta(v)u - g(u,v)\xi, \quad (\nabla_u x\eta)(v) = g(\varphi u, v),$$
$$\nabla_u\xi = \varphi u, \quad \nabla_u g = 0$$
$$(\nabla_u d\eta)(v,z) = 2\eta(v)g(u,z) - 2\eta(z)g(u,v) = 2\eta(R(v,z)u).$$

Among Sasakian manifolds one can characterizes these ones of constant φ-sectional curvature using the previous results. More precisely, we have

Proposition 5.5. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$, $m \geq 3$ be a Sasakian manifold. Then $K(R) = Q_5(K(R)) \in W_5$ if and only if it has constant φ-sectional curvature $c \neq -3$ $(M^{2m+1} \neq R^{2m+1}(-3))$.* □

Corollary 5.6. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$, $m \geq 3$ be a Sasakian manifold $\neq R^{2m+1}(-3)$. Then $K(R) = Q_5(K(R)) \in W_5$, if and only if it is C-projectively flat.* □

We refer to [**78**] for characterization of other classes of Sasakian manifolds using the decomposition of curvature tensors and the corresponding examples (see also [**8**], [**9**], [**66**] etc.).

A normal almost contact metric manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$ has *a cosymplectic structure* if the fundamental 2-form Ω defined by $\Omega(u, v) = 2g(\varphi u, v)$ and the 1-form η are closed on M^{2m+1}. Examples of cosymplectic manifolds are provided by the products $\bar{M} \times S^1$, where $\bar{M}$ is any Kähler manifold. For a cosymplectic manifold Matzeu [**78**] has proved

(i) $K(R) = R = Q_5(K(R)) \in W_5$ if and only if it has constant φ-sectional curvature $c = \frac{\tau(K)}{m(m+1)}$,

(ii) $K(R) = R = Q_5(K(R)) + Q_9(K(R))$ if and only if it is η-Einstein, i.e. $\rho(K) = a(g - \eta \otimes \eta)$, where $a = \frac{\tau(K)}{2m}$ is constant.

We refer also to [**78**] for the studying of real hypersurfaces on complex space forms in this spirit.

IV. THE CHARACTERISTIC CLASSES

IV.1. Some basis notions and definitions

Let $GL(m, \mathbb{R})$ be the full general linear group and $\mathrm{gl}(m, \mathbb{R})$ be the Lie algebra of $GL(m, \mathbb{R})$; this is the Lie algebra of real $m \times m$ matrices. A map $Q : \mathrm{gl}(m, \mathbb{R}) \to \mathbb{C}$ is invariant if $Q(gAg^{-1}) = Q(A)$ for all $A \in \mathrm{gl}(m, \mathbb{R})$ and for all $g \in GL(m, \mathbb{R})$. Let $\mathcal{Q}$ be the ring of invariant polynomials. One can decompose $\mathcal{Q} = \oplus \mathcal{Q}_\nu$ as a graded ring, where $\mathcal{Q}_\nu$ is the subspace of invariant polynomials which are homogeneous of degree ν. Let

$$\mathrm{Ch}(A) := \sum_\nu \mathrm{Ch}_\nu \quad \text{for } \mathrm{Ch}_\nu(A) := \mathrm{Tr}\Big\{\Big(\frac{\sqrt{-1}}{2\pi}A\Big)^\nu\Big\},$$

$$C(A) := \det\Big(I + \frac{\sqrt{-1}}{2\pi}A\Big) = 1 + C_1(A) + \cdots + C_m(A)$$

define *the Chern character* and *total Chern polynomial*; $\mathrm{Ch}_\nu \in \mathcal{Q}_\nu$ and $C_\nu \in \mathcal{Q}_\nu$. The Chern characters and the Chern polynomials generate the *characteristic ring*: $\mathcal{Q} = \mathbb{C}[C_1, \ldots, C_m]$ and $\mathcal{Q} = \mathbb{C}[\mathrm{Ch}_1, \ldots, \mathrm{Ch}_m]$. If $Q \in \mathcal{Q}_\nu$ we polarize Q to define a multilinear form $Q(A_1, \ldots, A_\nu)$ so that $Q(A) = Q(A, \ldots, A)$ and $Q(A_1, \ldots, A_\nu) = Q(gA_1g^{-1}, \ldots, gA_\nu g^{-1})$.

We shall restrict our attention to the tangent bundle TM henceforth; let ∇ be an arbitrary connection on TM and R the corresponding curvature tensor. If $\{e_i\}$ is a local frame for TM, then $R(e_i, e_j)e_k = R_{ijk}{}^l e_l$. We shall let

$$\mathcal{R} = \mathcal{R}_k^l := \frac{1}{2} R_{ijk}{}^l e^i \wedge e^j$$

be the associated 2-form valued endomorphism. As we are not assuming that a metric is given, we do not restrict to orthonormal frames. Thus the structure group is the full general linear group $GL(m, \mathbb{R})$ and not the orthogonal group $O(m)$.

If $Q \in \mathcal{Q}_\nu$, we define

$$Q(\nabla) := Q(\mathcal{R}, \ldots, \mathcal{R}) \in C^\infty(\Lambda^{2\nu} M)$$

by substitution; the value is independent of the frame chosen and associates a closed differential form of degree 2ν to any connection ∇ on TM. The corresponding cohomology class $[Q(\nabla)] \in H^{2\nu}(M; \mathbb{C})$ is independent of the connection ∇ chosen; as we shall see later. These are the characteristic forms and classes. For more details one can use also [**35**], [**40**], [**61**].

From now on we deal with complex manifolds.

We express now C_1, C_1^2 and C_2 by using a suitable chosen frame of TM. Let $E_1, JE_1, \ldots, E_m, JE_m$, be a real base for tangent space T_pM and $w^1, \bar{w}^1, \ldots, w^m, \bar{w}^m$ the corresponding dual base for T_p^*M. Then we will write $E_{m+s} = JE_s = E_{\bar{s}}$ and similarly $w^{m+r} = \bar{w}^r$, $1 \le s, r \le m$. We suppose summation for every pair of repeated indexes. We use also the following ranges for indexes $i, j, s, r = 1, 2, \ldots, m$, and $I, J, S, R = 1, 2, \ldots, 2m$. We denote $JE_S = E_{\bar{S}}$ and $R(u, v)E_S = R_{uvS}{}^R E_R$. For $u = E_I$; $v = E_J$ we simplify notation and write $R_{E_I E_J S}{}^R = R_{IJS}{}^R$.

It will be useful for our consideration of Chern classes to introduce the following traces:

$$\mu(u, v) = \frac{1}{2} \operatorname{tr}\{w \mapsto R(u, v)w\} = R_{uvi}{}^i, \tag{1.1}$$

$$\bar{\mu}(u, v) = \frac{1}{2}\{w \mapsto J \circ R(u, Jv)w\} = R_{uJv\bar{i}}{}^i, \tag{1.2}$$

for $u, v \in T_pM \otimes \mathbb{C}$ and $w \in T_pM$. After some computations one can express these traces in terms of the Ricci tensor as follows

$$2\bar{\mu}(u, v) = \rho(u, v) + \rho(Jv, Ju),$$
$$2\mu(u, v) = \rho(v, u) - \rho(u, v).$$

We put $\mathcal{R}_I^J(u, v) = R_{uvI}{}^J$, i.e., $\mathcal{R}_I{}^J = R_{RSI}{}^J \omega^R \wedge \omega^S$ and

$$\Theta_i^j(u, v) = -(\mathcal{R}_i^j(u, v) - \sqrt{-1}\mathcal{R}_{\bar{i}}^j(u, v)),$$

for $u, v \in T_pM \otimes \mathbb{C}$. Then (Θ_i^j) is a matrix of complex 2-forms and

$$\det\left(\delta_i^j - \frac{1}{2\pi\sqrt{-1}}\Theta_i^j\right) = 1 + C_1 + \cdots + C_m$$

is a globally defined closed form which represents *the total Chern class* of M via de Rham's theorem (see [**61**, vol. II, p. 307]). Chern classes determined by C_1, C_2 are denoted by c_1, c_2 respectively. In particular, the Chern forms C_1, C_2, and C_1^2 are given by

$$\begin{aligned}
C_1 &= \frac{\sqrt{-1}}{2\pi}\sum \Theta_i^i = \frac{\sqrt{-1}}{2\pi}(\mathcal{R}_i^i - \sqrt{-1}\mathcal{R}_{\bar i}^i),\\
C_1^2 &= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m} \Theta_i^i \wedge \Theta_j^j\\
&= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m}\{(\mathcal{R}_i^i\wedge\mathcal{R}_j^j - \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_{\bar j}^j) - \sqrt{-1}(\mathcal{R}_i^i\wedge\mathcal{R}_{\bar j}^j + \mathcal{R}_i{}^{\bar i}\wedge\mathcal{R}_j^j)\},\\
C_2 &= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m}\{\Theta_i^i\wedge\Theta_j^j - \Theta_i^j\wedge\Theta_j^i\}\\
&= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m}\{(\mathcal{R}_i^i\wedge\mathcal{R}_j^j - \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_{\bar j}^j - \mathcal{R}_i^j\wedge\mathcal{R}_j^i + \mathcal{R}_{\bar i}^j\wedge\mathcal{R}_{\bar j}^i)\\
&\qquad\qquad - \sqrt{-1}(\mathcal{R}_i^i\wedge\mathcal{R}_{\bar j}^j + \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_j^j - \mathcal{R}_i^j\wedge\mathcal{R}_{\bar j}^i - \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_j^i)\}.
\end{aligned}$$

We consider Chern numbers $\gamma_2(M) = \int_M C_2$ and $\gamma_1^2(M) = \int_M C_1^2$ for a compact complex surface M and similarly $\gamma_1^m = \int_M C_1^m$ for an arbitrary complex compact m-dimensional manifold.

Let $A \in \sigma(m)$ be a skew-symmetric matrix. Then $C_{2\nu+1}(A) = 0$ and we define $P_\nu(A) = (-1)^\nu C_{2\nu}(A)$; $P = \sum_\nu P_\nu(A)$ is *the total Pontrjagin polynomial.* The $\{P_\nu\}$ for $2\nu \le m$ generate *the characteristic ring of the orthogonal group* $O(m)$. We can always choose a Riemannian metric g for M and use the associated Levi-Civita connection $\nabla(g)$ to compute the characteristic classes of the tangent bundle. This reduces the structure group to $O(m)$ and shows that only the Pontrjagin classes are relevant in the study of the primary characteristic classes of TM. From the point of view of cohomology, the connection plays an unessential role; however, in many geometrical applications one must work with differential forms not cohomology classes. We illustrate it by the following facts. Let dx be the volume element of compact 4-dimensional orientable M, where M is without boundary. The Chern-Gauss-Bonnet formula [**36**] and the Atiyah-Patodi-Singer formula [**1**] yields formulas for *the Euler-Poincaré characteristic* $\chi(M)$ and *the signature* $\mathrm{Sign}(M)$:

$$\chi(M) = \int_M E_4(\nabla(g))dx, \quad \mathrm{Sign}(M) = \frac{1}{3}\int_M P_1(\nabla(g)),$$

where

$$\begin{aligned}
E_4(\nabla(g)) &= \frac{1}{32\pi^2}(R_{ijji}R_{kllk} - 4R_{ijjk}R_{illk} + R_{ijkl}R_{ijkl}),\\
P_1(\nabla(g)) &= -\frac{1}{32\pi^2}R_{ijk_1k_2}R_{jik_3k_4}e^{k_1}\wedge e^{k_2}\wedge e^{k_3}\wedge e^{k_4}.
\end{aligned}$$

The interior integrands E_4 and P_1 are primary characteristic forms, not characteristic classes. But to express $\chi(M)$ and $\mathrm{Sign}(M)$ of compact 4-dimensional orientable manifold M with smooth boundary $\partial M \neq \phi$ we need also secondary characteristic forms.

We introduce firstly relative secondary characteristic forms and later absolute ones.

The space of all connections is an affine space; the space of torsion free connections is an affine subspace. If ∇_i are connections on TM, let $\nabla_t := t\nabla_1 + (1-t)\nabla_0$. Let $\Psi = \nabla_1 - \nabla_0$; Ψ is an invariantly defined 1-form valued endomorphism. Let $R(t)$ be the associated curvature. Let $Q \in \mathcal{Q}_\nu$. Let

$$\begin{aligned} TQ(\nabla_1, \nabla_0) : &= \nu \int_0^1 Q(\Psi, R(t), \dots, R(t))dt; \\ dTQ(\nabla_1, \nabla_0) &= Q(\nabla_1) - Q(\nabla_0). \end{aligned} \tag{1.3}$$

This shows that $[Q(\nabla_1)] = [Q(\nabla_0)]$ in de Rham cohomology. Note that we have:

$$TQ(\nabla_0, \nabla_1) + TQ(\nabla_1, \nabla_2) = TQ(\nabla_0, \nabla_2) + \text{exact form}.$$

Suppose now that M is a 4-dimensional Riemannian manifold with smooth nonempty boundary ∂M. Let g be a Riemannian metric on M. Let indices i, j, k and l range from 1 to 4 and index a local orthonormal frame $\{e_i\}$ for the tangent bundle. At a point of the boundary of M, we assume e_4 is the inward unit normal and let indices a, b, c range from 1 to 3. Let $L_{ab} := (\nabla(g)_{e_2} e_b, e_4)$ be the second fundamental form on ∂M. We choose $x = (y, t)$ to be local coordinates for M near ∂M so the curves $t \mapsto (y, t)$ are unit speed geodesics perpendicular to ∂M. This identifies a neighborhood of ∂M in M with a collared neighborhood $\mathcal{K} = \partial M \times (0, \epsilon)$ for some $\epsilon > 0$. Let h_0 be the associated product metric. We denote by ∇_1, ∇_0 the Levi-Civita connections of h, h_0 respectively. The $TP_1(\nabla_1, \nabla_0)$ is given by

$$TP_1(\nabla_1, \nabla_0) = TP_1(L, \nabla_t) := -\frac{1}{16\pi^2} L_{ab} R_{4acd} e^b \wedge e^c \wedge e^d,$$

and consequently

$$\operatorname{Sign}(M) = \frac{1}{3}\int_M P_1(\nabla(h)) - \frac{1}{3}\int_{\partial M} TP_1(L, \nabla(h)) - \eta(\partial M),$$

where the invariant $\eta(\partial M)$ is intrinsic to ∂M and we will not be concerned with this invariant here; see **[40]** for details.

To define absolute secondary characteristic forms we need the principal frame bundle $\pi : \mathcal{P} \to M$ for TM. A local section e to $\mathcal{P}$ is a frame $e = \{e_i\}$ for TM. Let g be the natural inclusion of $GL(m, \mathbb{R})$ in the Lie algebra $\mathfrak{gl}(m, \mathbb{R})$ of $m \times m$ real matrices. The Maurer-Cartan form dgg^{-1} on $GL(m, \mathbb{R})$ is a $\mathfrak{gl}(m, \mathbb{R})$ valued 1-form on $GL(m, \mathbb{R})$ which is invariant under right multiplication. Let ∇ be a connection on TM. Fix a local frame field e for TM; this is often called a choice of gauge. We denote by w the associated connection 1-form, $\nabla e_i = w_i^j e_j$. Let

$$\begin{aligned} \Theta : &= \Theta(\nabla) := dgg^{-1} + g\omega g^{-1}, \\ \Omega : &= \Omega(\nabla) := g(d\omega - \omega \wedge \omega)g^{-1} = g(\pi^* R)g^{-1}. \end{aligned}$$

These are Lie algebra valued forms on the principal bundle $\mathcal{P}$ which do not depend on the local frame field chosen. If $Q \in \mathcal{Q}_\nu$, then we have $Q(\Omega) = \pi^* Q(\nabla)$. We set $\Omega(t) = td\Theta - t^2\Theta \wedge \Theta = t\Omega + (t - t^2)\Theta \wedge \Theta$ and define

$$\mathcal{T}Q(\nabla) := \nu \int_0^1 Q(\Theta, \Omega(t), \dots, \Omega(t))dt. \tag{1.4}$$

We refer to Chern and Simons [**37**, Propositions 3.2, 3.7 and 3.8] for the proof of:

Theorem 1.1. *Let $Q \in \mathcal{Q}_\ni$ and $\tilde{Q} \in \mathcal{Q}_\mu$. (1) We have $d\mathcal{T}Q(\nabla) = \pi^*Q(\nabla)$.*
(2) We have $\mathcal{T}(Q\tilde{Q})(\nabla) = \mathcal{T}Q(\nabla) \wedge \pi^*\tilde{Q}(\nabla) +$ exact $= \pi^*Q(\nabla) \wedge \mathcal{T}\tilde{Q}(\nabla) +$ exact.
(3) Let ∇_ρ be a smooth 1 parameter family of connections. Let $A := \partial_\rho \nabla_\rho|_{\rho=0}$. Then $\partial_\rho \mathcal{T}Q(\nabla_\rho)|_{\rho=0} = \nu Q(A, \Omega_0, \dots, \Omega_0) +$ exact. □

Suppose M is parallelizable. Let e be a global frame for the principal frame bundle $\mathcal{P}$. Let ${}^e\nabla e = 0$ define the connection ${}^e\nabla$. We use equations (1.3) and (1.4) to see that

$$e^*\mathcal{T}Q(\nabla) = \int_0^1 Q(w_e, \mathcal{R}_t, \dots, \mathcal{R}_t) = TQ[\nabla, {}^e\nabla),$$

where $w_e = \nabla e$ and $\mathcal{R}_t = tdw_e - t^2 w_e \wedge w_e = t\mathcal{R} + (t - t^2)w_e \wedge w_e$.

We note that $\mathcal{R}_t$ is the curvature of the connection $t^e\nabla + (1-t)\nabla$. Fix $g \in GL(m, \mathbb{R})$. Since Q is GL invariant, we have $e^*\mathcal{T}Q(\nabla) = (ge)^*\mathcal{T}Q(\nabla)$.

Let $Q \in \mathcal{Q}_\nu$. Suppose that $Q(\nabla) = 0$. Then $e^*\mathcal{T}Q(\nabla)$ is a closed form on M of degree $2\nu - 1$ and $[e^*\mathcal{T}Q(\nabla)]$ in $H^{2\nu-1}(M; C)$ is independent of the homotopy class of e. We say that Q is integral if Q is the image of an integral class in the classifying space; see [**37**, §3] for details; the Pontrjagin polynomials are integral.

Theorem 1.2. *Let $Q \in \mathcal{Q}_\nu$. Assume that M is parallelizable and $Q(\nabla) = 0$.*
(1) If Q is integral, then $[e^*\mathcal{T}Q(\nabla)]$ is independent of e in $H^{2\nu-1}(M; \mathbb{C}/\mathbb{Z})$.
(2) If ν is odd, then $[e^*\mathcal{T}Q(\nabla)]$ is independent of e in $H^{2\nu-1}(M; \mathbb{C})$. □

IV.2. Characteristic classes and symmetries of a curvature tensor

The main purpose of this section is to study topology of a manifold endowed with a torsion free connection which curvature tensor has symmetries, invariant under the action of some classical groups in the spirit of Section II.

The relations between topology and the existence of some flat connection have been studied by Milnor [**87**], Auslander [**2**], Benzécri [**5**] etc.

The topological obstruction of the existence of a complex torsion free connection with skew-symmetric Ricci tensor has been studied in [**12**]. More precisely, we proved if ∇ is a complex torsion free connection on a Riemann surface M and the Ricci tensor ρ for ∇ is skew-symmetric then $\gamma_1(M) = \int_M C_1 = 0$.

Let us remark if M is a sphere S^2 we have $\gamma_1(M) \neq 0$. Therefore there is no a complex torsion free connection $\tilde{\nabla}$ with the skew-symmetric Ricci tensor globally defined on S^2. The local existence of this connection is proved by its construction. Namely, (S^2, g) is the standard sphere with the standard embedding into the Euclidean space $\mathbb{R}^3$ determined by

$$x = \cos\alpha \sin\beta, \quad y = \sin\alpha \sin\beta, \quad z = \cos\beta, \quad 0 < \alpha < 2\pi, \ 0 < \beta < \pi.$$

The Christoffel symbols for our complex connection are given by the following formulas

$$\begin{aligned} \tilde{\Gamma}^2_{22} &= \frac{1-\cos\beta}{\sin\beta}, \quad \tilde{\Gamma}^1_{22} = \frac{-1+\cos\beta}{\sin^2\beta}, \\ \tilde{\Gamma}^1_{12} &= \tilde{\Gamma}^1_{21} = \frac{1}{\sin\beta}, \quad \tilde{\Gamma}^2_{11} = -\sin\beta, \\ \tilde{\Gamma}^2_{12} &= \tilde{\Gamma}^2_{21} = \tilde{\Gamma}^1_{11} = 1 - \cos\beta; \end{aligned}$$

(see [**12**] for more details).

Having in mind the previously mentioned facts, torus T^2 is a good candidate to permite a globally defined torsion free connection $\tilde{\nabla}$ with the skew-symmetric Ricci tensor. Really, let $x_1 = \cos\alpha$, $x_2 = \sin\alpha$, $x_3 = \cos\beta$, $x_4 = \sin\beta$, $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq 2\pi$, be the standard embedding of the torus into the Euclidean space $\mathbb{R}^4$. Let $\tilde{\Gamma}^k_{ij}$ $(i,j,k = 1,2)$ be the Christoffel symbols for a complex torsion free connection $\tilde{\nabla}$. Then

$$\begin{aligned} \tilde{\Gamma}^1_{11} &= \tilde{\Gamma}^2_{12} = \tilde{\Gamma}^2_{21} = -\tilde{\Gamma}^1_{22} = -\cos\alpha\sin\beta, \\ \tilde{\Gamma}^1_{12} &= \tilde{\Gamma}^1_{21} = \tilde{\Gamma}^2_{22} = -\tilde{\Gamma}^2_{11} = \sin\alpha\cos\beta. \end{aligned}$$

We point out these connections belong to the class of affine conformal invariants, studied by Simon in [**123**].

The Chern characteristic classes of complex surfaces endowed with a holomorphic affine connection ∇ have been studied in [**14**]. The following theorem considers complex surfaces with all vanishing characteristic classes.

Theorem 2.1. *Let M be a complex surface* $(\dim M = 2)$ *endowed with a holomorphic affine connection ∇. Then its Chern characteristic classes C_2 and C_1^2 vanish. Moreover, if M is a complex equiaffine surface (∇ permits a parallel complex 2-form ${}^c w$) then C_1 also vanishes.* □

One can find in [**14**] the examples of nonflat holomorphic affine connections on the torus T^4 and the Euclidean space $\mathbb{R}^4$.

Let us assume for our complex torsion free connection ∇ to have a symmetric curvature operator, i.e. $R(x,y)$ satisfies the relation $g(R(x,y)z,v) = g(R(x,y)v,z)$. Then R satisfies also the following relations

$$R(Jx,Jy) = R(x,y), \quad \rho(x,y) = -\rho(y,x), \quad \rho(Jx,Jy) = \rho(x,y),$$

(see [**93**] for the proof). Now one can use the results from the section IV.1 to study the Chern characteristic classes of a Hermite surface M endowed with a complex torsion free connection ∇ with the symmetric curvature operator and conclude

$$[C_1(M)] = 0, \quad [C_1^2(M)] = 0, \quad [C_2(M)] = [\gamma\tilde{\delta}_2] = [\hat{\delta}_2],$$

where

$$\tilde{\delta}_2 = \frac{1}{16\pi^2}(2||\rho||^2 - ||R||^2)\Phi^2, \qquad \hat{\delta}_2 = \frac{1}{16\pi^2}(\tau^{*2} - ||R||^2)\Phi^2,$$

$||R||$, $||\rho||$ are the norms of the curvature and the Ricci tensor, i.e.

$$||R||^2 = \sum R_{PQIK}R_{PQIK}, \qquad ||\rho||^2 = \sum \rho_{PQ}\rho_{PQ}$$

and $\Phi = \sum w^i \wedge \bar{w}^i$ is the fundamental 2-form; assuming that $(w^i, \bar{w}^i)$ is the corresponding dual base for (E_i, JE_i). Moreover, we have also some geometrical consequences. More precisely we have

Theorem 2.2. *Suppose that a torsion free complex connection ∇ exists on a compact Hermite surface M with $\tau^* = 0$. Then $\gamma_2(M) \leq 0$. The equality holds if and only if ∇ is a flat connection.* □

Corollary 2.3. *Let (M, J) be a compact Hermite surface which admits a Kähler-Einstein metric. Then every complex torsion free connection with $\tau^* = 0$ on M is flat.* □

We refer to [**16**] for more details related to the symmetric curvature operators and topology. One can find also some examples of complex torsion free connections on reducible Hermite surface M with the generic $R \in \mathcal{R}(T_pM)$ or with R belonging to some vector subspaces of $\mathcal{R}(T_pM)$, which are invariant or, irreducible under the action of the unitary group $U(m)$. Some of examples show that the compactness of M is an essential assumption in Theorem 2.2 and Corollary 2.3.

IV.3. The relations between characteristic classes and projective geometry

If we are interested in relations between topology and geometry of a smooth manifold we must work with differential forms. The main purpose of this section is to study invariance of characteristic forms with respect to some group of transformations. We are interested in also does the topology of a manifold M determine the relations between the group of holomorphically projective transformations, the group of projective transformations and the group of affine transformations on M.

First, we are interested in the invariance of characteristic forms. Conformally equivalent metrics and projectively equivalent torsion free connections have the same characteristic forms. More precisely, it yields

Theorem 3.1. *Let $Q \in \mathcal{Q}_\nu$ and let $\beta \in C^\infty_+(M)$.*
(1) Let $\nabla(h)$ be the Levi-Civita connection of a semi-Riemannian metric. Then $Q(\nabla(h)) = Q(\nabla\beta(h))$, where $\beta(h) = \beta h$.
(2) Let ∇ and $\tilde{\nabla}$ be two projectively equivalent torsion free connections. Then $Q(\nabla) = Q(\tilde{\nabla})$. □

We refer [**3**], [**15**] for the proof of this theorem.

Matzeu [**77**] has studied the Chern algebra of the complex vector subbundle H of TM defined as $H = Ker\eta$, where M is normal almost contact manifold.

The corresponding conditions to have invariant Chern forms under C-projective transformations have been found. We refer to Section III.5. for basic notations. Suppose that ∇ is an adapted symmetric connection with symmetric Ricci tensor field and D is its restriction to the vector subbundle H of TM defined as $H = Ker\eta$. The D is the complex connection with the Ricci tensor $\rho(K)$ symmetric too. We refer to [**77**] for the proofs of the following theorems.

Theorem 3.2. *If the connection D with symmetric Ricci tensor field has the first Chern form proportional to $d\eta$, then all its Chern forms are C-projectively invariant.* □

Theorem 3.3. *All the Chern forms of a C-projectively flat adopted connection are C-projectively invariant.* □

Theorem 3.4. *The Chern classes of a C-projectively flat manifold are trivial.*□

One can use the traces $\mu, \bar{\mu}$ given by (1.1), (1.2) and Chern numbers to prove the following theorems related to the influence of topology of M in the group of projective transformations and its subgroup of affine transformations. We refer to [**13**] for details.

Theorem 3.5. *Let M, $\dim_{\mathbb{C}} M = m$, be a compact complex manifold with a complex symmetric connection ∇. If*
(i) $\rho(u,v) = \rho(v,u)$, and $\rho(Ju,Jv) = \rho(u,v)$,
(ii) $\bar{\mu}$ is a semi-definite bilinear form, of rank 0 or m, nonnegative if m is odd,
(iii) $\gamma_1^m(M) \leq 0$ then the group of all projective diffeomorphisms of the connection ∇ coincides with the group of all affine diffeomorphisms of the same connection. □

Theorem 3.6. *Let M be a surface of general type with a complex symmetric connection ∇. If*
(i) $\rho(v,u) = \rho(u,v)$, and $\rho(Ju,Jv) = \rho(u,v)$,
(ii) $\bar{\mu}$ is a semi-definite bilinear form of rank 0 or m, nonnegative if m is odd,
(iii) $\gamma_2(M) \leq 0$, then the group of all projective diffeomorphisms of the connection ∇ coincides with the group of all affine diffeomorphisms of the same connection. □

Under the assumptions of Theorems 3.5. or 3.6. one can prove the group of holomorphically projective transformations coincides with the group of affine transformations of ∇.

In the general case the group of projective diffeomorphisms, the group of affine diffeomorphisms and the isometry group do not coincide for a Riemannian manifold (M, g). Nagano [**89**] has proved that if M is a complete Riemannian manifold with parallel Ricci tensor then the largest connected group of projective transformations of M coincides with the largest connected group of affine transformations of M unless M is a space of positive constant sectional curvature. For Kähler manifolds problems of this type have been studied in [**13**]. So we have

Theorem 3.7. *Let M be a compact Kähler manifold of complex dimension $m > 1$. If: (i) τ = constant, (ii) $c_1 = 0$, then* *(a) the group of all projective diffeomorphisms of the Levi-Civita connection coincides with the group of all affine diffeomorphisms of the same connection;*

(b) the identity component of the group of all holomorphically projective diffeomorphisms of the Levi-Civita connection coincides with the identity component of its group of isometries. □

Kobayashi and Ochiai have studied in **[62]** holomorphic normal projective connections on complex manifolds and classified all compact complex analytic surfaces which admit flat holomorphic projective connections. They have proved in **[63]** a complex analytic surface of general type, which admits a holomorphic (normal) projective connection, is covered by a unit ball $B^2 \subset \mathbb{C}^2$ without ramification.

IV.4. Characteristic classes and affine differential geometry

Let us recall, if x is a nondegenerate embedding of a manifold M as a hypersurface in affine space, we let (x, X, y) be a relative normalization. This defines a triple (∇, h, ∇^*) on M, where h is a semi-Riemannian metric, and where ∇ and ∇^* are torsion free connections on the tangent bundle TM. If Q is an invariant polynomial, then $Q(\nabla) = 0$, $Q(\nabla(h)) = 0$ and $Q(\nabla^*) = 0$. Moreover, the secondary characteristic forms of the connections $\nabla, \nabla^*, \nabla(h)$ vanish. To be more precise we introduce a decomposable invariant polynomial Q by the relation $Q = \sum_i Q_{i,1}Q_{i,2}$, where $0 \neq Q_{i,j} \in \mathcal{Q}_{\nu(i,j)}$ and $\nu(i,j) > 0$. For the proofs of following lemma and theorems we refer **[15]**.

Lemma 4.1. *Let (∇, h, ∇^*) be the conjugate triple defined by a relative normalization (x, X, y) of an affine embedding of an orientable manifold M. Let $Q \in \mathcal{Q}_\nu$.*

(1) If Q is decomposable, then $[\mathcal{T}_x Q(\nabla)] = 0$, $[\mathcal{T}_x Q(\nabla(h))] = 0$, and $[\mathcal{T}_x Q(\nabla^*)] = 0$ in $H^{2\nu-1}(M, \mathbb{C})$.

(2) The classes $[\mathcal{T} Q(\nabla)]$, $[\mathcal{T}_x Q(\nabla^*)]$, and $[\mathcal{T} Q(\nabla(h))]$ in $H^{2\nu-1}(M, \mathbb{C})$ are affine invariants; these cohomology classes are independent of the relative normalization chosen. □

Theorem 4.2. *Let (∇, h, ∇^*) be the conjugate triple defined by a relative normalization (x, X, y) of an affine embedding of an orientable manifold M. Let $Q \in \mathcal{Q}_\nu$.*

(1) We have $[\mathcal{T}_x Q(\nabla)] = 0$ in $H^{2\nu-1}(M; \mathbb{C})$.

(2) If Q is integral and if ν is even, then $[\mathcal{T} Q(\nabla^*)] = 0$ in $H^{2\nu-1}(M; \mathbb{C}/\mathbb{Z})$.

(3) If ν is odd, then $[\mathcal{T}_x Q(\nabla^*)] = 0$ in $H^{2\nu-1}(M; \mathbb{C})$.

(4) If ν is even, then $[\mathcal{T}_x Q(\nabla(h))] = 0$ in $H^{2\nu-1}(M; \mathbb{C})$.

(5) If ν is odd and if h is definite, then $[\mathcal{T}_x Q(\nabla(h))] = 0$ in $H^{2\nu-1}(M, \mathbb{C})$. □

One can apply these results to 3-dimensional affine differential geometry to construct obstructions to realizing the conformal class of a Riemannian metric as the second fundamental form of an embedding; this generalizes work of Chern and Simons **[37]**.

To state the corresponding theorem we have in mind that if M is a compact orientable 3-dimensional manifold, then M is parallelizable. Hence we can choose a global frame f for TM. If $Q \in \mathcal{Q}_2$, then $Q = cP_1$ + decomposable, so we need only to study $[\mathcal{T}_x P_1]$, where P_1 is the first Pontrjagin form. Since P_1 is a real integral differential form, we define

$$\Phi(\nabla) = \int_M f^* \mathcal{T} P_1(\nabla) \in \mathbb{R}/\mathbb{Z}.$$

One can prove that $\Phi(\nabla)$ is independent of the particular parallelization f which is chosen. Consequently, Theorem 4.2 implies

Theorem 4.3. *Let (M, g_o) be a 3-dimensional Riemannian manifold.*

(1) If there exists an immersion $x : M \to \mathbb{R}^4$ so that g_0 is conformally equivalent to the first fundamental form of x, then $\Phi(\nabla(g)) = 0$ in $\mathbb{R}/\mathbb{Z}$.

(2) If there exists an immersion $x : M \to \mathbb{R}^4$ so that g_0 is conformally equivalent to the second fundamental form of x, then $\Phi(\nabla(g)) = 0$ in $\mathbb{R}/\mathbb{Z}$. □

We refer to [**15**] for details of the proof of this theorem and also for other references related to other applications of the secondary characteristic forms in 3-dimensional geometry and in mathematical physics.

V. DIFFERENTIAL OPERATORS OF LAPLACE TYPE

The main purpose of this Section is to study the second order differential operators of Laplace type which are naturally appeared in differential geometry. Of course, the most interesting for us are these operators which depend on a torsion free connection, and relations between the spectrum of operators from one side and geometry and topology of a manifold from other side. To study these problems we explore the heat equation method. We refer to monographs [**6**], [**34**], [**44**], and expository papers [**30**], [**39**], [**58**] [**84**] etc. for more details.

V.1. Definitions and basic notations

Let M be a compact Riemannian manifold of dimension m. *The Laplace-Beltrami operator* (shorter *the Laplacian*) is an operator

$$\Delta(f) = - \div (\operatorname{grad} f), \tag{1.1}$$

where $f \in C^\infty(M)$, i.e. in coordinates $\Delta = -\sum_{i,j} g^{-1}\partial_i(gg^{ij}\partial_j)$, for $g = \sqrt{\det(g_{ij})}$. For example if the metric is given by $ds^2 = h(dx^2 + dy^2)$, then $\Delta = g^{-1}(\partial_x^2 + \partial_y^2)$.

Let $f_t(x)$ denote the temperature in a time t and a point $x \in M$. If we assume the heat trasfers into the coolest direction, then $f_t(x)$ satisfies the equation

$$\frac{\partial}{\partial t} f + \Delta f = 0. \tag{1.2}$$

We say $f_0(x)$ is *an eigenfunction* of Δ with *the eigenvalue* $\lambda \in \mathbb{R}$ if it yields $\Delta(f_0) = \lambda \cdot f_0$. One can check then $f_t(x) = e^{-\lambda t} f_0(x)$ satisfies the heat equation (1.2). Therefore, f_t may be interpreted as "a heat wave" with "the frequency" $e^{-\lambda t}$.

The theory of partial differential operators implies that there exist countable set of eigenvalues λ_i and for every λ_i the finite-dimensional family of eigenfunctions f_i, such that we have $\Delta(f_i) = \lambda_i f_i$. Furthermore, λ_i are positive, and $\lambda_i \to \infty$ when $i \to \infty$. The collection $\{\lambda_i\}$, together with the multiplicities of each λ_i, is *the spectrum of a manifold* M.

If one struck M with a mallet than λ_i may be interpreted as the sounds emitted by M, assuming that sound satisfies a similar equation to that of heat.

Weyl [**137**] has proved that the spectrum of M determine one of significant geometrical invariant - volume of M. This was a reason to believe that the spectrum determines completely the geometry of Kac [**58**] formulated this problem in a lovely question: "Can one hear a shape of a drum". The example of two 16th dimensional non-isometric torus [**86**] with the same spectrum have shown that expectations were excessively strong. Many examples have been constructed later on (see [**30**], [**84**] etc.) using different methods to show the same things.

We say manifolds M_1 and M_2 are *isospectral* if they have the same spectrum.

A function $H_t(x, y)$ is *a fundamental solution* of the heat equation (or *a heat kernel*) if

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) H = 0, \tag{1.3}$$

$$\lim_{t \to 0} \int_M H_t(x, y) f(y) dy = f(x), \tag{1.4}$$

for any $f \in C^\infty(M)$. One can use (1.3) and (1.4) to check that the general solution $f_t(x)$ of the heat equation with initial equation $f_0(x) = f$ is given by the formula

$$f_t(x) = \int_M H_t(x, y) f(y) dy.$$

We look for H_t to fulfill the following conditions

(i) $H_t(x, y)$ is uniquely determined by (1.3) and (1.4).

(ii) If M is a compact manifold and $\{f_i\}$ is an orthonormal base of eigenfunctions with corresponding eigenvalues $\{\lambda_i\}$, then

$$H_t(x, y) = \sum_i e^{-\lambda_i t} f_i(x) f_i(y).$$

Now we use (ii) to eliminate $f_i's$ and describe $\lambda_i's$. Therefore, we put

$$\operatorname{tr}(H_t) = \int_M H_t(x, x) dx.$$

Consequently

$$\begin{aligned}\operatorname{tr}(H_t) = \operatorname{tr}_{L^2} e^{-t\Delta} &= \int_M \sum e^{-\lambda_i t} f_i^2(x)dx \\ &= \sum e^{-\lambda_i t} \int_M f_i^2 dx = \sum e^{-\lambda_i t}.\end{aligned}$$

If $t \to 0^+$ then there is a power serious expansion, asymptoticaly equivalent to $\sum e^{-\lambda_i t}$, i.e.

$$\sum_i e^{-t\lambda_i} \sim \sum_{n=0}^{\infty} a_n(\Delta) t^{(n-m)/2},$$

where $a_n(\Delta)$ are spectral invariants determined by local geometry of M. If M is a manifold with boundary, i.e. $\partial M \neq \phi$, then $a_{2k+1}(\Delta) \neq 0$ and they depend on the boundary conditions

$$\begin{aligned}\mathcal{B}_D f &= f|_{\partial M} = 0 && \text{(Dirichlet boundary condition)} \quad \text{or} \\ \mathcal{B}_N^S f &= (\partial_\nu + S)|_{\partial M} && \text{(modified Neumann boundary condition).}\end{aligned}$$

These results may be generalized for a partial differential operator D of order $d > 0$ on a smooth vector bundle. We assume the leading symbol of D is self-adjoint and positive definite. If the boundary of M is non-empty, we impose boundary conditions $\mathcal{B}$ and let $\operatorname{Domain}(D_{\mathcal{B}}) = \{w \in C^\infty(V) : \mathcal{B}w = 0\}$. We assume the boundary conditions $\mathcal{B}$ are strongly elliptic; see Gilkey [**44**, §1.11].

Let $f \in C^\infty(M)$ be an auxiliary test function. Then there is an asymptotic series at $t \downarrow 0^+$ of the form

$$\operatorname{tr}_{L^2}(fe^{-tD_{\mathcal{B}}}) \sim \sum_{n=0}^{\infty} a_n(f, D, \mathcal{B}) t^{(n-m)/d};$$

see Gilkey [**44**, Theorem 1.11.4] for details. The global invariants $a_n(f, D, \mathcal{B})$ are locally computable. Let $\partial_m^\nu f$ be the ν^{th} normal covariant derivative of f. Then there exists local measure valued invariants $A_n(x, D)$ defined for $x \in M$ and $\mathcal{A}_{n,\nu}^{bd}(y, D, \mathcal{B})$ defined for $y \in \partial M$ such that

$$a_n(f, D, \mathcal{B}) = \int_M f A_n(x, D) + \sum_{0 \le \nu \le n-1} \int_{\partial M} (\partial_m^\nu f) \mathcal{A}_{n,\nu}^{bd}(y, D, \mathcal{B}). \tag{1.5}$$

From now on we study the local geometry of operators of Laplace type. Let $D = -(g^{\nu\mu}\partial_\nu\partial_\mu + A^\sigma\partial_\sigma + B)$ be an operator of Laplace type on $C^\infty(M)$, for $A^\sigma \in \operatorname{End}(M)$ and $B \in \operatorname{End}(M)$. One can note that Dirichlet and modified Neumann boundary conditions are strongly elliptic for second order operators of Laplace type. We refer to [**43**] for the proof of

Lemma 1.1. *There exists a unique connection ∇_D on $C^\infty(M)$ and a unique function $E_D \in C^\infty(M)$ so that $D = -(\mathrm{tr}(\nabla_D^2) + E_D)$. If w_D is the connection 1-form of ∇_D, then*

$$w_{D,\delta} = \frac{1}{2} g_{\nu\delta}(A^\nu + g^{\mu\sigma}\Gamma_{g,\mu\sigma}{}^\nu), \quad \text{and}$$
$$E_D = B - g^{\nu\mu}(\partial_\mu w_{D,\nu} + w_{D,\nu} w_{D,\mu} - w_{D,\sigma}\Gamma_{g,\nu\mu}{}^\sigma). \qquad \square$$

We set $f = 1$ in (1.5) to recover the invariants $a_n(D, \mathcal{B})$ for an operator of Laplace type. We use now these invariants to express $\mathcal{A}_n(x, D)$. Let $\Omega_{D,ij}$ be the curvature of the connection ∇_D on $C^\infty(M)$ and let ';' be multiple covariant differentiation with respect to the Levi-Civita connection. We refer to Gilkey [**43**], [**44**] for the proof of the following theorem:

Theorem 1.2. *Let $D = D(\nabla_D, E_D)$ on $C^\infty(M)$.*
(a) $\mathcal{A}_0(x, D) = (4\pi)^{-m/2}$.
(b) $\mathcal{A}_2(x, D) = 6^{-1}(4\pi)^{-m/2}(\tau_g + 6E_D)$.
(c) $\mathcal{A}_4(x, D) = 360^{-1}(4\pi)^{-m/2}\{60(E_D);kk + 60\tau_g E_D + 180(E_D)^2 + 30\Omega_{D,ij}\Omega_{D,ij} + 12(\tau_g)_{;kk} + 5(\tau_g)^2 - 2|\rho_g|^2 + 2|Rg|^2\}$.

We suppose given some auxiliary geometric structure $\mathcal{J}$ on which $C_+^\infty(M)$ also acts. For $g \in \mathfrak{C}$ and $s \in \mathcal{J}$ we assume given a natural operator $D = D\{g, s\}$ on M which is of Laplace type. Let $D \mapsto {}_\beta D := D({}_\beta g, {}_\beta s)$, where $g \mapsto {}_\beta g = \beta g$, $\beta \in C_+^\infty(M), g \in \mathfrak{C}$, and let $\mathcal{M}(\beta)$ be function multiplication. An operator D is said to transform conformally if ${}_\beta D = \mathcal{M}(\beta^a) \circ D \circ \mathcal{M}(\beta^b)$ for $a + b = -1$. If D transforms conformally, the conformal index theorem of Branson and Orsted [**29**] and Parker and Rosenberg [**104**] shows that $a_m(D) = a_m({}_\beta D)$.

V.2. Asymptotics of Laplacians defined by torsion free connections

In this section we present the heat equation asymptotics of the Laplacians defined by torsion free connections.

V.2.1. Laplacians on the tangent bundle of a manifold without boundary. We assume that (M^m, g) is a compact Riemannian manifold without boundary of dimension $m > 1$. We choose a local coordinates to have ∂_i and dx^i as local coordinate frames for the tangent TM^m and cotangent T^*M^m bundles respectively. If ∇ is a torsion free connection on TM^m we denote by $w_i \in \mathrm{End}(TM^m)$ the connection 1-form of ∇

$$\nabla\partial_j = dx^i \otimes w_i(\partial_j) = w_{ij}{}^k dx^i \otimes \partial_k.$$

Since ∇ is torsion free it follows $w_{ij}{}^k = w_{ji}{}^k$. Let ${}^\nabla R$ be the curvature of ∇. Let ${}^g\nabla = \nabla(g)$ be the Levi-Civita connection corresponding to the metric g. Then

${}^g\nabla(\partial_j) = \Gamma_{ij}{}^k dx^i \otimes \partial_k$ and $\theta = \nabla - {}^g\nabla$ is tensorial and $\theta_{ij}{}^k = w_{ij}{}^k - \Gamma_{ij}{}^k$. We introduce also the tensor $\mathcal{F} \in TM^m$ by contracting the first two indices of θ:

$$\mathcal{F}^i = g^{jk}\theta_{jk}{}^i = g^{jk}(w_{jk}{}^i - \Gamma_{jk}{}^i).$$

The dual connection ∇^* on T^*M^m is defined by $d(u,\alpha) = (\nabla u, \alpha) + (u, \nabla^*\alpha)$, for any smooth vector field u and smooth covector field α. Consequently

$$\nabla^*(dx^j) = -dx^i \otimes w_i^*(\partial_j) = -w_{ik}{}^j dx^i \otimes dx^k.$$

If $\nabla^* \otimes 1 + 1 \otimes \nabla$ is the tensor product connection on $T^*M^m \otimes TM^m$, then

$$\mathcal{P} = \mathcal{P}(\nabla) = -g^{ij}\{\nabla^* \otimes 1 + 1 \otimes \nabla\}_i \nabla_j \quad \text{on} \quad C^\infty(TM^m) \tag{2.1}$$

is a second order *PDO* of Laplace type.

We shall use Roman indices for a coordinate frame and Greek indices for an orthonormal frame. We refer to [**26**] for the proof of:

Theorem 2.1. *Let* $\mathcal{P} = \mathcal{P}(\nabla)$ *be a PDO of Laplace type given by (2.1). Then*
(a) $a_0(\mathcal{P}) = m \cdot \mathrm{vol}(M)$,
(b) $a_2(\mathcal{P}) = \frac{m}{12}\int_M (2\tau - 3\mathcal{F}_\nu\mathcal{F}_\nu)$,
(c) $a_4(\mathcal{P}) = \frac{1}{360}\int_M \{m\{5\tau^2 - 2\rho^2 - 2\rho^2 + 2R^2 + 15\tau(2\mathcal{F}_{\nu;\nu} - \mathcal{F}_\nu\mathcal{F}_\nu)$
$+ \frac{45}{4}(2\mathcal{F}_{\nu;\nu} - \mathcal{F}_\nu\mathcal{F}_\nu)^2 + \frac{15}{2}(\mathcal{F}_{\mu;\nu} - \mathcal{F}_{\nu;\mu})^2\} + \mathrm{Tr}(30\Omega^2)\}$. □

We define the Hessian H_∇ for a torsion free connection ∇ on TM by

$$(H_\nabla f)(u,v) = u(v(f)) - \nabla_u v(f).$$

One can check easily that $(H_\nabla f)(u,v) = (H_\nabla f)(v,u)$ and H_∇ is tensorial in X and Y. The normalized Hessian $\mathcal{H}(f) := H_\nabla(f) + (m-1)^{-1} f\rho_\nabla$ arises naturally in the study of Codazzi equations; see [**108**] for details. We contract the normalized Hessian for a torsion free connection with symmetric Ricci tensor ρ_∇ to define an operator of Laplace type

$$Df = D(g,\nabla)f := -\operatorname{tr}_g\{H_\nabla(f) + (m-1)^{-1} f\rho_\nabla\}. \tag{2.2}$$

In general case, D need not be self-adjoint.

If $\tilde\nabla$ and ∇ are projectively equivalent, as in (1.1) Section II, then we may choose a local primitive ϕ so $d\phi = \pi$. Then

$$\mathcal{H}_{\tilde\nabla} = e^\phi \mathcal{H}_\nabla e^{-\phi}; \tag{2.3}$$

i.e. the operators $\mathcal{H}_{\tilde\nabla}$ and $\mathcal{H}_\nabla$ are locally conjugate. Furthermore, if $\tilde g = e^{2\psi}g$, then (2.3) implies

$$D(\tilde g, \tilde\nabla) = e^{-2\psi+\phi} D(g,\nabla) e^{-\phi}. \tag{2.4}$$

If $[\pi] = 0$ in the first cohomology group $H^1(M)$, then ϕ is globally defined and $D(g, \tilde{\nabla})$ and $D(g, \nabla)$ are conjugate and hence isospectral. We refer to **[17]**–**[19]**, **[21]** for more details related to this operator.

Let $\mathcal{K}$ be a Codazzi structure, let $(g, {}^*\nabla) \in \mathcal{K}$ and ${}^w\nabla$ be the Weyl connection defined by $\mathcal{K}$. Then we use (2.2) and define

(i) Let ${}^*D := D\{g, {}^*\nabla\}$ be the trace of the normalized Hessian of ${}^*\nabla$.
(ii) Let ${}^wD := D\{g, {}^w\nabla\}$ be the trace of the normalized Hessian of ${}^w\nabla$.
(iii) Let ${}^w\Delta := -\operatorname{tr}_g {}^w\nabla d$ be the scalar Laplacian of ${}^w\nabla$.
(iv) Let ${}^g\Box := -\operatorname{tr}_g \delta_g d + (m-2)\tau(g)/4(m-1)$ be the conformal Laplacian.

V.3. Geometry reflected by the spectrum

As we already know torsion free connections arise naturally in affine differential geometry and Weyl geometry. The main purpose of this section is to study geometry of a manifold in the framework previously mentioned, reflected by the spectrum of an differential operator of Laplace type. One can find more details in **[17]**–**[21]** and **[26]**.

V.3.1. Affine differential geometry reflected by the spectrum. In this subsection we deal with smooth hypersurfaces M^m immersed into an affine space $\mathcal{A}^{m+1}$. Because of the convinience reason througout this subsection we denote by ${}^1\nabla$, ${}^2\nabla$ the induced connection and the conormal connection. We suppose the Blaschke metric G positive definite henceforth; this means that the immersed hypersurface $x(M^m)$ is locally strongly convex. Let $\mathcal{P} = \mathcal{P}({}^1\nabla, G) = \mathcal{P}(x, X, y)$ on $C^\infty(TM^m)$ be defined by (2.1). The spectral geometry of $\mathcal{P}$ should play an important role in affine geometry. Since the Blaschke metric G and first affine connection ${}^1\nabla$ are defined by expressions which are invariant under the group of affine transformations, the operator $\mathcal{P}$ and its spectrum are affinely invariant.

One can compute the heat equation invariants.

Lemma 3.1. *(a)* $C_{ij}{}^k = \theta_{ij}{}^k$, *(b)* $w_{ij}{}^k = {}^1\Gamma_{ij}{}^k$. *(c)* $\mathcal{F}_\nu = m\tilde{T}_\nu$,
(d) $\operatorname{tr}(\Omega^2) = {}^1R_{ijk}{}^l\,{}^1R_{ijl}{}^k = -2\{m(m-1)H^2 - \frac{1}{m}\sum_{i<j}(\lambda^i - \lambda^j)^2\}$. □

We combine now Lemma 3.1 and Theorem 2.1 with results from Section I.2. to prove the following theorems.

Theorem 3.2. *Let x and $\bar{x} : M^m \to \mathcal{A}^{m+1}$ define hyperovaloids with the same regular relative spherical indicatrix $y = \bar{y}$ which are $\mathcal{P}$ isospectral. Then x and $\bar{x}$ are translation equivalent.* □

Theorem 3.3. *Let x and $\bar{x} : M^2 \to \mathcal{A}^3$ be ovaloids with centroaffine normalization which are $\mathcal{P}$ isospectral. If $x(M^2)$ is an ellipsoid, then $\bar{x}(M^2)$ is an ellipsoid.* □

Theorem 3.4. *Let M_i be ovaloids with equiaffine normalization and M_1 an ellipsoid. If for $M_1 := M$, $M_2 := \bar{M}$*
(i) $\int_M H = \int_{\bar{M}} \bar{H}$, $\int_M \operatorname{tr}(1) = \int_{\bar{M}} \operatorname{tr}(1)$ *or*

(ii) $\int_M H^2 = \int_{\bar M} \bar H^2$, $\int_M \operatorname{tr}(1) = \int_{\bar M} \operatorname{tr}(1)$, $H^2 > 0$
(H^2 is the second elementary curvature function)
then $\bar M = M_2$ is an ellipsoid. □

We study now affine geometry reflected by the spectrum of the operator D given by (2.2). Since the ${}^i\nabla$ are torsion free, Ricci symmetric connections, we can apply the results of Section V.2. to this setting. Let

$$^1D := D(G, {}^1\nabla), \quad {}^2D := D(G, {}^2\nabla) \quad \text{and} \quad {}^GD := D(G, {}^G\nabla)$$

be the associated operators of Laplace type on $C^\infty(M)$; these operators and their spectra are affine invariants of (x, X, y).

We now compute the expressions of Lemma 1.1, which we need to obtain the coefficients in the corresponding heat equation asymptotics.

Lemma 3.5. *Let $D = {}^1D$ and let $\epsilon = 1$ or let $D = {}^2D$ and let $\epsilon = -1$. Then:*

$$^D\theta = \epsilon C, \quad w_D = -\frac{1}{2}\epsilon m\tilde T, \quad \Omega_D = 0,$$
$$E_D = mH - \frac{1}{4}m^2|\tilde T|^2 + \frac{1}{2}\epsilon m\tilde T i; i. \quad \square$$

Theorem 3.6. *Let $x : M \to \mathcal{A}$ be a hyperovaloid. Let $D = {}^1D$ and let $\epsilon = 1$ or let $D = {}^2D$ and let $\epsilon = -1$. Then:*
(a) $\mathcal{A}_0(x, D) = (4\pi)^{-m/2}$. $a_0(D) = (4\pi)^{-m/2}\int_M 1$.
(b) $\mathcal{A}_2(x, D) = (4\pi)^{-m/2}\{\frac{1}{6}\tau_g + mH - \frac{1}{4}m^2|\tilde T|^2 + \frac{1}{2}\epsilon m\tilde T_{i;i}\}$
$a_2(D) = (4\pi)^{-m/2}\int_M\{\frac{1}{6}\tau_g + mH - \frac{1}{4}m^2|\tilde T|^2\}$.
(c) $\mathcal{A}_4(x, D) = (4\pi)^{-m/2}360^{-1}\{60\tau_G(mH - \frac{1}{4}m^2|\tilde T|^2 + \frac{1}{2}\epsilon m\tilde T_{i;i}) + 180(mH - \frac{1}{4}m^2|\tilde T|^2 + \frac{1}{2}\epsilon m T_{i;i})^2 + 60(mH - \frac{1}{4}m^2|\tilde T|^2 + \frac{1}{2}\epsilon m\hat T_{i;i})_{;jj} + 12\tau_{G;kk} + 5\tau_G^2 - 2|\rho_G|^2 + 2|R_G|^2\}$. □

One can combine Lemma 3.5 and Theorem 3.6 with results from Section I.2. to prove the following theorems, which consider affine geometry reflected by the spectrums of ${}^iD$.

Theorem 3.7. *Let $D = {}^1D$ or $D = {}^2D$.*
(a) Let (x, X, y) be a relative normalization. Then

$$(4\pi)^{m/2}\Big\{a_2(D) - \frac{m-1}{m+5}a_2({}^GD)\Big\} \le m\int_M H.$$

Equality holds if and only if the normalization is equiaffine.
(b) If the normalization is equiaffine, then

$$m(m+5)\int_M H \le 6(4\pi)^{m/2}a_2(D) \le m(m+5)\int_m (H + J).$$

Equality holds on the left or on the right hand side if and only if the hyperovaloid is an ellipsoid. □

Theorem 3.8. *Let $D = {}^1D$ or $D = {}^2D$. With the centroaffine normalization:*
(a) $a_0(D) = (4\pi)^{-m/2}\int_M 1.\quad a_2(D) = (4\pi)^{-m/2}\int_M(\frac{1}{6}\tau_G + \boldsymbol{m} - \frac{1}{4}m^2|\tilde{T}|^2)$.
(b) $6(4\pi)^{m/2}\{a_2(D) - ma_0(D)\} \leq \int_m \tau_G$ *with equality if and only if the hyperovaloid is an ellipsoid.*
(c) Let $x(M^2)$ and $\tilde{x}(M^2)$ be ovaloids in 3 space with centroaffine normalization which are D isospectral. Then x is an ellipsoid if and only if $\tilde{x}$ is an ellipsoid. □

As we have mentioned already the operators ${}^iD$ are not self-adjoint in general case. The following theorem deals with the conditions to have self-adjoint operators ${}^iD$.

Theorem 3.9. *(1) Let $\{x, X, y\}$ be the Euclidean normalization. Then the following assertions are equivalent*
1-a) The Gauss-Kronecker curvature $K = K_n$ is constant.
1-b) We have ${}^1D = {}^2D$.
1-c) The operator ${}^1D$ or the operator ${}^2D$ is self-adjoint.
(2) Let x be a compact centroaffine hypersurface with non-empty boundary. The following assertions are equivalent:
2-a) We have ${}^1D = {}^2D$.
2-b) We have that ${}^1D$ or ${}^2D$ is self-adjoint.
2-c) We have that x is a proper affine sphere.
(3) Let x be a compact centroaffine hypersurface without boundary. Then the following assertions are equivalent:
3-a) We have ${}^1D ={}^2 D$.
3-b) We have that ${}^1D$ or ${}^2D$ is self-adjoint.
3-c) We have that x is a hyperovalloid. □

Dirichlet boundary conditions on affine hypersurface with boundary were considered by Schwenk [**113**], [**128**] and Simon [**116**]. Their methods are different from this one.

V.3.2. Projective geometry reflected by the spectrum. In general, constructing projective invariants is quite difficult. One such example is the projective curvature tensor of H. Weyl (see Section II.1., especialy the formula (1.2)). This subsection deals with spectral invariants which are also projectively invariant. More precisely we have

Theorem 3.10. *Let $\nabla, \tilde{\nabla}$ be torsion free projectively equivalent connections on a Riemannian manifold (M, g). Let $D = D(g, \nabla)$ and $\tilde{D} = D(g, \tilde{\nabla})$.*
(a) $\mathcal{A}_n(x, \tilde{D}) = \mathcal{A}_n(x, D)$ *and* $\mathcal{A}_n^{bd}(y, \tilde{D}, \mathcal{B}) = \mathcal{A}_n^{bd}(y, D, \mathcal{B})$.
(b) $a_n(\tilde{D}, \mathcal{B}) = a_n(D, \mathcal{B})$. □

If m is odd, and if the boundary of M is empty, there is a global spectral invariant called *the functional determinant* which can be defined in this context.

For $\mathrm{Re}(s) \gg 0$, let $\zeta(s, D) := \mathrm{tr}_{L^2}(D^{-s})$, where we project on the complement of the kernel of D to avoid the 0-spectrum. This has a meromorphic extension to

$\mathbb{C}$ with isolated simple poles on the real axis. The origin is a regular value and $\zeta'(0) := -\log(\det(D))$ is a global invariant of D.

Theorem 3.11. *Let $\nabla, \tilde{\nabla}$ be torsion free projectively equivalent connections on a Riemannian manifold (M, g) and $\tilde{g} \in \mathfrak{C}(g)$. If the boundary of M is empty and if $m = \dim M$ is odd, then $\zeta'(0, D) = \zeta'(0, \tilde{D})$.* □

V.3.3. Invariants of Codazzi and Weyl structures.

The relation (2.4) informs us that the operator D, given by (2.2) transforms conformally. This implies one can apply the conformal index theorem of Branson and Orsted [**29**] and Parker and Rosenberg [**104**] to prove the following lemma.

Lema 3.12. *(i) Let D be an operator of Laplace type given by (2.2). Then $a_m(D(\tilde{g}, \tilde{\nabla})) = a_m(D(g, \nabla))$.*

(ii) We have that $a_m(^*D), a_m(^wD), a_m(^w\Delta)$, and $a_m(^g\Box)$ are gauge invariants of a Codazzi structure $\mathcal{K}$. □

One can compute the endomorphism E and the curvature Ω for four natural operators defined in Section V.2.

Lemma 3.13. We have

(i) $E\{^*D\} = \{(m+2)\tau(g, {}^w\nabla) - (m-2)\tau(g)\}/4(m-1)$.
(ii) $\Omega\{^*D\} = -(m+2)^wF/2$.
(iii) $E\{^wD\} = -(m-2)\delta_g\hat{\theta}/2 - (m-2)||\hat{\theta}||_g^2/4 + (m-1)^{-1}\tau(g, {}^w\nabla)$.
(iv) $\Omega\{^wD\} = -(m-2)^wF/2$
(v) $E\{^w\Delta\} = -(m-2)\delta_g\hat{\theta}/2 - (m-2)^2||\hat{\theta}||_g^2/4$.
(vi) $\Omega(^w\Delta) = -(m-2)^wF/2$.
(vii) $E\{^g\Box\} = -(m-2)\tau(g)/4(m-1)$ and $\Omega\{^g\Box\} = 0$. □

We refer to [**20**] for the proof of this Lemma and more details related to the operators $^*D$, ^wD, $^w\nabla$ and $^g\Box$.

We use now Lemma 3.13 and Theorem 1.2 in dimensions $m = 2$ and $m = 4$. Let $\chi(M)$ be the Euler-Poincare characteristic of M. The Chern Gauss Bonnet theorem yields

$$\chi(M^2) = (4\pi)^{-1} \int_M \tau(g)(x) d\nu_g(x),$$

$$\chi(M^4) = (32\pi^2)^{-1} \int_M \{||^gR||_g^2 - 4||^g\rho||_g^2 + \tau(g)^2\}(x) d\nu_g(x).$$

Theorem 3.14. *Let $\dim(M) = 2$. Then*

(i) $a_2(^*D) = \chi(M)/6 + (4\pi)^{-1} \int_M \tau(g, {}^w\nabla) d\nu_g(x)$.
(ii) $a_2(^wD) = \chi(M)/6 + (4\pi)^{-1} \int_M \tau(g, {}^w\nabla)(x) d\nu_g(x)$.
(iii) $a_2(^w\Delta) = \chi(M)/6$.
(iv) $a_2(^g\Box) = \chi(M)/6$. □

Theorem 3.15. *Let $\dim(M) = 4$. Let ^gW be the Weyl conformal curvature. Then*

(i) $a_4({}^*D) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M\{3\|{}^gW\|_g^2 + 270\|{}^wF\|_g^2 + 45\tau(g,{}^w\nabla)^2\}d\nu_g(x)$.

(ii) $a_4({}^wD) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M\{3\|{}^gW\|_g^2 + 30\|{}^wF\|_g^2 + 45\tau(g,{}^w\nabla)^2\}d\nu_g(x)$.

(iii) $a_4({}^w\Delta) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M\{3\|{}^gW\|_g^2 + 30\|{}^wF\|_g^2 + 5\tau(h,{}^w\nabla^2\}d\nu_g(x)$.

(iv) $a_4({}^g\Box) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M\{3\|{}^gW\|_g^2\}d\nu_g(x)$. □

If f is a scalar invariant, let $f[M] := \int_M f(x)d\nu_g(x)$. The Euler characteristic is a topological invariant of M which does not depend on the Codazzi structure. Then we use Theorem 3.15 to prove the following Corollary:

Corollary 3.16. *(i) The invariants $\tau(g,{}^w\nabla)^2[M]$, $\|{}^wF\|_g^2[M]$ and $\|{}^gW\|_g^2[M]$ of a Weyl structure on M are determined by $\chi(M)$ and by the spectrum of the operators ${}^*D$, ${}^wD$, and ${}^w\Delta$.*

(ii) We have $32\pi^2\chi(M^4) \geq 45\tau(g,{}^w\nabla)^2[M] + 270\|{}^wF\|_g^2[M] - (4\pi)^2 360 a_4({}^*D)$ with equality if, and only if, the class $\mathfrak{C}$ is conformally flat.

(iii) We have $32\pi^2\chi(M^4) \geq 45\tau(g,{}^w\nabla)^2[M] + 3\|{}^gW\|_g^2[M] - (4\pi)^2 360 a_4({}^*D)$ with equality if, and only if, the length curvature ${}^wF = 0$. □

Bibliography

[1] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry, and Riemannian geometry I, II, III*, Math. Proc. Cambridge Phil. Soc. **77** (1975), 43–69, **78** (1975), 405–432, **79** (1976), 71–99.

[2] L. Auslander, *On the Euler characteristic of compact locally affine spaces*, Comment. Math. Helv. **35** (1961), 25–27.

[3] A. Avez, *Remarques sur les formes de Pontrjagin*, C.R. Acad. Sci. Paris **270** (1970), 1248.

[4] J.P. Benzécri, *Sur les variété s localement affines et localement projectives*, Bull. Soc. Math. France **88** (1960), 229–332.

[5] J.P. Benzécri, *Sur la classe d'Euler de fibré s affins plats*, C.R. Acad. Sci. Paris **260** (1965), 5442–5444.

[6] M. Berger, P. Gauduchon and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math. **194**, Springer-Verlag, 1971.

[7] D.E. Blair, *Contact manifolds in Riemannian Geometry*, Lecture Notes in Math. **509**, Springer-Verlag, Berlin, New York, 1976.

[8] D.E. Blair, L. Vanhecke, *New characterization of φ-symmetric spaces*, Kodai Math. J. **10** (1987), 102–107.

[9] D.E. Blair, L. Vanhecke, *Symmetries and φ-symmetric spaces*, Tohoku Math. J. **39** (1987), 373–382.

[10] W. Blaschke, *Vorlesungen über Differentialgeometrie, II. Affine Differential geometrie*, Springer-Verlag, Berlin, 1923.

[11] W. Blaschke, *Gesammelte Werke, Vol. 4. Affine Differentialgeometrie, Differentialgeometrie der Kreis- und Kugelgruppen*, Thales Verlag, Essen, 1985.

[12] N. Blažić and N. Bokan, *Compact Riemann surfaces with the skew-symmetric Ricci tensor*, Izvestiya VUZ Matematika and Soviet Math. **9(376)** (1993), 8–12.

[13] N. Blažić and N. Bokan, *Projective and affine transformations of a complex symmetric connection*, Bull. Austral. Math. Soc. **50** (1994), 337–347.

[14] N. Blažić and N. Bokan, *Invariance theory and affine differential geometry*, in: J. Janyska (ed.) et al., *Differential Geometry and Applications. Proc. 6th Internat. Conf., Brno*, Masaryk University, 1996, pp. 249–260.

[15] N. Blažić, N. Bokan and P.B. Gilkey, *Pontrjagin forms, Chern Simons classes, Codazzi transformations and affine hypersurfaces*, J. Geom. Phys. **27** (1998), 333–349.

[16] N. Blažić and N. Bokan, *Some topological obstructions to symmetric curvature operators*, Publ. Math. Debrecen **42** (1993), 357–368.

[17] N. Bokan, P. Gilkey and U. Simon, *Applications of spectral geometry to affine and projective geometry*, Beiträge Algebra Geom. **35** (1994), 283–314.

[18] N. Bokan, P. Gilkey and U. Simon, *Second order local affine invariants*, Izvestiya VUZ Matematika and Soviet Math. **5** (1995), 11–15.

[19] N. Bokan, P. Gilkey and U. Simon, *Relating the second and fourth order terms in the heat equation*, J. Geom. **59** (1997), 61–65.

[20] N. Bokan, P. Gilkey and U. Simon, *Geometry of differential operators on Weyl manifolds*, Proc. Roy. Soc. London Ser. A **453** (1996), 2527–2536.

[21] N. Bokan, P. Gilkey and U. Simon, *Spectral invariants of affine hypersurfaces*, Publ. Inst. Math. (Beograd) **64(78)** (1998), 133–145.

[22] N. Bokan, K. Nomizu, U. Simon, *Affine hypersurfaces with parallel cubic forms*, Tohoku Math. J. **42** (1990), 101–108.

[23] N. Bokan, *On the complete decomposition of curvature tensors of Riemannian manifolds with symmetric connection*, Rend. Circ. Mat. Palermo (3) **39** (1990), 331–380.

[24] N. Bokan, *The decomposition theory and its application*, in: G. M. Rassias (ed.), *The Mathematical Heritage of C. F. Gauss*, World Scientific, Singapore, 1991, pp. 66–99.

[25] N. Bokan, *C-holomorphically projective transformations in a normal almost contact manifold*, Tensor N.S. **33** (1979), 189–199.

[26] N. Bokan and P. Gilkey, *Asymptotics of Laplacians defined by Symmetric Connections*, Results in Math. **20** (1991), 589–599.

[27] N. Bokan, *Curvature tensors on Hermitian manifolds*, Colloquia Mathematica Societatis Janos Bolyai 46, Topics in Diff. Geom. Debrecen (Hungary (1984), 213–239.

[28] N. Bokan and S. Nikčević, *A characterization of projective and holomorphic projective structures*, Arch. Math. **62** (1994), 368-377.

[29] T. Branson and B. Orsted, *Conformal indices of Riemannian manifolds*, Comp. Math. **60** (1986), 261–293.

[30] R. Brooks, *Constructing Isospectral Problems*, Amer. Math. Monthly Nov. (1988), 823–839.

[31] W. Burau, U. Simon, *Blaschkes Beiträge zur affinen Differentialgeometrie*, In [11], 11–34.

[32] E. Calabi, *Complete affine hyperspheres I*, Symposia Math. **10** (1972), 19–38.

[33] E. Calabi, *Hypersurfaces with maximal affinely invariant area*, Amer. J. Math. **104** (1982), 91–126.

[34] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

[35] S.S. Chern, *Geometry of characteristic classes*, Canadian Math. Congress, Proc. 13th. Biennial Seminar **1** (1971), 1–40.

[36] S.S. Chern, *On the curvatura integra in a Riemannian manifold*, Ann. of Math. **46** (1945), 674–684.

[37] S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. **99** (1974), 48–69.

[38] F. Dilen, L. Verstraelen and L. Vrancken, *Complex affine differential geometry*, Preprint Reihe Mathematik FB 3 TU Berlin, No. **153**, 1987.

[39] J. Dodziuk, *Eigenvalues of the Laplacian and the heat equation*, Amer. Math. Monthly **88** (1981), 686–695.

[40] T. Eguchi, P.B. Gilkey, and A.J. Hanson, *Gravitation, gauge theory, and differential geometry*, Physics Reports **6G** (1980), 213–393

[41] L. Einsenhart, *Non-Riemannian Geometry*, 5th ed., Colloq. Publ. **8**, AMS, Providence, RI, 1964.

[42] G.B. Folland, *Weyl manifolds*, J. Diff. Geom. **4** (1970), 145–153.

[43] P.B. Gilkey, *The spectral geometry of a Riemannian manifold*, J. Diff. Geom. **10** (1975), 601–618.

[44] P.B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, CRC Press, 1994.

[45] S.I. Goldberg, *Curvature and Homology*, Academic Press, New York, 1962.

[46] A. Gray and L.M. Hervella, *The sixteen classes of almost Hermitian manifolds*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[47] A. Gray, *Invariants of curvature operators of four-dimensional Riemannian manifolds*, Proc. Thirteenth Biennial Seminar Canad. Math. Congres, Halifax **2** (1971), 42–63.

[48] A. Gray and I.M. Hervella, *The sixteen classes of almost Hermitian manifolds*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[49] G.S. Hall, *Weyl manifolds and connections*, J. Math. Phys. **33** (1992), 2633-2638.

[50] G.S. Hall and Barry M. Haddow, *Some remarks on metric and Weyl connections*, J. Math. Phys. **36** (1995), 5938–5948.

[51] G.S. Hall, B.M. Haddow and J.R. Pulham, *Weyl connections, curvature and generalized gauge changes*, submitted.

[52] T. Higa, *Weyl manifolds and Einstein-Weyl manifolds*, Comm. Math. Univ. Sancti Pauli **42** (1993), 143–160.

[53] T. Higa, *Curvature tensors and curvature conditions in Weyl geometry*, Comment. Math. Univ. Sancti Pauli **43** (1994), 139–153.

[54] N.S. Hitchin, *Complex manifolds and Einstein's equation*, Lecture Notes in Math. 970, Springer-Verlag, Berlin (1982), 73–99.

[55] S. Ishihara, *Holomorphically projective changes and their groups in an almost complex manifold*, Tohoku Math. J. **9** (1957), 273–297.

[56] S. Ishihara, *Groups of projective transformations and groups of conformal transformations*, J. Math. Soc. Japan **9(2)** (1957), 195–227.

[57] D. I. Jonson, *Sectional curvature and curvature normal forms*, Michigan Math. J. **27** (1980), 275–294.

[58] M. Kac, *Can one hear the shape of a drum?* Amer. Math. Monthly **73** (1966), 1–23.

[59] S. Kobayashi and T. Ochiai, *Holomorphic projective structures on compact complex surfaces*, Math. Ann. **249** (1980) 75–94.

[60] S. Kobayashi and T. Nagano, *On Projective Connections*, J. Math. Mech. **13(2)** (1964), 215–235.

[61] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, vol. 1,2, Interscience, 1969.

[62] S. Kobayashi and T. Ochiai, *Holomorphic projective structures on compact complex surfaces*, Math. Ann. **249** (1980), 75–94.

[63] S. Kobayashi and T. Ochiai, *Holomorphic projective structures on compact complex surfaces.* II Math. Ann. **255** (1981), 519–521.

[64] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin, 1972.

[65] S. Kobayashi and C. Horst, *Complex differential geometry*, (DMV seminar: band 3) Birkhäuser, Basel, 1983.

[66] O. Kowalski, S. Wegrzynowski, *A classification of five-dimensional φ-symmetric spaces*, Tensor N.S. **46** (1987).

[67] O. Kowalski, *Partial curvature structures and conformal geometry of submanifolds*, J. Diff. Geom. (1) **8** (1973) 53–70.

[68] M. Kozlowski, *Affine maximal surfaces*, Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **124** (1987), 137-139.

[69] M. Kozlowski, *Surfaces defined by solutions of the Euler-Lagrange-Equation*. Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **125** (1988), 71–73.

[70] M. Kozlowski, *Improper affine spheres*. Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **125** (1988), 95–96.

[71] M. Kozlowski, *A class of affine maximal hypersurfaces*. Rend. Circ. Mat. Palermo **38** (1989), 444–448.

[72] M. Kozlowski, *The Monge-Ampère equation in affine differential geometry*. Anz. Österreich. Akad. Wiss. **126** (1989), 21–24.

[73] R.S. Kulkarni, *On the Bianchi identities*, Math. Ann. **199** (1972), 175–204.

[74] A.M. Li, G. Penn, *Uniqueness theorems in affine differential geometry*, Part II. Results in Mathematics **13** (1988), 308–317.

[75] M. Magid, P. Ryan, *Flat affine spheres*, Geom. Dedicata **33** (1990), 277–288.

[76] M. Magid, P. Ryan, *Affine 3-Spheres with Constant Affine curvature*, Transactions AMS (to appear).

[77] P. Matzeu, *Chern forms and C-projective curvature of normal almost contact manifolds*, An. Stiint. Univ. "Al. I. Cuza" Iasi, Sect. Ia Mat. **42** (1996), 193–200.

[78] P. Matzeu, *Decomposition of curvature tensors on normal almost contact manifolds*, Rend. Sem. Mat. Univ. Politecn. Torino **47(2)** (1989), 179–210.

[79] P. Matzeu, *Su una caratterizzazione delle varieta normali*, Bull. Unione Mat. Ital. VI **4** (1985), 153–160.

[80] P. Matzeu, *On the Geometry of Almost Contact Structures*, Univ. "Al. I. Cuza", doctoral thesis, 1990.

[81] P. Matzeu and S. Nikčević, *Linear algebra of curvature tensors on Hermitian manifolds*, Annal. Sciint. Univ. "Al. I. Cuza", **37** (1991), 71–86.

[82] A.S. Mischenko and A.T. Fomenko, *Kurs differencialnoj geometrii i topologii*, MGU, Moskva, 1980.

[83] J. Микеш, *Геодезические F-планарные и голоморфно-проективные отображения римановых пространств*, диссерт., Оломоуц, 1995.

[84] R.S. Millman, *Manifolds with the same spectrum*, Amer. Math. Monthly **90** (1983), 553–555.

[85] R. Milman, G. Parker, *Elements of Differential geometry*, Englewood Cliffs, Prentice Hall, 1997.

[86] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 542.

[87] J. Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223.

[88] H. Mori, *On the decomposition of generalized K-curvature tensor fields*, Tohoku Math. J. **25** (1973), 225–235.

[89] T. Nagano, *The projective transformation on a space with parallel Ricci tensor*, Kodai Math. Sem. Rep. **11** (1959), 131–138.

[90] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. Math. **65** (1954), 391–404.

[91] S. Nikčević, *Conformal and projective invariants in Riemannian geometry*, preprint.

[92] S. Nikčević, *On the decomposition of curvature tensor fields on Hermitian manifolds*, Colloq. Math. Soc. Janos Bolyai **56** (1992), 555–568.

[93] S. Nikčević, *Induced representation of unitary group in tensor spaces of Hermite manifolds*, (preprint).

[94] S. Nikčević, *On the decomposition of curvature tensor*, Collection Sci. Papers Fac. Sci. Kragujevac **16** (1996), 61–68.

[95] K. Nomizu, *A survey of Recent Results in Affine Differential Geometry*, In: L. Verstraelen, A. West (Eds.), *Geometry and Topology of Submanifolds III*, Conf. Leeds 1990, World Scientific, London, Singapore, etc., 1991, pp. 227–256.

[96] K. Nomizu and F. Podestá, *On affine Kähler structures*, Bull. Soc. Math. Belg. **41** (1989), 275–282.

[97] K. Nomizu and T. Sasaki, *Affine Differential Gometry*, Cambridge Univ. Press, 1993.

[98] A.P. Norden, *Spaces of Desquartes composition*, Izvestiya VUZ **4** (1963), 117–128.

[99] A.P. Norden, *On structure of connections on line manifolds of noneuclidean spaces*, Izvestiya VUZ **12** (1972), 84–94.

[100] A.P. Norden, *Prostranstva afinoy svyznosti*, Nauka, Moskva, 1976.

[101] M. Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Japan J. Math. **26** (1956), 43–77.

[102] V. Oliker and U. Simon, *Affine Geometry and Polar Hypersurfaces*, In: *Analysis and Geometry: Trends in Research and Teaching* (B. Fuchssteiner and W.A.J. Luxemburg, eds.), 87–112, Bibl. Inst. Mannheim–Zuerich, 1992.

[103] V. Oproiu, *The C-projective curvature tensor on 3-dimensional normal almost contact manifolds*, Tensor N.S. **46** (1987), 250–257.

[104] T. Parker and S. Rosenberg, *Invariants of conformal Laplacians*, J. Diff. Geom. **25** (1987), 199–222.

[105] E.M. Patterson, *A characterization of Kähler manifolds in terms of parallel fields of planes*, J. London Math. Soc. **28** (1953), 260–269.

[106] H. Pedersen, A. Swann, *Riemannian submersions, four manifolds and Einstein-Weyl geometry*, Proc. London Math. Soc. **66** (1991), 381–399.

[107] R. Penrose, W. Rindler, Spinors and spacetime. Two-spinor calculus and relativistic fields, Cambridge Monographs on Mathematical Physics, vol. 1. Cambridge University Press, 1984.

[108] U. Pinkall, A. Schwenk–Schellschmidt, and U. Simon, *Geometric methods for solving Codazzi and Monge–Ampére equations*, Math. Ann. **298** (1994), 89–100.

[109] W.A. Poor, *Differential Geometric Structures*, McGraw-Hill New York, 1981.

[110] P.K. Rasewskij, *Riemanowa geometria i tenzornij analiz*, Gostehizdat, Moskva, 1953.

[111] P.A. and A.P. Schirokow, *Affine Differentialgeometrie*, Teubner, Leipzig, 1962, (Zentralblatt für Mathematik 106.147; Russ. Original Zbl. 85.367).

[112] J.A. Schouten, *Ricci-Calculus*, 2nd ed., Springer-Verlag, Berlin, 1954.

[113] A. Schwenk, *Eigenwertprobleme des Laplace-Operators und Anwendungen auf Untermannigfaltigkeiten*, Dissertation, TU, Berlin, 1984.

[114] A. Schwenk, U. Simon, *Hypersurfaces with constant equiaffine mean curvature*. Archiv Math. 46 (1986), 85–90.

[115] U. Simon, *Kennzeichnungen von Sphären*, Math. Ann., **175**, (1968). 81–88.

[116] U. Simon, *Dirichlet problems and the Laplacian in affine hypersurface theory*, Lecture Notes Math. **1369**, Springer-Verlag, Berlin, 1989, pp. 243–260.

[117] U. Simon, *Zur Entwicklung der affinen Differentialgeometrie*. In W. Blaschke, *Gesammelte Werke, Vol. 4, Affine Differentialgeometrie. Differentialgeometrie der Kreis- und Kugelgrupen*. Thales Verlag, Essen 1985.

[118] U. Simon, *Connections and conformal structure in affine differential geometry*. In: D. Krupka, A. Svec (Eds); *Differential Geometry and its Applications*, J.E. Pyrkine Univ., CSSR, 1987, 315–327.

[119] U. Simon, *Dirichlet problems and the Laplacian in affine hypersurface theory*. Lecture Notes Math. **1369**, Springer Verlag, Berlin, 1989, 243–260.

[120] U. Simon, *Local classification of two-dimensional affine spheres with constant curvature metric*, Diff. Geom. Appl. 1, (1991), 123–132.

[121] U. Simon, *Hypersurfaces in equiaffine differential geometry*, Geom. Dedicata **17** (1984), 157–168.

[122] U. Simon, C.P. Wang, *Local Theory of Affine 2-Spheres*, Proc. 1990 Summer Inst. Diff. Geom (R.E. Greene, S.T. Yau; eds). Proc. Symp. Pure Math., **54**, part 3, (1993), 585–598.

[123] U. Simon, *Connections and conformal structure in affine differential geometry*, in *Differential geometry and its applications*, Proc. of the Conference, Brno, August 24-30, (D. Reidel, Dordrecht, 1986, pp. 315–328.

[124] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science University Tokyo, 1991.

[125] I.M. Singer and J.A. Thorpe, *The curvature of 4-dimensional Einstein space*, Global Analysis (paper in honour of K. Kodaira), Univ. of Tokyo Press, Tokyo (1969), 355–365.

[126] M. Sitaramayya, *Curvature tensors in Kähler manifolds*, Tran. Amer. Math. Soc. **183** (1973), 341–351.

[127] R.S. Strichartz, *Linear algebra of curvature tensors and their covariant derivatives*, Canad. J. Math. **40** (1988), 1105–1143.

[128] A. Schwenk, *Affinsphären mit ebenen Schattengrenzen*. In: D. Ferus, R.B. Gardner, S. Helgason, U. Simon (Eds), *Global Differential Geometry and Global Analysis 1984*, Lecture Notes Math. **1156**, Springer-Verlag, 1985, pp. 296–315.

[129] Y. Tashiro, *On a holomorphically projective correspondences in an almost complex space*, Math. J. Okayama Univ. **6** (1957), 147–152.

[130] K.P. Tod, *Compact three-dimensional Einstein-Weyl structures*, J. London Math. Soc. **45** (1993), 341–351.

[131] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. (2) **267** (1981), 365–398.

[132] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc., Lect. Notes **83**, Cambridge Univ. Press, 1983.

[133] L. Vrancken, A.-M. Li, U. Simon, *Affine spheres with constant sectional curvature*. Math. Z. **206** (1991), 651–658.

[134] Ch. Wang, *Some examples of complete hyperbolic affine 2-spheres in*$\mathbb{R}^3$. In: *Global Differential Geometry Global Analysis*, Proceedings, Berlin 1990. Lecture Notes Math. **1481**, Springer Verlag, Berlin etc. 1991, 272–280.

[135] H. Weyl, *Classical Groups, Their Invariants and Representations*, Princenton Univ. Press, Princeton, N. J., 1946.

[136] H. Weyl, *Zur Infinitesimalgeometrie, Einordnung der projetiven und der konformen Auffassung*, Gottinger Nachrichten, (1921), 99–112.

[137] H. Weyl, *Über die Asymptötische Verteilung der Eigenwerte*, Nachr. Königl. Ges. Wiss. Göttingen (1911), 110–117.

[138] H. Weyl, *Space-time Matter*, Dover, 1922.

[139] K. Yano, *Differential Geometry of Complex and Almost Complex Spaces*, Pergamon Press, New York, 1965.

[140] K. Yano and S. Bochner, *Curvature and Betti Numbers*, Princeton University Press, Pricenton, New Yersey, 1953.

Eberhard Malkowsky
and
Vladimir Rakočević

AN INTRODUCTION INTO THE THEORY OF SEQUENCE SPACES AND MEASURES OF NONCOMPACTNESS

CONTENTS

Preface

This paper gives a self-contained, comprehensive treatment of the theories of sequence spaces and measures of noncompactness, as well as a survey of some of the authors' recent research results in these fields. It contains subjects of lectures at the universities of Niš, Novi Sad and Belgrade, Giessen (Germany) and Irbid (Jordan), and talks given by the authors at various international conferences in the Czech Republic, Germany, Hungary, India, Italy, Jordan, Poland and Yugoslavia.

For the first time, methods from the fields of summability, in particular of sequence spaces and matrix transformations on one hand, and of measures of noncompactness on the other are successfully linked on a large scale to obtain necessary and sufficient conditions for matrix maps between certain sequence spaces of a general class to be compact operators. The original idea for research in this field dates back to the classical paper of L. W. Cohen and N. Dunford [10]. In this paper they gave necessary and sufficient conditions for matrix transformations from l_1 to l_p, l_p to c_0 and l_p to l_1, and found the norm of these transformations. Furthermore they established necessary and sufficient conditions for these operators to be compact. Although the concept of measure of noncompactness is not explicitly mentioned in their paper, their studies and techniques are very closely related to our research.

These notes are addressed to both experts and nonexperts with an interest in getting acquainted with sequence spaces and measures of noncompactness. They could also be used as a guideline for research and teaching at graduate and post graduate levels.

Sections 1 and 2 deal with the necessary basic concepts and results of the theory of FK spaces, their duals, matrix transformations and measures of noncompactness. Although most of the results presented are well known and can be found, for instance, in [105, 108, 107, 91] concerning Section 1, and in [1, 7, 86] concerning Section 2, proofs are given in almost all cases to make the paper self-contained.

In Section 3, the authors give their own research results and apply the methods and results of the first two chapters to characterize matrix transformations between sequence spaces closely related to various concepts of summability, such as ordinary and strong summability, spaces of difference sequences of higher order and of strongly convergent and bounded sequences. Finally they apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for a matrix map between these spaces to be a compact operator.

Although there is a very wide range of problems for further research related to the presented topics, only the most closely related and possibly interesting will be mentioned here. We hope that the results presented here will be a useful introduction to further studies in the following fields. Concerning measures of noncompactness it seems most interesting to study when operators between sequence spaces are strictly singular [45, 104], Fredholm or semi-Fredholm [16, 17, 19, 21]. Results in this direction could also be applied in the perturbation theory of Fredholm and semi-Fredholm operators. The authors' research is also connected with A. Wilansky's results [106], and will be in this direction. Concerning the theory

(P.5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ (*continuity of multiplication by scalars*).

If p is a paranorm on X, then (X, p), or X for short, is called a *paranormed space*. A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total*. For any two paranorms p and q, p is called *stronger* than q if, whenever (x_n) is a sequence such that $p(x_n) \to 0$ $(n \to \infty)$, then also $q(x_n) \to 0$ $(n \to \infty)$. If p is stronger than q, then q is said to be *weaker* than p. If p is stronger than q and q is stronger than p, then p and q are called *equivalent*. If p is stronger than q, but p and q are not equivalent, then p is said to be *strictly stronger* than q, and q is called *strictly weaker* than p.

It is easy to see that *every totally paranormed space is a linear metric space*. The converse is also true. *The metric of any linear metric space is given by some total paranorm* (cf. [105, Theorem 10.4.2, p. 183]). A sequence of paranorms may be used to define a paranorm.

Theorem 1.2. *Let $(p_k)_{k=1}^{\infty}$ be a sequence of paranorms on a linear space X. We define the so-called Fréchet combination of (p_k) by*

$$p(x) = \sum_{n=0}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1 + p_k(x)} \quad \text{for all } x \in X. \tag{1.1}$$

Then:

(a) *p is a paranorm on X and satisfies*

$$p(x_n) \to 0 \ (n \to \infty) \text{ if and only if } p_k(x_n) \to 0 \ (n \to \infty) \quad \text{for each } k; \tag{1.2}$$

(b) *p is the weakest paranorm which is stronger than every p_k;*

(c) *p is total if and only if every p_k is total.*

Proof. (a) Conditions (P.1), (P.2) and (P.3) in Definition 1.1 are obvious, since every p_k is a paranorm. To prove condition (P.4), we observe that, for all reals a and b with $0 \le a \le b$, we have $a(1 + b) = a + ab \le b + ab = b(1 + a)$ and so $a/(1 + a) \le b/(1 + b)$. Applying this with $0 \le a = p_k(x + y) \le p_k(a) + p_k(y) = b$, we conclude

$$\frac{p_k(x+y)}{1 + p_k(x+y)} \le \frac{p_k(x) + p_k(y)}{1 + p_k(x) + p_k(y)} \le \frac{p_k(x)}{1 + p_k(x)} + \frac{p_k(y)}{1 + p_k(y)} \quad \text{for all } k,$$

and from this $p(x + y) \le p(x) + p(y)$. To prove the statement in (1.2), we first assume $p_k(x_n) \to 0$ $(n \to \infty)$ for each k. Since

$$0 \le \frac{p_k(x_n)}{1 + p_k(x_n)} \le 1 \quad \text{for all } n, k$$

and $\sum_{k=1}^{\infty} 1/2^k$ converges, the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x_n)}{1+p_k(x_n)}$$

converges uniformly in n. Thus $\lim_{n\to\infty} p(x_n) = 0$.

Conversely, we assume $p(x_n) \to 0$, $(n \to \infty)$ and fix k. Then

$$\frac{1}{2^k} \frac{p_k(x_n)}{1+p_k(x_n)} \leq p(x_n)$$

implies $p_k(x_n) \leq 2^k p(x_n) + 2^k p_k(x_n) p(x_n)$. Since $p(x_n) \to 0$ $(n \to \infty)$, it follows that $2^k p(x_n) < 1$ for all sufficiently large n, hence $p_k(x_n)(1 - 2^k p(x_n)) \leq 2^k p(x_n)$ for all sufficiently large n, and consequently

$$p_k(x_n) \leq \frac{2^k p(x_n)}{1 - 2^k p(x_n)} \quad \text{for all sufficiently large } n.$$

This implies $p_k(x_n) \to 0$ $(n \to \infty)$.

To prove condition (P.5), let $\lambda_n \to \lambda$ and $p(x_n - x) \to 0$ $(n \to \infty)$. By the statement in (1.2), $p_k(x_n - x) \to 0$ $(n \to \infty)$ for all k, and, since every p_k is a paranorm, this implies $p_k(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ for all k. Now it follows from the statement in (1.2) that $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$.

(b) Let q be a paranorm which is stronger than every p_k. Then $q(x_n) \to 0$ $(n \to \infty)$ implies $p_k(x_n) \to 0$ $(n \to \infty)$ for all k, and, by the statement in (1.2), $p(x_n) \to 0$ $(n \to \infty)$. Thus q is stronger than p.

(c) Part (c) is trivial. □

Let us recall that a subset S of a linear space X is said to be ***absorbing*** if for each $x \in X$ there is $\varepsilon > 0$ such that $\lambda x \in S$ for all scalars λ with $|\lambda| \leq \varepsilon$.

Remark 1.3. Let (X, p) be a paranormed space. Then the open neighbourhoods of 0, $N_r(0) = \{x \in X : p(x) < r\}$, are absorbing for all $r > 0$.

Proof. We assume that is $N_r(0)$ is not absorbing for some $r > 0$. Then there are $x \in X$ and a sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars with $\lambda_n \to 0$ $(n \to \infty)$ and $\lambda_n x \notin N_r(0)$ for all $n = 0, 1, \ldots$. But this means $p(\lambda_n x) \geq r$ for all n contradicting condition (P.5) in Definition 1.1. □

Example 1.4. The set $\mathbb{C}$ of complex numbers with the usual algebraic operations and $p = |\cdot|$, the modulus, is a totally paranormed space. If we put $d(z, w) = |z - w|$ for all $z, w \in \mathbb{C}$, then $(\mathbb{C}, d)$ is a Fréchet space.

By ω, we denote the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$ which becomes a linear space with $x + y = (x_k + y_k)_{k=0}^{\infty}$ and $\lambda x = (\lambda x_k)_{k=0}^{\infty}$ or all $x, y \in \omega$ and $\lambda \in \mathbb{C}$. As an immediate consequence of Theorem 1.2 and Example 1.4, we obtain

Theorem 1.5. *The set ω is a Fréchet space with respect to the metric d defined by*

$$d(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \quad \text{for all } x, y \in \omega. \tag{1.3}$$

Furthermore convergence in (ω, d) and coordinatewise convergence are equivalent, that is $x^{(n)} \to x$ $(n \to \infty)$ in (ω, d) if and only if $x_k^{(n)} \to x_k$ $(n \to \infty)$ for every k.

Now we introduce the concept of a *Schauder basis*. For further studies on bases we refer the reader to [46, 74].

Definition 1.6. A *Schauder basis* of a linear metric space X is a sequence (b_n) of vectors such that for each vector $x \in P$ there is a unique sequence (λ_n) of scalars with $\sum_{n=1}^{\infty} \lambda_n b_n = x$, that is $\lim_{m\to\infty} \sum_{n=1}^{m} \lambda_n b_n = x$.

For finite dimensional spaces, the concepts of Schauder and algebraic bases coincide. In most cases of interest, however, the concepts differ. Every linear space has an algebraic basis. But there are linear metric spaces without a Schauder basis, as we shall see later in this subsection.

Example 1.7. For each $n = 0, 1, \dots$, let $e^{(n)}$ be the sequence with $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. Then $(e^{(n)})_{n=0}^{\infty}$ is a Schauder basis of ω. More precisely, every sequence $x = (x_k)_{k=0}^{\infty} \in \omega$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$ that is $\lim_{m\to\infty} x^{[m]} = x$ for $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$, the *$m$-section of x.*

A metric space (X, d) is called *separable* if it has a *countable dense set.* That means there is a countable set $A \subset X$ such that for all $\varepsilon > 0$ and for all $x \in X$ there is an element $a \in A$ with $d(x, a) < \varepsilon$.

Theorem 1.8. *Every complex linear metric space X with Schauder basis is separable.*

Proof. Let (b_n) be a Schauder basis of X. For each $m \in \mathbb{N}$, we put

$$A_m = \left\{ \sum_{n=1}^{m} \rho_n b_n : \rho_n \in \mathbb{Q} + i\mathbb{Q} \ (n = 1, 2, \dots, m) \right\} \text{ and } A = \bigcup_{m=1}^{\infty} A_m.$$

Then A is a countable set in X and it is easy to see that A is dense in X. □

Example 1.9. The set $l_\infty = \{x \in \omega : \sup_k |x_k| < \infty\}$ of all bounded sequences is a Banach space with $\|x\|_\infty = \sup_k |x_k|$ $(x \in l_\infty)$ which has no Schauder basis.

Proof. The proof that $(l_\infty, \|\cdot\|_\infty)$ is a Banach space is standard and left to the reader. To show that l_∞ has no Schauder basis, we show that l_∞ is not separable and apply Theorem 1.8. We assume that l_∞ is separable. Then there is a countable dense set $A = \{a_n : n = 0, 1, \dots\} \subset l_\infty$. For every n, let $U_n = N_{1/3}(a_n) = \{x \in l_\infty : \|x - a_n\|_\infty < 1/3\}$. Since $A \subset l_\infty$ is dense, $l_\infty \subset \bigcup_{n=0}^{\infty} U_n$. The set

$$B = \{0, 1\}^{\mathbb{N}_0} = \{x \in \omega : x_k \in \{0, 1\} \text{ for all } k = 0, 1, \dots\} \subset l_\infty$$

is uncountable. Therefore there must be a set U_m which contains at least two distinct sequences x and x' of B. Then

$$\|x - x'\|_\infty \geq 1 \quad \text{and} \quad \|x - x'\|_\infty \leq \|x - a_m\|_\infty + \|a_m - x'\|_\infty < 2/3,$$

a contradiction. Therefore l_∞ cannot be separable. □

At the end of this subsection we study the so-called *classical sequence spaces*

$$\begin{aligned} l_\infty &= \{x \in \omega : \sup_k |x_k| < \infty\}, \\ c &= \{x \in \omega : \lim_{k\to\infty} x_k = l \text{ for some } l \in \mathbb{C}\}, \\ c_0 &= \{x \in \omega : \lim_{k\to\infty} x_k = 0\} \end{aligned}$$

of all *bounded*, *convergent* and *null sequences*, and

$$l_p = \left\{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\right\} \text{ for } 1 \leq p < \infty.$$

The following result gives the algebraic and topological properties of the sets l_∞, c, c_0 and l_p.

Theorem 1.10. (a) *Each of the sets l_∞, c_0 and c is a Banach space with $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \sup_k |x_k|$. Moreover $|x_k| \leq \|x\|_\infty$ for all $k = 0, 1, \dots$.*

(b) *The sets l_p are Banach spaces for $1 \leq p < \infty$ with $\|\cdot\|_p$ defined by $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$. Moreover $|x_k| \leq \|x\|_p$ for all $k = 0, 1, \dots$.*

(c) *The sequence $(e^{(n)})_{n=0}^\infty$ is a Schauder basis for each of the spaces c_0 and l_p for $1 \leq p < \infty$. More precisely, every sequence $x = (x_n)_{n=0}^\infty$ in any of these spaces has a unique representation $x = \sum_{n=0}^\infty x_n e^{(n)}$.*

(d) *Let e be the sequence with $e_k = 1$ for all $k = 0, 1, \dots$. We put $b^{(0)} = e$ and $b^{(n)} = e^{(n-1)}$ for $n = 1, 2, \dots$. Then the sequence $(b^{(n)})_{n=0}^\infty$ is a Schauder basis for c. More precisely, every sequence $x = (x_n)_{n=0}^\infty \in c$ has a unique representation $x = le + \sum_{n=0}^\infty (x_n - l)e^{(n)}$ where $l = l(x) = \lim_{n\to\infty} x_n$.*

(e) *The space l_∞ has no Schauder basis.*

Proof. Part (e) is Example 1.9. Parts (a) to (d) are standard and therefore left to the reader. (The triangle inequality for $\|\cdot\|_p$ follows by Minkowski's inequality (see appendix A.4.2).) □

1.2. Introduction into the theory of FK spaces. In this subsection, we shall give an introduction into the general theory of *FK spaces*. It is the most powerful tool for the solution of problems of various kinds in summability, in particular in the *characterization of matrix transformations between sequence spaces*. Most of the results of this subsection can be found in [108].

We saw in Theorem 1.5 that the set ω is a Fréchet space with the metric d defined in (1.3) and that convergence in ω and coordinatewise convergence are equivalent. Furthermore, by Theorem 1.10, the spaces l_∞, c_0, c and l_p $(1 \leq p < \infty)$ are Banach spaces with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$, and convergence in any one of these spaces implies coordinatewise convergence by the inequalities in Theorem 1.10 parts (a) and (b). Thus the metric generated by these norms is stronger than the metric of ω on them.

Definition 1.11. A Fréchet sequence space (X, d_X) is said to be an *FK space* if its metric d_X is stronger than the metric $d|_X$ of ω on X. A *BK space* is an FK space which is a Banach space.

Remark 1.12. (a) Some authors include *local convexity* in the definition of FK spaces. But much of the theory can be developed without local convexity.

(b) By definition, an FK space X is continuously embedded in ω, that is the *inclusion map* $\iota : (X, d_X) \mapsto (\omega, d)$ defined by $\iota(x) = x$ $(x \in X)$ is continuous. An FK space X is a Fréchet sequence space with continuous *coordinates* $P_k : X \mapsto \mathbb{C}$ defined by $P_k(x) = x_k$ $(k = 0, 1, \dots)$ for all $x \in X$.

Example 1.13. The space ω is an FK space with its natural metric d. The spaces l_∞, c, c_0 and l_p $(1 \leq p < \infty)$ are BK spaces with their natural norms.

Theorem 1.14. *Let (X, d_X) be a Fréchet space, (Y, d_Y) an FK space and $f : X \mapsto Y$ a linear map. Then $f : (X, d_X) \mapsto (Y, d|_Y)$ is continuous if and only if $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous.*

Proof. First we assume that $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous. Since Y is an FK space its metric d_Y is stronger than the metric $d|_Y$ of ω on Y. So $f : (X, d_X) \mapsto (Y, d|_Y)$ is continuous.

Conversely we assume that $f : (X, d_X) \mapsto (Y, d|_Y)$ is continuous. Since $(Y, d|_Y)$ is a Hausdorff space and f is continuous, the *graph of* f, $\text{graph}(f) = \{(x, f(x)) : x \in X\}$, is a closed set in $(X, d_X) \times (Y, d|_Y)$ by the *closed graph lemma* (see appendix A.4.4), hence a closed set in $(X, d_X) \times (Y, d_Y)$, since the FK metric d_Y is stronger than $d|_Y$. By the *closed graph theorem* (see appendix A.4.5), the map $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous. □

Corollary 1.15. *Let X be a Fréchet space, Y an FK space, $f : X \mapsto Y$ a linear map and P_n the n-th co-ordinate, that is $P_n(y) = y_n$ $(y \in Y)$ for all $n = 0, 1, \dots$. If each map $P_n \circ f : X \mapsto \mathbb{C}$ is continuous, so is $f : X \mapsto Y$.*

Proof. Since $P_n \circ f : X \mapsto \mathbb{C}$ is continuous for each n, the map $f : X \mapsto \omega$ is continuous by the equivalence of coordinatewise convergence and convergence in ω. By Theorem 1.14, $f : X \mapsto Y$ is continuous. □

By ϕ we denote the set of all *finite sequences* that is of sequences that terminate in zeros.

We shall frequently make use of the following result.

Remark 1.16. Let $X \supset \phi$ be an FK space and $a \in \omega$. If the series $\sum_{k=0}^\infty a_k x_k$ converges for each $x \in X$, then the linear functional $f_a : X \mapsto \mathbb{C}$ defined by

$$f_a(x) = \sum_{k=0}^\infty a_k x_k \quad \text{for all } x \in X$$

is continuous.

Proof. For each $n \in \mathbb{N}_0$, we define the linear functional $f_{a,n} : X \mapsto \mathbb{C}$ by $f_{a,n}(x) = \sum_{k=0}^{n} a_k x_k$ for all $x \in X$. Since X is an FK space, the coordinates $P_k : X \mapsto \mathbb{C}$ are continuous on X for all $k = 0, 1, \dots$, and so are the functionals $f_{a,n} = \sum_{k=0}^{n} a_k P_k$ $(n = 0, 1, \dots)$. For each $x \in X$, $f_a(x) = \lim_{n\to\infty} f_{a,n}(x)$ exists, and so $f_a : X \mapsto \mathbb{C}$ is continuous by the *Banach-Steinhaus theorem* (see appendix A.4.6). □

For the next result we shall need some notations.

Given any two subsets X and Y of ω and any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers, we shall write $A_n = (a_{n,k})_{k=0}^{\infty}$ for the sequence in the n-th row of A,

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (x \in X) \text{ for all } n = 0, 1, \dots$$

(provided the series converge) and

$$A(x) = (A_n(x))_{n=0}^{\infty}.$$

Furthermore let (X, Y) be the class of all matrices A that map X into Y, that is for which the series $A_n(x)$ converge for all $x \in X$ and for all n, and $A(x) \in Y$ for all $x \in X$.

Theorem 1.17. *Any matrix map between FK spaces is continuous.*

Proof. Let X and Y be FK spaces, $A \in (X, Y)$ and the map $f_A : X \mapsto Y$ be defined by $f_A(x) = A(x)$ for all $x \in X$. Since the maps $P_n \circ f_A : X \mapsto \mathbb{C}$ are continuous for all $n \in \mathbb{N}_0$ by Remark 1.16, the linear map f_A is continuous by Corollary 1.15. □

Definition 1.18. An FK space $X \supset \phi$ has *AK* if, for every sequence $x = (x_k)_{k=0}^{\infty} \in X$,

$$x = \sum_{k=0}^{\infty} x_k e^{(k)}, \quad \text{that is} \quad x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x \ (m \to \infty),$$

and X has *AD* if ϕ is dense in X. If an FK space has AK or AD we also say that it is an *AK* or *AD space*.

Remark 1.19. Every AK space has AD. The converse is not true in general.

Proof. The first part is trivial, and the second part can be found in [108, Example 5.2.5, p. 78]. □

Example 1.20. The spaces ω, c_0 and l_p $(1 \le p < \infty)$ all have AK by Example 1.7 and Theorem 1.10.

The FK metric of an FK space will turn out to be unique.

Theorem 1.21. *Let X and Y be FK spaces and $X \subset Y$. Then the metric d_X on X is stronger than the metric $d_Y|_X$ of Y on X. The metrics are equivalent if and only if X is a closed subspace of Y. In particular, the metric of an FK space is unique, this means there is at most one way to make a linear subspace of ω into an FK space.*

Proof. Let $\iota : (X, d_X) \mapsto (Y, d_Y)$ be the inclusion map. Since X is an FK space, $\iota : (X, d_X) \mapsto (Y, d|_Y)$ is continuous, and so is $\iota : (X, d_X) \mapsto (Y, d_Y)$ by Theorem 1.14. Thus d_X is stronger than $d_Y|_X$. The uniqueness of an FK space is shown in exactly the same way. Let X be closed in Y, then X becomes an FK space with $d_Y|_X$, and the uniqueness of an FK metric implies that d_X and $d_Y|_X$ are equivalent.

Conversely, if d_X and $d_Y|_X$ are equivalent, then X is a complete subspace of Y, hence a closed subspace of Y. □

Example 1.22. The BK spaces c_0 and c are closed subspaces of l_∞. Thus the BK norms on c_0, c and l_∞ must be the same. The BK space l_1 is a subspace of l_∞ which is not closed in l_∞. Thus its BK norm $\|\cdot\|_1$ is strictly stronger than the BK norm $\|\cdot\|_\infty$ on l_∞.

1.3. Matrix transformations into l_∞, c and c_0. In this subsection we shall apply the results of Subsection 1.2 to characterize classes (X, Y) where X is any FK space and Y is any of the spaces l_∞, c and c_0. We shall need some more notations.

If $X \subset \omega$ is a linear metric space with respect to d_X and $a, x_0 \in X$, then we shall write

$$S_\delta[x_0] = S_{X,\delta}[x_0] = \{x \in X : d_X(x, x_0) \le \delta\} \ (\delta > 0)$$
$$\|a\|_D^* = \|a\|_{X,D}^* = \sup\left\{\left|\sum_{k=0}^\infty a_k x_k\right| : x \in S_{1/D}[0]\right\} \ (D > 0)$$

provided the expression on the right exists and is finite. By Remark 1.6, this is the case whenever X is an FK space and the series $\sum_{k=0}^\infty a_k x_k$ converge for all $x \in X$. If X is a BK space we write

$$\|a\|^* = \|a\|_X^* = \sup\left\{\left|\sum_{k=0}^\infty a_k x_k\right| : \|x\| = 1\right\}.$$

Let X and Y be two Fréchet spaces. By $B(X, Y)$ we denote the set of all continuous linear operators $L : X \mapsto Y$, and we write $X' = B(X, \mathbb{C})$ for the set of all continuous linear functionals on X, the set X' is called the *continuous dual of* X. If X and Y are normed spaces and $L \in B(X, Y)$, then we write

$$\|L\| = \sup\{\|L(x)\| : \|x\| = 1\} \quad \text{for all } L \in B(X, Y). \tag{1.4}$$

for the *operator norm of* L; furthermore we write X^* for X' with the norm in (1.4), that is $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$ for all $f \in X'$.

Let A be an infinite matrix, D a positive real and X an FK space. Then we put

$$M^*_{A,D}(X, l_\infty) = \sup_n \|A_n\|^*_D$$

and, if X is a BK space, then we write

$$M^*_A(X, l_\infty) = \sup_n \|A_n\|^*.$$

Theorem 1.23. *Let X and Y be FK spaces.*

(a) *Then $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$ where $L_A(x) = A(x)$ for all $x \in X$.*

(b) *Then $A \in (X, l_\infty)$ if and only if*

$$\|A\|^*_D = M^*_{A,D}(X, l_\infty) < \infty \quad \textit{for some } D > 0. \tag{1.5}$$

If X is a BK space and $A \in (X, l_\infty)$, then $\|A\|^* = M^*_A(X, l_\infty) = \|L_A\| < \infty$.

(c) *If $(b^k)_{k=0}^\infty$ is a Schauder basis for X, and Y_1 a closed FK space in Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A(b^{(k)}) \in Y_1$ for all $k = 0, 1, \ldots$.*

Proof. Part (a) is Theorem 1.17. (b) First we assume that condition (1.5) holds. Then, for all $x \in S_{1/D}[0]$, the series $A_n(x)$ $(n = 0, 1, \ldots)$ converge and $A(x) \in l_\infty$. Since the set $S_{1/D}[0]$ is absorbing by Remark 1.3, we conclude that $A_n(x)$ converges for each $x \in X$ and $A(x) \in l_\infty$ for all $x \in X$, hence $A \in (X, l_\infty)$.

Conversely, we assume $A \in (X, l_\infty)$. Then L_A is continuous by part (a). Hence there exist a neighbourhood N of 0 in X and a real $D > 0$ such that $S_{1/D}[0] \subset N$ and $\|L_A(x)\| < 1$ for all $x \in N$. This implies condition (1.5). If X is a BK space, then $L_A \in B(X, Y)$ implies

$$\|A(x)\|_\infty = \sup_n |A_n(x)| = \|L_A(x)\|_\infty \le \|L_A\| \quad \text{for all } x \in X \text{ with } \|x\| = 1.$$

Thus $|A_n(x)| \le \|L_A$ for all n and for all $x \in X$ with $\|x\| = 1$, and, by the definition of the norm $\|\cdot\|^*$,

$$\|A\|^* = \sup_n \|A_n\|^* \le \|L_A\|. \tag{1.6}$$

Further, given $\varepsilon > 0$, there is $x \in X$ with $\|x\| = 1$ such that $\|A(x)\|_\infty \ge \|L_A\| - \varepsilon/2$, and there is $n(x) \in \mathbb{N}_0$ with $|A_{n(x)}(x)| > \|A(x)\|_\infty - \varepsilon/2$, consequently $|A_{n(x)}(x)| \ge \|L_A\| - \varepsilon$. Therefore $\|A\|^* = \sup_n \|A_n\|^* \ge \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|A\|^* \ge \|L_A\|$, and, with (1.6), we have $\|A\|^* = \|L_A\|$.

(c) The necessity of the conditions for $A \in (X, Y_1)$ is trivial.

Conversely, if $A \in (A, Y)$, then $L_A \in B(X, Y)$. Since Y_1 is a closed subspace of Y, the FK metrics of Y_1 and Y are the same by Theorem 1.21. Consequently, if S is any subset in Y_1, then, for its closures $\text{clos}_{Y_1}(S)$ and $\text{clos}_{Y|_{Y_1}}(S)$ with respect to the metrics d_{Y_1} and $d_{Y|_{Y_1}}$, we have

$$\text{clos}_{Y_1}(S) = \text{clos}_{Y|_{Y_1}}(S). \tag{1.7}$$

Let $x \in X$ and $SB = \left\{\sum_{k=0}^{m} \lambda_k b^{(k)} : m \in \mathbb{N}_0,\ \lambda_k \in \mathbb{C}\ (k = 0, 1, \dots)\right\}$ denote the *span of* $\{b^{(k)} : k = 0, 1, \dots\}$. Since $L_A(b^{(k)}) \in Y_1$ for all $k = 0, 1, ..$ and the metrics d_{Y_1} and $d_{Y|_{Y_1}}$ are equivalent, the map $L_A\big|_{SB} : (X, d_X) \mapsto (Y_1, d_{Y_1})$ is continuous. Further, since $(b^k)_{k=0}^{\infty}$ is a basis of X, we have $\overline{SB} = X$. Therefore, by (1.7) and the continuity of $L_A\big|_{SB}$, we have

$$L_A(X) = L_A(\overline{SB}) = \operatorname{clos}_{Y_1}\left(L_A\big|_{SB}(SB)\right) = \operatorname{clos}_{Y|_{Y_1}}\left(L_A\big|_{SB}(SB)\right)$$
$$\subset \operatorname{clos}_{Y|_{Y_1}}(Y_1) = Y_1$$

Thus $A(x) \in Y$ for all $x \in X$. □

1.4. The α– and β–duals of sets of sequences. In this subsection we shall study the so-called α–, β– and *continuous dual spaces* of sets of sequences. The first two kinds of dual spaces naturally arise in the study of absolute and ordinary convergence of sequences from a subset of ω.

Furthermore the conditions given in Subsection 1.3 for an infinite matrix A to be in the classes (X, l_∞), (X, c) and (X, c_0) for arbitrary FK spaces X involved the norm of the operator L_A defined by $L_A(x) = A(x)$. Since $A \in (X, Y)$ can only hold if $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ converges for all $x \in X$ and for all $n = 0, 1, \dots$, it is essential to know the set of all sequences $a \in \omega$ for which $\sum_{k=0}^{\infty} a_k x_k$ converges for all $x \in X$, the so-called *β-dual of* X. Finally, if X and Y are given FK spaces, then we intend to replace the operator norm in the conditions for $A \in (X, Y)$ by conditions for the entries of the matrix A. In many cases this can be achieved by replacing the operator norm by the natural norm on the β–dual of X.

The α– and β–duals are special cases of the so-called *multiplier spaces*.

Definition 1.24. Let X and Y be subsets of ω.

(a) For all $z \in \omega$, we write $z^{-1} * Y = \left\{x \in \omega : xz = (x_k z_k)_{k=0}^{\infty} \in Y\right\}$. The set $Z = M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \left\{a \in \omega : ax \in Y \text{ for all } x \in X\right\}$ is called the *multiplier space of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

(b) By cs and bs, we denote the set of all *convergent* and *bounded series*, respectively, that is $cs = \{x \in \omega : \sum_{k=0}^{\infty} x_k \text{ converges}\}$ and $bs = \{x \in \omega : (\sum_{k=0}^{n} x_k)_{n=0}^{\infty} \in l_\infty\}$, and we define the norm $\|\cdot\|_{bs}$ on cs and bs by $\|x\|_{bs} = \sup_n |\sum_{k=0}^{n} x_k|$. In the special case where $Y = l_1$ or $Y = cs$, the multiplier spaces $X^\alpha = M(X, l_1)$ and $X^\beta = M(X, cs)$ are called the $\boldsymbol{\alpha}$– or *Köthe–Toeplitz* and *$\boldsymbol{\beta}$–duals of* $\boldsymbol{X}$. If $\dagger$ denotes either of the symbols α or β, then $X \subset \omega$ is said to be $\dagger$*-perfect* if $X^{\dagger\dagger} = (X^\dagger)^\dagger = X$.

Lemma 1.25. *Let $X, Y, Z \subset \omega$ and $\{X_\delta : \delta \in A\}$ be any collection of subsets of ω. Then:*

(*i*) $X \subset M(M(X, Y), Y)$

(*ii*) $X \subset Z$ *implies* $M(Z, Y) \subset M(X, Y)$

(*iii*) $M(X, Y) = M(M(M(X, Y), Y), Y)$

(*iv*) $M\left(\bigcup_{\delta \in A} X_\delta, Y\right) = \bigcap_{\delta \in A} M(X_\delta, Y)$.

Proof. (i) If $x \in X$, then $ax \in Y$ for all $a \in M(X, Y)$, and consequently $x \in M(M(X, Y), Y)$.

(ii) Let $X \subset Z$. If $a \in M(Z,Y)$, then $ax \in Y$ for all $x \in Z$, hence $ax \in Y$ for all $x \in X$, since $X \subset Z$. Thus $a \in M(X,Y)$.

(iii) We apply (i) with X replaced by $M(X,Y)$ to obtain

$$M(X,Y) \subset M(M(M(X,Y),Y),Y).$$

Conversely, by (i), $X \subset M(M(X,Y),Y)$, and so (ii) with $Z = M(M(X,Y),Y)$ yields $M(M(M(X,Y),Y),Y) \subset M(X,Y)$.

(iv) First $X_\delta \subset \bigcup_{\delta \in A} X_\delta$ for all $\delta \in A$ implies $M\left(\bigcup_{\delta \in A} X_\delta, Y\right) \subset \bigcap_{\delta \in A} M(X_\delta, Y)$ by part (i).

Conversely, if $a \in \bigcap_\delta M(X_\delta, Y)$, then $a \in M(X_\delta, Y)$ for all $\delta \in A$, and so we have $ax \in Y$ for all $\delta \in A$ and for all $x \in X_\delta$. This implies $ax \in Y$ for all $x \in \bigcup_{\delta \in A} X_\delta$, hence $a \in M(\bigcup_{\delta \in A} X_\delta, Y)$. Thus $\bigcap_{\delta \in A} M(X_\delta, Y) \subset M(\bigcup_{\delta \in A} X_\delta, Y)$.
□

As an immediate consequence of Lemma 1.25 we obtain

Corollary 1.26. *Let $X, Y \subset \omega$ and $\{X_\delta : \delta \in A\}$ be a collection of subsets of ω. If $\dagger$ denotes either of the symbols α or β, then*

(i)	$X \subset X^{\dagger\dagger}$	*(ii)*	$X \subset Y$ *implies* $Y^\dagger \subset X^\dagger$
(iii)	$X^\dagger = X^{\dagger\dagger\dagger}$	*(iv)*	$\left(\bigcup_{\delta \in A} X_\delta\right)^\dagger = \bigcap_{\delta \in A} X_\delta^\dagger$.

A subset X of ω is said to be *normal* if $x \in X$ and $|\tilde{x}_k| \leq |x_k|$ $(k = 0, 1, \dots)$ together imply $\tilde{x} \in X$.

Remark 1.27. Obviously $X^\alpha \subset X^\beta$ for arbitrary $X \subset \omega$. If X is a normal subset of ω, then $X^\alpha = X^\beta$.

Proof. The first part is obvious. For the second part, we have to show $X^\beta \subset X^\alpha$. Let $a \in X^\beta$ and $x \in X$ be given. We define the sequence y by $y_k = \mathrm{sgn}(x_k)|x_k|$ for $k = 0, 1, \dots$. Then obviously $|y_k| \leq |x_k|$ for all k, and consequently $y \in X$, since X is normal, and so $ax \in cs$. Further, by the definition of the sequence y, $ay = (|a_k||x_k|)_{k=0}^\infty = |ax| \in cs$, hence $ax \in l_1$. Since $x \in X$ was arbitrary, $a \in X^\alpha$. This shows $X^\beta \subset X^\alpha$. □

Example 1.28. We have:
(i) $M(c_0, c) = l_\infty$, (ii) $M(c, c) = c$, (iii) $M(l_\infty, c) = c_0$.

Proof. (i) If $a \in l_\infty$, then $ax \in c$ for all $x \in c_0$, and so $l_\infty \subset M(c_0, c)$.

Conversely we assume $a \notin l_\infty$. Then there is a subsequence $(a_{k_j})_{j=0}^\infty$ of the sequence a such that $|a_{k_j}| > j + 1$ for all $j = 0, 1, \dots$. We define the sequence x by

$$x_k = \begin{cases} (-1)^j / a_{k_j} & \text{for } k = k_j \\ 0 & \text{for } k \neq k_j \end{cases} \quad (j = 0, 1, \dots). \tag{1.8}$$

Then $x \in c_0$ and $a_{k_j} x_{k_j} = (-1)^j$ for all $j = 0, 1, \dots$, hence $ax \notin c$. This shows $M(c_0, c) \subset l_\infty$.

(ii) If $a \in c$, then $ax \in c$ for all $x \in c$, and so $c \subset M(c, c)$.

Conversely we assume $a \notin c$. Since $e \in c$ and $ae = a \notin c$, we have $a \notin M(c,c)$. This shows $M(c,c) \subset c$.

(iii) If $a \in c_0$, then $ax \in c$ for all $x \in l_\infty$, and so $c_0 \subset M(l_\infty, c)$.

Conversely we assume $a \notin c_0$. Then there are a real $b > 0$ and a subsequence $(a_{k_j})_{j=0}^\infty$ of the sequence a such that $|a_{k_j}| > b$ for all $j = 0, 1, \ldots$. We define the sequence x as in (1.8). Then $x \in l_\infty$ and $a_{k_j} x_{k_j} = (-1)^j$ for $j = 0, 1, \ldots$, hence $a \notin M(l_\infty, c)$. This shows $M(l_\infty, c) \subset c_0$. □

Now we shall give the $\alpha-$ and β–duals of the classical sequence spaces.

Theorem 1.29. *Let $\dagger$ denote either of the symbols α or β. Then*

(a) *$\omega^\dagger = \phi$ and $\phi^\dagger = \omega$.*

(b) *$l_1^\dagger = l_\infty$, $l_p^\dagger = l_q$ for $1 < p < \infty$ and $q = p/(p-1)$, and for all $a \in l_p^\beta$, $\|a\|_{l_1}^* = \|a\|_\infty$ and $\|a\|_{l_p}^* = \|a\|_q$ for $1 < p < \infty$.*

(c) *$c_0^\dagger = c^\dagger = l_\infty^\dagger = l_1$ and $\|a\|_{c_0}^* = \|a\|_c^* = \|a\|_{l_\infty}^* = \|a\|_1$ for all $a \in l_\infty^\beta$.*

The multiplier space of two BK spaces will turn out to be a BK space.

Theorem 1.30. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be BK spaces with $X \supset \phi$ and $Z = M(X,Y)$. Then Z is a BK space with $\|\cdot\|$ defined by*

$$\|z\| = \|z\|_X^* = \sup\{\|xz\|_Y : \|x\|_X = 1\} \qquad \text{for all } z \in Z.$$

Proof. It is well known that $B = B(X,Y)$ is a Banach space. Each $z \in Z$ defines a diagonal matrix map $\hat{z} : X \mapsto Y$ where $\hat{z}(x) = xz$ for all $x \in X$ which is continuous by Theorem 1.17. This embeds Z in B, for if $\hat{z} = 0$, then $\hat{z}(e^{(n)}) = (z_n)_{n=0}^\infty = 0 = z$. To see that the coordinates are continuous, we fix $n \in \mathbb{N}_0$ and put $u = 1/\|e^{(n)}\|_X$ and $v = \|e^{(n)}\|_Y$. Then $\|ue^{(n)}\|_X = 1$ and

$$uv|z_n| = u\|z_n e^{(n)}\|_Y = u\|e^{(n)} z\|_Y = \|(ue^{(n)})z\|_Y \le \|z\|_X^* = \|z\| \quad \text{for all } n.$$

It remains to show that Z is a closed subspace of B. Let $(\hat{z}^{(m)})_{m=0}^\infty$ be a sequence in B with $\hat{z}^{(m)} \to T \in B$. For each fixed $X \in X$, we obtain $\hat{z}^{(m)}(x) \to T(x) \in Y$ $(m \to \infty)$ and since Y is a BK space, this implies $(\hat{z}^{(m)}(x))_k \to (T(x))_k$, that is $z_k^{(m)} x_k \to (T(x))_k$ $(m \to \infty)$ for each fixed k. We put $x = e^{(k)}$. Then $z_k^{(m)} \to (T(e^{(k)}))_k = t_k$ $(m \to \infty)$, and so $x_k z_k^{(m)} = (T(x))_k$ $(m \to \infty)$ and $x_k z_k^{(m)} \to (T(x))_k$ $(m \to \infty)$. Therefore $T(x) = xt$ for all $x \in X$, and so $T = \hat{t}$. □

Corollary 1.31. *The $\alpha-$ and $\beta-$ duals of a BK space X are BK spaces with respect to $\|a\|_\alpha = \|a\|_{X,\alpha} = \sup\{\|ax\|_1 = \sum_{k=0}^\infty |a_k x_k| : \|x\| \le 1\}$ and $\|a\|_\beta = \|a\|_{X,\beta} = \sup\{\|ax\|_{bs} = \sup_n |\sum_{k=0}^n a_k x_k| : \|x\| \le 1\}$.*

Example 1.32. Let X be any of the spaces l_∞, c, c_0 and l_p for $1 \le p < \infty$. Then the norms $\|\cdot\|_{X^\beta}$, $\|\cdot\|_X^*$, $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{X,\beta}$ are equivalent on X^β.

Proof. The norm $\|\cdot\|_X^*$ and the natural norm $\|\cdot\|_{X^\beta}$ are equal on X^β by Theorem 1.29. Since each set X^β is a BK space with its natural norm, $\|\cdot\|_{X^\beta}$ and

$\|\cdot\|_{X,\beta}$ are equivalent by Corollary 1.31 and Theorem 1.21. Finally, since $X^\alpha = X^\beta$ for each set X, the norms $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{X,\beta}$ are equivalent by Corollary 1.31 and Theorem 1.21. □

The analogues of Theorem 1.30 and Corollary 1.31 do not hold for FK spaces in general.

Remark 1.33. The space ω is an FK space and $\omega^\alpha = \omega^\beta = \phi$ and ϕ has no Fréchet metric (cf. [108, 4.0.5, p. 51]).

1.5. The continuous duals of the classical sequence spaces. In this subsection we shall give the continuous duals of the spaces l_p for $1 \le p < \infty$, c and c_0.

There is a close relation between the β-dual and the continuous dual of an FK space which is very useful in the determination of the continuous duals of the spaces l_p, c and c_0.

Theorem 1.34. *Let X be a BK space and $X \supset \phi$. Then there is a linear one-to-one map $T : X^\beta \mapsto X'$; we denote this by $X^\beta \subset X'$. If X has AK, then T is onto.*

Proof. We define the map T on X^β as follows. For every $a \in X^\beta$, let $Ta : X \mapsto \mathbb{C}$ be defined by $(Ta)(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$. Since $a \in X^\beta$, the series $\sum_{k=0}^\infty a_k x_k$ converge for all $x \in X$, and obviously Ta is linear. Further, since X is an FK space, $Ta \in X'$ for ach $a \in X^\beta$. Therefore $T : X^\beta \mapsto X'$. Further it is easy to see that T is linear.

To show that T is one-to-one, we assume $a, b \in X^\beta$ with $Ta = Tb$. This means $(Ta)(x) = (Tb)(x)$ for all $x \in X$. Since $\phi \subset X$, we may choose $x = e^{(k)}$ for each $k \in \mathbb{N}_0$ and obtain $(Ta)(e^{(k)}) = a_k = b_k = (Tb)(e^{(k)})$ for $k = 0, 1, \ldots$, and so $a = b$.

Now we assume that X has AK and $f \in X'$. We put $a_n = f(e^{(n)})$ for $n = 0, 1, \ldots$. Let $x \in X$ be given. Then $x = \sum_{k=0}^\infty x_k e^{(k)}$, since X has AK, and $f \in X'$ implies $f(x) = \sum_{k=0}^\infty x_k f(e^{(k)}) = \sum_{k=0}^\infty a_k x_k = (Ta)(x)$. As $x \in X$ was arbitrary and the series converge, $a \in X^\beta$ and $f = Ta$. This shows that T is onto X'. □

Now we shall give the continuous duals of the classical sequence spaces.

Two linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are called *norm isomorphic* if there is an isomorphism $T : X \mapsto Y$ such that $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$; we shall write $X \simeq Y$.

Theorem 1.35. *We have:*

(a) *$l_p^* \simeq l_\infty$ for $0 < p \le 1$ and $l_p^* \simeq l_q$ for $1 < p < \infty$ where $q = p/(p-1)$;*

(b) *$c_0^* \simeq l_1$;*

(c) *$f \in c^*$ if and only if $f(x) = l\chi_f + \sum_{k=0}^\infty a_k x_k$ with $a \in l_1$ where $l = \lim_{k\to\infty} x_k$ and $\chi_f = f(e) - \sum_{k=0}^\infty a_k$. Furthermore $\|f\|^* = |\chi_f| + \|a\|_1$.*

It is worth mentioning that the continuous dual of l_∞ is not isomorphic to a sequence space (cf. [40, 31.1, pp. 427, 428] or [105, Example 6.4.8, pp. 93, 94]).

For further studies concerning multiplier spaces, some important special cases, f- and continuous duals, we refer the reader to [108, 91, 22, 23].

1.6. Matrix transformations between some classical sequence spaces. We now apply the results of the previous subsections to characterize certain classes of matrix transformations between some classical sequence spaces by giving necessary and sufficient conditions on the entries of a matrix to belong to the respective class.

Let A be an infinite matrix. We write $q = p/(p-1)$ for $1 < p < \infty$, $q = \infty$ for $p = 1$ and $q = 1$ for $p = \infty$, put

$$M_A(l_p, l_\infty) = \begin{cases} \|A\|_\infty = \sup_{n,k} |a_{nk}| & (p = 1) \\ \|A\|_q = \sup_n \left(\sum_{k=0}^\infty |a_{nk}|^q\right) & (1 < p \leq \infty) \end{cases}$$

and consider the conditions

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 0, 1, \dots), \tag{1.9}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^\infty a_{nk}\right) = 0, \tag{1.10}$$

$$\lim_{n\to\infty} a_{nk} = l_k \quad \text{for some } l_k \in \mathbb{C} \ (k = 0, 1, \dots) \tag{1.11}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^\infty a_{nk}\right) = l \quad \text{for some } l \in \mathbb{C}. \tag{1.12}$$

Theorem 1.36. *We have*

(a) $(c_0, l_\infty) = (c, l_\infty) = (l_\infty, l_\infty)$ *and* $A \in (l_\infty, l_\infty)$ *if and only if*

$$M_A(l_\infty, l_\infty) = \sup_n \left(\sum_{k=0}^\infty |a_{nk}|\right) < \infty, \tag{1.13}$$

(b) $A \in (c_0, c_0)$ *if and only if conditions* (1.13) *and* (1.9) *hold;*

(c) $A \in (c, c_0)$ *if and only if conditions* (1.13), (1.9) *and* (1.10) *hold;*

(d) $A \in (c_0, c)$ *if and only if conditions* (1.13) *and* (1.11) *hold;*

(e) $A \in (c, c)$ *if and only if conditions* (1.13), (1.11) *and* (1.12) *hold.*

Proof. (a) We have $A \in (l_\infty, l_\infty)$ if and only if condition (1.13) holds by Theorems 1.23 and 1.29.

Further, if condition (1.13) holds, then $A \in (l_\infty, l_\infty) \subset (c_0, c)$, since $c_0 \subset l_\infty$.

Conversely, let $A \in (c_0, l_\infty)$. Then $\sup_n \|A_n\|_{c_0}^* < \infty$ by Theorem 1.23 (b). Since the series $A_n(x)$ converge for all x and n, we have $f_{A_n} \in c_0^*$ for all n where $f_{A_n}(x) = \sum_{k=0}^\infty a_{nk}x_k$ for all $x \in c_0$, hence $|f_{A_n}(x)| \leq \|f_{A_n}\| = \|A_n\|_{c_0}^*$. We fix $n \in \mathbb{N}_0$. Let $m \in \mathbb{N}_0$ be arbitrary. We define the sequence $x^{[m,n]}$ by $x^{[m,n]} = \sum_{k=0}^m \operatorname{sgn}(a_nk)e^{(k)}$. Then $x^{[m,n]} \in c_0$, $\|x^{[m,n]}\|_\infty \leq 1$ and $|f_{A_n}(x^{[m,n]})| = \sum_{k=0}^m |a_{nk}| \leq \|A_n\|_{c_0}^*$. Since $m \in \mathbb{N}_0$ was arbitrary, $\|A_n\|_1 = \sum_{k=0}^\infty |a_{nk}| \leq \|A_n\|_{c_0}^*$ for all $n = 0, 1, \dots$. Therefore condition (1.13) must hold. Finally $c_0 \subset c \subset l_\infty$ and $(c_0, l_\infty) = (l_\infty, l_\infty)$ together imply $(c, l_\infty) = (l_\infty, l_\infty)$.

Parts (b) to (e) follow from part (a), Theorem 1.23 (c) and Theorem 1.10. □

Similarly, we obtain

Theorem 1.37. *Let $1 < p < \infty$. Then:*
(a) *$A \in (l_p, l_\infty)$ if and only if*

$$M(l_p, l_\infty) < \infty; \tag{1.14}$$

(b) *$A \in (l_p, c_0)$ if and only if conditions* (1.14) *and* (1.9) *hold;*
(c) *$A \in (l_p, c)$ if and only if conditions* (1.14) *and* (1.11) *hold.*

2. Measures of concompactness

In Section 1 we developed and applied parts of the FK space theory to give necessary and sufficient conditions for $A \in (X, Y)$ for given sequence spaces. The most important result was that matrix transformations between FK spaces are continuous. It is quite natural to find conditions for a matrix map between FK spaces to define a compact operator. This can be achieved by applying the *Hausdorff measure of noncompactness*. The first *measure of noncompactness*, the function α, was defined and studied by Kuratowski [41] in 1930. It is surprising that later in 1955 Darbo [12] was the first who continued to use the function α. Darbo proved that if T is a continuous self-mapping of a nonempty, bounded, closed and convex subset C of a Banach space X such that $\alpha(T(Q)) \leq k\alpha(Q)$ for all $Q \subset C$, where $k \in (0,1)$ is a constant, then T has at least one fixed point in the set C. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and it includes the existence part of Banach's fixed point theorem.

Other measures were introduced by Goldenstein, Gohberg and Markus (the *ball measures of noncompactness, Hausdorff measure of noncompactness*) [19] in 1957 (later studied by Goldenstein and Markus [20] in 1968), Istrăţesku [30] in 1972 and others. Apparently Goldenstein, Gohberg and Markus were not aware of the work of Kuratowski and Darbo. It is surprising that Darbo's theorem was almost never noticed and applied, not till in the seventies mathematicians working in operator theory, functional analysis and differential equations begun to apply Darbo's theorem and developed the theory connected with measures of noncompactness.

The use of these measures is discussed for example in the monographs [1, 6, 7, 24, 25, 28, 31, 42, 86, 99, 100], Ph. D. theses [2, 4, 75, 77, 83, 102] and expository papers [47, 93, 104]. We refer the reader to these works with references given there.

2.1. Introduction. Let us recall some definitions and results which are probably well known. If M and S are subsets of a metric space (X, d) and $\epsilon > 0$, then the set S is called *ϵ-net* of M if for any $x \in M$ there exists $s \in S$, such that $d(x, s) < \epsilon$. If the set S is finite, then the ϵ-net S of M is called *finite ϵ-net*. The set M is said to be *totally bounded* if it has a finite ϵ-net for every $\epsilon > 0$. It is well known, that a subset M of a metric space X is compact if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of that subsequence is in M. The set M is said to be *relatively compact* if the closure $\overline{M}$ of M is a compact set. If the set M is relatively compact, then M is totally bounded. If the metric space (X, d) is complete, then the set M is relatively compact if and only if it is totally bounded. It is easy to prove that a subset M of a metric space X is relatively compact if and only if every sequence (x_n) in M has a convergent subsequence; in that case the limit of that subsequence need not be in M.

If $x \in X$ and $r > 0$, then the open ball with centre at x and radius r is denoted by $B(x,r)$, $B(x,r) = \{y \in X : d(x,y) < r\}$. If X is a normed space, then we denote by B_X the closed unit ball in X and by S_X the unit sphere in X. Let $\mathcal{M}_X$ (or simply $\mathcal{M}$) be the set of all nonempty and bounded subsets of a metric space (X,d), and let $\mathcal{M}_X^c$ (or simply $\mathcal{M}^c$) be the subfamily of $\mathcal{M}_X$ consisting of all closed sets. Further, let $\mathcal{N}_X$ (or simply $\mathcal{N}$) be the set of all nonempty and relatively compact subsets of (X,d). Let $d_H : \mathcal{M}_X \times \mathcal{M}_X \mapsto \mathbb{R}$ be the function defined by

$$(2.1) \qquad d_H(S,Q) = \max\{\sup_{x\in S} d(x,Q), \sup_{y\in Q} d(y,S)\} \quad (S,Q \in \mathcal{M}_X).$$

The function d_H is called *Hausdorff distance*, and $d_H(S,Q)$ (S, $Q \in \mathcal{M}_X$) is the *Hausdorff distance of sets S and Q*.

Let us remark that if $\emptyset \neq F \subset X$, $r > 0$ and

$$B(F,r) = \bigcup_{x\in F} B(x,r) = \{y \in X : d(y,F) < r\}$$

is the *open ball with centre in F and radius r*, then (2.1) is equivalent to

$$d_H(S,Q) = \inf\{\epsilon > 0 : S \subset B(Q,\epsilon) \quad \text{and} \quad Q \subset K(S,\epsilon)\}, \quad (S,Q \in \mathcal{M}_X).$$

It is well known that $(\mathcal{M}_X, d_H)$ is a pseudometric space and that $(\mathcal{M}_X^c, d_H)$ is a metric space.

Let X and Y be infinite-dimensional complex Banach spaces and denote the set of bounded linear operators from X into Y by $B(X,Y)$. We put $B(X) = B(X,X)$. For T in $B(X,Y)$, $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. A linear operator L from X to Y is called *compact* (or *completely continuous*) if $D(L) = X$ for the domain of L, and for every sequence $\{x_n\} \subset X$ such that $\|x_n\| \leq C$, the sequence $\{L(x_n)\}$ has a subsequence which converges in Y. A compact operator is bounded. An operator L in $B(X,Y)$ is of *finite rank* if $\dim R(L) < \infty$. An operator of finite rank is clearly compact. Let $F(X,Y)$, $K(X,Y)$ denote the set of all finite rank and compact operators from X to Y, respectively. Set $F(X) = F(X,X)$ and $K(X) = K(X,X)$.

Let X be a vector space over the field $\mathbb{F}$. A subset E of X is said to be *convex* if $\lambda x + (1-\lambda)y \in E$ for all $x, y \in E$ and for all $\lambda \in (0,1)$.

Clearly the intersection of any family of convex sets is a convex set. If F is a subset of X, then the intersection of all convex sets that contain F is called *convex cover* or *convex hull* of F denoted by $\mathrm{co}(F)$.

The vector subspace $\mathrm{lin}F$ is the set of all linear combinations of elements in F. We shall prove that there is an analogous representation of the set $\mathrm{co}(F)$. Let us mention that a *convex combination of elements* of the set F is an element of the form

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \left(x_i \in F,\ \lambda_i \geq 0\,(i = 1, \ldots, n),\ \sum_{i=1}^{n} \lambda_i = 1\,(n \in \mathbb{N}) \right).$$

Let us write $\mathrm{cvx}(F)$ for the set of all convex combinations of elements of the set F.

Theorem 2.1. *If X is a vector space over the field $\mathbb{F}$ and E, $E_1, \dots E_n$ are convex subsets of X and $F \subset X$, then*

$$\operatorname{cvx}(E) \subset E, \tag{2.2}$$

$$\operatorname{co}(F) = \operatorname{cvx}(F), \tag{2.3}$$

$$\operatorname{co}\Big(\bigcup_{i=1}^{n} E_i\Big) = \Big\{\sum_{i=1}^{n} \lambda_i E_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, i = 1, \dots, n\Big\}. \tag{2.4}$$

Proof. To prove (2.2), it suffices to show that for any $n \geq 2$

$$\begin{gathered} x_i \in E, \lambda_i \geq 0\, (i = 1, \dots n) \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{together} \\ \text{imply} \quad \lambda_1 x_1 + \dots + \lambda_n x_n \in E. \end{gathered} \tag{2.5}$$

We shall use the method of mathematical induction. For $n = 2$ the statement clearly is true. Suppose that the statement in (2.5) is true for a natural number $n > 2$, and let us prove the statement for $n+1$. If $x_i \in E$, $\lambda_i \geq 0$ $(i = 1, \dots, n+1)$ and $\sum_{i=1}^{n+1} \lambda_i = 1$, then there are two cases: first, if $\sum_{i=1}^{n} \lambda_i = 0$, then $\lambda_i = 0$ $(i = 1, \dots, n)$ and $\lambda_1 x_1 + \dots + \lambda_{n+1} x_{n+1} = x_{n+1} \in E$; second, if $\lambda \equiv \sum_{i=1}^{n} \lambda_i \neq 0$, then $\lambda_1 x_1 + \dots + \lambda_{n+1} x_{n+1} = \lambda(\lambda_1 \lambda^{-1} x_1 + \dots + \lambda_n \lambda^{-1} x_n) + \lambda_{n+1} x_{n+1} \in E$. Thus we have shown inclusion (2.2).

It follows from (2.2) that $\operatorname{cvx}(F) \subset \operatorname{co}(F)$. Hence, since $\operatorname{co}(F)$ is a convex subset of X, it suffices to show that $\operatorname{cvx}(F)$ is convex. Suppose that $\lambda \in (0,1)$, and $x, y \in \operatorname{cvx}(F)$. Then there exist $n, m \in \mathbb{N}$, α_i, x_i $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, β_j, y_j $(j = 1, \dots, m)$ with $\sum_{j=1}^{m} \beta_j = 1$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$ and $y = \sum_{j=1}^{m} \beta_j y_j$. Now $\sum_{i=1}^{n} \lambda\alpha_i + \sum_{j=1}^{m} (1-\lambda)\beta_j = \lambda + (1-\lambda) = 1$ implies $\lambda x + (1-\lambda) y \in \operatorname{cvx}(F)$. Hence we have proved (2.3).

We put $S = \{\sum_{i=1}^{n} \lambda_i E_i : \lambda_i \geq 0,\ (i = 1, \dots, n) \sum_{i=1}^{n} \lambda_i = 1\}$. By (2.2) it follows that $S \subset \operatorname{co}(\bigcup_{i=1}^{n} E_i)$. Since $\bigcup_{i=1}^{n} E_i \subset S$, to prove (2.4), it suffices to show that S is convex. Suppose that $\lambda \in (0,1)$ and $x, y \in S$. Now there exist $\alpha_i, x_i\, (i = 1, \dots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, $\beta_i, y_i\, (i = 1, \dots, n,)$ with $\sum_{i=1}^{n} \beta_i = 1$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$, $y = \sum_{i=1}^{n} \beta_i y_i$. We put $\gamma_i = \lambda\alpha_i + (1-\lambda)\beta_i\, (i = 1, \dots n)$. Since $E_1, \dots E_n$, are convex, there exist $z_i \in E_i\, (i = 1, \dots n)$ such that

$$\lambda\alpha_i x_i + (1-\lambda)\beta_i y_i = \gamma_i z_i \quad \text{for } i = 1, \dots, n. \tag{2.6}$$

Let us remark

$$\sum_{i=1}^{n} \gamma_i = \lambda \sum_{i=1}^{n} \alpha_i + (1-\lambda) \sum_{i=1}^{n} \beta_i = \lambda + (1-\lambda) = 1. \tag{2.7}$$

By (2.6) and (2.7) we have $\lambda x + (1-\lambda) y = \sum_{i=1}^{n} \gamma_i z_i \in S$. □

We continue with the study of convex sets in normed spaces.

Lemma 2.2. *Let Q be a bounded subset of a normed space X. Then for any $x \in X$*

$$\sup_{y \in \mathrm{co}(Q)} \|x - y\| = \sup_{z \in Q} \|x - z\|. \tag{2.8}$$

Proof. To prove (2.8), it suffices to show the inequality "$\leq$". If $y \in$ co (Q), then there exist $x_i \in Q$, $\lambda_i \geq 0\,(i = 1, \dots n)$ such that $\sum_{i=1}^n \lambda_i = 1$ and $y = \sum_{i=1}^n \lambda_i x_i$. From $x - y = \sum_{i=1}^n \lambda_i x - \sum_{i=1}^n \lambda_i x_i = \sum_{i=1}^n \lambda_i (x - x_i)$, it follows that $\|x - y\| \leq \sum_{i=1}^n \lambda_i \|x - x_i\| \leq \sup_{z \in Q} \|x - z\|$. □

Corollary 2.3. *Let Q be a bounded subset of a normed space X. Then the sets Q and $\mathrm{co}(Q)$ have equal diameter, that is $\mathrm{diam}(Q) = \mathrm{diam}(\mathrm{co}(Q))$.*

Proof. This follows by Lemma 2.2. □

Let Q be a nonempty and bounded subset of a normed space X. Then the *convex closure* of Q, is denoted by $\mathrm{Conv}(Q)$, and $\mathrm{Conv}(Q)$ is the smallest convex and closed subset of X that contains Q. It is easy to prove that $\mathrm{Conv}(Q) = \overline{\mathrm{co}(Q)}$.

Corollary 2.4. *Let Q be a bounded subset of a normed space X. Then the sets Q and $\mathrm{Conv}(Q)$ have equal diameters, that is $\mathrm{diam}(Q) = \mathrm{diam}(\mathrm{Conv}(Q))$.*

Proof. This follows from Corollary 2.3.

□

2.2. The Kuratowski measure of noncompactness. The notation of measure of noncompactness ($\alpha-$ measure or set-measure), introduced by Kuratowski [41], and the associated notion of an $\alpha-$ contraction, have proved useful in several areas of functional analysis, operator theory and differential equations (see for example, [1, 6, 7]). We start with some results from Kuratowski [41, 42].

Definition 2.5. Let (X, d) be a metric space and Q a bounded subset of X. Then the *Kuratowski measure of noncompactness of* $\boldsymbol{Q}$, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \epsilon$, that is

$$\alpha(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^n S_i,\ S_i \subset X,\ \mathrm{diam}(S_i) < \epsilon\ (i = 1, \dots, n;\ n \in \mathbb{N})\right\}. \tag{2.9}$$

The function α is called *Kuratowski's measure of noncompactness.* Clearly

$$\alpha(Q) \leq \mathrm{diam}(Q) \quad \text{for each bounded subset } Q \text{ of } X. \tag{2.10}$$

As an immediate consequence of Definition 2.5, we obtain.

Lemma 2.6. *Let Q, Q_1 and Q_2 be bounded subsets of a complete metric space (X, d). Then:*

$$\alpha(Q) = 0 \quad \text{if and only if} \quad \overline{Q} \text{ is compact,} \tag{2.11}$$

$$\alpha(Q) = \alpha(\overline{Q}), \tag{2.12}$$

$$Q_1 \subset Q_2 \quad \text{implies} \quad \alpha(Q_1) \leq \alpha(Q_2), \tag{2.13}$$

$$\alpha(Q_1 \cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}, \tag{2.14}$$

$$\alpha(Q_1 \cap Q_2) \leq \min\{\alpha(Q_1), \alpha(Q_2)\}. \tag{2.15}$$

Proof. The statements in (2.11) and (2.13) follow from Definition 2.5.

Clearly $\alpha(Q) \leq \alpha(\overline{Q})$. Let $\epsilon > 0$, S_i be a bounded subset of X with $\operatorname{diam}(S_i) < \epsilon$ for $i = 1, \dots, n$, and $Q \subset \bigcup_{i=1}^n S_i$. Then $\overline{Q} \subset \overline{\bigcup_{i=1}^n S_i} = \bigcup_{i=1}^n \overline{S_i}$. Since $\operatorname{diam}(S_i) = \operatorname{diam}(\overline{S_i})$, we conclude $\alpha(\overline{Q}) \leq \alpha(Q)$. This proves equality (2.12).

From (2.13), we have $\alpha(Q_1) \leq \alpha(Q_1 \cup Q_2)$ and $\alpha(Q_2) \leq \alpha(Q_1 \cup Q_2)$, and so

$$\max\{\alpha(Q_1), \alpha(Q_2)\} \leq \alpha(Q_1 \cup Q_2). \tag{2.16}$$

Let $\max\{\alpha(Q_1), \alpha(Q_2)\} = s$ and $\epsilon > 0$. By Definition 2.5 we know that Q_1 and Q_2 can be covered by a finite number of subsets of diameter smaller than $s + \epsilon$. Obviously, the union of these covers is a finite cover of $Q_1 \cup Q_2$. Hence, we have $\alpha(Q_1 \cup Q_2) \leq s + \epsilon$, and now we obtain (2.14) from (2.16). From $Q_1 \cap Q_2 \subset Q_1$ and $Q_1 \cap Q_2 \subset Q_2$ we obtain $\alpha(Q_1 \cap Q_2) \leq \alpha(Q_1)$ and $\alpha(Q_1 \cap Q_2) \leq \alpha(Q_2)$. Hence $\alpha(Q_1 \cap Q_2) \leq \min\{\alpha(Q_1), \alpha(Q_2)\}$. This proves inequality (2.15). □

The next theorem is a generalization of the well-known Cantor intersection theorem.

Theorem 2.7. (Kuratowski [41]) *Let (X, d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X such that $\lim_{n\to\infty} \alpha(F_n) = 0$, then the intersection $F_\infty = \bigcap_{n=1}^\infty F_n$ is a nonempty and compact subset of X.*

Proof. The set F_∞ is a closed subset of X. Since $F_\infty \subset F_n$ for all $n = 1, 2, \dots$, we obtain from (2.11) and (2.1.3) that F_∞ is a compact set. Now we show $F_\infty \neq \emptyset$. Let $x_n \in F_n$ ($n = 1, 2, \dots$) and $X_n = \{x_i : i \geq n\}$ for $n = 1, 2, \dots$. Since $X_n \subset F_n$, we obtain from (2.11), (2.13) and (2.14)

$$\alpha(X_1) = \alpha(X_n) \leq \alpha(F_n) \quad \text{for each } n. \tag{2.17}$$

The assumption of the theorem and (2.17) together imply $\alpha(X_1) = 0$, hence X_1 is a relatively compact set. Thus the sequence (x_n) has a convergent subsequence with limit $x \in X$, say. Since F_n is closed in X, we get $x \in F_n$ for all $n = 1, 2, \dots$, that is $x \in F_\infty$. □

If X is a normed space, then the function α has some additional properties connected with the vector (linear) structures of a normed space [12].

Theorem 2.8. (Darbo [12]) *Let Q, Q_1 and Q_2 be bounded subsets of a normed space X. Then:*

$$\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2), \tag{2.18}$$

$$\alpha(Q + x) = \alpha(Q) \quad \text{for each } x \in X, \tag{2.19}$$

$$\alpha(\lambda Q) = |\lambda|\alpha(Q) \quad \text{for each } \lambda \in \mathbb{F}, \tag{2.20}$$

$$\alpha(Q) = \alpha(\operatorname{co}(Q)). \tag{2.21}$$

Proof. Let S_i be a bounded subset of X with $\operatorname{diam}(S_i) < d$ for each $i = 1, \dots, n$ and $Q_1 \subset \bigcup_{i=1}^n S_i$. Furthermore, let G_j be a bounded subset of X with $\operatorname{diam}(G_j) < p$ for each $j = 1, \dots, m$ and $Q_2 \subset \bigcup_{j=1}^m G_j$. Then

$$Q_1 + Q_2 \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (S_i + G_j) \quad \text{and} \quad \operatorname{diam}(S_i + G_j) < d + p. \tag{2.22}$$

It follows from (2.22) that $\alpha(Q_1 + Q_2) < d + p$. This shows inequality (2.18). Let $x \in X$. By (2.18) it follows that

$$\alpha(Q + x) \leq \alpha(Q) + \alpha(\{x\}) = \alpha(Q), \tag{2.23}$$

and by the same argument we have

$$\alpha(Q) = \alpha((Q + x) + (-x)) \leq \alpha(Q + x) + \alpha(\{-x\}) = \alpha(Q + x). \tag{2.24}$$

Now we obtain (2.19) from (2.23) and (2.24).

For $\lambda = 0$, equality (2.20) is obvious. Let S_i be a bounded subset of X with $\operatorname{diam}(S_i) < d$ for $i = 1, \dots, n$ and $Q_1 \subset \bigcup_{i=1}^n S_i$. Then for any $\lambda \in \mathbb{F}$, $\lambda Q \subset \bigcup_{i=1}^n \lambda S_i$ and $\operatorname{diam}(\lambda S_i) = |\lambda| \operatorname{diam} S_i$. Hence it follows that $\alpha(\lambda Q) \leq |\lambda|\, \alpha(Q)$. If $\lambda \neq 0$, analogously we have $\alpha(Q) = \alpha(\lambda^{-1}(\lambda Q)) \leq |\lambda^{-1}|\, \alpha(\lambda Q)$, that is $|\lambda|\alpha(Q) \leq \alpha(\lambda Q)$. This proves (2.20).

Now we prove (2.21). Clearly $\alpha(Q) \leq \alpha(\operatorname{co} Q)$, and it suffices to show $\alpha(\operatorname{co} Q) \leq \alpha(Q)$. Let S_i be a bounded subset of X with $\operatorname{diam}(S_i) < d$ for each $i = 1, \dots, n$ and $Q = \bigcup_{i=1}^n S_i$. By Theorem 2.1 it follows that

$$\operatorname{co}(Q) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1,\, x_i \in \operatorname{co}(S_i)\, (i = 1, \dots, n) \right\}. \tag{2.25}$$

Let $\epsilon > 0$ and $\mathbf{S} = \{(\lambda_1, \dots, \lambda_n) : \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0\, (i = 1, \dots, n)\}$. Then $\mathbf{S}$ is a compact subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|(\lambda_1, \dots, \lambda_n)\|_\infty = \sup_{1 \leq i \leq n} |\lambda_i|$. We put $M = \sup\{\|x\| : x \in \bigcup_{i=1}^n \operatorname{co}(S_i)\}$. Let $\mathbf{T} = \{(t_{j,1}, \dots, t_{j,n}) : j = 1, \dots, m\} \subset \mathbf{S}$ be a finite $\epsilon/(Mn)$ -net for $\mathbf{S}$, with respect to the $\|\cdot\|_\infty$ -norm. Hence, if $\sum_{i=1}^n \lambda_i x_i$ is a convex combination of elements of Q, where we suppose that $x_i \in \operatorname{co}(S_i)$ for $i = 1, \dots, n$, then there exists $(t_{j,1}, \dots, t_{j,n}) \in \mathbf{T}$ such that

$$\|(\lambda_1, \dots, \lambda_n) - (t_{j,1}, \dots, t_{j,n})\|_\infty < \frac{\epsilon}{M} n. \tag{2.26}$$

Since

$$\sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} t_{j,i} x_i + \sum_{i=1}^{n} (\lambda_i - t_{j,i}) x_i, \tag{2.27}$$

it follows from (2.25), (2.26) and (2.27) that

$$\operatorname{co}(Q) \subset \bigcup_{j=1}^{m} \left\{ \sum_{i=1}^{n} t_{j,i} \operatorname{co}(S_i) \right\} + \frac{\epsilon}{Mn} \sum_{i=1}^{n} B_i, \tag{2.28}$$

where $B_i = \{x \in X : \|x\| \leq M\}$ for $i = 1, 2, \dots, n$. Now, by (2.4), (2.5), (2.18), (2.20), Corollary 2.3 and (2.28), we have

$$\begin{aligned}
\alpha(\operatorname{co}(Q)) &\leq \alpha\left(\bigcup_{j=1}^{m} \left\{ \sum_{i=1}^{n} t_{j,i} \operatorname{co}(S_i) \right\} \right) + \alpha\left(\frac{\epsilon}{Mn} \sum_{i=1}^{n} B_i \right) \\
&\leq \max_{1 \leq j \leq m} \alpha\left(\sum_{i=1}^{n} t_{j,i} \operatorname{co}(S_i) \right) + \frac{\epsilon}{Mn} \sum_{i=1}^{n} \alpha(B_i) \\
&< \max_{1 \leq j \leq m} \sum_{i=1}^{n} t_{j,i} \alpha(\operatorname{co}(S_i)) + \frac{\epsilon}{Mn} 2nM \\
&< d \max_{1 \leq j \leq m} \sum_{i=1}^{n} t_{j,i} + 2\epsilon < d + 2\epsilon.
\end{aligned}$$

□

Let us remark that Darbo [12] proved (2.21) and, then applied it in the proof of his famous fixed point theorem [12, 1, 7, 86, 100]. His fixed point theorem is a very important generalization of the Schauder fixed point theorem, and is the first important result with applications of Kuratowski's measure of noncompactness.

Let X be an infinite–dimensional normed space and B_X the closed unit ball in X. Then, clearly $\alpha(B_X) \leq 2$, but Furi and Vignoli [18] and Nussbaum [79] have shown more precisely:

Theorem 2.9. (Furi-Vignoli [18], Nussbaum [79]) *Let X be an infinite-dimensional normed space. Then $\alpha(B_X) = 2$.*

Proof. Clearly $\alpha(B_X) \leq 2$. If $\alpha(B_X) < 2$, then there exist bounded and closed subsets Q_i of X with $\operatorname{diam}(Q_i) < 2$ for $i = 1, \dots, n$ such that $B_X \subset \bigcup_{i=1}^{n} Q_i$. Let $\{x_1, \dots x_n\}$ be a linearly independent subset of X and E_n be the set of all linear combinations of elements of the set $\{x_1, \dots, x_n\}$ with real coefficients. Clearly, E_n is a real n-dimensional normed space (the norm on E_n, of course, being the restriction of the norm on X). By $S_n = \{x \in E_n : \|x\| = 1\}$, we denote the unit sphere of E_n. Let us mention that $S_n \subset \bigcup_{i=1}^{n} S_n \cap Q_i$, $\operatorname{diam}(S_n \cap Q_i) < 2$ and $S_n \cap Q_i$ is a closed subset of E_n for each $i = 1, \dots n$. This is a contradiction to the well-known Ljusternik-Šnirelman-Borsuk theorem (see the proof in [86] or in [15, pp. 303–307]: If S_n is the unit sphere of an n-dimensional real normed space E_n, F_i a closed subset of E_n for each $i = 1, \dots, n$ and $S_n \subset \bigcup_{i=1}^n F_i$, then there exists $i_0 \in \{1, \dots, n\}$ such that the set $S_n \cap F_{i_0}$ contains a pair of antipodial points, that is, there exists $x_0 \in S_n \cap F_{i_0}$, such that $\{x_0, -x_0\} \subset S_n \cap F_{i_0}$. □

2.3. The Hausdorff measure of noncompactness. Usually it is complicated to find the exact value of $\alpha(Q)$. Another measure of noncompactness, which is more applicable in many cases, were introduced and studied by Goldenstein, Gohberg and Markus (the *ball measures of noncompactness, Hausdorff measure of noncompactness*) [19] in 1957 (later studied by Goldenstein and Markus [20] in 1968), is given in the next definition.

Definition 2.10. Let (X, d) be a metric space and Q a bounded subset of X. Then the *Hausdorff measure of noncompactness of the set Q*, denoted by $\chi(Q)$ is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$, that is

$$(2.29)\quad \chi(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in X,\ r_i < \epsilon\, (i = 1, \dots n) n \in \mathbb{N}\}.$$

The function χ is called *Hausdorff measure of noncompactness.*

Let us remark that in the definition of the Hausdorff measure of noncompactness of the set Q it is not supposed that centres of the balls which cover Q belong to Q. Hence, (2.29) can equivalently be stated as follows:

$$(2.30)\qquad \chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{-net in } X\}.$$

The Hausdorff measure of noncompactness is often called *ball measure of noncompactness.* The next lemma and theorem could be proved analogously as in the case of the Kuratowski measure of noncompactness.

Lemma 2.11. *Let Q, Q_1 and Q_2 be bounded subsets of the metric space (X, d). Then*

$$\begin{aligned}
&\chi(Q) = 0 \quad \text{if and only if} \quad Q \text{ is totally bounded,}\\
&\chi(Q) = \chi(\overline{Q}),\\
&Q_1 \subset Q_2 \quad \text{implies} \quad \chi(Q_1) \le \chi(Q_2),\\
&\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\},\\
&\chi(Q_1 \cap Q_2) \le \min\{\chi(Q_1), \chi(Q_2)\}.
\end{aligned}$$

Proof. The proof is left as an exercise for the reader. □

Theorem 2.12. *Let Q, Q_1 and Q_2 be bounded subsets of the normed space X. Then*

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$
$$\chi(Q + x) = \chi(Q) \quad \text{for each } x \in X,$$
$$\chi(\lambda Q) = |\lambda|\chi(Q) \quad \text{for each } \lambda \in \mathbb{F}$$
$$\chi(Q) = \chi(\operatorname{co}(Q)). \tag{2.31}$$

Proof. The proof is left as an exercise for the reader. □

The next theorem shows that the functions α and χ are in some sense equivalent.

Theorem 2.13. *Let (X, d) be a metric space and Q be a bounded subset of X. Then*

$$\chi(Q) \leq \alpha(Q) \leq 2\chi(Q). \tag{2.32}$$

Proof. Let $\epsilon > 0$. If $\{x_1, \dots, x_n\}$ is an ϵ-net of Q, then $\{Q \cap B(x_i, \epsilon)\}_{i=1}^n$ is a cover of Q with sets of diameter $< 2\epsilon$. This shows $\alpha(Q) \leq 2\chi(Q)$. To prove the left side inequality in (2.32), let us suppose that $\{S_i\}_{i=1}^k$ is a cover of Q with sets of diameter $< \epsilon$ and $y_i \in S_i$ for $i = 1, \dots k$. Now $\{y_1, \dots, y_k\}$ is an ϵ-net of Q. This proves $\chi(Q) \leq \alpha(Q)$. □

Let us remark that the inequalities (2.32) are best possible in general, as an example shows. These measures are closely related to geometric properties of the space and it is possible to improve the inequality $\chi(Q) \leq \alpha(Q)$ in certain spaces (see e.g. Dominguez Benavides and Ayerbe [14], Webb and Weiyu Zhao [103]). For example (see [1], [7]) in Hilbert space, $\sqrt{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q)$, and in l^p for $1 \leq p < \infty$, $\sqrt[p]{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q)$.

Theorem 2.14. *Let X be an infinite-dimensional normed space and B_X be the closed unit ball of X. Then $\chi(B_X) = 1$.*

Proof. Obviously $\chi(B_X) \leq 1$. If $\chi(B_X) = q < 1$, then we choose $\epsilon > 0$ such that $q + \epsilon < 1$. Now there exists a $(q + \epsilon)$-net of B_X, say $\{x_1, \dots, x_k\}$. Hence

$$B_X \subset \bigcup_{i=1}^{k} \{x_i + (q + \epsilon)B_X\}. \tag{2.33}$$

Now it follows from Lemmas 2.11 and 2.12 that

$$q = \chi(B_X) \leq \max_{1 \leq i \leq k} \chi(\{x_i + (q + \epsilon)B_X\}) = (q + \epsilon)q. \tag{2.34}$$

Since $q + \epsilon < 1$, by (2.33) we have $q = 0$, that is B_X is a totally bounded set. But this is impossible since X is an infinite-dimensional space. Hence $\chi(B_X) = 1$. □

Let us remark that Theorem 2.14 follows from Theorems 2.9 and 2.13. (But we offer another proof.)

Now we shall show how to compute the Hausdorff measure of noncompactness in the spaces ℓ_p for $1 \leq p < \infty$ and c_0.

Theorem 2.15. *Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \leq p < \infty$ or c_0. If $P_n : X \mapsto X$ is the operator defined by $P_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ for $(x_1, x_2, \dots) \in X$; then*

$$\chi(Q) = \lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.35}$$

Proof. Clearly

$$Q \subset P_nQ + (I - P_n)Q. \tag{2.36}$$

It follows from Lemma 2.11, Theorem 2.12 and (2.36) that

$$\chi(Q) \leq \chi(P_nQ) + \chi((I - P_n)Q) = \chi((I - P_n)Q) \leq \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.37}$$

Since the limit in (2.35) clearly exists, we have by (2.37)

$$\chi(Q) \leq \lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.38}$$

Now we prove the converse inequality in (2.38). Let $\epsilon > 0$ and $\{z_1, \dots, z_k\}$ be a $[\chi(Q) + \epsilon]$-net of Q. It is easy to prove that

$$Q \subset \{z_1, \dots, z_k\} + [\chi(Q) + \epsilon]B_X. \tag{2.39}$$

It follows from (2.39) that for any $x \in Q$ there exist $z \in \{z_1, \dots, z_k\}$ and $s \in B_X$ such that $x = z + [\chi(Q) + \epsilon]s$. Hence

$$\sup_{x\in Q} \|(I - P_n)x\| \leq \sup_{1\leq i\leq k} \|(I - P_n)z_i\| + [\chi(Q) + \epsilon]. \tag{2.40}$$

Finally, (2.40) implies $\lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\| \leq \chi(Q) + \epsilon$. □

Concerning the space $\ell_\infty(\mathbb{R})$, to the best of our knowledge is the following theorem [13, Proposition 3.5].

Theorem 2.16. (Dominguez Benavides [13]) *Let ℓ_∞ be the real normed space of bounded sequences with* sup-norm *and Q be a bounded subset of ℓ_∞. Then $\alpha(Q) = 2\chi(Q)$.*

Proof. We know that $\alpha(Q) \leq 2\chi(Q)$. Let $\epsilon > 0$ and $Q_1, \dots, Q_n$ be subsets of $\ell_\infty(\mathbb{R})$ such that $Q \subset \bigcup_{i=1}^n Q_i$ and $\operatorname{diam} Q_i < \alpha(Q) + \epsilon$. For any $k \in \mathbb{N}$ we put $\alpha_{k,i} = \inf\{x_k : (x_j) \in Q_i\}$, $\beta_{k,i} = \sup\{x_k : (x_j) \in Q_i\}$, $c_{k,i} = (\alpha_{k,i} + \beta_{k,i})/2$, $B_i = B((c_{k,i})_{k=1}^\infty, (\alpha(Q) + \epsilon)/2)$ for $i = 1, \dots, n$. It is easy to prove that $Q_i \subset B_i$. Hence $\chi(Q) \leq (\alpha(Q) + \epsilon)/2$, that is $2\chi(Q) \leq \alpha(Q)$. □

We shall prove that the Hausdorff measure of noncompactness is connected with the Hausdorff distance.

Theorem 2.17. *Let (X,d) be a metric space. Then $(\mathcal{M}_X^c, d_H)$ is a metric space.*

Proof. Clearly $d_H(S,Q)=0$ if and only if $S=Q$, and $d_H(S,Q)=d_H(Q,S)$ for all $S,Q \in \mathcal{M}_X^c$.

To show the triangle inequality, suppose $S,Q,F \in \mathcal{M}_X^c$, $x \in S$, $y \in Q$ and $z \in F$. It is easy to prove $d(x,F) \le d(x,y)+d(y,F) \le d(x,y)+d_H(Q,F)$, and this implies

$$\begin{aligned} d(x,F) &\le \inf_{y\in Q} d(x,y)+d_H(Q,F) = d(x,Q)+d_H(Q,F) \\ &\le d_H(S,Q)+d_H(Q,F). \end{aligned} \tag{2.41}$$

Replacing x and F by z and S in (2.41), respectively, we obtain

$$d(z,S) \le d_H(F,Q)+d_H(Q,S) \tag{2.42}$$

Finally, (2.41) and (2.42) together imply $d_H(S,F) \le d_H(S,Q)+d_H(Q,F)$. □

Theorem 2.18. *Let (X,d) be a metric space, $Q,Q_1,Q_2 \in \mathcal{M}_X$, and $\mathcal{N}_X^c$ be the set of all nonempty and compact subsets of (X,d). Then*

$$|\chi(Q_1)-\chi(Q_2)| \le d_H(Q_1,Q_2), \tag{2.43}$$

$$\chi(Q) = d_H(Q,\mathcal{N}_X^c). \tag{2.44}$$

Proof. Let $\epsilon > 0$ and $d = d_H(Q_1,Q_2)$. Then it follows from (2.29) and (2.1) that there exists a finite set $S \subset X$, such that

$$Q_1 \subset B(Q_2, d+\epsilon) \quad \text{and} \quad Q_2 \subset B(S, \chi(Q_2)+\epsilon). \tag{2.45}$$

Furthermore, (2.45) implies

$$Q_1 \subset B(S, d+\chi(Q_2)+2\epsilon), \tag{2.46}$$

and so we conclude

$$\chi(Q_1) \le \chi(Q_2)+d+2\epsilon. \tag{2.47}$$

Now (2.43) clearly follows from (2.47).

To prove (2.44), let us remark that the inequality $\le$ in (2.44) follows from (2.43). Therefore it suffices to show the inequality $\ge$. If $\epsilon > 0$, then there exists a finite set $F \subset X$, such that

$$Q \subset B(F,\chi(Q)+\epsilon) \quad \text{and} \quad F \subset B(Q,\chi(Q)+\epsilon). \tag{2.48}$$

Now (2.48) and (2.1) together imply $d_H(Q,\mathcal{N}_X^c) \le d_H(Q,F) \le \chi(Q)+\epsilon$. □

Corollary 2.19. *Let $\mathcal{N}^c_X$ be the set of all nonempty and compact subsets of a complete metric space (X,d). Then $\mathcal{N}^c_X$ is a closed subset of $(\mathcal{M}^c_X, d_H)$.*

Proof. This is an immediate consequence of (2.44). □

If the centres of the balls in Definition 2.10 are in Q we have

Definition 2.20. Let (X,d) be a metric space and Q a bounded subset of X. Then the *inner Hausdorff measure of noncompactness* of the set Q, denoted by $\chi_i(Q)$ is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$ and centers in Q, that is

$$\chi_i(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in Q,\ r_i < \epsilon\,(i = 1, \dots\ n)\, n \in \mathbb{N}\right\}.$$

The function χ_i is called *inner Hausdorff measure of noncompactness.* Hence the formula in Definition 2.20 can equivalently be stated as follows:

$$\chi_i(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{–net in } Q\}.$$

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X,d), then

$$\begin{aligned}&\chi_i(Q) = 0 \quad \text{if and only if} \quad Q \quad \text{is totally bounded,}\\ &\chi_i(Q) = \chi_i(\overline{Q}),\end{aligned}$$

but in general

$$Q_1 \subset Q_2 \quad \text{does not imply} \quad \chi_i(Q_1) \le \chi_i(Q_2),$$

and

$$\chi_i(Q_1 \cup Q_2) \ne \max\{\chi_i(Q_1), \chi_i(Q_2)\}.$$

Let Q, Q_1 and Q_2 be bounded subset of the normed space X. Then

$$\begin{aligned}&\chi_i(Q_1 + Q_2) \le \chi_i(Q_1) + \chi_i(Q_2),\\ &\chi_i(Q + x) = \chi_i(Q) \quad \text{for each } x \in X,\\ &\chi_i(\lambda Q) = |\lambda|\chi_i(Q) \quad \text{for each } \lambda \in \mathbb{F},\end{aligned}$$

but in general

$$\chi_i(Q) \ne \chi_i(\operatorname{co}(Q)).$$

In the fixed point theory in normed space (or more generally in locally convex spaces) the relation $\alpha(Q) = \alpha(\operatorname{co}(Q))$ is of great importance. Let us remark that O. Hadžić [26], among other things, studied the inner Hausdorff measure of noncompactness in paranormed spaces. She proved under some additional conditions the inequality $\chi_i(\operatorname{co}(Q)) \le \varphi[\chi_i(Q)]$, where $\varphi : [0,\infty) \mapsto [0,\infty)$, and, then got some fixed point theorems for multivalued mappings in general topological vector spaces.

Istrăţescu's measure of noncompactness is closely related to the Hausdorff and Kuratowski measures of noncompactness. Before we give its definition, we need to recall that a bounded subset Q of a complete metric space (X,d) is to be said ϵ-*discrete* if $d(x,y) \geq \epsilon$ for all $x,y \in Q$ with $x \neq y$. Obviously, the set Q is relatively compact if and only if every ϵ-discrete set is finite for all $\epsilon > 0$.

Definition 2.21. (Istrăţescu, [30]) Let (X,d) be a complete metric space and Q a bounded subset of X. Then the *Istrăţescu measure of noncompactness (β-measure, I -measure)* of Q, is denoted by $\beta(Q)$, and defined by

$$\beta(Q) = \inf\{\epsilon > 0 : Q \text{ has no infinite } \epsilon\text{-discrete subsets}\}.$$

The function β is called *Istrăţesku's measure of noncompactness.* Let us remark [11] that β can be defined also by

$$\beta(Q) = \sup\{\epsilon > 0 : Q \text{ contains an infinite } \epsilon\text{-discrete set}\},$$

and the above mentioned properties of α are also valid for β (see e.g. [1, 7, 11]).

Theorem 2.22. (Daneš, [11]) *Let (X,d) be a metric space and Q be a bounded subset of X. Then*

$$\chi(Q) \leq \chi_i(Q) \leq \beta(Q) \leq \alpha(Q) \leq 2\chi(Q).$$

Hence, in particular, $\frac{1}{2}\alpha(Q) \leq \beta(Q) \leq \alpha(Q)$ *and* $\chi(Q) \leq \beta(Q) \leq 2\chi(Q)$.

Now we shall point out the well-known result of Goldenštein, Gohberg and Markus [19, Theorem 1] (see also [7, Theorem 6.1.1] or [1, 1.8.1]) concerning the Hausdorff measure of noncompactness in Banach spaces with Schauder basis. Let X be a Banach space with a Schauder basis $\{e_1, e_2, \dots\}$. Then each element $x \in X$ has a unique representation $x = \sum_{i=1}^{\infty} \phi_i(x)e_i$ where the functions ϕ_i are the basis functionals. Let $P_n : X \mapsto X$ be the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$, that is $P_n(x) = \sum_{i=1}^{n} \phi_i(x)e_i$. Then, in view of the Banach-Steinhaus theorem, all operators P_n and $I - P_n$ are equibounded. Now we shall prove

Theorem 2.23. (Goldenštein, Gohberg and Markus [19]) *Let X be a Banach space with a Schauder basis $\{e_1, e_2, \dots\}$, Q be a bounded subset of X, and $P_n : X \mapsto X$ the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$. Then*

$$\begin{aligned} &\frac{1}{a} \limsup_{n\to\infty} \Big(\sup_{x\in Q} \|(I - P_n)(x)\|\Big) \leq \chi(Q) \leq \\ &\leq \inf_n \sup_{x\in Q} \|(I - P_n)(x)\| \leq \limsup_{n\to\infty} \Big(\sup_{x\in Q} \|(I - P_n)(x)\|\Big), \end{aligned} \tag{2.49}$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

Proof. Clearly, for any natural number n we have

$$Q \subset P_nQ + (I - P_n)Q. \tag{2.50}$$

It follows from Lemma 2.11, Theorem 2.12 and (2.50) that

$$\chi(Q) \leq \chi(P_nQ) + \chi((I-P_n)Q) = \chi((I-P_n)Q) \leq \sup_{x \in Q} \|(I-P_n)(x)\|. \tag{2.51}$$

Now we obtain

$$\chi(Q) \leq \inf_n \sup_{x\in Q} \|(I-P_n)(x)\| \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right), \tag{2.52}$$

Hence it suffices to show the first inequality in (2.49). Let $\epsilon > 0$ and $\{z_1, \dots, z_k\}$ be a $[\chi(Q)+\epsilon]$-net of Q. It is easy to show that $Q \subset \{z_1, \dots, z_k\} + [\chi(Q)+\epsilon]B_X$. This implies that for any $x \in Q$ there exist $z \in \{z_1, \dots, z_k\}$ and $s \in B_X$ such that $x = z + [\chi(Q)+\epsilon]s$, and so

$$\sup_{x\in Q} \|(I-P_n)(x)\| \leq \sup_{1\leq i\leq k} \|(I-P_n)(z_i)\| + [\chi(Q)+\epsilon]\,\|(I-P_n)\|.$$

This implies

$$\limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right) \leq (\chi(Q)+\epsilon) \limsup_{n\to\infty} \|I-P_n\|.$$

□

Let us mention that concerning the number a in Theorem 2.23, if $X = c_0$, then $a = 1$, but if $X = c$, then $a = 2$ (see e.g. [7, p. 22]).

2.4. Operators. So far we "measured" the noncompactness of a bounded subset of a metric space. Now we "measure" the noncompactness of an operator.

Definition 2.24. Let κ_1 and κ_2 any of the measures of noncompactness defined above on the Banach spaces X and Y, respectively. An operator $L : X \mapsto Y$ is said to be (κ_1, κ_2)*-bounded* if

$$L(Q) \in \mathcal{M}_Y \quad \text{for each } Q \in \mathcal{M}_X \tag{2.53}$$

and there exists a real k with $0 \leq k < \infty$ such that

$$\kappa_2(L(Q)) \leq k\kappa_1(Q) \quad \text{for each } Q \in \mathcal{M}_X. \tag{2.54}$$

If an operator L is (κ_1, κ_2)*-bounded* then the number $\|K\|_{\kappa_1,\kappa_2}$ defined by

$$\|L\|_{\kappa_1,\kappa_2} = \inf\{k \geq 0 : \kappa_2(L(Q)) \leq k\kappa_1(Q) \quad \text{for each } Q \in \mathcal{M}_X\} \tag{2.55}$$

is called (κ_1, κ_2)*-operator norm of* L, or (κ_1, κ_2)*-measure of noncompactness of* L, or simply *measures of noncompactness of* L.

If $\kappa_1 = \kappa_2 = \kappa$, then we write $\|L\|_\kappa$ instead of $\|L\|_{\kappa,\kappa}$.

The next theorem is related to the Hausdorff measure of noncompactness.

Theorem 2.25. *Let X and Y be Banach spaces and $L \in B(X,Y)$. Then $\|L\|_\chi = \chi(L(S_X)) = \chi(L(B_X))$.*

Proof. We write $B = B_X$ and $S = S_X$. Since $\mathrm{co}(S) = B_X$ and $L(\mathrm{co}(S)) = \mathrm{co}(L(S))$, it follows from (2.31) that

$$\chi(L(B)) = \chi(L(\mathrm{co}(S)) = \chi(\mathrm{co}L(S)) = \chi(L(S)), \tag{2.56}$$

hence we have by (2.55) and Theorem 2.14 $\chi(L(B)) \leq \|L\|_\chi$. Now we show $\|L\|_\chi \leq \chi(L(B))$. Let $Q \in \mathcal{M}$ and $\{x_i\}_{i=1}^n$ be a finite r-net of Q. Then $Q \subset \bigcup_{i=1}^n B(x_i, r)$ and obviously

$$L(Q) \subset \bigcup_{i=1}^n L(B(x_i, r)). \tag{2.57}$$

It follows from (2.57), Lemma 2.11 and Theorem 2.12 that

$$\chi(L(Q)) \leq \chi\Big(\bigcup_{i=1}^n L(B(x_i, r))) = \chi(L(B(0,r)\Big) = r\chi(L(B)),$$

and we have $\chi(L(Q)) \leq \chi(Q)\chi(L(B))$ □

Corollary 2.26. *Let X, Y and Z be Banach spaces, $L \in B(X,Y)$, $\tilde{L} \in B(Y,Z)$ and $\|\cdot\|_K$ the quotient norm on the Banach space $B(X,Y)/K(X,Y)$. Then $\|\cdot\|_\chi$ is a seminorm on $B(X,Y)$ and*

$$\|L\|_\chi = 0 \quad \text{if and only if} \quad L \in K(X,Y), \tag{2.58}$$

$$\|L\|_\chi \leq \|L\|, \tag{2.59}$$

$$\|L + K\|_\chi = \|L\|_\chi, \text{ for each } K \in K(X,Y), \tag{2.60}$$

$$\|\tilde{L} \circ L\|_\chi \leq \|\tilde{L}\|_\chi \|L\|_\chi. \tag{2.61}$$

$$\|L\|_\chi \leq \|L\|_K. \tag{2.62}$$

Proof. The proof is left as an exercise to the reader. □

The following results will give a technique for the evaluation of the Hausdorff measure of noncompactness of an operator on the space l_1.

Theorem 2.27. *We have $L \in B(l_1, l_1)$ if and only if there exists an infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers such that*

$$\|A\| = \sup_k \sum_{n=0}^\infty |a_{nk}| < \infty \tag{2.63}$$

$$L(x) = A(x) \quad \text{for all } x \in l_1. \tag{2.64}$$

In this case

$$\|L\| = \|A\|, \tag{2.65}$$

and the operator L uniquely determines the matrix $A = (a_{nk})_{n,k}^{\infty}$. The operator L is said to be given (defined) by the matrix A.

Proof. First we assume $L \in B(X,Y)$. We write $L_n = P_n \circ L$ for all n where P_n denotes the n-th coordinate, and put $a_{nk} = L_n(e^{(k)})$ for all $n, k = 0, 1, \dots$. Since l_1 is a BK space, we have $L_n \in l_1^*$ for each n and so $L_n(x) = A_n(x)$ for each n by Theorem 1.35. This yields the representation in (2.64). If we choose $x = e^{(k)}$, then

$$\|L(e^{(k)})\|_1 = \sum_{n=0}^{\infty} |L_n(e^{(k)})| = \sum_{n=0}^{\infty} |a_{nk}| \leq \|L\| \|e^{(k)}\|_1 = \|L\| \quad \text{for all },$$

that is

$$\|A\| = \sup_k \sum_{n=0}^{\infty} |a_{nk}| \leq \|L\| < \infty \tag{2.66}$$

and (2.63) holds. Further

$$\|L(x)\|_1 = \sum_{k=0}^{\infty} |A_n(x)| \leq \sum_{k=0}^{\infty} |x_k| \sum_{n=0}^{\infty} |a_{nk}| \leq \|A\| \|x\|_1 \quad \text{for all } x \in l_1, \tag{2.67}$$

and so $\|L\| \leq \|A\|$. This and (2.66) together yield (2.65).

Conversely let condition (2.63) hold. Then obviously $\sup_k |a_{nk}| < \infty$ for all n, that is $A_n \in X^{\beta}$ for all n. Let $x \in l_1$. As in (2.67), we obtain $A(x) \in l_1$ by (2.63), whence $A \in (l_1, l_1)$. We define the linear operator $L : l_1 \mapsto l_1$ by (2.64). Then $L \in B(l_1, l_1)$ by Theorem 1.23 (a). □

Theorem 2.28. (Goldenštein, Gohberg and Markus [19]) *Let $L \in B(l_1, l_1)$ be given by an infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$. Then*

$$\|L\|_{\chi} = \lim_{m \to \infty} \sup_k \sum_{n=m}^{\infty} |a_{nk}|. \tag{2.68}$$

Proof. We write $S = S_{l_1}$. It follows from Theorems 2.15 and 2.27 that

$$\|L\|_{\chi} = \chi(L(S)) = \lim_{m \to \infty} \sup_{x \in S} \sum_{n=m}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right|. \tag{2.69}$$

The limit in (2.68) obviously exists. From

$$\sup_{x\in S}\sum_{n=m}^{\infty}\left|\sum_{k=0}^{\infty}a_{nk}x_k\right| \le \sup_{x\in S}\sum_{n=m}^{\infty}\sum_{k=0}^{\infty}|a_{nk}x_k| = \sup_{x\in S}\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}|a_{nk}||x_k|$$
$$\le \sup_k \sum_{n=m}^{\infty}|a_{nk}|$$

and (2.69) we obtain

$$\|L\|_\chi \le \lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}|. \tag{2.70}$$

To prove the converse inequality, we choose $x = e^{(k)} \in l_1$. Since $L(e^{(k)}) = A^k = (a_{nk})_{n=0}^{\infty}$, Theorem 2.15 implies

$$\chi(\{L(e^{(k)}) : k = 0, 1, \dots\}) = \lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}| \le \chi(L(S)).$$

This and inequality (2.70) together yield (2.68). □

As an immediate consequence of Theorem 2.28, we have

Corollary 2.29. *Let $L \in B(l_1, l_1)$ be given by the infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$. Then L is compact if and only if*

$$\lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}| = 0.$$

Let us mention that measures of noncompactness are of special interest in *spectral theory*, the theory of *Fredholm* and *semi-Fredholm operators* (see e.g. [8, 9, 17, 19, 21, 45, 78, 87, 88, 89, 93, 94, 101, 102, 110, 115, 116].

3. Matrix domains

In this section, we shall deal with sequence spaces related to the concepts of ordinary and strong summability, spaces of sequences of differences and sequences that are strongly convergent and bounded. We shall characterize matrix transformations between these spaces and apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for these matrix maps to be compact operators. This section contains some of our recent research results which can be found in [34, 64, 65, 68, 69, 70, 71, 72, 73] and in the survey articles [32, 66, 67].

Let A be an infinite matrix and $x = (x_k)_{k=0}^{\infty}$ be a sequence. The sequence x is said to be *A-summable to $l \in \mathbb{C}$* if

$$A_n(x) = \sum_{k=0}^{\infty}a_{nk}x_k \to l \quad (n\to\infty); \quad \text{we shall write } x \to l(A).$$

This means $A_n \in x^\beta = x^{-1} * cs$ for all n and $A(x) \in c$.

The sequence x is said to be *strongly summable A to $l \in \mathbb{C}$* if

$$\sum_{k=0}^{\infty} a_{nk}|x_k - l| \to 0 \quad (n \to \infty); \qquad \text{we shall write } x \to l[A].$$

The sequence x is said to be *absolutely summable A* if

$$\sum_{n=0}^{\infty} |A_n(x)| < \infty.$$

We shall mainly be interested in the first two concepts.

3.1. Ordinary and strong matrix domains. In this subsection, we define *ordinary* and *strong matrix domains* and study their topological properties.

Definition 3.1. Let X be a set of sequences and A an infinite matrix. The sets

$$X_A = \{x \in \omega : A(x) \in X\}$$

and

$$X_{[A]} = \{x \in \omega : A(|x|) = \Big(\sum_{k=0}^{\infty} a_{nk}|x_k|\Big)_{n=0}^{\infty} \in X\}$$

are called the *(ordinary) matrix domain* and *strong matrix domain of* $\boldsymbol{A}$. In the special case where $X = c$, the sets c_A and $c_{[A]}$ are called *convergence domain* and *strong convergence domain of* $\boldsymbol{A}$.

The sets c_A and $c_{[A]}$ are closely related to the concepts of ordinary and strong summability. Obviously $x \to l(A)$ if and only if $x \in c_A$ and $x \to l[A]$ if and only if $x - le \in (c_0)_{[A]}$.

It is known that the ordinary matrix domain of an FK space again is an FK space [108, Theorem 4.3.12, p. 63] or [91, Proposition 4.2.1, p. 101]. Since we shall here confine our studies to BK spaces and matrix domains of triangles, we shall only prove the special result. We need the following

Lemma 3.2. *Let X be a linear space, $(Y, \|\cdot\|)$ a normed space and $T : X \mapsto Y$ a linear one-to-one map. Then X becomes a normed space with $\|x\|_X = \|T(x)\|$. If, in addition, Y is a Banach space and T is onto Y, then $(X, \|\cdot\|_X)$ is a Banach space.*

Proof. The proof is elementary and left to the reader. □

Theorem 3.3. *Let T be a triangle and $(X, \|\cdot\|)$ be a BK space. Then X_T is a BK space with $\|x\|_T = \|T(x)\|$.*

Proof. We define the map $L_T : X_T \mapsto X$ by $L_T(x) = T(x)$ for all $x \in X_T$. Then L_T is linear, one-to-one, since T is a triangle, and onto X, since $X_T = L_T^{-1}(X)$ and L_T is one-to-one. By Lemma 3.2, X_T is a Banach space.

To show that the coordinates are continuous in X_T, let $x^{(n)} \to x$ in X_T. Then $y_k^{(n)} = T_k(x^{(n)}) \to y_k = T_k(x)$, since X is a BK space. Let S be the inverse of T, also a triangle. Then $x_k^{(n)} = \sum_{j=0}^k s_{kj} y_j^{(n)} \to \sum_{j=0}^k s_{kj} y_j = x_k$. This shows that the coordinates are continuous on X_T. □

As a special case of Theorem 3.3, we obtain

Corollary 3.4. [108, Theorem 4.3.13, p. 64] *Let T be a triangle. Then c_T is a BK space with $\|x\|_{T,\infty} = \|T(x)\|_\infty$.*

Theorem 3.5. [108, Theorem 4.3.14, p. 64] *If X is a closed subspace of Y, then X_A is a closed subspace of Y_A.*

Proof. Define the map $f : Y_A \mapsto Y$ by $f(y) = A(y)$, a continuous map. Then f_A is continuous by Theorem 1.17, and so $X_A = f^{-1}(X)$ is closed. □

A result similar to Theorem 3.3 holds for the strong matrix domains of triangles. We call a norm $\|\cdot\|$ a sequence space X *monotone*, if $|\tilde{x}_k| \leq |x_k|$ $(k = 0, 1, \dots)$ implies $\|\tilde{x}\| \leq \|x\|$.

Theorem 3.6. [34, Theorem 1] *Let X be a normal BK space with monotone norm $\|\cdot\|$, T a triangle and B a positive triangle. Then $X_{[B]}$ is a BK space with $\|x\|_{X_{[B]}} = \|B(|x|)\|$ for all $x \in X_{[B]}$.*

Proof. We write $\|\cdot\|' = \|\cdot\|_{X_{[B]}}$ for short. Obviously, $\|\cdot\|'$ is a norm on $X_{[B]}$. Further, since X is a BK space,

$$\|x^{(m)} - x\|' = \|B(|x^{(m)} - x|)\| \to 0 \quad (m \to \infty)$$

implies $B_n(|x^{(m)} - x|) = \sum_{k=0}^n b_{nk}|x_k^{(m)} - x_k| \to 0$ $(m \to \infty)$ for all n. Thus

$$|x_n^{(m)} - x_n \leq \frac{1}{b_{nn}} B_n(|x^{(m)} - x|) \to 0 \quad (m \to \infty) \quad \text{for all } n.$$

Hence the norm $\|\cdot\|'$ is stronger than the metric of ω on $X_{[B]}$. Let $(x^{(m)})_{m=0}^\infty$ be a Cauchy sequence in $X_{[B]}$, hence in ω by what we have just shown. Then there is $y \in \omega$ such that

$$x^{(m)} \to y \quad \text{in } \omega. \tag{3.1}$$

Further, by the completeness of X, there is $z \in X$ such that

$$B(|x^{(m)}|) \to z \quad \text{in } X. \tag{3.2}$$

From (3.1), we conclude $x_k^{(m)} \to y_k$ $(m \to \infty)$ for each fixed k, hence $B_n(|x^{(m)}|) \to B_n(|y|)$ $(m \to \infty)$ for all n, and consequently

$$B(|x^{(m)}|) \to B(|y|) \quad \text{in } \omega. \tag{3.3}$$

Finally (3.2) and (3.3) together imply $z = B(|y|) \in X$, that is $y \in X_{[B]}$. □

There is no general method to find the Schauder basis of a matrix domain X_A or $X_{[A]}$ from that of X not even when A is a positive triangle. We give a special result which will be applied later.

Theorem 3.7. [34, Theorem 2] (a) *Let X be a BK space with basis $(b^k)_{k=0}^\infty$, $\mathcal{U} = \{u \in \omega : u_k \neq 0$ for all $k\}$ $u \in \mathcal{U}$ and $c^{(k)} = (1/u) \cdot b^{(k)}$ $(k = 0, 1, \dots)$ where $1/u = (1/u_k)_{k=0}^\infty$. Then $(c^{(k)})_{k=0}^\infty$ is a basis for $Y = u^{-1} * X$.*

(b) *Let $u \in \mathcal{U}$ be a sequence such that $|u_0| \leq |u_1| \leq \dots$ and $|u_n| \to \infty$ for $n \to \infty$, and T a triangle with $t_{nk} = 1/u_n$ $(0 \leq k \leq n)$ and $t_{nk} = 0$ $(k > n)$ for all $n = 0, 1, \dots$. Then $(c_0)_T$ has AK.*

Proof. (a) Let $\|\cdot\|$ be the BK norm on X. Then Y is a BK space with $\|y\|_u = \|u \cdot y\|$ $(y \in Y)$ by Theorem 3.3. Further $u \cdot c^{(k)} = b^{(k)} \in X$ $(k = 0, 1, \dots)$ implies $c^{(k)} \in Y$ $(k = 0, 1, \dots)$. Finally let $y \in Y$ be given. Then $u \cdot y = x \in X$ and $x^{\langle m \rangle} = \sum_{k=0}^m \lambda_k b^{(k)} \to x$ $(m \to \infty)$ in X. We put $y^{\langle m \rangle} = (1/u) \cdot x^{\langle m \rangle}$. Then $u \cdot y^{\langle m \rangle} = x^{\langle m \rangle} \to x = u \cdot y$ in$X$, hence $y^{\langle m \rangle} \to y$ in Y, that is $y = \sum_{k=0}^\infty \lambda_k c^{(k)}$. Obviously, this representation is unique.

(b) $(c_0)_T$ is a BK space with respect to $\|x\|_{(c_0)_T} = \sup_n \left| \frac{1}{u_n} \sum_{k=0}^n x_k \right|$, by Theorem 3.3. Further $|u_n| \to \infty$ $(n \to \infty)$ implies $\phi \subset (c_0)_T$. Let $\varepsilon > 0$ and $x \in (c_0)_T$ be given. Then there is a nonnegative integer n_0 such that $|T_n(x)| < \varepsilon/2$ for all $n \geq n_0$. Let $m > n_0$. Then

$$\|x - x^{[m]}\|_{(c_0)_T} = \sup_{n \geq m+1} \left| \frac{1}{u_n} \sum_{k=m+1}^n x_k \right| \leq \sup_{n \geq m+1} |T_n(x) < +|T_m(x)| < \varepsilon.$$

Obviously, the representation is unique. □

3.2. Matrix transformations into matrix domains. In this subsection, we shall show that, for triangles T, the characterizations of the classes (X, Y) and $(X, Y_{[T]})$ can be reduced to that of (X, Y).

Theorem 3.8. [65, Theorem 1], [71, Proposition 3.4] *Let T be a triangle.*

(a) *Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*

(b) *Further, if X and Y are BK spaces and $A \in (X, Y_T)$, then*

$$\|L_A\| = \|L_B\|. \tag{3.4}$$

Proof. (a) The proof of part (a) is straightforward and can be found in [65, Theorem 1].

(b) Let $A \in (X, Y_T)$. Since Y is a BK space and T a triangle, Y_T is a BK space with

$$\|y\|_{Y_T} = \|T(y)\|_Y \quad (y \in Y_T) \tag{3.5}$$

by Theorem 3.3. Thus A is continuous by Theorem 1.17 and consequently

$$\|L_A\| = \sup\{\|L_A(x)\|_{Y_T} : \|x\| = 1\} = \sup\{\|A(x)\|_{Y_T} : \|x\| = 1\} < \infty. \tag{3.6}$$

Further, since B is continuous,

$$\|L_B\| = \sup\{\|L_B(x)\|_Y : \|x\| = 1\} = \sup\{\|B(x)\|_Y : \|x\| = 1\} < \infty. \tag{3.7}$$

Let $x \in X$. Since $A_n \in X^\beta$ for all $n = 0, 1, \ldots$, we have $x \in \omega_A$. Further $T_n \in \phi$ $(n = 0, 1, \ldots)$, since T is a triangle. Thus $B(x) = (TA)(x) = T(A(x))$ (cf. [108, Theorem 1.4.4, p. 8]), and (3.4) follows from (3.5), (3.6) and (3.7). □

For the characterization of the class $(X, Y_{[T]})$, we need the following lemma.

Lemma 3.9. [81] *Let $a_0, a_1, \ldots, a_n \in \mathbb{C}$. Then*

$$\sum_{k=0}^{n} |a_k| \le 4 \cdot \max_{N \in \{0,\ldots,n\}} \left| \sum_{k \in N} a_k \right|.$$

Proof. First we consider the case where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. We put $N^+ = \{k \in \{0, \ldots, n\} : a_k \ge 0\}$ and $N^- = \{k \in \{0, \ldots, n\} : a_k < 0\}$. Then

$$\sum_{k=0}^{n} |a_k| = \left| \sum_{k \in N^+} a_k \right| + \left| \sum_{k \in N^-} a_k \right| \le 2 \cdot \max_{N \in \{0,\ldots,n\}} \left| \sum_{k \in N} a_k \right|.$$

Now let $a_0, a_1, \ldots, a_n \in \mathbb{C}$. We write $a_k = \alpha_k + i\beta_k$ $(k = 0, 1, \ldots, n)$. For any subset N of $\{0, \ldots, n\}$, we write

$$x_N = \sum_{k \in N} \alpha_k, \; y_N = \sum_{k \in N} \beta_k \quad \text{and} \quad z_N = x_N + iy_N = \sum_{k \in N} a_k.$$

Now we choose subsets N_r, N_i and N_* of $\{0, \ldots, n\}$ such that

$$|x_{N_r}| = \max_{N \subset \{0,\ldots,n\}} |x_N|, \; |y_{N_i}| = \max_{N \subset \{0,\ldots,n\}} |y_N| \quad \text{and} \quad |z_{N_*}| = \max_{N \subset \{0,\ldots,n\}} |z_N|.$$

Then, for all $N \subset \{0, \ldots, n\}$, we have $|x_N|, |y_N| \le |z_{N_*}|$ and $|x_{N_r}| + |y_{N_i}| \le 2 \cdot |z_{N_*}|$. Thus, by the first part of the proof,

$$\sum_{k=0}^{n} |a_k| \le \sum_{k=0}^{n} |\alpha_k| + \sum_{k=0}^{n} |\beta_k| \le 2(|x_{N_r}| + |y_{N_i}|) \le$$
$$\le 4|z_{N_*}| = 4 \cdot \max_{N \subset \{0,\ldots,n\}} \left| \sum_{k \in N} z_k \right|.$$

□

Theorem 3.10. [70, Theorem 2] *Let A be an infinite matrix, B a positive triangle. For each $m \in \mathbb{N}_0$, let N_m be a subset of the set $\{0,1,\dots,m\}$, $N = (N_m)_{m=0}^{\infty}$ the sequence of the subsets N_m and $\mathcal{N}$ the set of all such sequences N. Furthermore, for each $N \in \mathcal{N}$, we define the matrix $S^N = S^N(A)$ by*

$$s_{mk}^N = \sum_{n \in N_m} b_{mn} a_{nk} \ (m,k = 0,1,\dots).$$

Then, for arbitrary subsets X of ω and any normal set Y of sequences, $A \in (X, Y_{[B]})$ if and only if $S^N(A) \in (X,Y)$ for all sequences N in $\mathcal{N}$.

Proof. First we assume $A \in (X, Y_{[B]})$. Then $A_n \in X^\beta$ $(n = 0,1,\dots)$ implies $S_m^N \in X^\beta$ for all m and all $N \in \mathcal{N}$. For each $x \in X$, we put $y = B(|(A(x)|)$. Then $A(x) \in Y_{[B]}$, that is $y \in Y$, and

$$|S_m^N(x)| = \left|\sum_{k=0}^{\infty} s_{mk}^N x_k\right| = \left|\sum_{n\in N_m} b_{mn} \sum_{k=0}^{\infty} a_{nk}x_k\right| \le |y_m| \ (m = 0,1,\dots)$$

for all $N \in \mathcal{N}$ together imply $S^N(x) \in Y$ for all $N \in \mathcal{N}$, since Y is normal. Thus $S^N \in (X,Y)$ for all $N \in \mathcal{N}$.

Conversely we assume $S^N \in (X,Y)$ for all $N \in \mathcal{N}$. Then $S_m^N \in X^\beta$ for all m and for all $N \in \mathcal{N}$, in particular, for $N = (\{m\})_{m=0}^{\infty}$, $S_m^N = b_{mm}A_m \in X^\beta$, hence $A_m \in X^\beta$, since $b_{mm} \neq 0$. Further, let $x \in X$ be given. For every $m = 0,1,\dots,$ we choose the set $N_m^{(0)} \subset \{0,\dots,\}$ such that

$$\left|\sum_{n\in N_m^{(0)}} b_{mn}A_n(x)\right| = \left|\max_{N_m \subset \{0,\dots,m\}} b_{mn}A_n(x)\right|.$$

Then, by Lemma 3.9,

$$|y_m| \le 4 \cdot \left|\sum_{n \in N_m^{(0)}} b_{mn}A_n(x)\right| = 4 \cdot \left|S^{N^{(0)}}(x)\right|.$$

By hypothesis, $S^{N^{(0)}}(x) \in Y$, and the normality of Y implies $y = B(|A(x)|) \in Y$, that is $A \in (X, Y_{[B]})$. □

3.3. Bounded and convergent difference sequences of order m. Now we apply the results of the previous subsections to *sets of of bounded and convergent sequences of order m* which may be considered as ordinary matrix domains of a certain triangle. We shall give their Schauder bases and their α– and β–duals. The results may be found in [69] and [39, 63] in the special case $m = 1$.

Let m denote a positive integer throughout and the operators $\Delta^{(m)}$, $\sum^{(m)} : \omega \mapsto \omega$ be defined by

$$(\Delta^{(1)}x)_k = \Delta^{(1)}x_k = x_k - x_{k-1}, \qquad \left(\overset{(1)}{\sum} x\right)_k = \sum_{j=0}^{k} x_j \qquad (k = 0,1,\dots),$$

$$\Delta^{(m)} = \Delta^{(1)} \circ \Delta^{(m-1)}, \qquad \overset{(m)}{\sum} = \overset{(1)}{\sum} \circ \overset{m-1}{\sum} \qquad (m \ge 2).$$

We shall write $\Delta = \Delta^{(1)}$ for short and use the convention that any term with a negative subscript is equal to naught. For any subset X of ω let

$$X(\Delta^{(m)}) = \{x \in \omega : \Delta^{(m)}x \in X\}.$$

We shall be interested in the cases where $X = c_0$, $X = c$ or $X = l_\infty$. The following results are well known and can be found in [27]:

$$(3.8)\quad (\Delta^{(m)}x)_k = \sum_{j=0}^{m}(-1)^j\binom{m}{j}x_{k-j} = \sum_{j=\max\{0,k-m\}}^{m}(-1)^{k-j}\binom{m}{k-j}x_j \quad (k = 0, 1, \dots),$$

$$(3.9)\qquad \left(\sum^{(m)} x\right)_k = \sum_{j=0}^{k}\binom{m+k-j-1}{k-j}x_j \qquad (k = 0, 1, \dots),$$

$$(3.10)\qquad \sum^{(m)} \circ \Delta^{(m)} = \Delta^{(m)} \circ \sum^{(m)} = \mathrm{id}, \text{ the identity on } \omega,$$

$$(3.11)\qquad \sum_{j=0}^{k}\binom{m+j-1}{j} = \binom{m+k}{k} \quad (k = 0, 1, \dots)$$

$$(3.12)\qquad \begin{cases} \text{there are positive constants } M_1, M_2 \text{ such that} \\ M_1 k^m \le \binom{m+k}{k} \le M_2 k^m \text{ for all } k = 1, 2 \dots \end{cases}$$

As an immediate consequence of Example 1.13 and Theorems 3.3 and 3.5, we obtain

Corollary 3.11. [69, Proposition 1] *Let m be a positive integer. Then the sets $l_\infty(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $c_0(\Delta^{(m)})$ are BK spaces with $\|\cdot\|$ defined by*

$$\|x\| = \sup_k \left|\left(\Delta^{(m)}x\right)_k\right| = \sup_k \left|\sum_{j=0}^{m}(-1)^j\binom{m}{j}x_{k-j}\right|$$

and $c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$ are closed subspaces of $l_\infty(\Delta^{(m)})$.

Now we shall give Schauder bases for the spaces $c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$.

Theorem 3.12. [69, Theorem 1] *Let m be a positive integer. We define the sequences $b^k(m)$ by*

$$b_n^{(-1)}(m) = \binom{m+n}{n} \quad (n = 0, 1, \dots),$$
$$b_n^{(k)}(m) = \begin{cases} 0 & (n \le k-1) \\ \binom{m+n-k-1}{n-k} & (n \ge k) \end{cases} \quad (k = 0, 1, \dots).$$

(a) *Then $(b^{(k)}(m))_{k=0}^\infty$ is a basis of $c_0(\Delta^{(m)})$. More precisely, every sequence $x = (x_k)_{k=0}^\infty \in c_0(\Delta^{(m)})$ has a unique representation*

$$x = \sum_{k=0}^\infty \lambda_k(m) b^{(k)}(m) \quad \text{where } \lambda_k(m) = (\Delta^{(m)}x)_k \quad (k = 0, 1, \dots). \tag{3.13}$$

(b) *Then $(b^{(k)}(m))_{k=-1}^\infty$ is a basis of $c(\Delta^{(m)})$. More precisely, every sequence $x = (x_k)_{k=0}^\infty \in c(\Delta^{(m)})$ has a unique representation*

$$x = l b^{(-1)}(m) + \sum_{k=0}^\infty (\lambda_k(m) - l) b^{(k)}(m) \quad \text{where } l = \lim_{k \to \infty} (\Delta^{(m)}x)_k. \tag{3.14}$$

Proof. (a) For $k = 0, 1, \dots$, we put

$$b^{(k)} = e - \sum_{p=0}^{k-1} e^{(p)}, \text{ that is } b_j^{(k)} = \begin{cases} 0 & (j \le k-1) \\ 1 & (j \ge k). \end{cases}$$

Then by (3.9) and (3.11),

$$\left(\sum^{(m-1)} b^{(k)}\right)_n = \sum_{j=0}^n \binom{m-1+n-j-1}{n-j} b_j^{(k)}$$
$$= \begin{cases} \sum_{j=k}^n \binom{m-1+n-j-1}{n-j} & (n \ge k) \\ 0 & (n \le k-1), \end{cases}$$

$$\sum_{j=k}^n \binom{m-1+n-j-1}{n-j} = \sum_{l=0}^{n-k} \binom{m-1+l-1}{l} = \binom{m-1+n-k}{n-k} \quad (n \ge k),$$

hence $b^{(k)}(m) = \sum^{(m-1)} b^{(k)} \quad (k = 0, 1, \dots)$, and by (3.10),

$$\Delta^{(m)} b^{(k)}(m) = \Delta b^{(k)} = e^{(k)} \in c_0 \quad (k = 0, 1, \dots) \tag{3.15}$$

Thus

$$b^{(k)}(m) \in c_0(\Delta^{(m)}) \quad (k = 0, 1, \dots). \tag{3.16}$$

Let $x = (x_k)_{k=0}^{\infty} \in c_0(\Delta^{(m)})$ be given. For every nonnegative integer p, we put $x^{\langle p \rangle} = \sum_{k=0}^{p} \lambda_k(m) b^{(k)}(m)$. Then by the linearity of $\Delta^{(m)}$ and by (3.15)

$$\Delta^{(m)} x^{\langle p \rangle} = \sum_{k=0}^{p} \lambda_k(m) \Delta^{(m)} b^{(k)}(m) = \sum_{k=0}^{p} (\Delta^{(m)} x)_k e^{(k)}$$

$$\Delta^{(m)}(x - x^{\langle p \rangle})_n = \begin{cases} 0 & (n \leq p) \\ (\Delta^{(m)} x)_n & (n \geq p+1). \end{cases}$$

Given $\varepsilon > 0$ there is an integer p_0 such that $|(\Delta^{(m)} x)_p| < \varepsilon/2$ for all $p \geq p_0$, hence

$$\|x - x^{\langle p \rangle}\| = \sup_{n \geq p} |(\Delta^{(m)} x)_n| \leq \sup_{n \geq p_0} |(\Delta^{(m)})_n| \leq \varepsilon/2 < \varepsilon$$

for all $p \geq p_0$. This proves the representation in (3.13). To show the uniqueness of this representation we assume $x = \sum_{k=0}^{\infty} \mu_k b^{(k)}$. Since $\Delta^{(m)} : c_0(\Delta^{(m)}) \to c_0$ obviously is a continuous linear operator, we have by (3.15)

$$(\Delta^{(m)} x)_n = \sum_{k=0}^{\infty} \mu_k (\Delta^{(m)} b^{(k)})_n = \sum_{k=0}^{\infty} \mu_k e_n^{(k)} = \mu_n \quad (n = 0, 1, \dots).$$

(b) First $b^{(-1)}(m) = \sum^{(m)} e$ implies $\Delta^{(m)} b^{(-1)}(m) = e \in c$, that is $b^{(-1)}(m) \in c(\Delta^{(m)})$. In view of (3.16) and the fact that $c_0(\Delta^{(m)}) \subset c(\Delta^{(m)})$, we have $b^{(k)}(m) \in c(\Delta^{(m)})$ for all $k = -1, 0, 1, \dots$. Let $x = (x_k)_{k=0}^{\infty} \in c(\Delta^{(m)})$ be given. Then there is a unique number l such that (3.14) holds. We put $y = x - l \cdot b^{(-1)}(m)$. Then

$$\Delta^{(m)} y = \Delta^{(m)}(x - l b^{(-1)}(m)) = \Delta^{(m)} x - le, \text{ that is } y \in c_0(\Delta^{(m)}),$$

and it follows from part (a) that x has a unique representation (3.14). □

Now we shall give the α-duals of the sets $c_0(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $l_\infty(\Delta^{(m)})$. If $u \in \mathcal{U}$, then obviously

(3.17) $$(u^{-1} * X)^{\dagger} = (1/u)^{-1} * X^{\dagger} \quad (\dagger \in \{\alpha, \beta\}) \text{ for every subset } X \text{ of } \omega.$$

Theorem 3.13. [69, Theorem 2] *Let m be a positive integer.*
(a) *We put $M^{\alpha}(m) = \{a \in \omega : \sum_{k=0}^{\infty} |a_k| k^m < \infty\}$. Then*

(3.18) $$\left(c_0(\Delta^{(m)})\right)^{\alpha} = \left(c(\Delta^{(m)})\right)^{\alpha} = \left(l_\infty(\Delta^{(m)})\right)^{\alpha} = M^{\alpha}(m).$$

(b) *We put $M^{\alpha\alpha}(m) = \{a \in \omega : \sup_{k \geq 1} |a_k| k^{-m} < \infty\}$. Then*

(3.19) $$\left(c_0(\Delta^{(m)})\right)^{\alpha\alpha} = \left(c(\Delta^{(m)})\right)^{\alpha\alpha} = \left(l_\infty(\Delta^{(m)})\right)^{\alpha\alpha} = M^{\alpha\alpha}(m).$$

Proof. (a) First we assume $a \in M^{\alpha}(m)$. Then

$$\sum_{k=0}^{\infty} |a_k| k^m < \infty. \tag{3.20}$$

Let $x \in l_\infty(\Delta^{(m)})$. Then there is a positive constant M such that $|(\Delta^{(m)}x)_k| \leq M$ $(k = 0, 1, \ldots)$, and by (3.10), (3.9), (3.11), (3.12) and (3.20)

$$\begin{aligned}\sum_{k=0}^{\infty} |a_k x_k| &= \sum_{k=0}^{\infty} |a_k| \left| \left(\sum^{(m)} \left(\Delta^{(m)} x \right) \right)_k \right| \\ &\leq \sum_{k=0}^{\infty} |a_k| \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} \left| \left(\Delta^{(m)} x \right)_j \right| \\ &\leq M \sum_{k=0}^{\infty} |a_k| \binom{m+k}{k} \leq M \cdot M_2 \sum_{k=0}^{\infty} |a_k| k^m < \infty.\end{aligned}$$

Thus we have shown

$$M^{\alpha}(m) \subset \left(l_\infty(\Delta^{(m)}) \right)^{\alpha}. \tag{3.21}$$

Conversely let $a \notin M^{\alpha}(m)$. By (3.12), there is a sequence $(k(s))_{s=0}^{\infty}$ of integers $0 = k(0) < k(1) < \ldots$ such that

$$\sum_{k=k(s)}^{k(s+1)-1} |a_k| \binom{m+k}{k} \geq s+1 \quad (s = 0, 1, \ldots). \tag{3.22}$$

We define the sequence x by

$$x_k = \sum_{l=0}^{s-1} \frac{1}{l+1} \sum_{j=k(l)}^{k(l+1)-1} \binom{m+k-j-1}{k-j} + \frac{1}{s+1} \sum_{j=k(s)}^{k} \binom{m+k-j-1}{k-j}$$
$$(k(s) \leq k \leq k(s+1) - 1; s = 0, 1, \ldots).$$

If we define the sequence $y \in c_0$ by $y_k = 1/(s+1)$ for $k(s) \leq k \leq k(s+1) - 1$ $(s = 0, 1, \ldots)$, then it easily follows from (3.9) that $x = \sum^{(m)} y$. Thus $\Delta^{(m)} x = y \in c_0$ and $x \in c_0(\Delta^{(m)})$. On the other hand by (3.11) and (3.22)

$$\begin{aligned}\sum_{k=k(s)}^{k(s+1)-1} |a_k x_k| &\geq \sum_{k=k(s)}^{k(s+1)-1} |a_k| \frac{1}{s+1} \sum_{j=0}^{k} \binom{m+j-1}{j} \\ &= \frac{1}{s+1} \sum_{k=k(s)}^{k(s+1)-1} |a_k| \binom{m+k}{k} \geq 1 \quad (s = 0, 1, \ldots).\end{aligned}$$

Thus $a \notin c_0(\Delta^{(m)})$, and we have shown

$$(c_0(\Delta^{(m)}))^\alpha \subset M^\alpha(m). \tag{3.23}$$

Since $c_0(\Delta^{(m)}) \subset c(\Delta^{(m)}) \subset l_\infty(\Delta^{(m)})$, (3.18) follows from (3.21) and (3.23).

(b) Since $y = (x_{k+1})_{k=0}^\infty \in M^\alpha(m)$ if and only if $y \in ((k+1)^m)_{k=0}^\infty)^{-1} * l_1$, and since $l_1^\alpha = l_\infty$, identity (3.19) follows from (3.17) and part (a). □

To determine the β–duals of the sets $c_0(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $l_\infty(\Delta^{(m)})$, we need a few results.

Lemma 3.14. [69, Lemma 1] *Let m be a positive integer. Then for arbitrary sequences a*

$$\left(a_k\binom{m+k}{k}\right)_{k=1}^\infty \in cs \quad \textit{if and only if} \quad (a_k k^m)_{k=1}^\infty \in cs.$$

Proof. We define the sequences b and c by

$$b_k = \frac{\binom{m+k}{k}}{k^m} \quad (k = 1, 2, \dots) \quad \text{and} \quad c = 1/b.$$

Since $cs^\beta = bv$ [108, Theorem 7.3.5(v), p. 110], it suffices to show that $b, c \in bv$. It is well known that $\lim_{k\to\infty} b_k = 1/m!$ [27, p. 97]. Therefore we have to show that b is monotone. We define the function f on $[0, 1/2]$ by $f(x) = (1+mx)(1-x)^m$. Then $f'(x) = -m(1-x)^{m-1}(m+1)x \leq 0$ for all $x \in [0, 1/2]$, whence $f(x) \leq f(0) = 1$ for all $x \in [0, 1/2]$. Thus

$$\frac{b_{k+1}}{b_k} = \frac{k+1+m}{k+1}\frac{k^m}{(k+1)^m} = \left(1+\frac{m}{k+1}\right)\left(1-\frac{1}{k+1}\right)^m = f\left(\frac{1}{k+1}\right) \leq 1$$

for all $k \geq 1$. □

Lemma 3.15. [63, Lemma 1] *Let (P_n) be a sequence of non decreasing positive reals. Then $y \in cs$ implies*

$$\lim_{n\to\infty}\left(P_n \sum_{k=1}^\infty \frac{y_{n+k-1}}{P_{n+k}}\right) = 0.$$

Proof. The proof can be found in [39, Lemma 3] and [63, Lemma 1]. □

Corollary 3.16. [63, Corollary 1] *Let $(P_n)_{n=1}^\infty$ be a sequence of nondecreasing positive reals. Then $a \in (P_n)^{-1} * cs$ implies $R \in (P_n)^{-1} * c_o$ where $R_n = \sum_{k=n+1}^\infty a_k$ $(n = 1, 2, \dots)$.*

Proof. Put $y_k = P_{k+1}a_{k+1}$ $(k = 1, 2, \dots)$ in Lemma 3.15. □

We shall frequently apply the following two versions of Abel's summation by parts: Let $b, c \in \omega$. We put

$$s = s(c) = \sum^{(1)} c \quad \text{and, if } c \in cs,\ R_k = R_k(c) = \sum_{j=k}^{\infty} c_k \ (k = 0, 1, \dots).$$

Then

$$\sum_{k=0}^{n} b_k c_k = -\sum_{k=0}^{n} s_k \Delta b_{k+1} + s_n b_{n+1} \quad (n = 0, 1, \dots) \tag{3.24}$$

$$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n} R_k \Delta b_k - b_n R_{n+1} \quad (n = 0, 1, \dots). \tag{3.25}$$

Theorem 3.17. [69, Theorem 3] *Let m be a positive integer.*

(a) *We put $R_k^1 = R_k = \sum_{j=k}^{\infty} a_j$, $R_k^{(m)} = \sum_{j=k}^{\infty} R_j^{(m-1)}$ $(k = 0, 1, \dots)$ for $m \geq 2$ and*

$$M_{\infty}^{\beta}(m) = \left\{ a \in \omega : \sum_{k=0}^{\infty} a_k k^m \text{ converges and } \sum_{k=0}^{\infty} |R_k^{(m)}| < \infty \right\}.$$

Then

$$\left(c(\Delta^{(m)})\right)^{\beta} = \left(l_{\infty}(\Delta^{(m)})\right)^{\beta} = M_{\infty}^{\beta}(m). \tag{3.26}$$

(b) *Further, let c_0^+ denote the set of all positive sequences in c_0. We put*

$$M_0^{\beta}(m) = \left\{ a \in \omega : \sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} v_j \text{ converges for all } v \in c_0^+ \right\}$$
$$\bigcap \left\{ a \in \omega : \sum_{k=0}^{\infty} |R_k^{(m)}| < \infty \right\}.$$

Then

$$\left(c_0(\Delta^{(m)})\right)^{\beta} = M_0^{\beta}(m). \tag{3.27}$$

Proof. (a) For all positive integers p let $s^{(p)} = \sum^{(p)} e$; we write $s = s^{(1)}$. First we assume $m = 1$ and write $M_{\infty}^{\beta} = M_{\infty}^{\beta}(1)$. Let $a \in M_{\infty}^{\beta}$. Then

$$R \in l_1 \tag{3.28}$$

$$as \in cs \tag{3.29}$$

Now condition (3.29) and Corollary 3.16 together imply

$$(3.30)\qquad \left(R_{n+1}s_n\right)_{n=0}^{\infty} \in c_0.$$

Let $x \in l_\infty(\Delta)$. From (3.25) with $b = x$ and $c = a$, we have

$$(3.31)\qquad \sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} R_k(\Delta x)_k - R_{n+1}x_n \ (n = 0, 1, \dots).$$

Since $\Delta x \in l_\infty$, condition (3.28) implies

$$(3.32)\qquad R\Delta x \in cs.$$

Further there is a constant $M > 0$ such that $|(\Delta x)_k| \leq M$ for all k and so $|x_n| \leq |(\sum^{(1)}(\Delta^{(1)}x))_k| \leq M(n+1) = M\dot{s}_n$ for all n. Now condition (3.30) implies

$$(3.33)\qquad \left(R_{n+1}x_n\right)_{n=0}^{\infty} \in c_0.$$

Finally (3.31), (3.32) and (3.33) together imply $ax \in cs$ for all $x \in l_\infty$, that is $a \in (l_\infty(\Delta))^\beta$.

Conversely, let $a \in (c(\Delta))^\beta$. Then $ax \in cs$ for all $x \in c(\Delta)$. First $e \in c(\Delta)$ implies $a = ae \in cs$, hence the sequence R is defined. Further, for $x = s$, we have $\Delta x = e \in c$, that is $x \in c(\Delta)$, and condition (3.29) holds. By Corollary 3.16, we have (3.30), and again this yields (3.33) for all $x \in c(\Delta)$. From (3.31) we conclude $R\Delta x \in cs$ for all $x \in c(\Delta)$, and so $R \in c^\beta = l_1$. Now we assume that identity (3.26) holds for some integer $m \geq 1$. Let $a \in M_\infty^\beta(m+1)$. Then by

$$(3.34)\qquad R^{(m+1)} = R^m(R) \in l_1.$$

and, by Lemma 3.14,

$$(3.35)\qquad as^{(m+1)} \in cs$$

Applying identity (3.24) with $b = R$ and $c = s^{(m)}$ we obtain

$$(3.36)\qquad \sum_{k=0}^{n} s_k^{(m)} R_k = \sum_{k=0}^{n} a_k s_k^{(m+1)} + R_{n+1}s_n^{(m+1)} \quad (n = 0, 1, \dots).$$

By Corollary 3.16, condition (3.35) implies

$$(3.37)\qquad \left(R_{n+1}s_n^{(m+1)}\right)_{n=0}^{\infty} \in c_0$$

and consequently by (3.36)

$$(3.38)\qquad s^{(m)}R \in cs$$

Now, by assumption, (3.34) and (3.38) together imply

$$R \in \left(l_\infty(\Delta^{(m)})\right)^\beta. \tag{3.39}$$

Let $x \in l_\infty(\Delta^{(m+1)})$ be given. Since $x \in l_\infty(\Delta^{(m+1)})$ if and only if $y = \Delta x \in l_\infty(\Delta^{(m)})$, condition (3.39) implies

$$R\Delta x \in cs \quad \text{for all } x \in l_\infty(\Delta^{(m+1)}). \tag{3.40}$$

Further there is a positive constant M such that $|(\Delta^{(m+1)}x)_j| \leq M$ $(j = 0, 1, \dots)$ and thus

$$\begin{aligned}
|x_k| = \left|\left(\sum^{(m+1)} (\Delta^{(m+1)}(x))\right)_k\right| &\leq \sum_{j=0}^{k} \binom{m+k-j}{k-j} \left|\left(\Delta^{(m+1)}x\right)_j\right| \\
&\leq M \cdot \sum_{j=0}^{k} \binom{m+k-j}{k-j} = M\left(\sum^{(m+1)} e\right)_k = M \cdot s_k^{(m+1)}
\end{aligned}$$

for $k = 0, 1, \dots$, and condition (3.37) implies

$$\left(R_{n+1}x_n\right)_{n=0}^\infty \in c_0. \tag{3.41}$$

Finally (3.31), (3.40) and (3.41) together imply $ax \in cs$ for all $x \in l_\infty(\Delta^{(m+1)})$, consequently $a \in \left(l_\infty(\Delta^{(m+1)})\right)^\beta$.

Conversely let $a \in \left(c(\Delta^{(m+1)})\right)^\beta$. Then $ax \in cs$ for all $x \in c(\Delta^{(m+1)})$. First, $e \in c(\Delta^{(m+1)})$ implies $a = ae \in cs$, hence the sequence R is defined. Further for $x = s^{(m+1)}$ we have $\Delta^{(m+1)}x = \Delta^{(m+1)}(\sum^{(m+1)}e) = e \in c$, that is $x \in c(\Delta^{(m+1)})$, and condition (3.35) is satisfied. By Corollary 3.16, we have (3.37) and again this yields (3.41) for all $x \in c(\Delta^{(m+1)})$. From (3.31) we conclude $R\Delta x \in cs$ for all $x \in c(\Delta^{(m+1)})$ and consequently $R \in (c(\Delta^{(m)}))^\beta$. This implies $R^{(m+1)} = R^{(m)}(R) \in l_1$ by assumption.

(b) For all positive integers p and all sequences $v \in c_0$, let $t^{(p)}(v) = \sum^{(p)}(v)$; we write $t(v) = t^{(1)}(v)$. The proof of part (b) is exactly the same as that of part (a) with s, $s^{(m)}$ and $s^{(m+1)}$ replaced by $t(v)$, $t^{(m)}(v)$ and $t^{(m+1)}(v)$. □

Remark 3.18. By [63, Theorem 2 (c)] it is obvious that $(c_0(\Delta^{(m)}))^\beta \neq (l_\infty(\Delta^{(m)}))^\beta$.

3.4. Matrix transformations in the spaces $c_0(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $l_\infty(\Delta^{(m)})$ and their measures of noncompactness. In this subsection we shall characterize matrix transformations between the spaces of bounded and convergent m–th order difference sequences and apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for these matrix maps to be compact operators.

Lemma 3.19. [69, Lemma 4] *Let m be a positive integer Then*

$$\|a\|^* = \|R^{(m)}\|_1 = \sum_{k=0}^{\infty} |R_k^{(m)}| \tag{3.42}$$

on any of the spaces $(c_0(\Delta^{(m)}))^\beta$, $(c(\Delta^{(m)}))^\beta$ and $(l_\infty(\Delta^{(m)}))^\beta$.

Proof. Let X be any of the sequences c_0, c or l_∞. If $m = 1$ and $a \in (X(\Delta^{(1)}))^\beta$, then

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k^{(k)} \Delta^{(1)} x_k \quad \text{for all } x \in X(\Delta^{(1)}) \tag{3.43}$$

by the proof of Theorem 3.17. Since $x \in X(\Delta^{(1)})$ if and only if $\Delta^{(1)}x \in X$, this implies $R^{(1)} \in X^\beta = l_1$. It is well known that $\|\cdot\|^* = \|\cdot\|_1$ on X^β, and (3.42) follows from the definition of the norm on $X(\Delta^{(1)})$.

Now we assume that (3.42) holds for some integer $m \geq 1$. Let $a \in X(\Delta^{(m+1)})$. Again, by the proof of Theorem 3.17, (3.43) holds for all $x \in X(\Delta^{(m+1)})$. Since $x \in X(\Delta^{(m+1)})$ if and only if $\Delta^{(1)} \in X(\Delta^{(m)})$, this implies $R^{(1)} \in (X(\Delta^{(m)}))^\beta$, and by assumption $\|a\|^* = \|R^{(m)}(R^{(1)})\|_1 = \|R^{(m+1)}\|_1$. □

Theorem 3.20. [69, Theorem 4] *Let m be a positive integer and A be an infinite matrix. For each n, we put $R_{nk}^{(1)} = R_{nk} = \sum_{j=k}^{\infty} a_{nj}$ and $R_n^{(m)} = \sum_{j=k}^{\infty} R_{nj}^{(m-1)}$ for $m \geq 2$.*

(a) *Then $A \in (l_\infty(\Delta^{(m)}), l_\infty)$ if and only if*

$$\sum_{k=0}^{\infty} k^m a_{nk} \quad \text{converges for all } n = 0, 1, \ldots \tag{3.44}$$

and

$$\sup_n \sum_{k=0}^{\infty} |R_{nk}^{(m)}| < \infty. \tag{3.45}$$

Further $(l_\infty(\Delta^{(m)}), l_\infty) = (c(\Delta^{(m)}), l_\infty)$.

(b) *Then $A \in (c_0(\Delta^{(m)}), l_\infty)$ if and only if condition (3.45) holds and*

$$\sum_{k=0}^{\infty} a_{nk} \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} v_j \quad \text{converges for all } v \in c_0^+ \tag{3.46}$$

and for all $n = 0, 1, \ldots$.

(c) *Then $A \in (c_0(\Delta^{(m)}), c_0)$ if and only if conditions (3.45) and (3.46) hold and*

$$\lim_{n\to\infty} \left(\sum_{j=k}^{\infty} \binom{m-1+j-k}{j-k} a_{nj} \right) = 0 \quad (k = 0, 1, \ldots). \tag{3.47}$$

(d) *Then* $A \in (c_0(\Delta^{(m)}), c)$ *if and only if conditions (3.45) and (3.46) hold and*

$$\lim_{n\to\infty}\left(\sum_{j=k}^{\infty}\binom{m-1+j-k}{j-k}a_{nj}\right) = l_k \quad (k = 0, 1, \ldots). \tag{3.48}$$

(e) *Then* $A \in (c(\Delta^{(m)}), c_0)$ *if and only if conditions (3.45), (3.46), (3.47) hold and*

$$\lim_{n\to\infty}\left(\sum_{j=0}^{\infty}\binom{m+j}{j}a_{nj}\right) = 0. \tag{3.48}$$

(f) *Then* $A \in (c(\Delta^{(m)}), c)$ *if and only if conditions (3.45), (3.46), (3.48) hold and*

$$\lim_{n\to\infty}\left(\sum_{j=0}^{\infty}\binom{m+j}{j}a_{nj}\right) = l_{-1}. \tag{3.50}$$

Proof. (a) Let $A \in (l_\infty(\Delta^{(m)}), l_\infty)$. Then $A_n \in (l_\infty(\Delta^{(m)}))^\beta$ for $n = 0, 1, \ldots$, and, by Theorem 3.17 (a), condition (3.44) holds for all n and $\sum_{n=0}^{\infty}|R_{nk}^{(m)}| < \infty$ $(n = 0, 1, \ldots)$. Further $\|A\|^* = \sup_n(\sum_{k=0}^{\infty}|R_{nk}^{(m)}|) < \infty$ by Theorem 1.23 (b) and Lemma 3.19. Conversely let conditions (3.44) and (3.45) hold. By Theorem 3.17 (a), this implies $A_n \in (l_\infty(\Delta^{(m)}))^\beta$ for all n, and again Theorem 1.23 (b) and Lemma 3.19 together imply $A \in (l_\infty(\Delta^{(m)}), l_\infty)$. Clearly $(l_\infty(\Delta^{(m)}), l_\infty) \subset (c(\Delta^{(m)}), l_\infty)$. If $A \in (c(\Delta^{(m)}), l_\infty)$, then condition (3.45) follows from Theorems 1.23 (b) and 3.17 (a) and Lemma 3.19. Further $A_n \in (c(\Delta^{(m)}))^\beta = (l_\infty(\Delta^{(m)}))^\beta$ and condition (3.44) holds (see Theorem 3.17 (a)). Therefore $A \in (l_\infty(\Delta^{(m)}), l_\infty)$ by what we have shown above.

(b) The proof of part (b) is exactly the same as that of the first part of part (a) with condition (4.3) and Theorem 3.17 (a) replaced by condition (3.46) and Theorem 3.17 (b).

Parts (c) to (f) follow from Theorem 1.23 (c) and parts (a) or (b), since $c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$ are closed subspaces of $l_\infty(\Delta^{(m)})$ by Corollary 3.11. □

As a corollary of Theorems 1.23 and 3.8, we have

Corollary 3.21. ([71, Corollary 3.5] *Let X be a BK space.*

(a) *Then* $A \in (X, l_\infty(\Delta^{(m)}))$ *if and only if*

$$M(X, l_\infty(\Delta^{(m)})) = \sup_n\left\|\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}A_l\right\|^* < \infty. \tag{3.51}$$

(b) *Further, if* $(b^k)_{k=0}^{\infty}$ *is a basis of* X, *then* $A \in (X, c_0(\Delta^{(m)}))$ *if and only if condition (3.51) holds and*

$$\lim_{n\to\infty}\left(\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}A_l(b^{(k)})\right) = 0 \quad \text{for each } k; \tag{3.52}$$

$A \in (X, c(\Delta^{(m)}))$ if and only if condition (3.51) holds and

$$\text{(3.53)} \quad \lim_{n\to\infty}\left(\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}A_l(b^{(k)})\right)=\alpha_k \quad \textit{for each } k=0,1,\dots;$$

Remark 3.22. (a) If $X = l_p$ $(1 \le p < \infty)$ and Y is any of the spaces $l_\infty(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $c_0(\Delta^{(m)})$, then the conditions for $A \in (X, Y)$ follow from the respective ones in Corollary 3.21 by replacing the norm $\|\cdot\|^*$ in condition (3.51) by the natural norm on the β–dual of l_p, that is on l_q $(q = p/(p-1), 1 < p < \infty; q = \infty, p = 1)$ which is norm isomorphic to l_p^*. Hence we have

$$M(l_p, l_\infty(\Delta^{(m)})) = \begin{cases} \sup_n\left(\sum_{k=0}^{\infty}\left|\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}a_{lk}\right|^q\right) & (1<p<\infty) \\ \sup_{n,k}\left|\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}a_{lk}\right| & (p=1). \end{cases}$$

(b) Let s be a nonnegative integer. If X is any of the spaces $l_\infty(\Delta^{(s)})$, $c(\Delta^{(s)})$ and $c_0(\Delta^{(s)})$, and Y is any of the spaces $l_\infty(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $c_0(\Delta^{(m)})$, then the conditions for $A \in (X, Y)$ are obtained from the respective ones in Theorem 3.20 by replacing the entries of the matrix A by those of the matrix $B = TA$, for instance

$$\sup_n \|B_n\|^* = \sup_n \|R^{(s)}(B_n)\|_1 < \infty$$

where

$$B_n = \sum_{l=\max\{0,n-m\}}(-1)^{n-l}\binom{m}{n-l}A_l.$$

Theorem 3.23. [71, Theorem 4] *Let A be as in Theorem 3.20, and for any integers m, n, r, $n > r$, set*

$$\text{(3.54)} \qquad \|A\|^{(r)} = \sup_{n>r}\|R^{(m)}(A_n)\|_1$$

Let X be either $c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, and let $A \in (X, c_0)$. Then we have

$$\text{(3.55)} \qquad \|L_A\|_\chi = \lim_{r\to\infty}\|A\|^{(r)}.$$

Let X be either $c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, and let $A \in (X, c)$. Then we have

$$\text{(3.56)} \qquad \frac{1}{2}\cdot\lim_{r\to\infty}\|A\|^{(r)} \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}.$$

Let X be either $l_\infty(\Delta^{(m)})$, $c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, and let $A \in (X, l_\infty)$. Then we have

$$\text{(3.57)} \qquad 0 \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}.$$

Proof. Let us remark that the limits in (3.55), (3.56) and (3.57) exist. We put $B = \{x \in X : \|x\| \leq 1\}$. In the case $A \in (X, c_0)$ for $X = c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, we have by Theorem 2.23

$$\|L_A\|_\chi = \chi(A(B)) = \lim_{r\to\infty}\left[\sup_{x\in B}\|(I-P_r)(A(x))\|\right], \tag{3.58}$$

where $P_r : c_0 \mapsto c_0$ for $r = 0, 1, \ldots$ is the projector on the first $r+1$ coordinates, that is $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for $x = (x_k) \in c_0$; (let us remark that $\|I - P_r\| = 1$ for $r = 0, 1, \ldots$). Further we have by Theorem 3.20

$$\|A\|^{(r)} = \sup_{x\in B}\|(I-P_r)(A(x))\|, \tag{3.59}$$

and by (3.58) we get (3.55).

To prove (3.56) let us remark that every sequence $x = (x_k)_{k=0}^{\infty} \in c$ has a unique representation $x = le + \sum_{k=0}^{\infty}(x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in c_0$. Let us define $P_r : c \mapsto c$ by $P_r(x) = le + \sum_{k=0}^{r}(x_k - l)e^{(k)}$ for $r = 0, 1, \ldots$. It is easy to prove that $\|I - P_r\| = 2$ for $r = 0, 1, \ldots$. Now the proof of (3.56) is similar as in the case (3.55), and we omit it.

To prove (3.57), we define $P_r : l_\infty \mapsto l_\infty$ by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for $x = (x_k) \in l_\infty$ and $r = 0, 1, \ldots$. It is clear that $A(B) \subset P_r(A(B)) + (I - P_r)(A(B))$. Now, by the elementary properties of function the χ we have

$$\begin{aligned}\chi(A(B)) &\leq \chi(P_r(A(B))) + \chi((I-P_r)(A(B))) = \chi((I-P_r)(A(B))\\ &\leq \sup_{x\in B}\|(I-P_r)(A(x))\|.\end{aligned} \tag{3.60}$$

Finally we get (3.57) by Theorem 3.20. □

As a corollary of the theorem above, we have

Corollary 3.24. [71, Corollary 4.3] *Let A be as in Theorem 3.23. Then if $A \in (X, c_0)$ for $X = c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, or if $A \in (X, c)$ for $X = c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, then in all cases we have*

$$L_A \quad \text{is compact if and only if} \quad \lim_{r\to\infty}\|A\|^{(r)} = 0. \tag{3.61}$$

Further, if $A \in (X, l_\infty)$ for $X = l_\infty(\Delta^{(m)})$, $X = c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, then we have

$$L_A \quad \text{is compact if} \quad \lim_{r\to\infty}\|A\|^{(r)} = 0. \tag{3.62}$$

The following example will show that it is possible for L_A in (3.62) to be compact in the case $\lim_{r\to\infty}\|A\|^{(r)} > 0$, and hence in general we have just "if" in (3.62).

Example 3.25. Let the matrix A be defined by $A_n = e^{(0)}$ $(n = 0, 1, \dots)$. Then obviously $R^{(m)}(A_n) = e^{(0)}$ for all n, and $A \in (l_\infty(\Delta^{(m)}), l_\infty)$. Further,

$$\|A\|^{(r)} = \sup_{n>r} \|R^{(m)}(A_n)\|_1 = \sup_{n>r} \|e^{(0)}\|_1 = 1 > 0 \quad \text{for all } r,$$

whence $\lim_{r\to\infty} \|A\|^{(r)} > 0$. Since $A(x) = x_0 e$ for all $x \in l_\infty(\Delta^{(m)})$, A is a compact operator.

Concerning Corollary 3.2.1 and the measures of noncompactness we have

Theorem 3.26. [71, Theorem 4.5] *Let X be a BK space and let A be as in Corollary 3.21 Then for all integers m, n, r, $n > r$, we put*

$$\|A\|_\Delta^{(r)} = \sup_{n>r} \left\| \sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} A_j \right\|^* . \tag{3.63}$$

Further, if X has a Schauder basis, and $A \in (X, c_0(\Delta^{(m)}))$, then we have

$$\|L_A\|_\chi = \lim_{r\to\infty} \|A\|_\Delta^{(r)}. \tag{3.64}$$

If X has a Schauder basis, and $A \in (X, c(\Delta^{(m)}))$, then we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \|A\|_\Delta^{(r)} \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|_\Delta^{(r)}. \tag{3.65}$$

Finally, if $A \in (X, l_\infty(\Delta^{(m)}))$, then we have

$$0 \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|_\Delta^{(r)}. \tag{3.66}$$

Proof. Let us remark that the limits in (3.64), (3.65) and (3.66) exist. We put $B = \{x \in X : \|x\| \le 1\}$. To prove (3.64), we have by Theorems 3.12 and 2.23

$$\|L_A\|_\chi = \chi(A(B)) = \lim_{r\to\infty} \left[\sup_{x\in B} \|(I - P_r)(A(x))\| \right], \tag{3.67}$$

where $P_r : c_0(\Delta^{(m)}) \mapsto c_0(\Delta^{(m)})$ $(r = 0, 1, \dots)$ is the projector defined by

$$P_r(x) = \sum_{k=0}^{r} \lambda_k(m) b^{(k)}(m), \tag{3.68}$$

for $x = \sum_{k=0}^{\infty} \lambda_k(m) b^{(k)}(m) \in c_0(\Delta^{(m)})$ and the Schauder basis $(b^{(k)}(m))_{k=0}^{\infty}$ of $c_0(\Delta^{(m)})$ (see Theorem 3.12). Let us remark that $\|I - P_r\| = 1$ for $(r = 0, 1, \dots)$. Further we have by Theorem 3.8

$$\|A\|_\Delta^{(r)} = \sup_{x\in B} \|(I - P_r)(A(x))\|, \tag{3.69}$$

To prove (3.65), let us remark (see Theorem 3.12) that $c(\Delta^{(m)})$ has the Schauder basis $b^{(k)}(m)$ $k = -1, 0, 1, \dots$, and every $x \in c(\Delta^{(m)})$ has a unique representation

$$x = lb^{(-1)}(m) + \sum_{k=0}^{\infty}(\lambda_k(m) - l)b^{(k)}(m) \quad \text{where} \quad l = \lim_{k\to\infty}(\Delta^{(m)}x)_k. \tag{3.70}$$

Now let us define $P_r : c(\Delta^{(m)}) \mapsto c(\Delta^{(m)})$ $(r = 0, 1, \dots)$ by

$$P_r(x) = lb^{(-1)}(m) + \sum_{k=0}^{r}(\lambda_k(m) - l)b^{(k)}(m). \tag{3.71}$$

It is easy to show that $\|I - P_r\| = 2$ for $r = 0, 1, \dots$. Now the proof of (3.65) is similar as in the case (3.64), and we omit it.

Finally in order to prove (3.66), we define $P_r : l_\infty(\Delta^{(m)}) \mapsto l_\infty(\Delta^{(m)})$, by $P_r(x) = (x_0, x_1, \dots, x_r, 0, 0, \dots)$ for $x = (x_k) \in l_\infty(\Delta^{(m)})$ and $r = 0, 1, \dots$. It is clear that $A(B) \subset P_r(A(B)) + (I - P_r)(A(B))$. Now, by the elementary properties of the function χ, we again have (3.60) and, then (3.66) by Theorem 3.8 and Corollary 3.21. □

As a corollary of the theorem above, we have

Corollary 3.27. [71, Corollary 4.6] *Let X be a BK space and let A and $\|A\|_\Delta^{(r)}$ be as in Theorem 3.26. If X has a Schauder basis, and either $A \in (X, c_0(\Delta^{(m)}))$ or $A \in (X, c(\Delta^{(m)}))$, then L_A is compact if and only if $\lim_{r\to\infty}\|A\|_\Delta^{(r)} = 0$. Further, if $A \in (X, l_\infty(\Delta^{(m)}))$, then L_A is compact if $\lim_{r\to\infty}\|A\|_\Delta^{(r)} = 0$.*

Finally we obtain several corollaries concerning Remark 3.22.

Corollary 3.28. [71, Corollary 4.7] *If either $A \in (l^p, c_0(\Delta^{(m)}))$ or $A \in (l^p, c(\Delta^{(m)}))$ $(1 < p < \infty)$, then*

L_A is compact if and only if

$$\lim_{r\to\infty}\sup_{n>r}\left(\sum_{k=0}^{\infty}\left|\sum_{j=\max\{0,n-m\}}^{n}(-1)^{n-j}\binom{m}{n-j}a_{jk}\right|^q\right) = 0, \quad q = p/(p-1).$$

Further, if either $A \in (l_\infty, c_0(\Delta^{(m)}))$ or $A \in (l_\infty, c(\Delta^{(m)}))$, then

L_A is compact if and only if

$$\lim_{r\to\infty}\sup_{n>r,k}\left|\sum_{j=\max\{0,n-m\}}^{n}(-1)^{n-j}\binom{m}{n-j}a_{jk}\right| = 0.$$

If $A \in (l^p, l_\infty(\Delta^{(m)}))$ for $1 < p < \infty$, then

L_A is compact if

$$\lim_{r\to\infty}\sup_{n>r}\left(\sum_{k=0}^{\infty}\left|\sum_{j=\max\{0,n-m\}}^{n}(-1)^{n-j}\binom{m}{n-j}a_{jk}\right|^q\right) = 0, \quad q = p/(p-1).$$

Finally, if $A \in (l_\infty, l_\infty(\Delta^{(m)}))$, *then*

$$L_A \text{ is compact if } \lim_{r\to\infty} \sup_{n>r,k} \left| \sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} a_{jk} \right| = 0.$$

From Corollary 3.24, Theorem 3.20 and Remark 3.22, we have

Corollary 3.29. [71, Corollary 4.8] *Let* s *and* m *be non negative integers. If* $A \in (X, c_0(\Delta^{(m)})$ *for* $X = c_0(\Delta^{(s)})$ *or* $X = c(\Delta^{(s)})$, *or if* $A \in (X, c(\Delta^{(m)})$ *for* $X = c_0(\Delta^{(s)})$ *or* $X = c(\Delta^{(s)})$, *then in all cases we have*

L_A *is compact if and only if*

$$\lim_{r\to\infty} \sup_{n>r} \left\| R^{(s)} \left(\sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} A_j \right) \right\|_1 = 0.$$

Further, if $A \in (X, l_\infty(\Delta^{(m)})$ *for* $X = l_\infty(\Delta^{(s)})$, $X = c_0(\Delta^{(s)})$ *or* $X = c(\Delta^{(s)})$, *then we have*

$$L_A \text{ is compact if } \lim_{r\to\infty} \sup_{n>r} \left\| R^{(s)} \left(\sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} A_j \right) \right\|_1 = 0.$$

3.5. Sequences of weighted means. In this subsection, we shall study sets of *weighted means sequences* , give their bases and determine their β- and continuous duals. The results can be found in [34].

Let $(q_k)_{k=0}^{\infty}$ be a positive sequence, Q be the sequence with $Q_n = \sum_{k=0}^{n} q_k$ $(n = 0, 1, \dots)$ and the matrix $\bar{N}_q$ be defined by

$$(\bar{N}_q)_{n,k} = \begin{cases} q_k/Q_n & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots).$$

Then we define the sets $(\bar{N}, q)_0 = (c_0)_{\bar{N}_q}$, $(\bar{N}, q) = c_{\bar{N}_q}$ and $(\bar{N}, q)_\infty = (l_\infty)_{\bar{N}_q}$ of sequences that are $(\bar{N}, q)$ summable to naught, summable and bounded, respectively.

For any $x \in X$, we write $\tau = \tau(x)$ for the sequence defined by

$$\tau_n = (\bar{N}_q)_n(x) = \frac{1}{Q_n} \sum_{k=0}^{n} q_k x_k \ (n = 0, 1, \dots),$$

and τ is called the sequence of the $\bar{N}_q$ or *weighted means of* x. As an immediate consequence of Example 1.13 and Theorems 3.3 and 3.5, we obtain

Proposition 3.30. (cf. [34, Corollary 1]) *Each of the sets $(\bar{N},q)_0$, $(\bar{N},q)$ and $(\bar{N},q)_\infty$ is a BK space with*

$$\|x\|_{\bar{N}_q} = \sup_n \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k x_k \right|.$$

Further, if $Q_n \to \infty$ $(n \to \infty)$, then $(\bar{N},q)_0$ has AK, and every sequence $x = (x_k)_{k=0}^\infty \in (\bar{N},q)$ has a unique representation $x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in (\bar{N},q)_0$.

We define the operator $\Delta^+ : \omega \mapsto \omega$ by $\Delta^+ x = ((\Delta^+ x)_k)_{k=0}^\infty = (x_k - x_{k+1})_{k=0}^\infty$.

Theorem 3.31. [34, Theorem 6] *Let $q = (q_k)_{k=0}^\infty$ be a positive sequence and Q the sequence with $Q_n = \sum_{k=0}^n q_k$ $(n = 0,1,\dots)$. We write $1/q$ for the sequence $(1/q_n)_{n=0}^\infty$, and put $M_1 = \{a \in \omega : Q(\Delta^+ a) \in l_1\}$, $\mathcal{N}_0 = (1/q)^{-1} * (M_1 \cap (Q^{-1} * l_\infty))$, $\mathcal{N} = (1/q)^{-1} * (M_1 \cap (Q^{-1} * c))$ and $\mathcal{N}_\infty = (1/q)^{-1} * (M_1 \cap (Q^{-1} * c_0))$. Then $(\bar{N},q)_0^\beta = \mathcal{N}_0$, $(\bar{N},q)^\beta = \mathcal{N}$ and $(\bar{N},q)_\infty^\beta = \mathcal{N}_\infty$.*

Proof. We put $X_1 = (1/q)^{-1} * M_1$ and observe that

$$x_k = \frac{1}{q_k}(Q_k\tau_k - Q_{k-1}\tau_{k-1}) \ (k = 0,1,\dots) \quad \text{for all } x \in \omega \tag{3.72}$$

and for all $n = 0,1,\dots$

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^n \frac{a_k}{q_k}\Delta(Q_k\tau_k) = \sum_{k=0}^{n-1}\left(Q_k\tau_k\Delta^+\left(\frac{a_k}{q_k}\right)\right) + \frac{a_nQ_n}{q_n}\tau_n. \tag{3.73}$$

Let $a \in X_1$, that is $Q\Delta^+(a/q) \in l_1 = l_\infty^\beta$. Thus $\tau \cdot (Q\Delta^+(a/q)) \in cs$ for all $\tau \in l_\infty$, hence for all $\tau \in c$ and $\tau \in c_0$. Further $a \in (Q/q)^{-1} * c_0$ implies $(aQ/q)\tau \in c_0$ for all $\tau \in l_\infty$. Since $\tau = \tau(x) \in l_\infty$ if and only if $x \in (\bar{N},q)_\infty$, $ax \in cs$ for all $x \in (\bar{N},q)_\infty$ by (3.73), that is $a \in (\bar{N},q)_\infty^\beta$. Similarly $a \in (Q/q)^{-1} * c$ or $a \in (Q/q) * l_\infty$ imply $a \in (\bar{N},q)^\beta$ or $a \in (\bar{N},q)_0^\beta$. Thus we have proved $\mathcal{N}_0 \subset (\bar{N},q)_0^\beta$, $\mathcal{N} \subset (\bar{N},q)$ and $\mathcal{N}_\infty \subset (\bar{N},q)_\infty^\beta$.

To prove the converse inclusions we first assume $ax \in cs$ for all $x \in (\bar{N},q)_0$. Then $ax \in c_0$ for all $x \in (\bar{N},q)_0$, hence $(a/q)\Delta(Q\tau) \in c_0$ for all $\tau = \tau(x) \in c_0$, whence

$$\frac{a_k}{q_k}\Delta(Q_k(-1)^k|\tau_k|) = (-1)^k\frac{a_k}{q_k}(Q_k|\tau_k| + Q_{k-1}|\tau_{k-1}|) \to 0 \quad \text{for all } \tau \in c_0.$$

This implies $(aQ/q)\tau \in c_0$ for all $\tau \in c_0$, and thus $aQ/q \in l_\infty$ by Example 1.28. From (3.73), we conclude $Q\Delta^+(a/q)\tau \in cs$ for all $\tau \in c_0$, that is $Q\Delta^+(a/q) \in c_0^\beta = l_1$. Thus $a \in X_1$, and we have proved $(\bar{N},q)_0^\beta \subset \mathcal{N}_0$.

Now let $ax \in cs$ for all $x \in (\bar{N},q)$. Then $ax \in cs$ for all $x \in (\bar{N},q)_0$, and consequently $a \in (\bar{N},q)_0^\beta \subset X_1$. Thus by (3.73), $(aQ/q)\tau \in c$ for all $\tau \in c$, hence $aQ/q \in c$ by Example 1.28. This proves $(\bar{N},q)_\infty \subset \mathcal{N}$.

Finally let $ax \in cs$ for all $x \in (\bar{N},q)_\infty$. Then again $a \in X_1$, and by (3.73), $(aQ/q)\tau \in c$ for all $\tau \in l_\infty$, hence $aQ/q \in c_0$ by Example 1.28. This proves $(\bar{N},q)^\beta \subset \mathcal{N}_\infty$. □

3.6. Matrix transformations in the spaces $(\bar{N},q)_0$, $(\bar{N},q)$ and $(\bar{N},q)_\infty$ and their measures of noncompactness. We need the following proposition

Proposition 3.32. [72, Proposition 3.1] *On any of the spaces $(\bar{N},q)_0^\beta$, $(\bar{N},q)^\beta$ and $(\bar{N},q)_\infty^\beta$, we have*

$$||a||^* = \sup_n \left(\sum_{k=0}^{n-1} Q_k \left| \frac{a_k}{q_k} - \frac{a_{k+1}}{q_{k+1}} \right| + \left| \frac{a_n Q_n}{q_n} \right| \right).$$

Proof. Given any sequence x we shall write $\tau^{[n]} = \tau(x^{[n]})$ $(n = 0, 1, \dots)$ where $x^{[n]}$ is the n-section of x. Let $a \in \mathcal{N}_0$ and n be a nonnegative integer. We define the sequence $b^{[n]}$ by

$$b_k^{[n]} = \begin{cases} Q_k \Delta^+(a/q)_k & (0 \le k \le n) \\ a_n Q_n / q_n & (k = n) \\ 0 & (k > n) \end{cases}$$

and put $||a||_{\mathcal{N}} = \sup_n ||b^{[n]}||_1 = \sup_n (\sum_{k=0}^\infty |b_k^{[n]}|)$. Then

$$\begin{aligned} \left| \sum_{k=0}^\infty a_k x_k^{[n]} \right| &= \left| \sum_{k=0}^n \frac{a_k}{q_k} \Delta (Q\tau^{[n]})_k \right| \le \sum_{k=0}^{n-1} \left| Q_k \tau_k^{[n]} \Delta^+(a/q)_k \right| + \left| \frac{a_n Q_n}{q_n} \right| |\tau_n^{[n]}| \\ &\le \sup_k |\tau_k^{[n]}| \cdot \left(\sum_{k=0}^{n-1} \left| Q_k \Delta^+(a/q)_k \right| + \left| \frac{a_n Q_n}{q_n} \right| \right) \\ &= ||x^{[n]}||_{\bar{N}_q} ||b^{[n]}||_1 = ||a||_{\mathcal{N}} ||x^{[n]}||_{\bar{N}_q}. \end{aligned}$$

Thus

$$||a||^* \le ||a||_{\mathcal{N}}. \tag{3.74}$$

To prove the converse inequality let n be an arbitrary integer. We define the sequence $x^{(n)}$ by $\tau_k(x^{(n)}) = \operatorname{sign}(b_k^{[n]})$ for $k = 0, 1, \dots$. Then $\tau_k(x^{(n)}) =$ for $k > n$, that is $x^{(n)} \in (\bar{N},q)_0$, $||x^{(n)}||_{\bar{N}_n} = ||\tau(x^{(n)})||_\infty \le 1$ and

$$\left| \sum_{k=0}^\infty a_k x_k^{(n)} \right| = \left| \sum_{k=0}^n b_k^{[n]} x_k^{(n)} \right| = \sum_{k=0}^n |b_k^{[n]}| \le ||a||^*.$$

Since n was arbitrary, we have $||a||_{\mathcal{N}} \le ||a||^*$. This and (3.74) together yield the conclusion. □

As a corollary of Theorems 1.23 and 3.31 and Proposition 3.32 we obtain

Corollary 3.33. [72, Corollary 3.4] *Let $q = (q_k)_{k=0}^{\infty}$ be a positive sequence and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ $(n \to \infty)$.*

(a) *Then $A \in ((\bar{N}, q)_\infty, l_\infty)$ if and only if*

$$M((\bar{N},q)_\infty, l_\infty) = \sup_{n,m}\left(\sum_{k=0}^{m-1} Q_k \left|\frac{a_{nk}}{q_k} - \frac{a_{n,k+1}}{q_{k+1}}\right| + Q_m \frac{|a_{nm}|}{q_m}\right) < \infty \tag{3.75}$$

and

$$A_n Q/q \in c_0 \quad \text{for all } n = 0, 1, \ldots. \tag{3.76}$$

(b) *Then $A \in ((\bar{N}, q), l_\infty)$ if and only if condition (3.75) holds and*

$$A_n Q/q \in c \quad \text{for all } n = 0, 1, \ldots. \tag{3.77}$$

(c) *Then $A \in ((\bar{N}, q)_0, l_\infty)$ if and only condition (3.75) holds.*

(d) *Then $A \in ((\bar{N}, q)_0, c_0)$ if and only condition (3.75) holds and*

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for all } k = 0, 1, \ldots. \tag{3.78}$$

(e) *Then $A \in ((\bar{N}, q)_0, c)$ if and only if condition (3.75) holds and*

$$\lim_{n\to\infty} a_{nk} = l_k \quad \text{for all } k = 0, 1, \ldots. \tag{3.79}$$

(f) *Then $A \in ((\bar{N}, q), c_0)$ if and only if conditions (3.75), (3.77) and (3.78) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0. \tag{3.80}$$

(g) *Then $A \in ((\bar{N}, q), c)$ if and only if conditions (3.75), (3.77) and (3.79) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = l. \tag{3.81}$$

As a corollary of Theorem 3.8 and Corollary 3.33, we obtain

Corollary 3.34. [72, Corollary 3.5] *Let X be a BK space, $(p_k)_{k=0}^{\infty}$ a positive sequence and $P_n = \sum_{k=0}^{n} p_k$ $(n = 0, 1, \ldots)$. Then $A \in (X, (\bar{N}, p)_\infty)$ if and only if*

$$M(X, (\bar{N}, p)_\infty) = \sup_m \left\| \frac{1}{P_m} \sum_{n=0}^{m} p_n A_n \right\|^* < \infty. \tag{3.82}$$

Further, if $(b^k)_{k=0}^{\infty}$ is a basis of X, then $A \in (X, (\bar{N},p)_0)$ if and only if condition (3.82) holds and

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n A_n(b^{(k)})\right) = 0 \quad \textit{for all } k = 0,1,\dots, \tag{3.83}$$

and $A \in (X, (\bar{N},p))$ if and only if condition (3.83) holds and

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n A_n(b^{(k)})\right) = l_k \quad \textit{for all} \quad k = 0,1,\dots. \tag{3.84}$$

Remark 3.35. (a) If $X = l_r$ $(1 \le r < \infty)$ and Y is any of the spaces $(\bar{N},p)_\infty$, $(\bar{N},p)$ and $(\bar{N},p)_0$, then the conditions for $A \in (X,Y)$ follow from the respective ones in Corollary 3.34 by replacing the norm $\|\cdot\|^*$ in condition (3.82) by the natural norm on l_s where $s = \infty$ for $r = 1$ and $s = r/(r-1)$ for $1 < r < \infty$, that is

$$M(l_r, (\bar{N},p)_\infty) = \begin{cases} \sup_{m,k}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right| & (r = 1) \\ \sup_m \sum_{k=0}^{\infty}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|^s & (1 < r < \infty), \end{cases}$$

and by replacing the terms $A_n(b^{(k)})$ in conditions (3.83) and (3.84) by the terms a_{nk}.

(b) We consider the conditions

$$\begin{aligned} &M((\bar{N},q)_\infty, (\bar{N},p)_\infty) \\ &= \sup_{m,n}\left(\sum_{k=0}^{n-1} Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m} p_l\left(\Delta^+ A_l/q\right)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m} p_l a_{ln}\right|\right) < \infty, \end{aligned} \tag{3.85}$$

$$\left(\frac{a_{nk}Q_k}{q_k}\right)_{k=0}^{\infty} \in c_0 \qquad (n = 0,1,\dots), \tag{3.86}$$

$$\left(\frac{a_{nk}Q_k}{q_k}\right)_{k=0}^{\infty} \in c \qquad (n = 0,1,\dots), \tag{3.87}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right) = 0 \qquad (k = 0,1,\dots), \tag{3.88}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right) = l_k \qquad (k = 0,1,\dots), \tag{3.89}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n\left(\sum_{k=0}^{\infty} a_{nk}\right)\right) = 0, \tag{3.90}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n\left(\sum_{k=0}^{\infty} a_{nk}\right)\right) = l. \tag{3.91}$$

Then

$$\begin{array}{lll}
A \in ((\bar{N},q)_\infty, (\bar{N},p)_\infty) & \text{and only if} & (3.85) \text{ and } (3.86); \\
A \in ((\bar{N},q), (\bar{N},p)_\infty) & \text{and only if} & (3.85) \text{ and } (3.87); \\
A \in ((\bar{N},q)_0, (\bar{N},q)_\infty) & \text{and only if} & (3.85); \\
A \in ((\bar{N},q)_0, (\bar{N},p)_0) & \text{and only if} & (3.85) \text{ and } (3.88); \\
A \in ((\bar{N},q)_0, (\bar{N},p)) & \text{and only if} & (3.85) \text{ and } (3.89); \\
A \in ((\bar{N},q), (\bar{N},p)_0) & \text{and only if} & (3.85), (3.87), (3.88) \text{ and } (3.90); \\
A \in ((\bar{N},q), (\bar{N},p)) & \text{and only if} & (3.85), (3.87), (3.89) \text{ and } (3.91).
\end{array}$$

Theorem 3.36. [72, Theorem 4.2] *Let A be as in Corollary 3.33, and for all integers n, r, $n > r$, set*

$$\|A\|^{(r)} = \sup_{\substack{m \\ n>r}} \left(\sum_{k=0}^{m-1} Q_k \left| \frac{a_{nk}}{q_k} - \frac{a_{n,k+1}}{q_{k+1}} \right| + Q_m \frac{|a_{nm}|}{q_m} \right) \tag{3.92}$$

Let X be either $(\bar{N},q)_0$ or $X = (\bar{N},q)$, and let $A \in (X, c_0)$. Then we have

$$\|L_A\|_X = \lim_{r\to\infty} \|A\|^{(r)}.$$

Let X be either $(\bar{N},q)_0$ or $X = (\bar{N},q)$, and let $A \in (X, c)$. Then we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \|A\|^{(r)} \le \|L_A\|_X \le \lim_{r\to\infty} \|A\|^{(r)}.$$

Let X be either $(\bar{N},q)_0$, $(\bar{N},q)$ or $X = (\bar{N},q)_\infty$, and let $A \in (X, l_\infty)$. Then we have $0 \le \|L_A\|_X \le \lim_{r\to\infty} \|A\|^{(r)}$.

Proof. The proof follows exactly the same lines as that of Theorem 3.26. □

As a corollary of the theorem above, we have

Corollary 3.37. [72, Corollary 4.3] *Let A be as in Theorem 3.36. Then if $A \in (X, c_0)$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, or if $A \in (X, c)$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, then in all cases we have L_A is compact if and only if $\lim_{r\to\infty} \|A\|^{(r)} = 0$. Further, if $A \in (X, l_\infty)$ for $X = (\bar{N},q)_0$, $X = (\bar{N},q)$ or $X = (\bar{N},q)_\infty$, then we have*

$$L_A \quad \text{is compact if} \quad \lim_{r\to\infty} \|A\|^{(r)} = 0. \tag{3.93}$$

The following example will show that it is possible for L_A in (3.93) to be compact in the case $\lim_{r\to\infty} \|A\|^{(r)} > 0$, and hence in general we have just " if" in (3.93).

Example 3.38. Let the matrix A be defined by $A_n = e^{(0)}$ $(n = 0, 1, \dots)$ and $q_n = 2^n$ for $n = 0, 1, 2, \dots$. Then $M((\bar{N},q)_\infty, l_\infty) = \sup_n [1 + (2 - 2^{-n})] < 3$, and by Corollary 3.33 we know that $A \in ((\bar{N},q)_\infty, l_\infty)$. Further

$$\|A\|^{(r)} = \sup_{n>r} [1 + (2 - \frac{1}{2^n})] = 3 - \frac{1}{2^{r+1}} \quad \text{for all} \quad r,$$

whence $\lim_{r\to\infty} \|A\|^{(r)} = 3 > 0$. Since $A(x) = x_0 e_0$ for all $x \in (\bar{N},q)_\infty$, L_A is a compact operator.

Now we continue with the following auxiliary result.

Lemma 3.39. [72, Lemma 4.5] *Let $q_k > 0$ $(k = 0, 1, \dots)$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ $(n \to \infty)$. Let $r \geq 0$ and the operators $B_0^{(r)} : (\bar{N}, q)_0 \mapsto (\bar{N}, q)_0$ and $B^{(r)} : (\bar{N}, q) \mapsto (\bar{N}, q)$ be defined by*

$$B_0^{(r)} x = \sum_{k=r+1}^{\infty} x_k e^{(k)}, \quad (x \in (\bar{N}, q)_0)$$

$$B^{(r)} x = \sum_{k=r+1}^{\infty} (x_k - l) e^{(k)}, \quad (x \in (\bar{N}, q))$$

where $l = \lim_{n\to\infty} \tau_n(x)$. Then

$$\|B_0^{(r)}\| = 1 + \frac{Q_r}{Q_{r+1}} \tag{3.94}$$

$$\|B^{(r)}\| = 2. \tag{3.95}$$

Proof. First we show identity (3.94). Let $x \in (\bar{N}, q)_0$. Since $\tau_n(B_0^{(r)}(x)) = 0$ for $0 \leq n \leq r$, and, for $n \geq r + 1$,

$$\begin{aligned} |\tau_n(B_0^{(r)}(x))| &= \left| \frac{1}{Q_n} \sum_{k=r+1}^{n} q_k x_k \right| = \left| \tau_n(x) - \frac{Q_r}{Q_n} \tau_r(x) \right| \\ &\leq \left(1 + \frac{Q_r}{Q_{r+1}} \right) \|x\|_{(\bar{N},q)_\infty}, \end{aligned}$$

it follows that

$$\left\|B_0^{(r)}(x)\right\|_{(\bar{N},q)_\infty} \leq \left(1 + \frac{Q_r}{Q_{r+1}} \right) \|x\|_{(\bar{N},q)_\infty},$$

and consequently

$$\|B_0^{(r)}\| \leq 1 + \frac{Q_r}{Q_{r+1}}. \tag{3.96}$$

Defining the sequence x by

$$x_k = \begin{cases} -1 & (0 \leq k \leq r) \\ \dfrac{Q_r + Q_{r+1}}{q_{r+1}} & (k = r+1) \\ -\dfrac{Q_r + Q_{r+1}}{q_{r+2}} & (k = r+2) \\ 0 & (k \geq r+3), \end{cases}$$

we conclude $\tau_n(x) = -1$ for $0 \leq n \leq r$

$$\tau_{r+1}(x) = -\frac{Q_r}{Q_{r+1}} + \frac{Q_r}{Q_{r+1}} + 1 = 1$$

$$\tau_n(x) = \frac{1}{Q_n} \Big(-Q_r + Q_r + Q_{r+1} - (Q_r + Q_{r+1}) \Big) = -\frac{Q_r}{Q_n} \quad (n \geq r+2).$$

Since $Q_n \to \infty$ $(n \to \infty)$, we have $x \in (\bar{N},q)_0$ and $\|x\|_{(\bar{N},q)_\infty} = 1$. Further

$$\tau_{r+1}(B_0^{(r)}(x)) = \frac{1}{Q_{r+1}}(Q_r + Q_{r+1}) = 1 + \frac{Q_r}{Q_{r+1}}$$

and $\tau_n(B_0^{(r)}(x)) = 0$ for $n \neq r+1$. Therefore

$$\|B_0^{(r)}(x)\|_{(\bar{N},q)_\infty} = 1 + \frac{Q_r}{Q_{r+1}} = \left(1 + \frac{Q_r}{Q_{r+1}}\right)\|x\|_{(\bar{N},q)_\infty}$$

and $\|B_0^{(r)}\| \geq 1 + Q_r/Q_{r+1}$. Now this and (3.96) together yield identity (3.94).

Now we prove identity (3.95). Let $x \in (\bar{N},q)$. Since $\tau_n(B^{(r)}(x)) = 0$ for $0 \leq n \leq r$ and, for $n \geq r+1$,

$$\begin{aligned}
|\tau_n(B^{(r)}(x))| &= \left|\frac{1}{Q_n}\sum_{k=r+1}^{n} q_k(x_k - l)\right| = \left|\tau_n(x) - \frac{Q_r}{Q_n}\tau_r(x) - l + \frac{Q_r}{Q_n}l\right| \\
&\leq \left|1 + \frac{Q_r}{Q_n}\right|\|x\|_{(\bar{N},q)_\infty} + \left|1 - \frac{Q_r}{Q_n}\right||l|,
\end{aligned}$$

since $|l| = \lim_{n\to\infty}|\tau_n(x)| \leq \|x\|_{(\bar{N},q)_\infty}$, it follows that $|\tau_n(B^{(r)}(x))| \leq 2\|x\|_{(\bar{N},q)_\infty}$ for $n \geq r+1$ and consequently

(3.97) $$\|B^{(r)}\| \leq 2.$$

Defining the sequence x by

$$x_k = \begin{cases} -1 & (0 \leq k \leq r) \\ 2Q_{r+1}/q_{r+1} - 1 & (k = r+1) \\ -1 & (k \geq r+2), \end{cases}$$

we conclude $\tau_n(x) = -1$ for $0 \leq n \leq r$,

$$\begin{aligned}
\tau_{r+1}(x) &= \frac{1}{Q_{r+1}}\left(-Q_r + 2Q_r - q_{r+1}\right) = 1 \\
\tau_n(x) &= \frac{1}{Q_n}\left(-Q_r + 2Q_{r+1} - \sum_{k=r+1}^{n} q_k\right) = \frac{1}{Q_n}\left(-Q_n + 2Q_{r+1}\right) \\
&= -1 + 2\frac{Q_{r+1}}{Q_n} \leq 1 \quad (n \geq r+2).
\end{aligned}$$

Hence $\|x\|_{(\bar{N},q)_\infty} = 1$ and $\lim_{n\to\infty}\tau_n(x) = -1$, that is $x \in (\bar{N},q)$. Finally $\tau_n(B^{(r)}(x)) = 0$ $(0 \leq n \leq r)$,

$$\begin{aligned}
\tau_{r+1}(B^{(r)}(x)) &= \frac{q_{r+1}}{Q_{r+1}}(x_{r+1} + 1) = 2 \\
\tau_n(B^{(r)}(x)) &= 2\frac{Q_{r+1}}{Q_n} \leq 2 \quad (n \geq r+2).
\end{aligned}$$

This implies $\|B^{(r)}\| \geq 2$, and together with (3.97) we obtain (3.95). □

Concerning Corollary 3.34 and the measures of noncompactness we have

Theorem 3.40. [72, Theorem 4.6] *Let X be a BK space, let A be as in Corollary 3.34, and let $P_m \to \infty\,(m \to \infty)$. Then for all integers m, r, $m > r$, we put*

$$\|A\|^{(r)}_{(\bar{N},p)_\infty} = \sup_{m>r} \left\| \frac{1}{P_m} \sum_{n=0}^{m} p_n A_n \right\|^* .$$

Further, if X has a Schauder basis, and $A \in (X, (\bar{N},p)_0)$, then we have

$$\frac{1}{b} \cdot \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty} \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty}, \tag{3.98}$$

where $b = \limsup_{n\to\infty}(2 - p_n/P_n)$.

If X has a Schauder basis, and $A \in (X, (\bar{N},p))$, then we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty} \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty}. \tag{3.99}$$

Finally, if $A \in (X, (\bar{N},p)_\infty)$, then we have

$$0 \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty}. \tag{3.100}$$

Proof. Let us remark that the limits in (3.98), (3.99) and (3.100) exist. We put $B = \{x \in X : \|x\| \le 1\}$. Suppose that $A \in (X, (\bar{N},p)_0)$. Let $B_0^{(r)} : (\bar{N},p)_0 \mapsto (\bar{N},p)_0$ be the projector defined in Lemma 3.39. Then by (3.94) we have that $\|B_0^{(r)}\| = 2 - p_r/P_r$. Now, to prove (3.98), we have by Theorem 2.23 and Proposition 3.30

$$\frac{1}{b} \limsup_{r\to\infty} \left(\sup_{x\in B} \|B_0^{(r)}(A(x))\| \right) \le \chi(A(B)) \le \limsup_{r\to\infty} \left(\sup_{x\in B} \|B_0^{(r)}(A(x))\| \right),$$

where $b = \limsup_{r\to\infty} \|B_0^{(r)}\|$. This proves (3.98), since

$$\sup_{x\in B} \|B_0^{(r)}(A(x))\| = \|A\|^{(r)}_{(\bar{N},p)_\infty}.$$

To prove (3.99) let us remark (see Proposition 3.30) that $(\bar{N},p)$ has the Schauder basis $e, e^{(k)}, k = 0, 1, \ldots$, and every $(x_k)_{k=0}^\infty \in (\bar{N},q)$ has a unique representation $x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$, where $l \in \mathbb{C}$ is such that $x - le \in (\bar{N},p)_0$. Let $B^{(r)} : (\bar{N},p)_0 \mapsto (\bar{N},p)_0$ be the projector defined by (see Lemma 3.39) $B^{(r)}(x) = \sum_{k=r+1}^\infty (x_k - l)e^{(k)}$. Then we have $\|B^{(r)}\| = 2$ by (3.95). Now the proof of (3.99) is similar as in the case (3.98), and we omit it. Let us prove (3.100). Now define $\mathcal{P}_r : (\bar{N},p)_\infty \mapsto (\bar{N},p)_\infty$, by $\mathcal{P}_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in (\bar{N},p)_\infty$ and $r = 1, 2, \ldots$. It is clear that $A(B) \subset \mathcal{P}_r(A(B)) + (I - \mathcal{P}_r)(A(B))$. By Remark 3.22 (b) it follows that $\mathcal{P}_r$ is a bounded operator, and since it is obviously a finite-rank, it is a compact operator. Now, by the elementary properties of function χ we have

$$\begin{aligned}\chi(A(B)) &\le \chi(\mathcal{P}_r(A(B))) + \chi((I - \mathcal{P}_r)(A(B))) = \chi((I - \mathcal{P}_r)(A(B)) \\ &\le \sup_{x\in B} \|(I - \mathcal{P}_r)(A(x))\| = \|A\|^{(r)}_{(\bar{N},p)_\infty}.\end{aligned}$$

□

As a corollary of the theorem above we have

Corollary 3.41. [72, Corollary 4.7] *Let X be a BK space and let A and $\|A\|^{(r)}_{(\bar{N},p)}$ be as in Theorem 4.6. If X has a Schauder basis, and either $A \in (X, (\bar{N},p)_0)$ or $A \in (X, (\bar{N},p))$, then L_A is compact if and only if $\lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)} = 0$. Further, if $A \in (X, (\bar{N},p)_\infty)$, then L_A is compact if $\lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)} = 0$.*

Now we get several corollaries concerning Remark 3.22.

Corollary 3.42. [72, Corollary 4.8] *If either $A \in (l^u, (\bar{N},p)_0)$ or $A \in (l^u, (\bar{N},p))$ for $1 < u < \infty$, then*

L_A is compact if and only if

$$\lim_{r\to\infty}\left[\sup_{m>r}\left(\sum_{k=0}^{\infty}\left|\frac{1}{P_m}\sum_{n=0}^{m}p_n a_{nk}\right|^v\right)^{1/v}\right] = 0 \quad \text{where } v = u/(u-1).$$

Further, if either $A \in (l^1, (\bar{N},p)_0))$ or $A \in (l^1, (\bar{N},p)))$, then

$$L_A \quad \text{is compact if and only if} \quad \lim_{r\to\infty}\left(\sup_{n>r,k}\left|\frac{1}{P_m}\sum_{n=0}^{m}p_n a_{nk}\right|\right) = 0.$$

If $A \in (l^u, (\bar{N},p))$ for $1 < u < \infty$, then

L_A is compact if

$$\lim_{r\to\infty}\left[\sup_{m>r}\left(\sum_{k=0}^{\infty}\left|\frac{1}{P_m}\sum_{n=0}^{m}p_n a_{nk}\right|^v\right)^{1/v}\right] = 0 \quad \text{where } v = u/(u-1).$$

Finally, if $A \in (l^1, (\bar{N},p))$, then

$$L_A \quad \text{is compact if } \lim_{r\to\infty}\left(\sup_{n>r,k}\left|\frac{1}{P_m}\sum_{n=0}^{m}p_n a_{nk}\right|\right) = 0.$$

From Corollary 3.41, Theorem 3.8 and Remark 3.22 (b), we have

Corollary 3.43. [72, Corollary 4.9] *If $A \in (X, (\bar{N},p)_0)$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, or if $A \in (X, (\bar{N},p))$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, then we have in all cases*

L_A is compact if and only if

$$\lim_{r\to\infty}\left[\sup_{m>r,n}\left(\sum_{k=0}^{n-1}Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m}p_l(\Delta^+ A_l/q)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m}p_l a_{ln}\right|\right)\right] = 0.$$

Further, if $A \in (X, (\bar{N},p)_\infty)$ for $X = (\bar{N},q)_\infty$, $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, then we have

L_A is compact if

$$\lim_{r\to\infty}\left[\sup_{m>r,n}\left(\sum_{k=0}^{n-1}Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m}p_l(\Delta^+ A_l/q)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m}p_l a_{ln}\right|\right)\right] = 0.$$

3.7. Spaces of strongly summable and convergent sequences. In this subsection, we shall study spaces of *strongly summable* and *strongly convergent* sequences and give their dual spaces. The results of this subsection can be found in [59, 64, 65, 70].

For $X \subset \omega$ and any real $p > 0$, we write

$$X_{[A]^p} = \{x \in \omega : A(|x|^p) \in X\}.$$

If $p = 1$, then we omit the index p and write $X_{[B]} = X_{[B]^1}$ for short.

Let C_1 be the Cesáro matrix of order 1, that is $(C_1)_{nk} = 1/n$ for $1 \leq k \leq n$ and 0 for $k > n$ $(n = 0, 1, \ldots)$. For $0 < p < \infty$, Maddox [51, 54] defined the sets

$$w_0^p = (c_0)_{[C_1]^p} = \left\{x \in \omega : \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^p\right) = 0\right\},$$
$$w_p = \{x \in \omega : x - le \in w_0^p \text{ for some complex number } l\},$$
$$w_\infty^p = (l_\infty)_{[C_1]^p}.$$

These sets are special cases of the so-called mixed normed spaces (see e.g. [38, 35, 36, 59, 22, 23]). Here we shall only deal with the cases $1 \leq p < \infty$. The following result is well known.

Proposition 3.44. [51, 59] *Each of the set w_0, w^p and w_∞^p is a BK space for $1 \leq p < \infty$ with*

$$\|x\| = \sup_{\nu\geq 0}\left(\frac{1}{2^\nu}\sum_{k=2^\nu}^{2^{\nu+1}-1}|x_k|^p\right)^{1/p}; \tag{3.101}$$

w_0^p has AK; every sequence $x = (x_k)_{k=1}^\infty \in w^p$ has a unique representation $x = le + \sum_{k=1}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in w_0^p$.

Let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. If $(n(\nu))_{\nu=0}^\infty$ is a sequence such that $0 = n(0) < n(1) < n(2) < \ldots$, then we denote the set of all integers k with $n(\nu) \leq k \leq n(\nu+1) - 1$ by $K^{\langle\nu\rangle}$, and we write $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all k in $K^{\langle\nu\rangle}$. We define the matrices $B = (b_{nk})_{n,k=0}^\infty$ and $\tilde{B} = (\tilde{b}_{\nu k})_{\nu,k=0}^\infty$ by

$$b_{nk} = \begin{cases} 1/\lambda_n & (0 \leq k \leq n) \\ 0 & (k < n) \end{cases} \quad\text{and}\quad \tilde{b}_{\nu k} = \begin{cases} 1/\lambda_{n(\nu+1)} & (k \in K^{<\nu>}) \\ 0 & (k \notin K^{<\nu>}). \end{cases}$$

Further, let $\Delta(\mu)$ be the matrix with

$$\Delta_{nk}(\mu) = \begin{cases} -\mu_{n-1} & (k = n-1) \\ \mu_n & (k = n) \\ 0 & (\text{otherwise}) \end{cases} \quad (n = 0, 1, \ldots) \quad \text{where } \mu_{-1} = 0.$$

We define the sets [76, 64]

$$c_0(\mu) = ((c_0)_{[B]})_{\Delta(\mu)}, \qquad \tilde{c_0}(\mu) = ((c_0)_{[\tilde{B}]})_{\Delta(\mu)},$$
$$c(\mu) = \{x \in \omega : x - le \in c_0(\mu)\}, \qquad \tilde{c}(\mu) = \{x \in \omega : x - le \in \tilde{c_0}(\mu)\},$$
$$c_\infty(\mu) = ((l_\infty)_{[B]})_{\Delta(\mu)}, \qquad \tilde{c_\infty}(\mu) = ((l_\infty)_{[\tilde{B}]})_{\Delta(\mu)}.$$

The following result is well known.

Proposition 3.45. [64, Theorem 2 (c)] *Let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. Then each of the spaces $c_0(\mu)$, $c(\mu)$ and $c_\infty(\mu)$ is a BK space*

$$\|x\|' = \|B(|\Delta(\mu)x|)\|_\infty = \sup_{n\geq 0}\left(\frac{1}{\mu_n}\sum_{k=0}^{n}|\mu_k x_k - \mu_{k-1}x_{k-1}|\right);$$

$c_0(\mu)$ has AK; every sequence $x = (x_k)_{k=0}^\infty \in c_\mu$ has a unique representation $x = le + \sum_{k=1}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in c_0(\mu)$.

A sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ of positive reals is called *exponentially bounded* if there is an integer $m \geq 2$ such that for all integers ν there is at least one λ_n in the interval $[m^\nu, m^{\nu+1})$. It is known (cf. [64, Lemma 1]) that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ of positive reals is exponentially bounded if and only if there are reals $s \leq t$ such that $0 < s \leq \lambda_{n(\nu)}/\lambda_{n(\nu+1)} \leq t < 1$ for some subsequence $(\lambda_{n(\nu+1)})_{\nu=0}^\infty$ for all $\nu = 0, 1, \ldots$; such a subsequence is called an *associated subsequence.*

The following result is well known.

Proposition 3.46. [64, Theorem 2] *Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu+1)})_{\nu=0}^\infty$ an associated subsequence. Then $c_0(\Lambda) = \tilde{c}_0(\Lambda)$, $c(\Lambda) = \tilde{c}(\Lambda)$ and $c_\infty(\Lambda) = \tilde{c}_\infty(\Lambda)$. The norms $\|x\|'$ and*

$$\|x\| = l\|\tilde{B}(|\Delta(\Lambda)x|)\|_\infty = \sup_{\nu\geq 0}\left(\frac{1}{\lambda_{n(\nu+1)}}\sum_{\nu}|\lambda_k x_k - \lambda_{k-1}x_{k-1}|\right)$$

are equivalent on $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$. Thus each of the spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ is a BK space with $\|\cdot\|$.

Proposition 3.47. [51] and [59, Theorems 4 and 6] *Let $K^{\langle\nu\rangle} = [2^\nu, 2^{\nu+1} - 1]$ $(\nu = 0, 1, \ldots)$. We put*

$$\mathcal{M}^p = \begin{cases} \left\{a \in \omega : \sum_{\nu=0}^{\infty} 2^{\nu/p}\max_\nu |a_k| < \infty\right\} & (p = 1) \\ \left\{a \in \omega : \sum_{\nu=0}^{\infty} 2^{\nu/p}\left(\sum_\nu |a_k|^q\right)^{1/q} < \infty\right\} & \left(1 < p < \infty;\ q = \dfrac{p}{p-1}\right) \end{cases}$$

$$\|a\|_{\mathcal{M}^p} = \begin{cases} \sum_{\nu=0}^{\infty} 2^{\nu/p}\max_\nu |a_k| & (p = 1) \\ \sum_{\nu=0}^{\infty} 2^{\nu/p}\left(\sum_\nu |a_k|^q\right)^{1/q} & (1 < p < \infty) \end{cases} \quad \text{for all } a \in \mathcal{M}^p.$$

Then $(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}^p$ and $\|a\|^* = \|a\|_{\mathcal{M}^p}$ on $\mathcal{M}^p$.

Proposition 3.48. [65, Lemma 2] *Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu+1)})_{\nu=0}^\infty$ an associated subsequence. We put*

$$\mathcal{C}(\Lambda) = \left\{ a \in \omega : \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right| < \infty \right\}$$

$$\|a\|_{\mathcal{C}(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right| \quad \text{for all } a \in \mathcal{C}(\Lambda).$$

Then $(c_0(\Lambda))^\beta = (c(\Lambda))^\beta = (c_\infty(\Lambda))^\beta = \mathcal{C}(\Lambda)$ and $\|a\|^* = \|a\|_{\mathcal{C}(\Lambda)}$ on $\mathcal{C}(\Lambda)$.

As an immediate consequence of Theorems 3.8 and 3.10 we obtain

Corollary 3.49. [70, Corollary 1] *Let X be an arbitrary FK space. Further, let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. We write $w_0 = w_0^1$ etc., for short and put*

$$M(X, w_\infty) = \sup_{m \geq 1} \left(\max_{N_m \subset \{1, \dots, m\}} \left\| \frac{1}{m} \sum_{n \in N_m} A_n \right\|_D^* \right)$$

$$M(X, c_\infty(\mu)) = \sup_{m \geq 0} \left(\max_{N_m \subset \{0, \dots, m\}} \left\| \frac{1}{\mu_m} \sum_{n \in N_m} (\mu_n A_n - \mu_{n-1} A_{n-1} \right\|_D^* \right).$$

(a) *Then $A \in (X, w_\infty)$ if and only if*

$$M(X, w_\infty) < \infty \quad \text{for some } D > 0. \tag{3.102}$$

Furthermore, if $(b^k)_{k=0}^\infty$ is a basis of X, then $A \in (X, w_0)$ if and only if condition (3.102) holds and

$$\lim_{m \to \infty} \left(\frac{1}{m} \sum_{n=1}^{m} |A_n(b^{(k)})| \right) = 0 \quad \text{for all } k = 0, 1, \dots; \tag{3.103}$$

$A \in (X, w)$ if and only if condition (3.102) holds and there are complex numbers l_k $(k = 0, 1, \dots)$ such that

$$\lim_{m \to \infty} \left(\frac{1}{m} \sum_{n=1}^{m} |A_n(b^{(k)}) - l_k)| \right) = 0 \quad \text{for all } k = 0, 1, \dots. \tag{3.104}$$

Finally, if X is a normed space and $A \in (X, Y)$ for $Y = w_0, w$ or w_∞, then, for

$$\|A\|_{w_\infty}^* = \sup_{m \geq 1} \left(\max_{N_m \subset \{1, \dots, m\}} \left\| \frac{1}{m} \sum_{n \in N_m} A_n \right\|^* \right),$$

we have

$$\|A\|^*_{w_\infty} \leq \|L_A\| \leq 4 \cdot \|A\|^*_{w_\infty}. \tag{3.105}$$

(b) *Then* $A \in (X, c_\infty(\mu))$ *if and only if*

$$M(X, c_\infty(\mu)) < \infty \quad \textit{for some } D > 0. \tag{3.106}$$

Further, if $(b^k)_{k=0}^\infty$ *is a basis of* X*, then* $A \in (X, c_0(\mu))$ *if and only if condition (3.106) holds and*

$$\lim_{m\to\infty}\left(\frac{1}{\mu_m}\sum_{n=0}^{m}|\mu_n A_n(b^{(k)}) - \mu_{n-1}A_{n-1}(b^{(k)})|\right) = 0 \tag{3.107}$$

for all $k = 0, 1, \ldots$; $A \in (X, c(\mu))$ *if and only if condition (3.107) holds and there are complex numbers* l_k *such that*

$$\lim_{m\to\infty}\left(\frac{1}{\mu_m}\sum_{n=0}^{m}|\mu_n(A_n(b^{(k)}) - l_k) - \mu_{n-1}(A_{n-1}(b^{(k)}) - l_k)|\right) = 0 \tag{3.108}$$

for all $k = 0, 1, \ldots$.

Finally, if X *is normed and* $A \in (X, Y)$ *for* $Y = c_0(\mu)$, $c(\mu)$ *or* $c_\infty(\mu)$*, then, for*

$$\|A\|^*_{c_\infty} = \sup_{m\geq 0}\left(\max_{N_m\subset\{0,\ldots,m\}}\left\|\frac{1}{\mu_m}\sum_{n\in N_m}(\mu_n A_n - \mu_{n-1}A_{n-1}\right\|^*\right),$$

we have

$$\|A\|^*_{c_\infty(\mu)} \leq \|L_A\| \leq 4 \cdot \|A\|^*_{c_\infty(\mu)}. \tag{3.109}$$

Proof. All we have to show are inequalities (3.105) and (3.109). Let $A \in (X, Y)$ where $Y = w_0$, w or w_∞. Then

$$\left|\frac{1}{m}\sum_{n\in N_m} A_n(x)\right| \leq \frac{1}{m}\sum_{n=1}^{m}|A_n(x)| \leq \|L_A\|$$

for all $m = 1, 2, \ldots$, all $N \subset N_m$ and all $\|x\| = 1$. This implies

$$\|A\|^*_{w_\infty} \leq \|L_A\|. \tag{3.110}$$

Further, given $\varepsilon > 0$ there is $x \in X$ with $\|x\| = 1$ such that

$$\|A(x)\| = \sup_{m\geq 1}\left(\frac{1}{m}\sum_{n=1}^{m}|A_n(x)|\right) \geq \|L_A\| - \varepsilon/2,$$

and there is an integer $m(x)$ such that

$$\frac{1}{m(x)}\sum_{n=1}^{m(x)}|A_n(x)| \geq \|A(x)\| - \varepsilon/2,$$

consequently

$$\frac{1}{m(x)}\sum_{n=1}^{m(x)}|A_n(x)| \geq \|L_A\| - \varepsilon.$$

By Lemma 3.9

$$4\cdot \max_{N_{m(x)}\subset\{1,\dots,m(x)\}}\left(\frac{1}{m(x)}\left|\sum_{n\in N_{m(x)}}A_n(x)\right|\right) \geq \frac{1}{m(x)}\sum_{n=1}^{m(x)}|A_n(x)| \geq \|L_A\| - \varepsilon$$

and so $4\cdot\|A\|^*_{w_\infty} \geq \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $4\cdot\|A\|^*_{w_\infty} \geq \|L_A\|$. Together with inequality (3.110) this yields (3.105). The inequalities in (3.109) are proved similarly. □

Remark 3.50. If X is a given BK space, and Y is any of the spaces w_0, w, w_∞, $c_0(\mu)$, $c(\mu)$ or $c_\infty(\mu)$, then the conditions for $A \in (X, Y)$ follow from the respective ones in Corollary 3.49 by replacing the norms $\|\cdot\|^*_D$ in conditions (3.102) and (3.106) by the natural norms on the β-duals of X. We shall write

$$\max_{N_m} \quad \text{for} \quad \begin{cases} \max\limits_{N_m\subset\{0,\dots,m\}} & \text{if } Y = w_0, w, w_\infty \\ \max\limits_{N_m\subset\{1,\dots,m\}} & \text{if } Y = c_0(\mu), c(\mu), c_\infty(\mu), \end{cases}$$

$q = p/(p-1)$ for $1 < p < \infty$, $\Delta_n(\mu_n a_{nk}) = \mu_n a_{nk} - \mu_{n-1}a_{n-1,k}$

$$\max_\nu \quad \text{for} \quad \max_{2^\nu\leq k\leq 2^{\nu+1}-1}, \quad \sum_\nu \quad \text{for} \quad \sum_{2^\nu\leq k\leq 2^{\nu+1}-1}.$$

(a) For $X = l_p$, we have

$$M(l_p, w_\infty) = \begin{cases} \sup\limits_m\left(\max\limits_{N_m}\left(\sup\limits_k\left|\frac{1}{m}\sum\limits_{n\in N_m}a_{nk}\right|\right)\right) & (p=1) \\ \sup\limits_m\left(\max\limits_{N_m}\left(\sum\limits_{k=1}^\infty\left|\frac{1}{m}\sum\limits_{n\in N_m}a_{nk}\right|^q\right)\right) & (1 < p \leq \infty), \end{cases}$$

$$M(l_p, c_\infty(\mu)) = \begin{cases} \sup\limits_m\left(\max\limits_{N_m}\left(\sup\limits_k\left|\frac{1}{\mu_m}\sum\limits_{n\in N_m}\Delta_n(\mu_n a_{nk})\right|\right)\right) & (p=1) \\ \sup\limits_m\left(\max\limits_{N_m}\left(\sum\limits_{k=1}^\infty\left|\frac{1}{\mu_m}\sum\limits_{n\in N_m}\Delta_n(\mu_n a_{nk})\right|^q\right)\right) & (1 < p < \infty). \end{cases}$$

(b) For $X = w_\infty^p$, we have

$$M(w_\infty^p, w_\infty) = \begin{cases} \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^{\infty} 2^{\nu/p} \max_\nu \left| \frac{1}{m} \sum_{n \in N_m} a_{nk} \right| \right)\right) & \text{for } p = 1 \\ \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_\nu \left| \frac{1}{m} \sum_{n \in N_m} a_{nk} \right|^q \right)^{1/q} \right)\right) & \text{for } 1 < p < \infty, \end{cases}$$

and

$$M(w_\infty^p, c_\infty(\mu)) = \begin{cases} \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^{\infty} 2^{\nu/p} \max_\nu \left| \frac{1}{\mu_m} \sum_{n \in N_m} \Delta_n(\mu_n a_{nk}) \right| \right)\right) & \text{for } p = 1 \\ \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_\nu \left| \frac{1}{\mu_m} \sum_{n \in N_m} \Delta_n(\mu_n a_{nk}) \right|^q \right)^{1/q} \right)\right) & \text{for } 1 < p < \infty. \end{cases}$$

(c) Let $\Lambda = (\lambda_k)_{k=0}^\infty$ be an exponentially bounded sequence of positive reals and $(\lambda_{k(\nu)})_{\nu=0}^\infty$ be an associated subsequence. We write $\max_\nu$ and $\sum_\nu$ for the maximum and the sum taken over all integers k such that $k(\nu) \le k \le k(\nu+1) - 1$. Then for $X = c_\infty(\Lambda)$ we have

$$M(c_\infty(\Lambda), w_\infty) = \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} \max_\nu \left| \sum_{j=k}^{\infty} \frac{1}{\lambda_j} \left(\frac{1}{m} \sum_{n \in N_m} a_{nj} \right) \right| \right)\right),$$

and

$$M(c_\infty(\Lambda), c_\infty(\mu)) = \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} \max_\nu \left| \sum_{j=k}^{\infty} \frac{1}{\lambda_j} \left(\frac{1}{\mu_m} \sum_{n \in N_m} \Delta_n(\mu_n a_{nk}) \right) \right| \right)\right).$$

The main result of this subsection is the following theorem (see Corollary 3.49).

Theorem 3.51. [70, Theorem 3] *Let A, X and Y be as in Corollary 3.49*
(a) *If X is a normed space and $A \in (X, Y)$ for $Y = w_0, w$ and w_∞, then, for*

$$\|A\|_{w_\infty}^{(m)} = \sup_{k>m} \left(\max_{N_{m,k} \subset \{m+1,\dots,k\}} \left\| \frac{1}{k} \sum_{i \in N_{m,k}} A_i \right\|^* \right)$$

we have

$$\lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} \qquad \text{if } Y = w_0, \tag{3.111}$$

$$\frac{1}{2} \cdot \lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} \qquad \text{if } Y = w, \tag{3.112}$$

$$0 \le \|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} \qquad \text{if } Y = w_\infty. \tag{3.113}$$

(b) *If X is a normed space and $A \in (X,Y)$ for $Y = c_0(\mu), c(\mu)$ and $c_\infty(\mu)$, then, for*

$$\|A\|_{c_\infty}^{(m)} = \sup_{k>m}\left(\max_{N_{m,k}\subset\{m+1,\dots,k\}}\left\|\frac{1}{\mu_k}\sum_{i\in N_{m,k}} \mu_i A_i - \mu_{i-1}A_{i-1}\right\|^*\right) \tag{3.114}$$

we have

$$\lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \le \|L_A\|_\chi \le 4\cdot \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \quad \text{if } Y = c_0(\mu), \tag{3.115}$$

(3.116)

$$\frac{1}{2}\cdot\lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \le \|L_A\|_\chi \le 4\cdot \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \quad \text{if } Y = c(\mu),$$

$$0 \le \|L_A\|_\chi \le 4\cdot \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \quad \text{if } Y = c_\infty(\mu). \tag{3.117}$$

Proof. Let us remark that the limits in (3.111) and (3.115) exist. We put $B = \{x \in X : \|x\| \le 1\}$. In the case $Y = w_0$ we have by Theorem 2.23

$$\|L_A\|_\chi = \chi(A(B)) = \lim_{m\to\infty}\left[\sup_{x\in B}\|(I-P_m)(A(x))\|\right], \tag{3.118}$$

where $P_m : w_0 \mapsto w_0$, $m = 1, 2, \dots$, is the projector on the first m coordinates, that is $P_m(x) = (x_1, x_2, \dots, x_m, 0, 0, \dots)$ for $x = (x_i) \in w_0$; (let us remark that $\|I - P_m\| = 1$ for $m = 1, 2, \dots$). For given $\epsilon > 0$ there is $x \in B$ such that

$$\|(I-P_m)(A(x))\| > \|(I-P_m)(A)\| - \epsilon/2. \tag{3.119}$$

Now there is an integer $k(x) > m$ such that

$$\frac{1}{k(x)}\sum_{i=m+1}^{k(x)} |A_i(x)| > \|(I-P_m)(A(x))\| - \frac{\epsilon}{2}. \tag{3.120}$$

Further by Lemma 3.9

$$4\cdot\left(\max_{N_{m,k(x)}\subset\{m+1,\dots,k(x)\}}\frac{1}{k(x)}\left|\sum_{i\in N_{m,k(x)}} A_i(x)\right|\right) \ge \frac{1}{k(x)}\sum_{i=m+1}^{k(x)}|A_i(x)|.$$

Now, by (3.119) and (3.120) we get

$$4\cdot\left(\max_{N_{m,k(x)}\subset\{m+1,\dots,k(x)\}}\frac{1}{k(x)}\left|\sum_{i\in N_{m,k(x)}} A_i(x)\right|\right) \ge \|(I-P_m)(A)\| - \epsilon. \tag{3.121}$$

Since $\epsilon > 0$ was arbitrary and $x \in B$, from (3.121) we have for each m

$$\|(I - P_m)(A)\| \le 4 \cdot \left[\sup_{k>m}\left(\max_{N_{m,k}\subset\{m+1,\dots,k\}}\left\|\frac{1}{k}\sum_{i\in N_{m,k}} A_i\right\|\right)\right]. \tag{3.122}$$

Hence, by (3.118) and (3.122) we get the right inequality in (3.111). To prove the left inequality in (3.111), suppose that m is an integer, $k > m$, $N_{m,k} \subset \{m+1, \dots, k\}$ and $x \in B$. Then

$$\left|\frac{1}{k}\sum_{i\in N_{m,k}} A_i(x)\right| \le \frac{1}{k}\sum_{i\in N_{m,k}} |A_i(x)| \le \frac{1}{k}\sum_{i=m+1}^{k} |A_i(x)| \le \|(I - P_m)(A(x))\|.$$

Thus for all m and $k > m$ we have

$$\left\|\frac{1}{k}\sum_{i\in N_{m,k}} A_i\right\| \le \|(I - P_m)(L_A)\|,$$

and by (3.118) we get the left inequality in (3.111).

To prove (3.112) we recall that every sequence $x = (x_k)_{k=1}^{\infty} \in w$ has a unique representation $x = le + \sum_{k=1}^{\infty}(x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in w$. Let us define $P_m : w \mapsto w$ by $P_m(x) = le + \sum_{k=1}^{m}(x_k - l)e^{(k)}$ for $m = 1, 2, \dots$. It is easy to prove that $\|I - P_m\| = 2$ for $m = 1, 2, \dots$. Now the proof of(3.112) is similar as in the case (3.111), and we omit it.

Let us prove (3.113). Now define $P_m : w_\infty \mapsto w_\infty$ by $P_m(x) = (x_1, x_2, \dots, x_m, 0, \dots)$ for all $x = (x_i) \in w_\infty$ and $m = 1, 2, \dots$. It is clear that $A(B) \subset P_m(A(B)) + (I - P_m)(A(B))$. Now, by the elementary properties of the function χ we have

$$\begin{aligned}\chi(A(B)) &\le \chi(P_m(A(B))) + \chi((I - P_m)(A(B))) = \chi((I - P_m)(A(B)) \\ &\le \sup_{x\in B}\|(I - P_m)(A(x))\|.\end{aligned} \tag{3.123}$$

Since the limit in (3.113) obviously exists, by (3.123) and from the proof of the right inequality in (3.111) we get (3.113).

Let us mention that inequalities (3.115), (3.116) and (3.117) are proved similarly as the inequalities (3.111), (3.112) and (3.113). □

Now as a corollary of the theorem above we have

Corollary 3.52. [70, Corollary 2] *Let A, X and Y be as in Theorem 3.51. Then for $A \in (X, Y)$ we have*

A is compact if and only if $\|A\|_{w_\infty} < \infty$ and

$$\lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

$$A \text{ is compact if } \|A\|_{w_\infty} < \infty \text{ and}$$
$$\lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} = 0, \qquad \text{if } Y = w_\infty,$$

$$A \text{ is compact if and only if } \|A\|_{c_\infty(\mu)} < \infty \text{ and}$$
$$\lim_{m\to\infty} \|A\|_{c_\infty(\mu)}^{(m)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

$$A \text{ is compact if } \|A\|_{c_\infty(\mu)} < \infty \text{ and}$$
$$\lim_{m\to\infty} \|A\|_{c_\infty(\mu)}^{(m)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

Now, concerning Remark 3.50, we get several corollaries.

Corollary 3.53. [70, Corollary 3] *Let A, X and Y be as in Theorem 3.51 and in Remark 3.50* (a). *We shall write $\max_{N_{m,k}}$ for $\max_{N_{m,k}\subset\{m+1,\ldots,k\}}$. For $A \in (X,Y)$ and $X = l_p$, we set for each m*

$$M(l_p, w_\infty)^{(m)} = \begin{cases} \sup_{k>m}\left(\max_{N_{m,k}}\left(\sup_j \left|\frac{1}{k}\sum_{i\in N_{m,k}} a_{ij}\right|\right)\right) & \text{for } p = 1 \\ \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{j=1}^{\infty}\left|\frac{1}{k}\sum_{i\in N_{m,k}} a_{ij}\right|^q\right)^{1/q}\right) & \text{for } 1 < p < \infty, \end{cases}$$

and

$$M(l_p, c_\infty(\mu))^{(m)} = \begin{cases} \sup_{k>m}\left(\max_{N_{m,k}}\left(\sup_j \left|\frac{1}{\mu_k}\sum_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|\right)\right) & \text{for } p = 1 \\ \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{j=1}^{\infty}\left|\frac{1}{\mu_k}\sum_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|^q\right)^{1/q}\right) & \text{for } 1 < p < \infty. \end{cases}$$

Now we have

$$A \text{ is compact if and only if } M(l_p, w_\infty) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

$$A \text{ is compact if } M(l_p, w_\infty) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_\infty,$$

$$A \text{ is compact if and only if } M(l_p, c_\infty(\mu)) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

$$A \text{ is compact if } M(l_p, c_\infty(\mu)) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

Corollary 3.54. [70, Corollary 4] *Let A, X and Y be as in Theorem 3.51 and in Remark 3.50 (b). We shall write $\max_{N_{m,k}}$ for $\max_{N_{m,k}\subset\{m+1,\dots,k\}}$. For $A \in (X,Y)$ and $X = w_\infty^p$, we set for each m*

$$M(w_\infty^p, w_\infty)^{(m)} = \begin{cases} \sup\limits_{k>m}\left(\max\limits_{N_{m,k}}\left(\sum\limits_{\nu=0}^{\infty} 2^{\nu/p} \max_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{k}\sum\limits_{i\in N_{m,k}} a_{ij}\right|\right)\right) \\ \text{for } \quad p=1 \\ \sup\limits_{k>m}\left(\max\limits_{N_{m,k}}\left(\sum\limits_{\nu=0}^{\infty} 2^{\nu/p}\left(\sum\limits_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{k}\sum\limits_{i\in N_{m,k}} a_{ij}\right|^q\right)^{1/q}\right)\right) \\ \text{for } \quad 1<p<\infty, \end{cases}$$

and

$$M(w_\infty^p, c_\infty(\mu))^{(m)}$$
$$= \begin{cases} \sup\limits_{k>m}\left(\max\limits_{N_{m,k}}\left(\sum\limits_{\nu=1}^{\infty} 2^{\nu/p} \max\limits_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{\mu_k}\sum\limits_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|\right)\right) \\ \text{for } p=1 \\ \sup\limits_{k>m}\left(\max\limits_{N_{m,k}}\left(\sum\limits_{\nu=1}^{\infty} 2^{\nu/p}\left(\sum\limits_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{\mu_k}\sum\limits_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|^q\right)^{1/q}\right)\right) \\ \text{for } 1<p<\infty. \end{cases}$$

Now we have

A is compact if and only if $M(w_\infty^p, w_\infty) < \infty$ and
$$\lim_{m\to\infty} M(w_\infty^p, w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

A is compact if $M(w_\infty^p, w_\infty) < \infty$ and
$$\lim_{m\to\infty} M(w_\infty^p, w_\infty)^{(m)}, \qquad \text{if } Y = w_\infty,$$

A is compact if and only if $M(w_\infty^p, c_\infty(\mu)) < \infty$ and
$$\lim_{m\to\infty} M(w_\infty^p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

A is compact if $M(w_\infty^p, c_\infty(\mu)) < \infty$ and
$$\lim_{m\to\infty} M(w_\infty^p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

Corollary 3.55. [70, Corollary 5] *Let A, X and Y be as in Theorem 3.51 and in Remark 3.50 (c). We shall write* $\max_{N_{m,k}}$ *for* $\max_{N_{m,k}\subset\{m+1,\dots,k\}}$. *For $A \in (X,Y)$, if $X = c_0(\Lambda)$, $c(\Lambda)$ or $c_\infty(\Lambda)$, we set for each m*

$$M(c_\infty(\Lambda), w_\infty)^{(m)} = \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=0}^{\infty}\lambda_{r(\nu+1)}\max_{r(\nu)\le r\le r(\nu+1)-1}\left|\sum_{j=r}^{\infty}\frac{1}{\lambda_j}\left(\frac{1}{k}\sum_{i\in N_{m,k}}a_{ij}\right)\right|\right)\right)$$

and

$$M(c_\infty(\Lambda), c_\infty(\mu))^{(m)} = \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=0}^{\infty}\lambda_{r(\nu+1)}\max_{r(\nu)\le r\le r(\nu+1)-1}\left|\sum_{j=r}^{\infty}\frac{1}{\lambda_j}\left(\frac{1}{\mu_k}\sum_{i\in N_{m,k}}\Delta_i(\mu_i a_{ij})\right)\right|\right)\right)$$

Now we have

$$A \text{ is compact if and only if } M(c_\infty(\Lambda), w_\infty) < \infty \text{ and } \lim_{m\to\infty} M(c_\infty(\Lambda), w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

$$A \text{ is compact if } M(c_\infty(\Lambda), w_\infty) < \infty \text{ and } \lim_{m\to\infty} M(c_\infty(\Lambda), w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_\infty,$$

$$A \text{ is compact if and only if } M(c_\infty(\Lambda), c_\infty(\mu)) < \infty \text{ and } \lim_{m\to\infty} M(c_\infty(\Lambda), c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

$$A \text{ is compact if } M(c_\infty(\Lambda), c_\infty(\mu)) < \infty \text{ and } \lim_{m\to\infty} M(c_\infty(\Lambda), c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

3.8. Further results. In this subsection, we shall give the characterizations of the classes (X, Y) where $X = l_1$ and $Y = w_\infty^p$, w^p, w_0^p $(1 \le p < \infty)$, or $X = w_0$, w, w_∞ and $Y = l_p$ $(1 \le p < \infty)$, or $X = w_0$, w, w_∞ and $Y = w_0^p$, w^p and w_∞^p $(1 \le p < \infty)$. Furthermore we shall apply the Hausdorff measure of compactness to give necessary and sufficient conditions for a linear operator between these spaces to be compact. The results can be found in [73].

Let $a \in \omega$. Then we write

$$\|a\|^{**} = \|a\|^{*}_{\mathcal{M}^p} = \sup\left\{\left|\sum_{k=1}^{\infty} a_k x_k\right| : \|x\|_{\mathcal{M}^p} = 1\right\}.$$

Lemma 3.56. [73, Lemma 1] *Let* $1 \le p < \infty$.

(a) *Then* $(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}^p$ *and* $\|a\|^* = \|a\|_{\mathcal{M}^p}$ *on* $\mathcal{M}^p$ *(cf. [49] or [65, Lemma 2]). Further the sets* $\mathcal{M}^p$ *are* BK *spaces with the norms* $\|\cdot\|_{\mathcal{M}^p}$ *(cf. [59, Theorem 2 (a)]), and it is easy to see that the spaces* $\mathcal{M}^p$ *have* AK.

(b) *Then* w_∞^p *is* β*-perfect, that is* $(w_\infty^p)^{\beta\beta} = w_\infty^p$ *and* $(w_0^p)^{\beta\beta} = (w^p)^{\beta\beta} = w_\infty^p$ *(cf. [59, Theorem 4 (b) and (c)]) and* $\|a\|^{**} = \|a\|_{w_\infty^p}$ *on* $(\mathcal{M}^p)^\beta = w_\infty^p$ *(cf. [63, Theorem 6 (b)]).*

If A is an infinite matrix, then we write A^T for its transpose.

Theorem 3.57. [73, Theorem 1] *Let* $1 \le p < \infty$. *Then*

(a) $A \in (l_1, w_\infty^p)$ *if and only if*

$$M(l_1, w_\infty^p) = \sup_{m,k}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk}|^p\right) < \infty; \tag{3.124}$$

(b) $A \in (l_1, w_0^p)$ *if and only if condition (3.124) holds and*

$$\lim_{m\to\infty}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk}|^p\right) = 0 \quad \text{for all } k; \tag{3.125}$$

(c) $A \in (l_1, w^p)$ *if and only if condition (3.124) holds and there is a sequence* $(\lambda_k)_{k=1}^\infty \in \omega$ *such that*

$$\lim_{m\to\infty}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk} - \lambda_k|^p\right) = 0 \quad \text{for all } k. \tag{3.126}$$

Proof. (a) condition (3.124) follows from [108, Example 8.4.1, p. 126] with $Y = w_\infty^p$.

(b) Parts (b) and (c) follow from part (a) and [70, Theorem 1 (c)]. □

By T we denote the set of all strictly increasing sequences $(t_\nu)_{\nu=0}^\infty$ of integers such that for each ν there is one and only one t_ν with $2^\nu \le t_\nu \le 2^{\nu+1}-1$. We put

$$M(w_0, l_p) = \begin{cases} \sup\limits_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}}\left(\sum\limits_{\nu=0}^{\infty} 2^\nu \max\limits_{\nu}\left|\sum\limits_{n\in N} a_{nk}\right|\right) & (p=1)\\ \sup\limits_{N\subset\mathbb{N}_0}\left(\sup\limits_{t\in T}\left(\sum\limits_{n=1}^{\infty}\left|\sum\limits_{\nu\in N} 2^\nu a_{n,t_\nu}\right|^p\right)\right) & (1<p<\infty)\\ \sup\limits_{N\subset\mathbb{N}_0}\left(\sup\limits_{t\in T}\left(\sup\limits_{n}\left|\sum\limits_{\nu\in N} 2^\nu a_{n,t_\nu}\right|\right)\right) & (p=\infty). \end{cases}$$

Theorem 3.58. [73, Theorem 2] *Let $1 \leq p \leq \infty$. Then*

$$(w_0, l_p) = (w, l_p) = (w_\infty, l_p); \tag{3.127}$$

further $A \in (w_0, l_p)$ if and only if

$$M(w_0, l_p) < \infty. \tag{3.128}$$

Proof. The case $p = 1$ follows from Lemma 3.56 (a) and from [63, Theorem 1] with $X = w_0, w, w_\infty$.

For $1 < p \leq \infty$, we apply [108, Theorem 8.3.9, p.124] with $X = w_0$ and $Z = l_q$ where $q = 1$ for $p = \infty$ and $q = p/(p-1)$ for $1 < p < \infty$. Since X and Z are BK spaces with AK, we obtain $(w_0, l_p) = (w_0^{\beta\beta}, l_p) = (w_\infty, l_p)$. (The second equality holds in view of Lemma 3.56 (b).) Since $w_0 \subset w \subset w_\infty$, we have established the identities in (3.127). Further, by [108, Theorem 8.3.9], $A \in (w_0, l_p)$ if and only if $A^T \in (l_q, w_0^\beta) = (l_q, \mathcal{M}^1)$, by Lemma 3.56 (a). Finally, by [59, Theorem 7], $A^T \in (l_q, \mathcal{M}^1)$ if and only if $M(w_0, l_p) < \infty$. □

Let us remark that an application of [70, Theorem 1 (b)] and Lemma 3.56 (a) yields $A \in (w_0, l_\infty)$ if and only if $\sup_n \sum_{\nu=0}^\infty 2^\nu \max_\nu |a_{nk}| < \infty$, a condition equivalent to condition (3.128) in Theorem 3.58 for $p = \infty$.

We write $N^{\langle\mu\rangle}$ for the set of all integers n with $2^\mu \leq n \leq 2^{\mu+1} - 1$, and we put

$$M(w_0, w_\infty^p) = \begin{cases} \sup\limits_{\mu \in \mathbb{N}_0} \left(\max\limits_{N_\mu \subset N^{\langle\mu\rangle}} \left(\sum\limits_{\nu=0}^{\infty} 2^\nu \max\limits_\nu \left| \frac{1}{2^\mu} \sum\limits_{n \in N_\mu} a_{nk} \right| \right) \right) & (p = 1) \\ \sup\limits_{N \subset \mathbb{N}_0} \left(\sup\limits_{t \in T} \left(\sup\limits_m \left(\frac{1}{m} \sum\limits_{n=1}^{m} \left| \sum\limits_{\nu \in N} \frac{1}{2^\nu} a_{n,t_\nu} \right|^p \right) \right) \right) & (1 < p < \infty). \end{cases}$$

Theorem 3.59. [73, Theorem 3] *Let $1 \leq p < \infty$.*
(a) *Then*

$$(w_0, w_\infty^p) = (w, w_\infty^p) = (w_\infty, w_\infty^p); \tag{3.129}$$

further $A \in (w_0, w_\infty^p)$ if and only if

$$M(w_0, w_\infty^p) < \infty; \tag{3.130}$$

(b) *$A \in (w_0, w_0^p)$ if and only if conditions (3.130) and (3.125) hold; $A \in (w_0, w^p)$ if and only if conditions (3.130) and (3.126) hold; $A \in (w, w_0^p)$ if and only if conditions (3.130) and (3.125) hold and*

$$\lim_{m \to \infty} \left(\frac{1}{m} \sum_{n=1}^m \left| \sum_{k=1}^\infty a_{nk} \right|^p \right) = 0; \tag{3.131}$$

$A \in (w, w^p)$ if and only if conditions (3.130) and (3.126) hold and

$$\lim_{m\to\infty}\left(\frac{1}{m}\sum_{n=1}^{m}\left|\sum_{k=1}^{\infty}a_{nk}-\lambda\right|^p\right)=0 \quad \textit{for some complex number } \lambda. \tag{3.132}$$

Proof. (a) For $p = 1$, part (a) is an immediate consequence of [63, Korollar 2] and Lemma 3.56 (a).

For $1 < p < \infty$, the identities in (3.129) follow by an argument similar to that in the proof of Theorem 3.58. We apply [108, Theorem 8.3.9] with $X = w_0$ and $Z = \mathcal{M}^p$ to conclude $A \in (w_0, w^p_\infty)$ if and only if $A^T \in (\mathcal{M}^p, w^p_0) = (\mathcal{M}^p, \mathcal{M}^1)$. Finally, by [59, Theorem 7], $A^T \in (\mathcal{M}^p, \mathcal{M}^1)$ if and only if $M(w_0, w^p_\infty) < \infty$.

(b) Part (b) follows from [70, Theorem 1 (c)], the fact that w_0 has AK and the representation for sequences in w given in Proposition 3.45. □

Now we shall give estimates for the operator norm $\|L_A\|$. We put

$$M^*_A(l_1, \tilde{w}^p_\infty) = \sup_{m,k}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk}|^p\right)^{1/p} \quad (1 \le p < \infty),$$

and for any BK space X

$$\begin{aligned}
M^*_A(X, l_1) &= \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}}\left\|\sum_{n\in N}A_n\right\|^*,\\
M^*_A(X, l_\infty) &= \sup_n \|A_n\|^*,\\
M^*_A(X, w_\infty) &= \sup_\mu\left(\max_{N_\mu\subset N^{(\mu)}}\left\|\frac{1}{2^\mu}\sum_{n\in N_\mu}A_n\right\|^*\right)\\
M^*_A(X, \mathcal{M}^1) &= \sup_{N\subset\mathbb{N}_0}\left(\left\|\sum_{\mu\in N}2^\mu A_{t_\mu}\right\|^*\right).
\end{aligned}$$

Theorem 3.60. [73, Theorem 4] (a) *Let $1 \le p < \infty$, $\|\cdot\|_{\tilde{w}^p_\infty}$ the norm on w^p_∞ defined by*

$$\|x\|_{\tilde{w}^p_\infty} = \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^p\right)^{1/p} \tag{3.133}$$

and $A \in (l_1, w^p_\infty)$. Then

$$\|L_A\| = M^*_A(l_1, \tilde{w}^p_\infty). \tag{3.134}$$

(b) *Let X be an arbitrary BK space. If $A \in (X, l_1)$, then*

$$M^*_A(X, l_1) \le \|L_A\| \le 4 \cdot M^*_A(X, l_1). \tag{3.135}$$

If $A \in (X, l_\infty)$, *then*

$$\|L_A\| = M_A^*(X, l_\infty). \tag{3.136}$$

If $\|\cdot\|_{w_\infty}$ *is the norm on* w_∞ *defined in (3.101) and* $A \in (X, w_\infty)$, *then*

$$M_A^*(X, w_\infty) \le \|L_A\| \le 4 \cdot M_A^*(X, w_\infty). \tag{3.137}$$

(c) Let X *be an arbitrary BK space and* $A \in (X, \mathcal{M}^1)$, *then*

$$M_A^*(X, \mathcal{M}^1) \le \|L_A\| \le 4 \cdot M_A^*(X, \mathcal{M}^1). \tag{3.138}$$

Proof. (a) Let $A \in (l_1, w_\infty^p)$, $x \in l_1$ with $\|x\|_1 = \sum_{k=1}^\infty |x_k| = 1$ and $m \in \mathbb{N}$ be given. Then we have by Minkowski's inequality

$$\begin{aligned}\left(\frac{1}{m}\sum_{n=1}^m |A_n(x)|^p\right)^{1/p} &= \left(\frac{1}{m}\sum_{n=1}^m \left|\sum_{k=1}^\infty a_{nk}x_k\right|^p\right)^{1/p} \le \sum_{k=1}^\infty |x_k| \left(\frac{1}{m}\sum_{n=1}^m |a_{nk}|^p\right)^{1/p} \\ &\le M_A^*(l_1, \tilde{w}_\infty^p),\end{aligned}$$

hence, since m was arbitrary, $\|A(x)\|_{\tilde{w}_\infty^p} \le M_A^*(l_1, \tilde{w}_\infty^p)$ and consequently

$$\|L_A\| = \sup\{\|A(x)\|_{\tilde{w}_\infty^p} : \|x\|_1 = 1\} \le M_A(l_1, \tilde{w}_\infty^p). \tag{3.139}$$

Now let $x = e^{(k)}$ $(k = 1, 2, \dots)$. Then $x \in l_1$, $\|x\|_1 = 1$ and

$$\|A(x)\|_{w_\infty^p} = \sup\left(\frac{1}{m}\sum_{n=1}^m |a_{nk}|^p\right)^{1/p} \le \|L_A\|$$

together imply

$$M_A^*(l_1, \tilde{w}_\infty^p) \le \|L_A\|. \tag{3.140}$$

Finally, from (3.139) and (3.140), we conclude (3.134).

(b) First we show (3.135). Let $A \in (X, l_1)$, $x \in X$ with $\|x\| = 1$ and $m \in \mathbb{N}$ be given. Then

$$\sum_{n=1}^m |A_n(x)| \le 4 \cdot \max_{N \subset \{1,\dots,m\}} \left|\sum_{k=1}^\infty \left(\sum_{n\in N} a_{nk}\right) x_k\right| \le 4 \cdot M_A^*(X, l_1).$$

Since m was arbitrary, we conclude $\|A(x)\|_1 \le 4 \cdot M_A^*(X, l_1)$ and consequently

$$\|L_A\| \le 4 \cdot M_A^*(X, l_1). \tag{3.141}$$

Conversely, let $N \subset \mathbb{N}$ be an arbitrary finite set. Then given $\varepsilon > 0$ there is a sequence $x = x(N, \varepsilon) \in X$ such that $\|x\| = 1$ and $\left\|\sum_{n\in N} A_n\right\|^* \le \left|\sum_{n\in N} A_n(x)\right| + \varepsilon$.

Therefore

$$\left\|\sum_{n\in N} A_n\right\|^* \leq \sum_{n=1}^{\infty} |A_n(x)| + \varepsilon \leq \|A(x)\|_1 + \varepsilon \leq \|L_A\| + \varepsilon.$$

Since $N \subset \mathbb{N}$ and $\varepsilon > 0$ were arbitrary,

$$M_A^*(X, l_1) \leq \|L_A\|. \tag{3.142}$$

Finally, from (3.141) and (3.142), we conclude (3.135) Equality (3.136) is Theorem 1.23 (b). The inequalities in (3.137) are shown in exactly the same way as those in [70, Corollary 1 (a), (2.8)] with m, $N_m \subset \{1, \ldots, m\}$ and $1/m$ replaced by μ, $N_\mu \subset N^{\langle\mu\rangle}$ and $1/2^\mu$.

(c) Let $A \in (X, \mathcal{M}^1)$, $x \in X$ with $\|x\| = 1$ and $\mu_0 \in \mathbb{N}_0$ be given. We choose $n_\mu \in N^{\langle\mu\rangle}$ $(\mu = 0, 1, \ldots)$ such that $|A_{n_\mu}(x)| = \max_{n\in N^{\langle\mu\rangle}} |A_n(x)|$. Then we have

$$\begin{aligned}
\sum_{\mu=0}^{\mu_0} 2^\mu |A_{n_\mu}(x)| &\leq 4 \cdot \max_{N\subset\{0,\ldots,\mu_0\}} \left|\sum_{\mu\in N} 2^\mu A_{n_\mu}(x)\right| \\
&= 4 \cdot \max_{N\subset\{0,\ldots,\mu_0\}} \left|\sum_{k=1}^{\infty}\left(\sum_{\mu\in N} 2^\mu a_{n_\mu,k}\right) x_k\right| \\
&\leq 4 \cdot \sup_{N\in\mathbb{N}_0}\left(\sup_{t\in T}\left\|\sum_{\mu\in N} 2^\mu A_{t_\mu}\right\|^*\right) = 4 \cdot M_A^*(X, \mathcal{M}^1).
\end{aligned}$$

Since this holds for all $\mu_0 \in \mathbb{N}_0$, we conclude $\|A(x)\|_{\mathcal{M}^1} \leq 4 \cdot M_A^*(X, \mathcal{M}^1)$ and consequently

$$\|L_A\| \leq 4 \cdot M_A^*(X, \mathcal{M}^1) \tag{3.143}$$

Conversely, let $N \in \mathbb{N}_0$, $t \in T$ and $\varepsilon > 0$ be given. Then there is a sequence $x = x(N, t, \varepsilon) \in X$ such that $\|x\| = 1$ and $\|\sum_{\mu\in N} 2^\mu A_{t_\mu}\|^* \leq |\sum_{\mu\in N} 2^\mu A_{t_\mu}(x)| + \varepsilon$. Therefore

$$\left\|\sum_{\mu\in N} 2^\mu A_{t_\mu}\right\|^* \leq \sum_{\mu=0}^{\infty} 2^\mu \max_{n\in N^{\langle\mu\rangle}} |A_n(x)| + \varepsilon = \|A(x)\|_{\mathcal{M}^1} + \varepsilon \leq \|L_A\| + \varepsilon.$$

Since $N \in \mathbb{N}_0$, $t \in T$ and $\varepsilon > 0$ were arbitrary, we have $M_A^*(X, \mathcal{M}^1) \leq \|L_A\|$. Finally, from this and (3.143), we conclude (3.138). □

Now we apply the previous results to estimate the operator norms of the matrix transformations characterized in Theorems 3.57, 3.58 and 3.59.

Let X be any of the spaces w_0, w and w_∞. We put

$$M_A^*(X,l_1) = \sup_{\substack{N\subset \mathbb{N}_0\\ N \text{ finite}}} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left|\sum_{n\in N} a_{nk}\right|\right),$$

$$M_A^*(X,l_\infty) = \sup_n \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu |a_{nk}|\right),$$

$$M_A^*(X,w_\infty) = \sup_\mu \left(\max_{N_\mu \subset N^{(\mu)}} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left|\frac{1}{2^\mu} \sum_{n\in N_\mu} a_{nk}\right|\right)\right),$$

and, for $1 < p < \infty$ and $q = p/(p-1)$,

$$M_{A^T}^*(l_q,\mathcal{M}^1) = \sup_{N\subset \mathbb{N}_0} \left(\sup_{t\in T}\left(\sum_{n=1}^{\infty}\left|\sum_{\nu\in N} 2^\nu a_{n,t_\nu}\right|^p\right)^{1/p}\right),$$

$$M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1) = \sup_{N\subset \mathbb{N}_0} \left(\sup_{t\in T}\left(\sup_\mu\left(\frac{1}{2^\mu}\sum_{n\in N^{(\mu)}}\left|\sum_{\nu\in N} 2^\nu a_{n,t_\nu}\right|^p\right)^{1/p}\right)\right).$$

Corollary 3.61. [73, Corollary 1] *Let X be any of the spaces w_0, w and w_∞ and $\|\cdot\|_{w_\infty^p}$ the norm defined in (3.101).*
If $A \in (X,l_1)$, then $M_A^*(X,l_1) \le \|L_A\| \le 4\cdot M_A^*(X,l_1)$.
If $A \in (X,l_\infty)$, then $\|L_A\| = M_A^*(X,l_\infty)$.
If $A \in (X,l_p)$ $(1<p<\infty,\ q=\frac{p}{p-1})$, then $M_{A^T}^*(l_q,\mathcal{M}^1) \le \|L_A\| \le 4\cdot M_{A^T}^*(l_q,\mathcal{M}^1)$.
If $A \in (X,w_\infty)$, then $M_A^*(X,w_\infty) \le \|L_A\| \le 4\cdot M_A^*(X,w_\infty)$.
If $A \in (X,w_\infty^p)$ $(1<p<\infty)$, then $M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1) \le \|L_A\| \le 4\cdot M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1)$.

We need with the following auxiliary lemma.

Lemma 3.62. [73, Lemma 2] (a) *Let $P_m : w_0^p \mapsto w_0^p$ for $1 \le p < \infty$ and $m = 1,2,\dots$ be the projector on the first m coordinates, that is $P_m(x) = (x_1,x_2,\dots,x_m, 0,0,\dots)$ for $x=(x_i)\in w_0^p$. Then $\|I-P_m\| = 1, m=1,2,\dots$.*

(b) *For $x \in w^p$, we use the representation in Proposition 3.44 and define $P_m : w^p \mapsto w^p$ by $P_m(x) = le + \sum_{k=1}^m (x_k - l)e^{(k)}$ for $m=1,2,\dots$. Then $\|I-P_m\| = 2$ for $m=1,2,\dots$.*

Proof. (a) It is clear that $\|I-P_m\| \le 1$. Since $I-P_m \ne O$ is a bounded linear operator and projector, we have $\|I-P_m\| \ge 1$. This proves (a).

(b) Let $x=(x_k)_{k=1}^\infty \in w^p$. Then x has the representation in Proposition 3.44, and we obtain

$$\|(I-P_m)(x)\| = \|\underbrace{(0,\dots,0}_{m},x_{m+1}-l,x_{m+2}-l,\dots\| \le \|x\|+|l| \le 2\|x\|.$$

Hence $\|I-P_m\| \le 2$, $m=1,2,\dots$. To prove that $\|I-P_m\| \ge 2$, let $\epsilon > 0$. Then, since

$$2\left(\frac{k}{m+k}\right)^{1/p} \to 2 \quad (k\to\infty),$$

there exists $k_0 \in \mathbb{N}$ such that

$$2\left(\frac{k_0}{m+k_0}\right)^{1/p} > 2-\epsilon.$$

Let $u_0 \in w^p$ be defined by

$$u_0 = (\underbrace{1,\ldots,1}_{m},\underbrace{-1,\ldots,-1}_{k_0},1,1,1,\ldots).$$

Then $\|u_0\| = 1$, $l = 1$ and

$$\|(I-P_m)(u_0)\| \geq \left(\frac{1}{m+k_0}\cdot 2^p k_0\right)^{1/p} = 2\left(\frac{k_0}{m+k_0}\right)^{1/p} > 2-\epsilon.$$

Hence $\|I-P_m\| > 2-\epsilon$, that is $\|I-P_m\| \geq 2$. □

Theorem 3.63. [73, Theorem 5] *Let $1 \leq p < \infty$, $\|\cdot\|_{\tilde{w}^p_\infty}$ the norm on w^p_0, w^p and w^p_∞ defined in (3.133). We put*

$$M^*_A(l_1,\tilde{w}^p_\infty)_{(m)} = \sup_{\substack{k\\ u>m}}\left(\frac{1}{u}\sum_{n=m+1}^{u}|a_{nk}|^p\right)^{1/p}.$$

(a) *If $A \in (l_1, w^p_0)$, then*

$$\|L_A\|_\chi = \lim_{m\to\infty} M^*_A(l_1,\tilde{w}^p_\infty)_{(m)}. \tag{3.144}$$

(b) *If $A \in (l_1, w^p)$, then*

$$\frac{1}{2}\cdot\lim_{m\to\infty} M^*_A(l_1,\tilde{w}^p_\infty)_{(m)} \leq \|L_A\|_\chi \leq \lim_{m\to\infty} M^*_A(l_1,\tilde{w}^p_\infty)_{(m)}. \tag{3.145}$$

(c) *If $A \in (l_1, w^p_\infty)$, then*

$$0 \leq \|L_A\|_\chi \leq \lim_{m\to\infty} M^*_A(l_1,\tilde{w}^p_\infty)_{(m)}. \tag{3.146}$$

Proof. Let us remark that the limits in (3.144), (3.145) and (3.146) exist. We put $B = \{x \in l_1 : \|x\| \leq 1\}$. In the case (a) we have by the inequality in Theorem 2.23

$$\|L_A\|_\chi = \chi(A(B)) = \lim_{m\to\infty}\left[\sup_{x\in B}\|(I-P_m)(A(x))\|\right], \tag{3.147}$$

where $P_m : w^p_0 \mapsto w^p_0$ for $m = 1,2,\ldots$ is the projector on the first m coordinates, that is $P_m(x) = (x_1,x_2,\ldots,x_m,0,0,\ldots)$ for $x = (x_k) \in w^p_0$. Let us recall that by

Lemma 3.62 (c) we have $\|I - P_m\| = 1, m = 1, 2, \dots$. Let $A_{(m)} = (\tilde{a}_{nk})$ be the infinite matrix defined by $\tilde{a}_{nk} = 0$ if $1 \le n \le m$ and $\tilde{a}_{nk} = a_{nk}$ if $m < n$. Now, by (3.134) we have

$$\sup_{x \in B} \|(I - P_m)(A(x))\| = \|L_{A_{(m)}}\| = M^*_{A_{(m)}}(l_1, \tilde{w}^p_\infty)_{(m)} = M^*_A(l_1, \tilde{w}^p_\infty)_{(m)}$$

Part (a) now follows from this and (3.147).

(b) Let $x = (x_k)_{k=1}^\infty \in w_0^p$. Then x has the representation in Proposition 3.44, and we define $P_m : w^p \mapsto w^p$ by $P_m(x) = le + \sum_{k=1}^m (x_k - l)e^{(k)}$ for $m = 1, 2, \dots$. By Lemma 3.62 (b) we know that $\|I - P_m\| = 2$ for $m = 1, 2, \dots$. Now the proof of (b) is similar as in the case (a), and we omit it.

Let us prove (3.146). Now define $P_m : w^p_\infty \mapsto w^p_\infty$ by $P_m(x) = (x_1, x_2, \dots, x_m, 0, \dots)$ for $x = (x_i) \in w^p_\infty$ and $m = 1, 2, \dots$. It is clear that $A(B) \subset P_m(A(B)) + (I - P_m)(A(B))$. Now, by the elementary properties of the function χ we have

$$\begin{aligned} \chi(A(B)) &\le \chi(P_m(A(B))) + \chi((I - P_m)(A(B))) = \chi((I - P_m)(A(B)) \\ &\le \sup_{x \in B} \|(I - P_m)(A(x))\| = \|L_{A_{(m)}}\|. \end{aligned} \tag{3.148}$$

Since the limit in (3.146) obviously exists, by (3.148) and (3.135) we get (3.146). □

Now as a corollary of the above theorem we have

Corollary 3.64. [73, Corollary 2] *If either $A \in (l_1, w_0^p)$ or $A \in (l_1, w_0^p)$, then*

$$L_A \quad \textit{is compact if and only if} \lim_{m \to \infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = 0.$$

If $A \in (l_1, w^p_\infty)$, then

$$L_A \quad \textit{is compact if} \lim_{m \to \infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = 0. \tag{3.149}$$

The following example will show that it is possible for L_A in (3.149) to be compact in the case $\lim_{m\to\infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} > 0$, and hence in general we have just "if" in (3.149).

Example 3.65. [73, Example 1] Let the matrix A be defined by $a_{nk} = 1$ if $n = 1$ and $a_{nk} = 0$ if $n \ne 1$. Then $M^*_A(l_1, \tilde{w}^p_\infty) = 1$ and $A \in (l_1, \tilde{w}^p_\infty)$. Further

$$M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = \sup_{k \ge 1, u > m} \left(\frac{1}{u} \sum_{n=m+1}^{u} |a_{nk}|^p \right)^{1/p} = \sup_{k \ge 1, u > m} \left(\frac{u - m}{u} \right)^{1/p} = 1.$$

Whence $\lim_{m\to\infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = 1 > 0$. Since $A(x) = x_1 e$ for all $x \in l_1$, L_A is a compact operator.

Now, concerning Corollary 3.61 we continue to study the measures of noncompactness of operators when the final spaces are the spaces l_p and w_∞^p. Let X be any of the spaces w_0, w and w_∞. For $m \in \mathbb{N}$ we put

$$M_A^*(X,l_1)_{(m)} = \sup_{\substack{N \subset \mathbb{N} \setminus \{1,2,\dots,m\} \\ N \text{ finite}}} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left| \sum_{n \in N} a_{nk} \right| \right),$$

$$M_A^*(X,l_\infty)_{(m)} = \sup_{n>m} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu |a_{nk}| \right),$$

$$M_A^*(X,w_\infty)_{(m)} = \sup_{\mu>m} \left(\max_{N_\mu \subset N^{(\mu)}} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left| \frac{1}{2^\mu} \sum_{n \in N_\mu} a_{nk} \right| \right) \right),$$

and, for $1 < p < \infty$ and $q = p/(p-1)$,

$$M_{A^T}^*(l_q,\mathcal{M}^1)_{(m)} = \sup_{N \subset \mathbb{N}_0} \left(\sup_{t \in T} \left(\sum_{n=m+1}^{\infty} \left| \sum_{\nu \in N} 2^\nu a_{n,t_\nu} \right|^p \right)^{1/p} \right),$$

$$M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1)_{(m)} = \sup_{N \subset \mathbb{N}_0 \setminus \{1,2,\dots,m\}} \left(\sup_{t \in T} \left(\sup_\mu \left(\frac{1}{2^\mu} \sum_{n \in N^{(\mu)}} \left| \sum_{\nu \in N} 2^\nu a_{n,t_\nu} \right|^p \right)^{1/p} \right) \right).$$

Theorem 3.66. [73, Theorem 6] *Let X be any of the spaces w_0, w and w_∞ and $\|\cdot\|_{w_\infty^p}$ the norm defined in (3.101). If $A \in (X, l_1)$, then*

(3.150) $$\lim_{m\to\infty} M_A^*(X,l_1)_{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_A^*(X,l_1)_{(m)}.$$

If $A \in (X, l_\infty)$, then

(3.151) $$\|L_A\|_\chi \le \lim_{m\to\infty} M_A^*(X,l_\infty)_{(m)}.$$

If $A \in (X, l_p)$ $(1 < p < \infty, q = p/(p-1))$, then

(3.152) $$\lim_{m\to\infty} M_{A^T}^*(l_q,\mathcal{M}^1)_{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_{A^T}^*(l_q,\mathcal{M}^1)_{(m)}.$$

If $A \in (X, w_\infty)$, then

(3.153) $$\|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_A^*(X,w_\infty)_{(m)}.$$

If $A \in (X, w_\infty^p)$ $(1 < p < \infty)$, then

(3.154) $$\|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1)_{(m)}.$$

Proof. Let us remark that the limits in (3.150) to (3.154) exist. Let $P_m : l^p \mapsto l^p$ for $m = 1,2,\dots$ and $1 \le p < \infty$ be the projector on the first m coordinates, that is $P_m(x) = (x_1, x_2, \dots, x_m, 0, 0, \dots)$ for $x = (x_i) \in l^p$. It is easy to check that $\|I - P_m\| = 1, m = 1,2,\dots$. Now the proof of (3.150) and (3.152) (when final spaces have a basis) can be given by the method of proof of Theorem 3.63 (a), while in the proof of (3.151), (3.153), and (3.154) (when final spaces have no basis) we can use the method of the proof of Theorem 3.63 (c). □

Now as a corollary of the theorem above we have

Corollary 3.67. [73, Corollary 3] *Let X be any of the spaces w_0, w and w_∞ and $\|\cdot\|_{w_\infty^p}$ the norm defined in (3.101). If $A \in (X, l_1)$, then*

$$L_A \quad \text{is compact if and only if} \quad \lim_{m\to\infty} M_A^*(X, l_1)_{(m)} = 0. \tag{3.155}$$

If $A \in (X, l_\infty)$, then

$$L_A \quad \text{is compact if} \quad \lim_{m\to\infty} M_A^*(X, l_\infty)_{(m)} = 0. \tag{3.156}$$

If $A \in (X, l_p)$ $(1 < p < \infty, q = p/(p-1))$, then

$$L_A \quad \text{is compact if and only if} \quad \lim_{m\to\infty} M_{A^T}^*(l_q, \mathcal{M}^1)_{(m)} = 0. \tag{3.157}$$

If $A \in (X, w_\infty)$, then

$$L_A \quad \text{is compact if} \quad \lim_{m\to\infty} M_A^*(X, w_\infty)_{(m)} = 0. \tag{3.158}$$

If $A \in (X, w_\infty^p)$ $(1 < p < \infty)$, then

$$L_A \quad \text{is compact if} \quad \lim_{m\to\infty} M_{A^T}^*(\mathcal{M}^p, \mathcal{M}^1)_{(m)} = 0. \tag{3.159}$$

Let us remark that it is possible for L_A in (3.156), (3.158) and (3.159) to be compact in the cases $\lim_{m\to\infty} M_A^*(X, l_\infty)_{(m)} > 0$, $\lim_{m\to\infty} M_A^*(X, w_\infty)_{(m)} > 0$ and $\lim_{m\to\infty} M_{A^T}^*(\mathcal{M}^p, \mathcal{M}^1)_{(m)} > 0$, respectively. This can be proved by Example 3.65.

4 Appendix

In this appendix, we collect the results from Functional Analysis needed in the previous sections.

4.1. Inequalities.

Theorem A.4.1. (Hölder's inequality) *Let $1 < p < \infty$, $q = p/(p-1)$ and $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n \in \mathbb{C}$. Then*

$$\sum_{k=0}^{n} |x_k y_k| \le \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} \left(\sum_{k=0}^{\infty} |y_k|^q\right)^{1/q}.$$

If $x \in l_p$ and $y \in l_q$, then $xy = (x_k y_k)_{k=0}^\infty \in l_1$ and $\|xy\|_1 \le \|x\|_p \|y\|_q$.

Theorem A.4.2. (Minkowski's inequality) *Let $1 \le p < \infty$ and $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n \in \mathbb{C}$. Then*

$$\left(\sum_{k=0}^{n} |x_k + y_k|^p\right)^{1/p} \le \left(\sum_{k=0}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=0}^{n} |y_k|^p\right)^{1/p}$$

If $x, y \in l_p$, then $x + y \in l_p$ and $\|x + y\|_p \le \|x\|_p + \|y\|_p$.

Theorem A.4.3. (Jensen's inequality) *Let $p > 0$ and $x_0, x_1, \dots, x_n \in \mathbb{C}$. Then*

$$\left(\sum_{k=0}^{n} |x_k|^p\right)^{1/p} \quad \text{is a decreasing function in } p,$$

that is, if $r > s > 0$, then

$$\left(\sum_{k=0}^{n} |x_k|^r\right)^{1/r} \leq \left(\sum_{k=0}^{n} |x_k|^s\right)^{1/s}.$$

If $p > p'$, then $l_{p'} \subset l_p$.

4.2. The closed graph theorem and the Banach–Steinhaus theorem.

Theorem A.4.4. (Closed graph lemma) *Any continuous map into a Hausdorff space has closed graph* [105, Theorem 11.1.1, p. 195].

Theorem A.4.5. (Closed graph theorem) *If X and Y are Fréchet spaces and $f : X \mapsto Y$ is a linear map with closed graph, then f is continuous* [105, Theorem 11.2.2, p. 200].

Theorem A.4.6. (Banach–Steinhaus theorem) *Let $(f_n)_{n=0}^{\infty}$ be a pointwise convergent sequence of continuous linear functionals on a Fréchet space X. Then f defined by*

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \in X,$$

is continuous [105, Corollary 11.2.4, p. 200].

Bibliography

[1] Р. Р. Ахмеров, М. И. Каменский, А. С. Потапов и др., *Меры некомпактности и уплотняющие операторы*, Наука, Новосибирск, 1986.

[2] A. G. Aksoy, *Approximation schemes, related s-numbers and applications*, Ph.D. thesis, University of Michigan, 1984.

[3] J. Arias-de-Reyna and T. Dominguez Benavides, *On a measure of noncompactness in Banach spaces with Schauder basis*, Boll. Un. Mat. Ital. A **7** (1993), 77–86.

[4] K. Astala, *On measures of noncompactness and ideal variations in Banach spaces*, Ann. Acad. Sci. Fenn. Ser. A. I Math. Dissertationes **29** (1980), 1–42.

[5] K. Astala and H-O. Tylli, *On the bounded compact approximation property and measures of noncompactness*, J. Funct. Anal. **70** (1987), 388–401.

[6] J. Banás, *Applications of measures of noncompactness to various problems*, Zeszyty Naukowe Politechniki Rzeszowskiej, Nr 34, Matematyka i fizyka z.5, Matematyka, z.5, Rzeszów, 1987.

[7] J. Banás and K. Goebl, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York and Basel, 1980.

[8] J. J. Buoni, R. Harte and T. Wickstead, *Upper and lower Fredholm spectra*, Proc. Amer. Math. Soc. **66** (1977), 309–314.

[9] S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces*, Marcel Dekker, New York, 1974.

[10] L. W. Cohen and N. Dunford, *Transformations on sequence spaces*, Duke Math. J. **3** (1937), 689-701.

[11] J. Daneš, *On the Istrăţesku's measure of noncompactness*, Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S) **16** (1972), 403–406.

[12] G. Darbo, *Punti uniti in transformazioni a condominio non compatto*, Rend. Sem. Math. Univ. Padova **24** (1955), 84–92.

[13] T. Dominguez Benavides, *Set-contractions and ball contractions in some class of spaces*, J. Math. Anal. Appl. **136** (1988), 131–140.

[14] T. Dominguez Benavides and J. M. Ayerbe, *Set-contractions and ball contractions in L^p-spaces*, J. Math. Anal. Appl. **159** (1991), 500–506.

[15] J. Dugundji, *Topology*, Allyn and Bacon, Boston, London, Sydney and Toronto, 1966.

[16] А. С. Файнштейн. *О мерах некомпактности линейных операторов и аналогах минимального модуля для полуфредгольмовых операторов*, Спектральная теория операторов и еѐ приложения, Элм, Баку, **6** (1985), 182–195.

[17] K.-Förster and E.-O. Libetrau, *Semi-Fredholm operators and sequence spaces*, Manuscripta Math. **44** (1983), 35–44.

[18] M. Furi and A. Vignoli, *On a property of the unit sphere in a linear normed space*, Bull. Polish Acad. Sci. Math. **18** (1970), 333–334.

[19] Л. С. Гольденштейн, И.Ц. Гохберг и А.С. Маркус, *Исследование некоторых свойств линейных ограниченных операторов в связи с их q-нормой*, Уч. зап. Кишинев. гос. унив. **29** (1957), 29–36.

[20] Л. С. Гольденштейн, А. С. Маркус, *О мере некомпактности ограниченных множеств и линейных операторов*, In: *Исследование по алгебре и математическому анализу*, Картя Молдавеняске, Кишинев 1965, pp. 45–54.

[21] M. González and A. Martinón, *Operational quantities characterizing semi-Fredholm operators*, Studia Math. **114** (1995), 13–27.

[22] K.G. Grosse–Erdmann, *The blocking technique, weighted mean operators and Hardy's inequality*, preprint.

[23] K. G. Grosse–Erdmann, *Strong weighted mean summability amd Kuttner's theorem*, preprint

[24] O. Hadžić, *Osnovi teorije nepokretne tačke*, Institut za matematiku, Novi Sad, 1978.

[25] O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, Institute of Mathematics, University of Novi Sad, Novi Sad, 1984.

[26] O. Hadžić, *Some properties of measures of noncompactness in paranormed spaces*, Proc. Amer. Math. Soc. **102** (1988), 843–849.

[27] G. H. Hardy, *Divergent Series*, Oxford University Press, 1973.

[28] R. Harte, *Invertibility and Singularuty for Bounded Linear Operators*, Marcel Dekker, New York and Basel, 1988.

[29] R. Harte and A. Wickstead, *Upper and lower Fredholm spectra II*, Math. Z. **154** (1977), 253–256.

[30] V. Istrăţesku, *On a measure of noncompactness*, Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S) **16** (1972), 195–197.

[31] V. Istrăţesku, *Fixed Point Theory, An Introduction*, Reidel, Dordrecht, Boston and London, 1981.

[32] P. K. Jain and E. Malkowsky, *Advances in Sequence Spaces*, Narosa, Delhi, Madras, Bombay, Calcutta, London, 1999.

[33] A. Jakimovski and D. C. Russell, *Matrix mappings between BK spaces*, Bull. London Math. Soc. **4** (1972), 345–353.

[34] A. M. Jarrah and E. Malkowsky, *BK spaces, bases and linear operators*, Rend. Circ. Mat. Palermo **(2)** Suppl. **52** (1998), 177–191.

[35] I. Jovanović and V. Rakočević, *Multipliers of mixed-norm sequence spaces and measures of noncompactness*, Publ. Inst. Math. (Beograd) (N.S.) **56(70)** (1994), 61–68.

[36] I. Jovanović and V. Rakočević, *Multipliers of mixed-norm sequence spaces and measures of noncompactness II*, Mat. Vesnik **49** (1997), 197–206.

[37] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Marcel Dekker, New York, 1981.

[38] C. N. Kellog, *An extension of the Hausdorff-Young theorem*, Michigan Math. J. **18** (1971), 121–127.

[39] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. **24** (1981), 169–176.

[40] G. Köthe, *Topologische lineare Räume I*, Springer-Verlag, 1966.

[41] K. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.

[42] K. Kuratowski, *Topologie*, Warsaw, 1958.

[43] C. G. Lascarides, *A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer*, Pacific J. Math. **38** (1971), 487–500.

[44] C. G. Lascarides and I. J. Maddox, *Matrix transformations between some classes of sequences*, Proc. Cambridge Philos. Soc. **68** (1970), 99–104.

[45] A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Funct. Anal. **7** (1971), 1–26.

[46] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, sequence spaces*, Springer-Verlag, 1977.

[47] R. Lowen, *Kuratowski's measure of noncompactness revisited*, Quart. J. Math. Oxford Ser. (2) **39** (1988), 235–254.

[48] Y. Luh, *Toeplitz-Kriterien für Matrixtransformationen zwischen paranormierten Folgenräumen*, Diplomarbeit, Giessen 1985.

[49] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser. (2) **18** (1967), 345–355.

[50] I. J. Maddox, *Paranormed sequence spaces generated by infinite matrices*, Proc. Cambridge Philos. Soc. **63** (1968), 335–340.

[51] I. J. Maddox, *On Kuttner's theorem*, J. London Math. Soc. **43** (1968), 285–290.

[52] I. J. Maddox, *Some properties of paranormed sequence spaces*, J. London Math. Soc. **(2),1** (1969), 316–322.

[53] I. J. Maddox, *Continuous and Köthe-Toeplitz duals of certain sequence spaces*, Proc. Cambridge Philos. Soc. **65** (1969), 431–435.

[54] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, London and New York, 1970.

[55] I. J. Maddox, *Operators on the generalized entire sequences*, Proc. Cam. Phil. Soc., **71** (1972), 491–494.

[56] I. J. Maddox, M. A. L. Willey, *Continuous operators on paranormed sequence spaces and matrix transfromations*, Pacific J. Math. **58** (1974), 217–228.

[57] E. Malkowsky, *Matrixabbildungen in paranormierten FK-Räumen*, Analysis **7** (1987), 275–292.

[58] E. Malkowsky, *Matrix transformations in certain spaces of strongly Cesàro summable and bounded sequences*, Analysis **8** (1998), 325–345.

[59] E. Malkowsky, *Matrix transformations between spaces of absolutely and strongly summable sequences*, Habilitationsschrift, Giessen, 1988.

[60] E. Malkowsky, *Matrix transformations in some spaces of strongly bounded sequences*, Colloq. Math. Soc. Janos Bolyai **58** Approximation Theory, Kecskemet, (1990), 498–511.

[61] E. Malkowsky, *A characterization of linear operators in certain paranormed sequence spaces*, Quaestiones Math. **18** (1995), 295–307.

[62] E. Malkowsky, *A study of the α-duals for $w_\infty(p)$ and $w_0(p)$*, Acta Sci. Math Szeged **60** (1995), 559–570.

[63] E. Malkowsky, *A note on the Köthe-Toeplitz duals of generalized sets of bounded and convergent difference sequences*, J. Anal. **4** (1995), 81–91.

[64] E. Malkowsky, *The continuous duals of the spaces $c_0(\Lambda)$ and $c(\Lambda)$ for exponentially bounded sequences Λ*, Acta Sci. Math. Szeged **61** (1995), 241–250.

[65] E. Malkowsky, *Linear Operators in certain BK spaces*, Bolyai Soc. Math. Stud. **5** (1996), 259–273.

[66] E. Malkowsky, *On a new class of sequence spaces related to absolute and strong summablility*, in: Fourier Analysis, Approximation Theory and Applications, New Age International, New Delhi, 1997, 145–154.

[67] E. Malkowsky, *A survey of recent results concerning a general class of sequence spaces and matrix transformations*, in: Fourier Analysis, Approximation Theory and Applications, New Age International, New Delhi, 1997.

[68] E. Malkowsky, *Recent results in the theory of matrix transformations in sequence spaces*, Mat. Vesnik **49** (1997), 187–196.

[69] E. Malkowsky, S. D. Parashar, *Matrix transformations in spaces of bounded and convergent difference sequences of order* m, Analysis **17** (1997), 87–97.

[70] E. Malkowsky and V. Rakočević, *The measure of noncompactness of linear operators between certain sequence spaces,* Acta Sci. Math. Szeged **64** (1998), 151–170.

[71] E. Malkowsky and V. Rakočević, *The measure of noncompactness of linear operators between spaces of* m^{th} *-order difference sequences,* Studia Sci. Math. Hungar. **35** (1999), 381-395.

[72] E. Malkowsky and V. Rakočević, *Measure of noncompactness of linear operators between spaces of sequences that are* $(\overline{N}, q)$ *summable or bounded,* Czechoslovak Math. J., to appear.

[73] E. Malkowsky and V. Rakočević, *The measure of noncompactness of linear operators between spaces of strongly* C_1 *summable and bounded sequences,* Acta Math. Hungar. **89**(2000), to appear

[74] J. M. Marti, *Introduction to the Theory of Bases*, Springer-Verlag, 1967.

[75] A. Martinon, *Cantidades operacionales en teoria de Fredholm*, Doctoral thesis, University of La Laguna, 1989.

[76] F. Mòricz, *On* Λ*-strong convergence of numerical sequences and Fourier series*, Acta Math. Hungar. **54** (1989), 319–327.

[77] R. G. Nussbaum *The fixed point index and fixed point theorems for* k *-set contractions*, Ph. D. dissertation, Univ. Chicago, 1969.

[78] R. G. Nussbaum, *The radius of the essential spectrum*, Duke Math. J. **38** (1970), 473–478.

[79] R. G. Nussbaum, *The fixed point index for local condensing maps* Ann. Mat. Pura. Appl. **(4)** Ser. **89** (1971), 217-258.

[80] W. V. Petryshin, *Fixed point theorems for various classes of 1-set contractive and 1-ball-contractive mappings in Banach spaces*, Trans. Amer. Math. Soc. **182** (1973), 323–352.

[81] A. Peyerimhoff, *Über ein Lemma von Herrn Chow,* J. London Math. Soc. **32** (1957), 33–36.

[82] A. Peyerimhoff, *Lectures on Summability*, Lecture Notes in Mathematics 107, Springer-Verlag, 1969

[83] V. Rakočević *Esencijalni spektar i Banachove algebre,* Doktorska disertacija, Univerzitet u Beogradu, Beograd, 1983.

[84] V. Rakočević, *Spectral radius formulae in quotient* C^**-algebras*, Proc. Amer. Math. Soc. **113** (1991), 1039–1040.

[85] V. Rakočević, *On a formula for the jumps in the semi-Fredholm domain,* Rev. Mat. Univ. Complut. Madrid **5** (1992), 225–232.

[86] V. Rakočević, *Funkcionalna analiza*, Naučna knjiga, Beograd, 1994.

[87] V. Rakočević, *Semi-Fredholm operators with finite ascent or descent and perturbations,* Proc. Amer. Math. Soc. **123** (1995), 3823–3825.

[88] V. Rakočević, *Semi-Browder operators and perturbations,* Studia Math. **122** (1997), 131–137.

[89] V. Rakočević and J. Zemánek, *Lower s-numbers and their asymptotic behaviour*, Studia Math. **91** (1988), 231–239.

[90] K. C. Rao, *Martix transformations of some sequence spaces*, Pacific J. Math. **31** (1969), 1717–173.

[91] W. Ruckle, *Sequence Spaces*, Pitman, London, 1981.

[92] W. Rudin, *Functional Analysis*, Mc-Graw Hill, New York, 1973.

[93] Б. Н. Садовский, *Предельно компактные и уплотняющие операторы*, Успехи мат. наук **27** (1972), 81–146.

[94] M. Schechter, *Quantities related to strictly singular operators*, Indiana Univ. Math. J. **21** (1972), 1061–1071.

[95] S. Simons, *The sequence spaces $l(p_\nu)$ and $m(p_\nu)$*, Proc. London Math. Soc. **(3) 15** (1965), 422–436.

[96] B. Stanković and S. Pilipović, *Teorija Distribucija*, Prirodno–Matematički Fakultet, Institut za Matematiku, Novi Sad, 1983.

[97] M. Stieglitz and H. Tietz, *Matrixtransformationen in Folgenräumen. Eine Ergebnisübersicht*, Math. Z. **154** (1977), 1–6.

[98] M.R. Tasković, *On an equivalent of the axiom of choice and its applications*, Math. Japon. **31** (1986), 979–991.

[99] M.R. Tasković, *Osnovi teorija fiksne tačke*, Zavod za udžbenike i nastavna sredstva, Beograd, 1986.

[100] M. R. Tasković, *Nelinearna funkcionalna analiza, Prvi deo, Teorijske osnove*, Zavod za udžbenike i nastavna sredstva, Beograd, 1993.

[101] H-O. Tylli, *On the asymptotic behaviour of some quantities related to semifredholm operators*, J. London Math. Soc. **(2) 31** (1985), 340–348.

[102] H-O. Tylli, *On semi-Fredholm operators , Calkin algebras and some related quantities*, Department of Mathematics, University of Helsinki, Academic dissertation, Helsinki, 1986.

[103] J.R.L. Webb and Weiyu Zhao, *On connections between set and ball measures of noncompactness*, Bull. London Math. Soc. **22** (1990), 471–477.

[104] L.W. Weis, *On the computation of some quantities in the theory of Fredholm operators*, Proc. 12 Winter School in Abstract Analysis, Rend. Circ. Mat. Palermo **(2)** Suppl. 5, (1984), 109–133.

[105] A. Wilansky, *Functional Analysis*, Blaisdell, New York, 1964.

[106] A. Wilansky, *Semi-Fredholm maps of FK spaces*, Math. Z. **144** (1975), 9–12.

[107] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw Hill, 1978.

[108] A. Wilansky, *Summability through Functional Analysis*, Mathematics Studies **85**, North-Holland, Amsterdam, 1984.

[109] N. A. Yerzakova, *Non-linear Superposition Operators on Space $C([0,1],E)$*, J. Math. Anal. Appl. **181** (1994), 385–391.

[110] N. A. Yerzakova, *The measure of noncompactness of Sobolev embeddings*, Integral Equations Operator Theory **19** (1994), 349–359.

[111] K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. **53** (1951), 463–487.

[112] K. Zeller, *Abschnittskonvergenz in FK-Räumen*, Math Z. **55** (1951), 55–70.

[113] K. Zeller, *Matrixtransformationen von Folgenräumen*, Univ. Rend. Mat. **12** (1954), 340–346.

[114] K. Zeller, W. Beekmann, *Theorie der Limitierungsverfahren*, Springer–Verlag, 1968.

[115] J. Zemánek, *The Semi-Fredholm Radius of a linear Operator*, Bull. Polish Acad. Sci. Math. **32** (1984), 67–76.

[116] J. Zemánek, *Geometric characteristics of semi-Fredholm operators and their asymptotic behaviour*, Studia Math. **80** (1984), 219–234.

Index

List of symbols

Milan Merkle

TOPICS IN WEAK CONVERGENCE OF PROBABILITY MEASURES

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

CONTENTS

1. Introduction

Let $\mathcal{X}$ be a topological space and let P_n $(n = 1, 2, \ldots)$ and P be probability measures defined on the Borel sigma field generated by open subsets of $\mathcal{X}$. We say that the sequence $\{P_n\}$ converges *weakly* to P, in notation $P_n \implies P$ if

$$\lim_{n\to+\infty} \int_{\mathcal{X}} f(x)\, dP_n(x) = \int_{\mathcal{X}} f(x)\, dP(x) \tag{1}$$

for every continuous and bounded real valued function $f : \mathcal{X} \mapsto \boldsymbol{R}$. In terms of random variables, let X_n $(n = 1, 2, \ldots)$ and X be $\mathcal{X}$-valued random variables defined on a common probability space and let P_n and P be corresponding distributions, that is, $P_n(B) = \mathrm{Prob}(X_n \in B)$, where B is a Borel set in $\mathcal{X}$. Then we say that the sequence X_n converges weakly to X and write $X_n \implies X$ if and only if $P_n \implies P$.

As we shall see in Section 4, there are many stronger convergence concepts than the introduced one. However, the weak convergence is a very powerful tool in Probability Theory, partly due to its comparative simplicity and partly due to its natural behavior in some typical problems. The weak convergence appears in Probability chiefly in the following classes of problems.

- Knowing that $P_n \implies P$ we may replace P_n by P for n large enough. A typical example is the Central Limit Theorem (any of its versions), which enables us to conclude that the properly normalized sum of random variables has approximately a unit Gaussian law.
- Conversely, if $P_n \implies P$ then we may approximate P with P_n, for n large enough. A typical example of this sort is the approximation of Dirac's delta function (understood as a density of a point mass at zero) by, say triangle-shaped functions.
- In some problems, like stochastic approximation procedures, we would like to have a strong convergence result $X_n \to X$. However, the conditions required to prove the strong convergence are usually very complex and the proofs are difficult and very involved. Then, one usually replaces the strong convergence with some weaker forms; one is often satisfies with $X_n \implies X$.
- It is not always easy to construct a measure with specified properties. If we need to show just its existence, sometimes we are able to construct a sequence (or a net) of measures which can be proved to be weakly convergent and that its limit satisfy the desired properties. For example, this procedure is usually applied to show the existence of the Wiener measure.

The concept of weak convergence is so well established in Probability Theory that hardly any textbook even mention its topological heritage. It, indeed, is not too important in many applications, but a complete grasp of the definition of the weak convergence is not possible without understanding its rationale. The first part of this paper (Sections 2 and 3) is an introduction to weak convergence of probability measures from the topological point of view. Since the set of probability measures is not closed under weak convergence (as we shall see, the limit of a net of probability measures need not be a probability measure), for a full understanding of the complete concept, one has to investigate a wider structure, which turns out to be the set of all finitely additive Radon measures. In this context we present results concerning the Baire field and sigma field, which are usually omitted when discussing probability measures. In Section 4 we consider weak convergence of probability measures and present classical results regarding metrics of weak convergence. In Section 5 we show that the set of probability measures is not closed and effectively show the existence of a finitely, but not countably additive measure in the closure of the set of probability measures. Section 6 deals with the famous Prohorov's theorem on metric spaces. In Section 7 we consider weak convergence of probability measures on Hilbert spaces. Here we observe a separable Hilbert space equipped with weak and strong topology and in both cases we give necessary and sufficient conditions for relative compactness of a set of probability measures.

2. Weak convergence in topology

2.1. Topology induced by a subset of algebraic dual. Let $\mathcal{X}$ be a vector space over a field F, where F stands for $\boldsymbol{R}$ or $\boldsymbol{C}$. Let $\mathcal{X}'$ be the set of all linear maps $\mathcal{X} \mapsto F$ (so called algebraic dual space). Let $Y \subset \mathcal{X}'$ be a subspace such that Y separates points in $\mathcal{X}$, i.e., if $\varphi(x) = \varphi(y)$ for all $\varphi \in Y$ then $x = y$. Define Y-topology on $\mathcal{X}$ by the sub-base

$$\{\varphi^{-1}(V) \mid \varphi \in Y,\ V \text{ open set in } F\}$$

The base of Y-topology is obtained by taking finite intersections of sub-base elements. Equivalently, a base at zero for the Y-topology is consisted of sets

$$O_{\varphi_1,\dots,\varphi_n} = \{x \in \mathcal{X} \mid \varphi_j(x) < 1 \text{ for } j = 1, \dots, n\},$$

where $\{\varphi_1, \dots, \varphi_n\}$ is an arbitrary finite set of elements in Y.

This topology is a Hausdorff one, since we assumed that Y separates point of $\mathcal{X}$. That is, if $x \neq y$ are points in $\mathcal{X}$, then there is a $\varphi \in Y$ so that $\varphi(x) \neq \varphi(y)$ and consequently there are disjoint open sets V_x and V_y in F so that $\varphi(x) \in V_x$ and $\varphi(y) \in V_y$, hence $\varphi^{-1}(V_x) \cap \varphi^{-1}(V_y) = \emptyset$.

The convergence in Y-topology may be described as

$$x_d \to x \iff \varphi(x_d) \to \varphi(x) \quad \text{for all } \varphi \in Y,$$

where $\{d\}$ is a directed set. It is important to know that Y-topology may not be metrizable, even in some simple cases, as we shall see later. So, sequences must not be used as a replacement for nets.

If $Y_1 \subset Y_2 \subset \mathcal{X}'$, then the Y_1 topology is obviously weaker (contains no more open sets) than the Y_2 topology. Therefore, if $x_d \to x$ in Y_2-topology, then it also converges in Y_1 topology, and the converse is not generally true.

2.2. Weak topology. Now we observe only locally convex Hausdorff (LC) topological vector spaces (TVS) $\mathcal{X}$, i.e., those that have a basis for the topology consisted of convex sets. Let $\mathcal{X}^*$ be the topological dual of $\mathcal{X}$, i.e., the space of all continuous linear functionals $\mathcal{X} \mapsto F$. By one version of the Hahn-Banach theorem, $\mathcal{X}^*$ separates points in $\mathcal{X}$, if $\mathcal{X}$ is a LC TVS. Then $\mathcal{X}^*$-topology on $\mathcal{X}$ is called the weak topology. Since for every $\varphi \in \mathcal{X}^*$ we have that

$$x_d \to x \text{ in the original topology of } \mathcal{X} \implies \varphi(x_d) \to x,$$

we see that the weak topology is weaker than the original (strong) topology of $\mathcal{X}$.

The space $\mathcal{X}$ equipped with the weak topology will be denoted by $\mathcal{X}_w$.

2.3. Example. Let $\mathcal{X}$ be a real separable infinitely dimensional Hilbert space, with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Then x_n converges weakly to x if and only if $\langle y, x_n \rangle \to \langle y, x \rangle$ for any $y \in \mathcal{X}$. Let $x_n = e_n$ be an orthonormal base for $\mathcal{X}$. Then since $\|y\|^2 = \sum \langle y, e_n \rangle^2 < +\infty$, we see that $\langle y, e_n \rangle \to 0$ for any $y \in \mathcal{X}$ and so the sequence e_n converges weakly to 0. However, since $\|e_n - e_m\|^2 = 2$, this sequence does not converge in the norm topology of $\mathcal{X}$. □

On finitely dimensional TVS, the weak and the strong topology coincide. However, on infinitely dimensional spaces, the weak topology exhibits some peculiar properties, as we shall see in the next subsection.

2.4. How weak is the weak topology? Let us firstly grasp some clues to understand the weak topology. We start with kernels of linear functionals and we prove the following theorem.

Theorem. *If* $\dim \mathcal{X} > 1$, *then there is no linear functional* $\varphi \in \mathcal{X}'$ *with* $\ker \varphi = \{0\}$.

Proof. Suppose that $\ker \varphi = \{0\}$ and let x_1, x_2 be arbitrary elements in $\mathcal{X}$, $x_1, x_2 \neq 0$. Then let $\lambda = \varphi(x_1)/\varphi(x_2)$, which is well defined, since $\varphi(x_2) \neq 0$ by assumptions. Let $y = x_1 - \lambda x_2$. Then $\varphi(y) = \varphi(x_1) - \lambda\varphi(x_2) = 0$, hence $y = 0$, i.e., $x_1 = \lambda x_2$. Since x_1, x_2 are arbitrary, the dimension of $\mathcal{X}$ is **1**. □

Let $\operatorname{sp} A$ denote the set of all finite linear combinations of elements of the set A.

2.5. Theorem. *Let* $\mathcal{X}$ *be a vector space over* F *and let* $\varphi_1, \dots, \varphi_n \in \mathcal{X}'$. *Then (i) and (ii) below are equivalent:*

(i) $\varphi \in \operatorname{sp}\{\varphi_1, \dots, \varphi_n\}$ *(ii)* $\bigcap_{i=1}^n \ker \varphi_i \subset \ker \varphi$

Proof. Suppose that (i) holds, that is, $\varphi(x) = \sum_{i=1}^n \alpha_i \varphi_i(x)$. Then clearly, $\varphi_i(x) = 0$ for all i implies that $\varphi(x) = 0$, which proves (ii). Conversely, assume that (ii) holds. Define a mapping $T: \mathcal{X} \mapsto F^n$ by $T(x) = (\varphi_1(x), \dots, \varphi_n(x))$ and define $S(T(x)) = \varphi(x)$. Then S is well defined on the range of T, since if $T(x) = T(y)$ then $x - y$ is in $\ker \varphi_i$ for all i, hence $x - y$ is in $\ker \varphi$ and so

$S(x) = S(y)$. Clearly, S is a linear map and its extension to F^n must be of the form $F(t_1, \dots, t_n) = \alpha_1 t_1 + \dots + \alpha_n t_n$, which means that

$$\varphi(x) = S(\varphi_1(x), \dots, \varphi_n(x)) = \alpha_1\varphi_1(x) + \dots + \alpha_n\varphi_n(x),$$

which was to be proved. □

From Theorem 2.5 we get an immediate generalization of Theorem 2.4:

2.6. Corollary. *If $\mathcal{X}$ is an infinitely dimensional TVS and if $\varphi_1, \dots, \varphi_n$ are arbitrary linear functionals, then $\bigcap_{i=1}^n \ker\varphi_i \neq \{0\}$.*

Proof. Suppose that $\bigcap_{i=1}^n \ker\varphi_i = \{0\}$. Then for any $\varphi \in \mathcal{X}'$ we have that $\{0\} \subset \ker\varphi$ and then, by Theorem 2.5, $\varphi \in \operatorname{sp}\{\varphi_1, \dots, \varphi_n\}$, which means that $\mathcal{X}'$ is finite dimensional, and so is $\mathcal{X}$, which contradicts the assumption. □

The next theorem describes a fundamental weakness of the weak topology.

2.7. Theorem. *If $\mathcal{X}$ is an infinitely dimensional TVS then each weakly open set contains a non-trivial subspace.*

Proof. Let $U \subset \mathcal{X}$ be a weakly open set. Without loss of generality, assume that $0 \in U$ (otherwise, do a translation). Then U must contain a set of the form

$$O_{\varphi_1,\dots,\varphi_n} = \{x \in \mathcal{X} \mid \varphi_1(x) < 1, \dots, \varphi_n(x) < 1\},$$

for some $\varphi_1, \dots, \varphi_n \in \mathcal{X}^*$. Then clearly $\bigcup_{i=1}^n \ker\varphi_i \subset O_{\varphi_1,\dots,\varphi_n} \subset U$ and according to Corollary 2.6 $\bigcup_{i=1}^n \ker\varphi_i$ is a non-trivial subspace.

2.8. Corollary. *Let $\mathcal{X}$ be an infinitely dimensional normed space. Then an open ball of $\mathcal{X}$ is not weakly open.*

Proof. Let B be an open ball in $\mathcal{X}$. If it were open in the weak topology, then (by Theorem 2.7) it would have contained a nontrivial subspace, which is not possible (for instance, it is not possible that $\|\alpha x\| < r$ for all scalars α). □

So, the next theorem may come as a surprise.

2.9. Theorem. *Let $\mathcal{X}$ be a LC TVS. Then $\mathcal{X}$ and $\mathcal{X}_w$ have the same closed convex sets. For each convex $S \subset \mathcal{X}$ we have that $\bar{S}^w = \bar{S}$, where $\bar{S}^w$ is the closure of S in the weak topology.* □

2.10. Example. Let $\mathcal{X}$ be a separable metric space. Denote by $\mathcal{B}_s$ the Borel sigma field generated by norm-open sets and let $\mathcal{B}_w$ be the Borel sigma field generated by weakly open sets. Since any weakly open set is also norm-open, we generally have that $\mathcal{B}_w \subset \mathcal{B}_s$, but not conversely. In this special case, each strongly open set is a countable union of closed balls, which are, by Theorem 2.9 also weakly closed. So, $\mathcal{B}_s \subset \mathcal{B}_w$, which gives that, in a separable metric space, $\mathcal{B}_s = \mathcal{B}_w$. □

From Theorem 2.9 it follows that a closed ball in a normed space $\mathcal{X}$ is also weakly closed. But from Theorem 2.7 we see that the weak interior of any ball in an infinitely dimensional normed space is an empty set!

2.11. Weak star topology on a dual space. We are now going to introduce a yet weaker than the weak topology. Let $\mathcal{X}$ be a LC TVS and let $\mathcal{X}^*$ be its topological dual. Define a mapping $\Phi : \mathcal{X} \mapsto (\mathcal{X}^*)'$ by $\Phi_x(\varphi) = \varphi(x)$, where $x \in \mathcal{X}$ and $\varphi \in \mathcal{X}^*$. This map is linear and one-to one (the one-to one property follows from the fact that $\mathcal{X}^*$ separates points). Now we can observe the $\Phi(\mathcal{X})$-topology on $\mathcal{X}^*$. It is customary to identify $\Phi(\mathcal{X})$ with $\mathcal{X}$ itself (especially in the case when $\mathcal{X}$ is a normed space, since then the natural topologies on $\mathcal{X}$ and $\Phi(\mathcal{X})$ coincide). So, the $\mathcal{X}$-topology on $\mathcal{X}^*$ is called the weak-$*$ (weak star) topology. In fact, this is the topology of pointwise convergence of functionals, since

$$\varphi_d \to \varphi \quad (w-*) \iff \varphi_d(x) \to \varphi(x) \quad \text{for every } x \in \mathcal{X}.$$

2.12. Three topologies on duals of normed spaces. Let $\mathcal{X}$ be a normed space. Then its topological dual $\mathcal{X}^*$ is also normed, with $\|\varphi\| = \sup_{\|x\|\le 1} |\varphi(x)|$. This norm defines the strong topology of $\mathcal{X}^*$. Further, the weak topology on $\mathcal{X}^*$ is defined as $\mathcal{X}^{**}$-topology and the weak star is $\mathcal{X}$-topology on $\mathcal{X}^*$. Since $\mathcal{X} \subset \mathcal{X}^{**}$, the weak star topology is weaker than the weak one, which is in turn weaker than the strong topology. Due to the order between topologies, it is not possible that a sequence (or a net) converges to one limit in one of mentioned topologies and to another limit in other topology. So, for instance, if a sequence converges to some x in the, say, weak star topology, then in the strong topology it either converges to x or does not converge at all.

2.13. Example. Let c_0 be the set of all real sequences converging to zero, with the norm $\|x\| = \sup_n |x_n|$. Then it is well known that $c_0^* = l_1$ and $c_0^{**} = l_1^* = l_\infty$, where l_1 is the space of sequences with the norm $\|x\|_{l_1} = \sum |x_n| < +\infty$ and l_∞ is the space of bounded sequences with $\|x\|_\infty = \sup_n |x_n|$. Linear maps are realized via so called duality pairing $\langle x, y\rangle$, acting like inner products with one component from $\mathcal{X}$ and the other one from $\mathcal{X}^*$. Observe a sequence in l_1, $x_n = \{x_{k,n}\}$ and let $y = \{y_k\}$ be an element in l_1. Then x_n converges to y:

- Strongly, if $\|x_n - y\| = \sup_k |x_{k,n} - y_k| \to 0$ as $n \to +\infty$.
- Weakly, if $\langle \xi, x_n\rangle = \sum_k \xi_k x_{k,n} \to \sum_k \xi_k y_k$, for any $\xi = \{\xi_k\} \in l_\infty$.
- Weak-star, if $\sum_k \xi_k x_{k,n} \to \sum_k \xi_k y_k$, for any $\xi = \{\xi_k\} \in c_0$.

Now observe the sequence $e_n = (0, 0, \dots, 0, 1, 0, \dots) \in l_1$ (with 1 as the n-th component). Then $\langle \xi, e_n\rangle = \xi_n$ and if $\xi \in c_0$ then $\langle \xi, e_n\rangle \to 0$, hence e_n converges to 0 weak star. However, if $\xi \in l_\infty$, then $\langle \xi, e_n\rangle$ need not converge, so e_n does not converge in the weak topology. Further, in the norm topology e_n does not converge to zero, because $\|e_n\| = 1$ for all n; therefore, $\{e_n\}$ is not convergent in the strong topology of l_1.

2.14. Canonical injections. Let $\mathcal{X}$ be a normed space, let $\mathcal{X}^*$ be its topological dual space and let $\mathcal{X}^{**} = (\mathcal{X}^*)^*$ be its second dual. If $\|x\|$ is a norm on $\mathcal{X}$, then the norm on $\mathcal{X}^*$ is defined by $\|\varphi\| = \sup_{\|x\|\le 1} \|\varphi(x)\|$. The norm on $\mathcal{X}^{**}$ is then defined by $\|\Phi\| = \sup_{\|\varphi\|\le 1} |\Phi(\varphi)|$. Observe the canonical mapping $\mathcal{X} \mapsto \mathcal{X}^{**}$ which is defined, as in 2.11 by

$$\Phi_x(\varphi) = \varphi(x).$$

Then for each $x \in \mathcal{X}$, Φ_x is a continuous linear functional defined on $\mathcal{X}^*$, and so it is a member of $\mathcal{X}^{**}$ with the norm

(2)

$$\|\Phi_x\| = \sup_{\|\varphi\|\le 1} |\Phi_x(\varphi)| = \sup\Big\{\Big|\varphi\Big(\frac{x}{\|x\|}\Big)\Big| \cdot \|x\| \;\Big|\; \varphi \in \mathcal{X}^*, \sup_{\|x\|\le 1} |\varphi(x)| \le 1\Big\} \le \|x\|.$$

On the other hand, by one version of the Hahn-Banach theorem, if $\mathcal{X}$ is a normed space, for each $x \in \mathcal{X}$ there exists $\varphi_0 \in \mathcal{X}^*$ with $\|\varphi_0\| = 1$ and $\varphi_0(x) = \|x\|$. Therefore,

$$\|\Phi_x\| = \sup_{\|\varphi\|\le 1} |\Phi_x(\varphi)| \ge \Phi_x(\varphi_0) = \varphi_0(x) = \|x\|. \tag{3}$$

From (2) and (3) it follows that $\|\Phi_x\| = \|x\|$. So, the canonical mapping $x \mapsto \Phi_x$ is bicontinuous, i.e.,

$$x_n \to x \iff \Phi_{x_n} \to \Phi_x$$

Further, as we already observed in 2.11, this mapping is linear and one-to one injection from $\mathcal{X}$ to $\mathcal{X}^{**}$. Since $\Phi(\mathcal{X})$ is isomorphic and isometric to $\mathcal{X}$, it can be identified with $\mathcal{X}$ in the algebraic and topological sense. This fact is usually denoted as $\mathcal{X} \subset \mathcal{X}^{**}$. If $\Phi(\mathcal{X}) = \mathcal{X}^{**}$, we say that $\mathcal{X}$ is a reflexive space, usually denoted as $\mathcal{X} = \mathcal{X}^{**}$.

2.15. Example. Each Hilbert space is reflexive. Due to the Riesz representation theorem, any linear functional in a Hilbert space H is of the form $\varphi_y(x) = \langle y, x\rangle$, where $y \in H$ and also $\|\varphi_y\| = \|y\|$. Hence, we may identify φ_y with y and write $H^* = H$. This equality means, in fact, that there exists a canonical injection (in fact, bijection) $H \mapsto H^*$ realized by the mapping $y \mapsto \varphi_y$.

From $H^* = H$ it follows that $H^{**} = (H^*)^* = H$, i.e., H is reflexive.

The space c_0 introduced in the example 2.13 is not reflexive, since $c_0^{**} = l_\infty$. However, $c_0 \subset l_\infty$. □

2.16. Inclusions. Now suppose that $\mathcal{X}_1 \subset \mathcal{X}_2$ are vector spaces with the same norm $\|\cdot\|$. Let φ be a continuous linear functional defined on $\mathcal{X}_2$. Then clearly, the restriction of φ to $\mathcal{X}_1$ is a continuous linear functional on $\mathcal{X}_1$ and therefore we have that $\mathcal{X}_2^* \subset \mathcal{X}_1^*$. For the second duals we similarly find that $\mathcal{X}_1^{**} \subset \mathcal{X}_2^{**}$. Hence,

$$\mathcal{X}_1 \subset \mathcal{X}_2 \implies \mathcal{X}_1^* \supset \mathcal{X}_2^* \implies \mathcal{X}_1^{**} \subset \mathcal{X}_2^{**}.$$

A paradoxical situation may arise if we have two Hilbert spaces $H_1 \subset H_2$. Then by canonical injection we have $H_1^* = H_1$ and $H_2^* = H_2$, which would lead to $H_2 \subset H_1$! This example shows that we have to be cautious while using equality as a symbol for canonical injection.

2.17. Weak star compact sets. For investigation of convergence, it is important to understand the structure of compact sets. Let $\mathcal{X}$ be a normed space. It is well known that a closed ball of $\mathcal{X}$ is compact in the strong topology if and only if $\mathcal{X}^*$ is finitely dimensional. Since $\mathcal{X}^*$ is also a normed space, the same holds for $\mathcal{X}^*$. However, in the weak star topology, we have the following result.

2.17. Theorem (Banach–Alaoglu). *Let $\mathcal{X}$ be an arbitrary normed space. A closed ball of $\mathcal{X}^*$ is weak star compact.*

Proof. Without a loss of generality, observe a closed unit ball of $\mathcal{X}^*$, call it B. Hence, B contains all linear continuous mappings φ from $\mathcal{X}$ to F such that $|\varphi(x)| \leq \|x\|$ for all $x \in \mathcal{X}$. For any $x \in \mathcal{X}$, define $D_x = \{t \in F \mid |t| \leq \|x\|\}$ and $K = \prod_{x\in\mathcal{X}} D_x$, with a product topology on K. If f is an element of K and $f(x)$ its co-ordinate in K, then f is a function $f : \mathcal{X} \mapsto F$. The product topology is the topology of pointwise convergence: $f_d \to f$ if and only if $f_d(x) \to f(x)$ for any $x \in \mathcal{X}$. So, B with the weak topology on it is a subset of K. Since each D_x is compact, Tychonov's theorem states that K is also compact, so we just need to show that B is closed in K. To this end, let φ_d be a net in B which converges to some $f \in K$. Then it is trivial to show that f must be linear; then by $|\varphi_d(x)| \leq \|x\|$ it follows that f is also continuous and that $\|f\| \leq 1$. Therefore, $f \in B$ and B is closed, hence compact. □

2.19. Remark. Tychonov's theorem states that the product space $\prod_i \mathcal{X}_i$ in the product topology as explained above, is compact if and only if each of $\mathcal{X}_i$ is compact. The proof of Banach-Alaoglu theorem relies on Tychonov's theorem, and the proof of the latter, in the part which is used here, relies on the Axiom of Choice (more precisely, Zorn's lemma, cf. [**9, 15, 35, 36**]). □

This theorem implies that any bounded sequence in $\mathcal{X}^*$ must have a convergent subnet. Unfortunately, such a subnet need not be a sequence, since the weak star topology on $\mathcal{X}^*$ need not be metrizable. However, the next theorem claims that in one special case we can introduce a metric.

2.20. Theorem. *Let $\mathcal{X}$ be a separable normed vector space. Then the $w-*$ topology on a closed ball of $\mathcal{X}^*$ is metrizable.*

Proof. Assume, without a loss of generality that B is the closed unit ball (centered at the origin) of the dual $\mathcal{X}^*$ of a separable normed vector space $\mathcal{X}$. The metrization of B can be realized, for instance, as follows. Let $\varphi_1, \varphi_2 \in B$, so

$$\sup_{\|x\|\leq 1} \varphi_i(x) \leq 1, \qquad i = 1, 2.$$

Let $\{x_n\}$ be a dense countable set in the unit ball of $\mathcal{X}$. Define

$$d(\varphi_1, \varphi_2) = \sum_n \frac{|\varphi_1(x_n) - \varphi_2(x_n)|}{2^n}.$$

Then $|\varphi_1(x_n) - \varphi_2(x_n)| \leq \|\varphi_1 - \varphi_2\| \cdot \|x_n\| \leq 2$ and the series converges, so d is a well defined function (even on the whole space $\mathcal{X}^*$). It is now a matter of an exercise to show that the d-topology on B coincides with the $w-*$ topology.

2.21. Corollary. *Let $\mathcal{X}$ be a separable normed vector space and let $\mathcal{X}^*$ be its topological dual space. Then every bounded sequence $\{\varphi_n\} \in \mathcal{X}^*$ has a weak star convergent subsequence.*

Proof. Every bounded sequence is contained in some closed ball B, which is, by Theorem 2.18, weak star compact. By Theorem 2.20, the weak star topology on B is metrizable, i.e., there is a metric d such that

$$\varphi_d \to \varphi \quad (w-*) \iff d(\varphi_d, \varphi) \to 0.$$

In a metric space, compactness is equivalent to sequential compactness, so, any sequence in B has a convergent subsequence. □

2.22. Example. Let H be a separable Hilbert space. Since it is reflexive, weak and weak star topology coincide. Define a metric

$$d(x, y) = \sum_n \frac{|\langle x - y, e_n\rangle|}{2^n}$$

where $\{e_n\}$ is an orthonormal base in H. We shall prove that this metric also generates the weak topology on the unit ball of H. Suppose that $x_n \to x$ weakly, i.e., $\langle x_n, y\rangle \to \langle x, y\rangle$ for any $y \in H$, where $\|x_n\| \le 1, \|x\| \le 1$. Since

$$|\langle x_n - x, e_k\rangle| \le \|x_n - x\| \cdot \|e_k\| \le 2,$$

the series

$$d(x_n, x) = \sum_k \frac{|\langle x_n, e_k\rangle - \langle x, e_k\rangle|}{2^k}$$

converges uniformly in n and so, by evaluating limits under the sum, we conclude that

$$\lim_{n\to+\infty} d(x_n, x) = 0.$$

Conversely, let $d(x_n, x) \to 0$ as $n \to +\infty$, where $\|x_n\| \le 1$ and $\|x\| \le 1$. Then it follows that $\langle x_n - x, e_k\rangle \to 0$ for every k. Now for any $y \in H$,

$$\langle x_n, y\rangle - \langle x, y\rangle = \sum_k \langle x_n - x, e_k\rangle\langle y, e_k\rangle.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\Big|\sum_{k=m}^{+\infty} \langle x_n - x, e_k\rangle\langle y, e_k\rangle\Big| &\le \sum_{k=m}^{+\infty} |\langle x_n - x, e_k\rangle| \cdot |\langle y, e_k\rangle| \\
&\le \Big(\sum_{k=m}^{+\infty} \langle x_n - x, e_k\rangle^2 \sum_{k=m}^{+\infty} \langle y, e_k\rangle^2\Big)^{1/2} \\
&\le \|x_n - x\| \cdot \Big(\sum_{k=m}^{+\infty} \langle y, e_k\rangle^2\Big)^{1/2} \\
&\le 2\Big(\sum_{k=m}^{+\infty} \langle y, e_k\rangle^2\Big)^{1/2}
\end{aligned}$$

and therefore, the series $\sum_k \langle x_n - x, e_k\rangle\langle y, e_k\rangle$ converges uniformly with respect to n. Hence,

$$\lim_{n\to+\infty}\langle x_n, y\rangle - \langle x, y\rangle = \sum_k \lim_{n\to+\infty}\langle x_n - x, e_k\rangle\langle y, e_k\rangle = 0.$$

So, $\{x_n\}$ converges weakly to x.

However, the metric described here does not generate the weak topology on the whole H. To see this, let $x_n = ne_n$. Then $d(x_n, 0) \to 0$ as $n \to +\infty$, but $\langle x_n, y\rangle = n\langle e_n, y\rangle$, which need not converge.

3. Finitely additive measures and Radon integrals

3.1. Spaces of measures as dual spaces. In general, it might be very hard to find the dual space of a given space, i.e., to represent it (via canonical injections) in terms of some well known structure. We are particularly interested in spaces of measures; it turns out that they can be viewed as dual spaces of some spaces of functions. The functionals on spaces of functions are expressed as integrals:

$$\varphi(f) = \int f(t)\, d\mu(t)$$

where μ is a measure which determines a functional. Then by a canonical injection, we can identify functionals and corresponding measures. There are several results in various levels of difficulty, depending on assumptions that one imposes on the underlying space X on which we observe measures. In this section we will present the most general result [**1**] regarding an arbitrary topological space. It turns out that finitely additive measures are the key notion in this general setting.

Although a traditional probabilist works solely with countably additive measures on sigma fields, their presence in Probability has a purpose to make mathematics simpler and is by no means natural. As Kolmogorov [**19**, p. 15] points out, "*dots* in describing any observable random process we can obtain only finite fields of probability. Infinite fields of probability occur only as idealized models of real random processes". Finitely additive measures have recently arose an increasing interest in Probability, so the exposition which follows may be interesting in its own rights.

3.2. Fields and sigma fields. Let X be a set and $\mathcal{F}$ a class of its subsets such that

1) $X \in \mathcal{F}$,
2) $B \in \mathcal{F} \implies B' \in \mathcal{F}$,
3) $B_1, B_2 \in \mathcal{F} \implies B_1 \cup B_2 \in \mathcal{F}$ Then we say that $\mathcal{F}$ is a field. If 3) is replaced by stronger requirement

3') $B_1, B_2, \ldots \in \mathcal{F} \implies \bigcup_{i=1}^{\infty} B_i \in \mathcal{F}$, then we say that $\mathcal{F}$ is a sigma field.

It is easy to see that a field is closed under finitely many set operations of any kind. Further, let $\mathcal{F}_i$, $i \in I$, be fields on X. Then $\mathcal{F} = \bigcap_{i\in I}\mathcal{F}_i$ is also a field, where

I is any collection of indices. This follows trivially by verification of conditions 1)–3) above.

Given any collection of sets $\mathcal{A}$ which are subsets of X, there is a field which contains $\mathcal{A}$: it is the family of all subsets of X. The intersection of all fields that contain $\mathcal{A}$ is called the field generated by $\mathcal{A}$. Obviously, a field $\mathcal{F}$ generated by $\mathcal{A}$ is the smallest field that contains $\mathcal{A}$, in the sense that there is no field which is properly contained in $\mathcal{F}$ and contains $\mathcal{A}$.

A sigma field is closed under countably many set operations. We define a sigma field generated by a collection of sets in much the same way as in the case of fields.

3.3. Borel field. Let X be a topological space. The field generated by the collection of all open sets is called the Borel field. Since the complement of an open set is a closed set, the Borel field is also generated by the collection of all closed sets.

Borel sigma field is the sigma field generated by open or closed sets. In separable metric spaces, the Borel sigma field is also generated by open or closed balls, since any open set can be expressed as a countable union of such balls.

Specifically, on the real line, Borel sigma field is generated by open and closed intervals of any kind. However, Borel field *is not generated by intervals*, since an arbitrary open set need not be represented as a finite union of intervals.

3.4. Baire field. Let X be a topological space and let $\boldsymbol{C}(X)$ be the collection of all bounded and continuous real valued functions defined on X. The Baire field is the field generated by the collection of sets

$$\mathcal{A} = \{Z \subset X \mid Z = f^{-1}(C)\},$$

where f is any function in $\boldsymbol{C}(X)$ and C is any closed set of real numbers.

Boundedness of functions in $\boldsymbol{C}(X)$ is not relevant, but is assumed here for the purposes of this paper. Indeed, for any continuous function $f : X \mapsto \boldsymbol{R}$, the function $g(x) = \operatorname{arctg} f(x)$ is a continuous bounded function defined on X and the collection of all $g^{-1}(C)$ coincides with the collection of all $f^{-1}(C)$, where C runs over closed subsets of $\boldsymbol{R}$.

It is well known that for any closed set $C \subset \boldsymbol{R}$ there is a continuous bounded function g_C such that $g_C^{-1}(\{0\}) = C$ (this is a consequence of a more general result that holds, for instance, on metric spaces, see [**6**, Theorem 1.2]. For an $f \in \boldsymbol{C}(X)$ and a closed set $C \subset \boldsymbol{R}$, define $F(x) = g_C(f(x))$. Then $\boldsymbol{F} \in \boldsymbol{C}(X)$ and $F^{-1}(\{0\}) = f^{-1}(C)$. Therefore, we may think of the Baire field as being generated by sets of the form $f^{-1}(\{0\})$, for $f \in \boldsymbol{C}(X)$.

Let us recall that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$; also $f^{-1}(A') = (f^{-1}(A))'$ if the complement is taken with respect to the domain of f. Hence, we have:

$$f^{-1}(C_1) \cup f^{-1}(C_2) = f^{-1}(C_1 \cup C_2); \quad (f^{-1}(C))' = f^{-1}(C')$$

and also $X = f^{-1}(\boldsymbol{R})$, for any f. Therefore, the Baire field is also generated by the collection of the sets $f^{-1}(O)$, where O is an open set of real numbers and

$f \in C(X)$. Further,

$$(f^{-1}(C))' = f^{-1}(O), \quad \text{where } O = C' \text{ is an open set.}$$

Now it is clear that if A belongs to the Borel field in $\boldsymbol{R}$ and if $f \in C(X)$, then $f^{-1}(A)$ belongs to the Baire field in X.

From now on, sets of the form $f^{-1}(C)$, where C is closed in $\boldsymbol{R}$, will be called Z-sets, and the sets of the form $f^{-1}(O)$, with O being an open set in $\boldsymbol{R}$, will be called U-sets.

Since the inverse image (with a continuous function) of any open (resp. closed) set is again an open (resp. closed) set, we see that Z-sets are closed and U-sets are open in X. Hence, the Baire field is a subset of Borel field; the same relation holds for the sigma fields. The converse is not generally true, since a closed set need not be a Z-set. With some restrictions on topology of X, the converse becomes true, for instance, in metric spaces. In general, in normal spaces in which every closed set can be represented as a countable intersection of open sets (so called G_δ set), every closed set is a Z-set (cf. [**15**, Corollary 1.5.11]) and so the Baire and the Borel field coincide.

3.5. Theorem. *The family of Z-sets is closed under finite unions and countable intersections. The family of U-sets is closed under countable unions and finite intersections.*

Proof. By 3.4, a set is a Z-set if and only if it is of the form $f^{-1}(\{0\})$ for some $f \in C(X)$. So, let $Z_1 = f_1^{-1}(\{0\})$, $Z_2 = f_2^{-1}(\{0\})$. If $g(x) = f_1(x)f_2(x)$, then $g^{-1}(\{0\}) = Z_1 \cup Z_2$, so the union of two Z-sets is again a Z-set. Let $Z_1, Z_2, \dots$ be Z-sets. Then there are continuous and bounded functions $f_1, f_2, \dots$ such that $Z_n = f_n^{-1}(\{0\}), n = 1, 2, \dots$. Define the function

$$F(x) = \sum_{n=1}^{+\infty} \frac{f_n^2(x)}{2^n \|f_n\|^2},$$

where $\|f_n\| = \sup_{x \in X} |f(x)|$. Since the above series is uniformly convergent on X, F is a continuous and bounded function; moreover, $F(x) = 0$ if and only if $f_n(x) = 0$ for all $n \geq 1$. Hence $F^{-1}(\{0\}) = \bigcap_{n=1}^{\infty} Z_n$, which proves that any countable intersection of Z-sets is a Z-set.

Statements about U-sets can be proved by taking complements.

3.6. Measures and regularity. Let X be a topological space. Let μ be a non-negative and finitely additive set function on some field or a sigma field $\mathcal{F}$ of subsets of X, with values in $[0, +\infty]$ (allowing $+\infty$ if not specified otherwise). Such a function will be called a measure.

We say that a set $A \in \mathcal{F}$ is μ-regular if

$$\mu(A) = \sup\{\mu(Z) \mid Z \subset A\} = \inf\{\mu(U) \mid A \subset U\}, \tag{4}$$

where Z and U are generic notations for Z-sets and U-sets respectively.

If all sets in $\mathcal{F}$ are μ-regular, we say that the measure μ is regular.

Note that a prerequisite for regularity is that all Z-sets and U-sets must be measurable, which is the case if $\mathcal{F}$ contains the Baire field. In the next theorem we give alternative conditions for regularity.

3.7. Theorem. *Let X be a topological space, $\mathcal{F}$ a field which contains the Baire field and μ a measure on $\mathcal{F}$. A set $A \in \mathcal{F}$ with $\mu(A) < +\infty$ is μ-regular if and only if either of the following holds:*

(i) For each $\varepsilon > 0$ there exists a Z-set Z_ε and an U-set U_ε so that

$$Z_\varepsilon \subset A \subset U_\varepsilon \quad \text{and} \quad \mu(U_\varepsilon \setminus Z_\varepsilon) < \varepsilon. \tag{5}$$

(ii) There are Z-sets $Z_1, Z_2, \ldots$ and U-sets $U_1, U_2, \ldots$ such that

$$Z_1 \subset Z_2 \subset \cdots \subset A \subset \cdots U_2 \subset U_1$$

and

$$\mu(A) = \lim_{n\to+\infty} \mu(Z_n) = \lim_{n\to+\infty} \mu(U_n).$$

Proof. (i) is straightforward, using properties of the infimum and the supremum. (ii) Suppose that A is μ-regular. Then for each n there is a Z-set Z_n^* such that $Z_n^* \subset A$ and $\mu(A) - 1/n < \mu(Z_n^*) < \mu(A)$. Let $Z_n = Z_1^* \cup \cdots \cup Z_n^*$, for $n = 1, 2, \ldots$. Then Z_n are Z-sets by Theorem 3.5. Further, $Z_1 \subset Z_2 \subset \cdots \subset A$ and $\mu(A) - 1/n < \mu(Z_n) < \mu(A)$, hence $\lim \mu(Z_n) = \mu(A)$. The part regarding U_n can be proved similarly.

Conversely, if there exist Z_n and U_n as in the statement of the theorem, then for a fixed $\varepsilon > 0$ there is an n such that $Z_n \subset A$ and $0 < \mu(A) - \mu(Z_n) < \varepsilon$, hence $\mu(A)$ is the least upper bound for $\mu(Z)$ over all Z-subsets of A. Similarly, it follows that $\mu(A)$ is the greatest lower bound for $\mu(U)$, over all U-sets that contain A.

3.8. Remark. The previous theorem does not imply either the countable additivity or continuity of μ. Also it holds regardless whether μ is defined on a sigma field or just on a field.

3.9. Theorem. *Suppose that μ is a countably additive measure defined on a sigma field $\mathcal{F}$ which contains the Baire field. Then a set $A \in \mathcal{F}$, $\mu(A) < +\infty$, is μ-regular if and only if there are Z-sets $Z_1, Z_2, \ldots$ and U-sets $U_1, U_2, \ldots$ such that*

$$Z_1 \subset Z_2 \subset \cdots \subset A \subset \cdots U_2 \subset U_1$$

and

$$\mu\Big(A \setminus \bigcup_{n=1}^{+\infty} Z_n\Big) = 0, \quad \mu\Big(\bigcap_{n=1}^{+\infty} U_n \setminus A\Big) = 0.$$

Proof. By the previous theorem, A is μ-regular if and only if $\mu(A) = \lim \mu(Z_n) = \lim \mu(U_n)$; by continuity property of sigma additive measures we have that $\lim \mu(Z_n) = \mu(\bigcup_n Z_n)$ and $\lim \mu(U_n) = \mu(\bigcap_n U_n)$, which ends the proof.

3.10. Theorem *Let X be a topological space and $\mathcal{F}$ a field which contains the Baire field. Let μ be a measure on $\mathcal{F}$, with $\mu(X) < +\infty$. Then the family $\mathcal{R}$ of all μ-regular sets in $\mathcal{F}$ is a field.*

Proof. Since $X = f^{-1}(\boldsymbol{R})$ and $\boldsymbol{R}$ is open and closed, it follows that both conditions in (4) hold and so $X \in \mathcal{R}$.

Suppose that $A \in \mathcal{R}$. Then for a fixed $\varepsilon > 0$ there are sets Z_ε and U_ε such that (5) holds. Taking complements we get

$$U_\varepsilon' \subset A' \subset Z_\varepsilon', \quad Z_\varepsilon' \setminus U_\varepsilon' = U_\varepsilon \setminus Z_\varepsilon,$$

which implies that A' is also μ-regular.

Finally, suppose that $A_1, A_2, \ldots A_n \in \mathcal{R}$. By Theorem 3.7(i), for any given $\varepsilon > 0$, there are Z-sets Z_i and U-sets U_i such that

$$Z_i \subset A_i \subset U_i \quad \text{and} \quad \mu(U_i \setminus Z_i) < \frac{\varepsilon}{2^i}, \quad i = 1, 2, \ldots$$

Let $A = \bigcup_{i=1}^n A_i$, $Z = \bigcup_{i=1}^n Z_i$ and $U = \bigcup_{i=1}^n U_i$. Then Z is a Z-set and U is a U-set and we have

$$Z \subset A \subset U \quad \text{and} \quad \mu(U \setminus Z) \le \sum_i \mu(U_i \setminus Z_i) \le \varepsilon, \tag{6}$$

so $A \in \mathcal{R}$.

3.11. Theorem. *Let X be a topological space, $\mathcal{F}$ a sigma field that contains the Baire field. Let μ be a countably additive measure on $\mathcal{F}$, with $\mu(X) < +\infty$. Then the family $\mathcal{R}$ of all μ-regular sets in $\mathcal{F}$ is a sigma field.*

Proof. In the light of Theorem 3.10, we need to prove only that a countable union of μ-regular sets is μ-regular.

Let $A_1, A_2, \ldots$ be μ-regular sets; for any $\varepsilon > 0$, there are Z-sets Z_i and U-sets U_i such that

$$Z_i \subset A_i \subset U_i \quad \text{and} \quad \mu(U_i \setminus Z_i) < \frac{\varepsilon}{2^i}.$$

Let $A = \bigcup_{i=1}^\infty A_i$, $Z = \bigcup_{i=1}^\infty Z_i$ and $U = \bigcup_{i=1}^\infty U_i$. Then U is a U-set (Theorem 3.5) and Z can be approximated by a finite union $Z^{(n)} = \bigcup_{i=1}^n Z_i$, where n is chosen in such a way that $\mu(Z \setminus Z^{(n)}) < \varepsilon$ (continuity of the countably additive measure). So, we have that

$$\mu(U \setminus Z^{(n)}) \le \mu(U \setminus Z) + \mu(Z \setminus Z^{(n)}) < 2\varepsilon,$$

which ends the proof.

3.12. Theorem. *Let X be a topological space, $\mathcal{F}$ the Baire sigma field and μ a countably additive measure on $\mathcal{F}$, with $\mu(X) < +\infty$. Then μ is regular.*

Proof. By Theorem 3.11, all μ-regular sets make a sigma field $\mathcal{R}$. We need to show that $\mathcal{R} = \mathcal{F}$, which will be accomplished if we show that each Z-set is

μ-regular. So, let Z be a Z-set. Then there is a function $f \in C(X)$ such that $Z = f^{-1}(0)$. Let $O_n = (-1/n, 1/n)$ and $U_n = f^{-1}(O_n)$. Then $U_1 \supset U_2 \supset \cdots \supset Z$ and $\bigcap_n U_n = Z$. By continuity of countably additive measure μ we have that $\mu(Z) = \lim_n \mu(U_n)$, so the condition of Theorem 3.7(ii) holds (with $Z_n = Z$ for all n), hence Z is μ-regular.

3.13. Remark. Theorem 3.12 implies that only non-countably additive measures may be non-regular. The condition of regularity as defined here obviously turns out to be natural for Baire fields. However, in Borel fields, one often uses a different concept of regularity, which is the approximation by closed sets rather than by sets of the form $f^{-1}(C)$. In spaces in which any closed set is G_δ, any countably additive measure is regular (on Borel sigma field) in the latter sense, cf. [**27**].

3.14. Radon measures and Radon integrals. Let X be a topological space and let $\mathcal{F}$ be the Baire field on X. Let $\mathcal{M}^+(X)$ be the set of all non-negative, finite, finitely additive and regular measures on $\mathcal{F}$. A generalized measure (or a Radon finitely additive measure) is any set function on $\mathcal{F}$ which can be represented as $m(A) = m_1(A) - m_2(A)$, where $m_1, m_2 \in \mathcal{M}^+(X)$. The set of all generalized measures will be denoted by $\mathcal{M}(X)$. It is a linear vector space; a norm can be introduced by the so called total variation of a measure:

$$|m| = m^+(X) + m^-(X), \tag{7}$$

where $m^+(X) = \sup\{m(B) \mid B \in \mathcal{F}\}$, $m^-(X) = -\inf\{m(B) \mid B \in \mathcal{F}\}$. $\mathcal{M}(X)$ with the norm (7) is a Banach space.

We are now ready to define an integral of a bounded function with respect to a generalized measure. Let f be an $\mathcal{F}$-measurable function and suppose that $\|f\| = K < +\infty$. Let $A_1, A_2, \ldots, A_n$ be any partition of the interval $[-K, K]$ into disjoint intervals (or, in general, sets from the Borel field on $\boldsymbol{R}$) and let $B_i = f^{-1}(A_i)$. In each A_i choose a point y_i and make the integral sum

$$S_d = \sum_{i=1}^{n} y_i m(B_i), \quad \text{where } d = (A_1, \ldots, A_n, y_1, \ldots, y_n).$$

If we direct the set $\{d\}$ in a usual way, saying that $d_1 \prec d_2$ if the partition in d_2 is finer than the one in d_1, then we can prove that S_d is a Cauchy net, hence there is a finite limit, which is the integral of f with respect to the finitely additive measure m, $\int f(x)\, dm(x)$.

3.15. Theorem (Aleksandrov [**1**]). *For an arbitrary topological space X, any linear continuous functional on $C(X)$ is of the form*

$$\varphi(f) = \langle f, m\rangle = \int f(x)\, dm(x), \tag{8}$$

Moreover,

$$\sup_{\|f\| \le 1} \left| \int f(x)\, dm(x) \right| = |m|.$$

There is an isometrical, isomorphical and one to one mapping between the space of all continuous linear functionals on $C(X)$ and the space $\mathcal{M}(X)$; in that sense we write $C(X)^* = \mathcal{M}(X)$. □

In some special cases, $C(X)^*$ has a simpler structure. For example, if X is a compact topological space, then $C(X)^*$ can be identified with the set of all Baire countably additive $\boldsymbol{R}$-valued measures on the Baire sigma-field of X. If, in addition, X is a compact metric space, then $C(X)$ is a separable normed vector space and $C(X)^*$ is the set of all Borel $\boldsymbol{R}$-valued countably additive measures on X.

4. Weak convergence in probability

4.1. Convergence of probability measures. Let now X be a metric space and let $\mathcal{B}$ be the sigma field of Borel (= Baire) subsets of X. Let $\mathcal{M}_1(X)$ be the set of all probability measures on X. Then according to 3.15, $\mathcal{M}_1(X)$ is a subset of the unit ball in $C(X)^*$. The structure of the second dual $C(X)^{**}$ is too complex, but it is well known that $B(X)$ - the set of all bounded Borel-measurable functions is a subset of $C(X)^{**}$. So, we have the following inclusions:

$$\begin{gathered}\text{Original space: } C(X)\\ \text{Dual space: } \mathcal{M}(X); \qquad \mathcal{M}_1(X) \subset \mathcal{M}(X)\\ \text{Second dual: } C(X) \subset B(X) \subset C(X)^{**}.\end{gathered}$$

Let $\langle f, \mu\rangle$ be defined as in (8). On $\mathcal{M}(X)$ we may observe the following topologies:

- The uniform topology [struk], with the norm $\sup|\langle f, \mu\rangle|$, where the supremum is taken over the unit ball in $B(X)$.
- The strong topology, defined by $\sup|\langle f, \mu\rangle|$, where the supremum is taken over the unit ball in $C(X)$.
- The weak topology defined by $\langle f, \mu\rangle$, for $f \in \mathcal{M}^*(X) = C(X)^{**}$.
- The $B(X)$-topology, defined by $\langle f, \mu\rangle$, for $f \in B(X)$.
- The weak star topology, defined by $\langle f, \mu\rangle$, where $f \in C(X)$.

First four topologies are too strong, and they do not respect a topological structure of X, as the following example shows.

Example. Let δ_x, δ_y be point masses at x and y respectively. Then $\langle f, \delta_x\rangle - \langle f, \delta_y\rangle = f(x) - f(y)$. If x and y are close in X, then $f(x)$ and $f(y)$ need not be close unless f is continuous. So, in this example, the weak or $B(X)$-topology are inadequate, but the weak star topology preserves the closedness of x and y. □

The convergence in the weak star topology is usually called the weak convergence in the probabilistic literature. This does not lead to a confusion, since the true weak convergence is never studied.

If μ_d converges weakly to μ, we write $\mu_d \implies \mu$.

The weak star convergence of probability measures is well investigated. We shall firstly give equivalent bases for weak star topology on the whole set $\mathcal{M}^+(X)$. So, the next theorem is not restricted to probability measures.

Let us recall that we say that A is a continuity set for a measure μ on a Borel algebra $\mathcal{B}$ if $\mu(\partial A) = 0$, or, equivalently, if $\mu(A) = \mu(\bar{A}) = \mu(A^\circ)$, where ∂A is the boundary, $\bar{A}$ is the closure and A° is the interior of A. On a Baire algebra we will say that A is a continuity set for μ if there is an U-set U and a Z-set Z such that $U \subset A \subset Z$ and $\mu(Z \setminus U) = 0$.

4.2. Theorem [34, p. 56]. *Let W be the weak star topology on $\mathcal{M}^+(X)$, where X is a topological space. Then the following families of sets make a local base of W around some measure $\mu_0 \in \mathcal{M}^+(X)$:*

$$\begin{aligned}
B_0 &= \{\mu \mid |\langle f_i,\mu\rangle - \langle f_i,\mu_0\rangle| < \varepsilon,\ i = 1,\dots k\}, && f_i \in C(X)\\
B_1 &= \{\mu \mid \mu(F_i) < \mu_0(F_i) + \varepsilon,\ i = 1,\dots,k\}, && F_i \text{ are } Z\text{-sets in } X\\
B_2 &= \{\mu \mid \mu(G_i) > \mu_0(G_i) - \varepsilon,\ i = 1,\dots,k\}, && G_i \text{ are } U\text{-sets in } X\\
B_3 &= \{\mu \mid |\mu(A_i) - \mu_0(A_i)| < \varepsilon,\ i = 1,\dots,k\}, && A_i \text{ are continuity sets for } \mu_0,
\end{aligned}$$

If X is a metric space, then we deal with the Borel algebra and so F_i above can be taken to be closed and G_i to be open sets.

As a straightforward consequence, we get the following

4.3. Theorem. *Measures with a finite support are dense in $\mathcal{M}^+(X)$.*

Proof. Let $\mu_0 \in \mathcal{M}^+(X)$ and let $B(\mu_0)$ be its neighborhood of the form

$$B(\mu_0) = \{\mu \mid \mu(F_i) < \mu_0(F_i) + \varepsilon,\ i = 1,\dots,k\},$$

where F_i are fixed Z-sets. The family of sets F_i together with their intersections and the complement of their union defines a finite partition of X. In each set B of this partition choose a point x_B and define $\mu_1(x_B) = \mu_0(B)$. The measure μ_1 is with a finite support (hence, countably additive!) and clearly $\mu_1(F_i) = \mu_0(F_i)$; so $\mu_1 \in B(\mu_0)$.

4.4. Theorem. *Let μ_d be a net of measures in $\mathcal{M}^+(X)$ and let $\mu_0 \in \mathcal{M}^+(X)$. The following statements are equivalent* [**6**, **30**, **34**]*:*

(i) $\mu_d \implies \mu_0$, *i.e.*, $\lim_d \int f\,d\mu_d = \int f\,d\mu_0$, *for each* $f \in C(X)$.

(ii) $\overline{\lim}\,\mu_d(F) \le \mu_0(F)$ *for any Z-set $F \subset X$ and* $\lim \mu_d(X) = \mu_0(X)$.

(iii) $\underline{\lim}\,\mu_d(G) \ge \mu_0(G)$ *for each U-set $G \subset X$ and* $\lim \mu_d(X) = \mu_0(X)$.

(iv) $\lim \mu_d(A) = \mu_0(A)$ *for each continuity set for* μ_0.

In a special case when we have probability measures on a metric space X, there is a richer structure that yields additional equivalent conditions. To proceed we need some facts on semicontinuous functions.

4.5. Semicontinuous functions. Let X be a metric space. A function $f : X \mapsto \boldsymbol{R}$ is called upper semicontinuous if $\overline{\lim}\, f(x_n) \le f(x)$ for each sequence $\{x_n\}$ such that $x_n \to x$. The function f is lower semicontinuous if $\underline{\lim}\, f(x_n) \ge f(x)$ for each sequence $x_n \to x$.

An important property of semicontinuous functions is that for each $M \in \boldsymbol{R}$ the set $\{x \mid f(x) < M\}$ is open for an upper semicontinuous function and the set $\{x \mid f(x) > M\}$ is open for a lower semicontinuous functions.

4.6. Theorem. *Let μ_d be a net of probability measures on a metric space X and let μ_0 be a probability measure on X. The following statements are equivalent* [**6, 30**]:

(i) $\mu_d \Longrightarrow \mu_0$, *i.e.*, $\lim_d \int f\,d\mu_d = \int f\,d\mu_0$, *for each* $f \in C(X)$.

(ii) $\lim_d \int f\,d\mu_d = \int f\,d\mu_0$ *for each* $f \in C_u(X)$ *(uniformly continuous and bounded functions).*

(iii) $\overline{\lim}\,\mu_d(F) \leq \mu_0(F)$ *for any closed set* $F \subset X$.

(iv) $\underline{\lim}\,\mu_d(G) \geq \mu_0(G)$ *for each open set* $G \subset X$.

(v) $\lim \mu_d(A) = \mu_0(A)$ *for each continuity set for* μ_0.

(vi) $\overline{\lim} \int f\,d\mu_d \leq \int f\,d\mu_0$ *for each upper semicontinuous and bounded from above function* $f : X \mapsto \boldsymbol{R}$.

(vii) $\underline{\lim} \int f\,d\mu_d \geq \int f\,d\mu_0$ *for each lower semicontinuous and bounded from below function* $f : X \mapsto \boldsymbol{R}$.

(viii) $\lim \int f\,d\mu_d = \int f\,d\mu_0$ *for each* μ_0 a.e. *continuous function* $f : X \mapsto \boldsymbol{R}$.

4.7. Vague convergence. In [**32**], a concept of so called vague convergence is introduced as follows. Let $K(X)$ be the set of all continuous functions with a compact support defined on X. Then we say that μ_d converges vaguely to μ if $\langle f, \mu_d\rangle \to \langle f, \mu\rangle$ for each $f \in K(X)$. This kind of convergence is clearly weaker than the weak star convergence. For example, the sequence δ_n converges vaguely to 0, although it does not converge in the weak star sense.

4.8. Metrics of weak convergence. By Theorem 2.20, the weak star topology on the closed unit ball of $\mathcal{M}(X)$ is metrizable if $C(X)$ is a separable metric space, which is the case if and only if X is a compact space. However, even if the weak star topology of the unit ball of $\mathcal{M}$ is not metrizable, this topology on the set of all probability measures may be metrizable; as a matter of fact, it probably is always metrizable, as we shall see in the subsequent discussion.

4.9. Theorem. *Let X be a separable metric space. Then the weak star topology on $\mathcal{M}_1(X)$ is metrizable by the metric*

$$d(P,Q) = \inf\{\varepsilon > 0 \mid Q(B) \leq P(B^\varepsilon) + \varepsilon,\ \ P(B) \leq P(Q^\varepsilon) + \varepsilon,\ \ B \in \mathcal{B}\}, \tag{9}$$

where $B^\varepsilon = \{x \in S \mid d(x,B) < \varepsilon\}$, and $\mathcal{B}$ is the Borel sigma algebra. □

The metric (9) is known as Lévy's metric or Prohorov's metric [**6, 30**]. Although the proof of Theorem 4.9 relies on separability of X, it has to be noted that the metrizability of $\mathcal{M}_1(X)$ is related to the so called problem of measure [**6, 12**] and that the examples of non-metrizable $\mathcal{M}_1(X)$ are not known. So, there is a strongly founded conjecture that for any metric space X, the topology of the weak star convergence on the set $\mathcal{M}_1(X)$ is metrizable and one metric is given by (9).

Moreover, it is known that, if X is a complete separable metric space (Polish space), then so is $\mathcal{M}_1(X)$.

There is another metric of weak star convergence [**30**, p. **117**], similar to the one introduced in Theorem 2.20.

4.10. Theorem. *Let X be a separable metric space. Then there is a countable set $\{f_1, f_2, \dots\}$ of uniformly continuous bounded real valued functions with values* *in* $[0,1]$ *so that the span of this set is dense in the set of all uniformly continuous and bounded functions on* X. *Now define*

$$\rho(P,Q)=\sum_{n=1}^{+\infty}\frac{|\langle f_n,P\rangle-\langle f_n,Q\rangle|}{2^n}.$$

Then ρ *is a metric on* $\mathcal{M}_1(X)$, *which is topologically equivalent to Prohorov's metric.*

This theorem can be proved by noticing that the condition $\rho(P_d,Q)\to 0$ is equivalent to the condition (ii) of Theorem 4.6 applied to P_d and Q. Since the condition (ii) holds (as an equivalent condition to weak star convergence) only for countably additive measures, we conclude that the metric of Theorem 4.10 can not generally be extended to the unit ball of $\mathcal{M}(X)$; hence, the unit ball of $\mathcal{M}(X)$ generally is not metrizable.

5. Finitely, but not countably additive measures in the closure of the set of probability measures

In this section we discuss topics of relative weak star compactness and closedness of the set of all probability measures. We will show that in a non-compact topological space X, under slight additional assumptions (say, if X is a metric s-pace) the set of probability measures $\mathcal{M}_1$ is not closed under the weak star limits. We actually show the existence of an additive, but not countably additive measure in the closure of $\mathcal{M}_1$. The fact that $\mathcal{M}_1$ is not closed is the main rationale for Prohorov's theorem, which will be presented in the next section.

5.1. Nets and filters. Nets and filters are introduced in Mathematics as generalizations of sequences. Nets were defined and discussed in papers of Moore [moore] in a context of determining a precise meaning of the limit of integral sums; early developments of nets can be found in papers [**8, 18, 25, 26, 33**]. Filters were introduced by Cartan [**10, 11**] in the second decade of 20th century. The theory of both filters and nets was completed by the mid of 20th century. We will give here a brief account of basic definitions and theorems, largely taken from [**35**, Sections 11 and 12].

A set D is called a *directed set* if there is a relation $\leq$ on D such that

(i) $x\leq x$ for all $x\in D$
(ii) If $x\leq y$ and $y\leq z$ then $x\leq z$
(iii) For any $x,y\in D$ there is a $z\in D$ so that $x\leq z$ and $y\leq z$.

A *net* in a set X is any mapping of a directed set D into X, usually denoted by x_d, say, like sequences.

Let D and E be directed sets and let φ be a function $E\to D$ such that:

(i) $a\leq b \implies \varphi(a)\leq\varphi(b)$ for each $a,b\in E$;
(ii) For each $d\in D$ there is an $e\in E$ so that $d\leq\varphi(e)$.

Then $x_{\varphi(e)}$ is a *subnet* of the net x_d; more often denoted by x_{d_e}.

Let X be a topological space. We say that a net $x_d, d\in D$ converges to some point $x\in X$ if for each neighborhood U of x there is a $d_0\in D$ so that $x_d\in U$ whenever $d\geq d_0$.

Let $S \subset X$. We say that $x_d, d \in D$ is *eventually* (or residually) in S if there is a $d_0 \in D$ so that $x_d \in S$ whenever $d \geq d_0$. Hence, a net x_d converges to x iff it is eventually in every neighborhood of x.

A net x_d is called an *ultranet* if for each $S \subset X$ it is eventually in S or eventually in S'.

Let S be any nonempty set. A collection $\mathcal{F}$ of non-empty subsets of S is called a *filter* if

(i) $S \in \mathcal{F}$.

(ii) If $B_1, B_2 \in \mathcal{F}$ then $B_1 \cap B_2 \in \mathcal{F}$.

(iii) If $B_1 \in \mathcal{F}$ and $B_1 \subset B_2 \subset S$, then $B_2 \in \mathcal{F}$.

A subcollection $\mathcal{F}_0 \subset \mathcal{F}$ is a *base* for the filter $\mathcal{F}$ if $\mathcal{F} = \{B \subset S \mid B \supset B_0$ for some $B_0 \in \mathcal{F}_0\}$, that is, if $\mathcal{F}$ is consisted of all supersets of sets in $\mathcal{F}_0$. Any collection $\mathcal{F}_0$ can be a base for some filter $\mathcal{F}$ provided that given any two sets $A, B \in \mathcal{F}_0$ there is a $C \in \mathcal{F}_0$ so that $C \subset A \cap B$.

In a topological space X, the set of all neighborhoods of some fixed point x is a filter, called the *neighborhood filter*. Its base is the neighborhood base at x.

A filter $\mathcal{F}_1$ is *finer* than the filter $\mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$.

We say that a filter $\mathcal{F}$ in a topological space X *converges* to $x \in X$ if $\mathcal{F}$ is finer than the neighborhood filter at x.

A filter $\mathcal{F}$ is called *principal* or *fixed* if $\bigcap_{B \in \mathcal{F}} B \neq \emptyset$; otherwise it is called *non-pricipal* or *free*.

A filter $\mathcal{F}$ on S is called an *ultrafilter* if there no filter on S which is strictly finer than $\mathcal{F}$. It can be shown [**35**, Theorem 12.11] that a filter $\mathcal{F}$ is an ultrafilter iff for any $B \in S$ either $B \in \mathcal{F}$ or $B' \in \mathcal{F}$. For example, the family of all sets that contain a fixed point $x \in X$ is an ultrafilter on X.

5.2. Relation between nets and filters. Both nets and filters are used to describe convergence and related notions. In fact, there is a close relationship between nets and filters.

Let $x_d, d \in D$ be a net in X. The sets $B_{d_0} = \{x_d \mid d \geq d_0\}$ make a base for a filter $\mathcal{F}$; we say that the filter $\mathcal{F}$ *is generated* by the net x_d.

Conversely, let $\mathcal{F}$ be a filter on a set S. Let D be the set of all pairs (x, F), where F runs over $\mathcal{F}$ and $x \in F$. Define the order by $(x_1, F_1) \leq (x_2, F_2) \iff F_1 \supset F_2$. Then the mapping $(x, F) \mapsto x$ is a *net based on* $\mathcal{F}$.

5.3. Conditions for compactness. A topological space X is called *compact* if every open cover has a finite subcover. The following conditions are equivalent [**35**, Theorem 17.4]:

a) X is compact

b) each family of closed subsets of X with the finite intersection property has an non-empty intersection,

c) for each filter in X there is a finer convergent filter,

d) each net in X has a convergent subnet,

e) each ultrafilter in X is convergent,

f) each ultranet in X is convergent.

5.4. Space of probability measures is not closed in $\mathcal{M}$. Since $\mathcal{M}(X)$, the space of generalized measures introduced in 3.14, is the dual space of the normed space $C(X)$, by Theorem 2.18 its unit ball $B_1 = \{m \in \mathcal{M} \mid |m| = 1\}$ is compact in the weak star topology. If $\mathcal{M}_1(X)$, the space of all probability measures, were closed in $\mathcal{M}(X)$, then it would have been also compact, being a subset of B_1. Then (at least if X is a separable metric space), since the topology on $\mathcal{M}_1(X)$ is metrizable, any sequence of probability measures would have had a weak star convergent subsequence and Prohorov's theorem and the notion of tightness (Section 6) would not be of any interest. However, this is not generally true. For example, for $X = \boldsymbol{R}$, the sequence $\{P_n\}$ of point masses at $n = 1, 2, \dots$ clearly does not have any weak star convergent subsequence. However, it must have a convergent subnet, and the limiting measure is in $\mathcal{M} \setminus \mathcal{M}_1$.

Here we give a rather general result [**20**] which proves the existence of a measure in the closure of $\mathcal{M}_1(X)$, which is not a probability measure (not countably additive). Before we proceed, we need a lemma concerning normal spaces. Recall that a topological space X is called normal if for any two disjoint closed sets A and B in X there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Equivalently, a space X is normal if and only if for any two disjoint sets A and B there is an $f \in C(X)$ such that $f(A) = \{0\}$, $f(B) = \{1\}$ and $0 \leq f(x) \leq 1$ for all $x \in X$ (Urysohn's lemma).

5.5. Lemma. *Let X be a normal space which contains an infinite sequence $S = \{x_1, x_2, \dots\}$ with no cluster points. Then for any infinite proper subset $S_0 \subset S$ there is a function $f \in C(X)$ such that $f(x) = 0$ if $x \in S_0$ and $f(x) = 1$ if $x \in S \setminus S_0$.*

Proof. Let S_0 be any infinite proper subset of S and let $S_1 = S \setminus S_0$. Then S_0 and S_1 are closed sets (no cluster points), hence by normality, the desired function exists.

5.6. Theorem. *Let X be a normal topological space and suppose that it contains a countable subset $S = \{x_1, x_2, \dots\}$ with no cluster points. Let P_n be point masses at x_n, that is, $P_n(B) = 1$ if $x_n \in B$ and $P_n(B) = 0$ otherwise. Then there exists a $w - *$ limit of a subnet P_{n_d} of the sequence of point masses P_n. Any such limit ψ satisfies:*

(i) For any set $B \subset X$ it holds either $\psi(B) = 0$ or $\psi(B) = 1$, with $\psi(S) = 1$.
(ii) ψ is a finitely (but not countably) additive set function
(iii) For every finite or empty set $B \subset X$, $\psi(B) = 0$.

The corresponding subnet n_d is the net based on the filter of sets of ψ-measure 1.

Proof. From the previous considerations it follows that $\{P_n\}$ has a cluster point. Clearly, we must have a directed set D and a net $x_d \in S$ such that

$$\lim f(x_d) = \int f(x)\, d\psi(x), \tag{10}$$

for some measure ψ in the unit ball of $\mathcal{M}(X)$ and for all $f \in C(X)$. Then ψ is additive; further, it is a straightforward consequence of (10) that ψ has to be concentrated on S, i.e., $\psi(S) = 1$ and also that $\psi(B) = 0$ for any finite set B (otherwise, one could modify f in such a way that the right hand side of (10) changes without affecting the left hand side). Since $\psi(S) = 1$ and $\psi(x_n) = 0$ for each n, ψ is not countably additive. Hence, (ii) and (iii) are proved. It remains to show that $\psi(B)$ must be 0 or 1 for any B. Suppose contrary, that there exists a set B with $\psi(B) = q$, $0 < q < 1$. Then $S \cap B$ is neither a finite nor a cofinite set. For a d_0 being fixed, infinitely many x_d with $d \geq d_0$ belong either to B or to B'. By Lemma 5.5, there is an $f \in C(X)$ which takes value 1 at $S \cap B$ and 0 at $S \cap B'$. Then the right hand side of (10) equals $\psi(B) = q$ and for any $\varepsilon > 0$ there is a d_0 so that for each $d \geq d_0$, $|f(x_d) - q| < \varepsilon$. Now suppose that B contains infinitely many x_d's for $d \geq d_0$, then we'd have $|1 - q| < \varepsilon$; otherwise $|q| < \varepsilon$ which are both impossible.

Therefore, we proved that any ψ which is a $w - *$ limit of a subnet $\{P_n\}$ satisfies conditions (i)–(iii). An analysis of the construction of Radon integral with respect to ψ, in 3.14, reveals that the subnet of convergence is the net based on the ultrafilter $\mathcal{F}$ consisted of sets with $\psi(B) = 1$.

5.7. Remarks. In any at least countable set X there exists a measure ψ which satisfies conditions (i)–(iii). This can be shown using theory of filters and the Axiom of Choice. The family of all sets $B \subset X$ with $\psi(B) = 1$ makes a non-principal ultrafilter. The existence of non-principal ultrafilters can be proved, but there is no concrete example of such a filter.

Theorem 5.6 holds, for instance, in any non-compact metric space.

6. Tightness and Prohorov's theorem

Although the notion of tightness can be defined in a more general context, in this section we observe only probability measures on metric spaces. Hence, X will denote a metric space, $\mathcal{B}$ a Borel sigma field and $\mathcal{M}_1(X)$ the set of all probability measures.

6.1. Definition. Let $\mathcal{P}$ be a set of probability measures on X. We say that $\mathcal{P}$ is *tight* if for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $\mu(K') \leq \varepsilon$.

The notion of tightness makes sense even if $\mathcal{P}$ is a singleton. In this case we have the following result.

6.2. Theorem. *If X is a complete and separable metric space, then each probability measure is tight.*

Proof. By separability of X, for each n there is a sequence $A_{n_1}, A_{n_2}, \ldots$ of open balls of radii $1/n$ that cover X. Choose i_n so that

$$P\Big(\bigcup_{i \leq i_n} A_{n_i}\Big) > 1 - \frac{\varepsilon}{2^n}$$

and let

$$K = \bigcap_{n \geq 1} \bigcup_{i \leq i_n} A_{n_i}.$$

Since $K \subset \bigcup_{i \leq i_n} A_{i_n}$ for each n, K is totally bounded set in a complete metric space and hence $\bar{K}$ is compact. Further,

$$P(K') \leq \sum_{n=1}^{+\infty} P\Big(\Big(\bigcup_{i \leq i_n} A_{n_i}\Big)'\Big) < \varepsilon \sum_{i=1}^{+\infty} \frac{1}{2^n} = \varepsilon,$$

hence $P(\bar{K}') \leq P(K') < \varepsilon$.

6.3. Definition. We say that a set $\mathcal{P}$ of probability measures is *relatively compact* if any sequence of probability measures $P_n \in \mathcal{P}$ contains a subsequence P_{n_k} which converges weak star to a probability measure in $\mathcal{M}_1(X)$. □

A precise topological term for relative compactness would be *relative sequential compactness in* $\mathcal{M}_1(X)$.

6.4. Remark. If X is compact, then from the previous section it follows that any set of probability measures is relatively compact. Otherwise, we need some conditions which are easier to check. One such condition is given in the next theorem. The proof presented here relies on the material of the previous section and departs from a classical presentation.

6.5. Theorem (Prohorov [**28**]). *Let X be an arbitrary metric space and let $\mathcal{P}$ be a tight set of measures. Then $\mathcal{P}$ is relatively compact.*

Proof. Let X be a metric space and let $\mathcal{P}$ be a tight set of Borel probability measures on it. Then for each $n \in \boldsymbol{N}$, let K_n be a compact subset of X such that $P(K_n) > 1 - 1/n$ for all $P \in \mathcal{P}$; we may assume that $K_1 \subset K_2 \subset \cdots$. A unit ball in any of spaces $\boldsymbol{C}(K_n)^*$ is compact and metrizable. For a given sequence $\{P_k\}$ of probability measures in $\mathcal{P}$, its restriction to a compact space K_n has a convergent subsequence. Then we can use a diagonal argument to show that there is a subsequence $P_{k'}$ such that

$$P_{k'} \implies P^{(n)} \quad \text{on } K_n, \quad n = 1, 2, \ldots \tag{11}$$

for some measures $P^{(n)}$ on K_n. Since K_n are increasing sets, the restriction of $P^{(n)}$ to K_{n-1} must coincide with $P^{(n-1)}$. Since $P^{(n)}$ is in the dual space of $\boldsymbol{C}(K_n)$, it is countably additive, and by (11), $P^{(n)}(K_n) \geq 1 - \varepsilon$.

Now if B is a Borel subset of X, define

$$P(B) = \lim_{n \to +\infty} P^{(n)}(B \cap K_n).$$

The limit here exists because of

$$P^{(n)}(B \cap K_n) \geq P^{(n)}(B \cap K_{n-1}) = P^{(n-1)}(B \cap K_{n-1}),$$

hence the sequence $\{P^{(n)}(B \cap K_n)\}$ is increasing and clearly is bounded from above by 1. To show that $P_{k'} \implies P$, we use the characterization of Theorem 4.6(iii). Let F be any closed set in X. From (11) we have that

$$\overline{\lim}\, P_{k'}(F \cap K_n) \leq P^{(n)}(F \cap K_n) \quad \text{for each } n = 1, 2, \ldots$$

Further,

$$P_{k'}(F) \le P_{k'}(F \cap K_n) + \frac{1}{n}$$

and so

$$\overline{\lim}\, P_{k'}(F) \le \overline{\lim}\, P_{k'}(F \cap K_n) + \frac{1}{n} \le P^{(n)}(F \cap K_n) + \frac{1}{n}.$$

Letting now $n \to +\infty$ we get that

$$\overline{\lim}\, P_{k'}(F) \le P(F),$$

that is, $P_{k'} \implies P$.

6.6. Theorem. *Let X be a complete separable metric space. If $\mathcal{P}$ is a relatively compact set of probability measures on X, then $\mathcal{P}$ is tight.*

Proof. By Theorem 4.9, the weak star topology on $\mathcal{M}_1$ is metrizable, so relative sequential compactness of $\mathcal{P}$ as defined in 6.3 becomes the topological compactness of $\bar{\mathcal{P}}$, that is, any open cover of $\bar{\mathcal{P}}$ has a finite subcover. Here we understand that the closure of $\mathcal{P}$ is in the metric space $\mathcal{M}_1$. Without loss of generality, we may and will assume that $\mathcal{P}$ itself is compact in $\mathcal{M}_1$.

Fix $\varepsilon > 0$ and $\delta > 0$. If $P \in \mathcal{P}$, then by Theorem 6.2 it is tight, so there is a compact set K_P such that $P(K_P) > 1-\varepsilon/2$. Being compact, K_P is totally bounded, that is, it can be covered with finitely many open δ-balls $B_{P,i}$, $i = 1, 2, \dots, k_P$. Let $G_P = \bigcup_{i=1}^{k_P} B_{P,i}$. By Theorem 4.2, there is a neighborhood of P (in the weak star topology of $\mathcal{M}(X)$) of the form

$$U_P = \{\mu \mid \mu(G_P) > P(G_P) - \varepsilon/2\}$$

The family $\{U_P\}_{P\in\mathcal{P}}$ makes an open cover of $\mathcal{P}$ and hence there is a finite subcover, say $U_{P_1}, \dots, U_{P_m}$. Then let $K_\delta = \bigcup_{j=1}^{m} G_{P_j}$. For any $Q \in \mathcal{P}$ we have that

$$Q(G_P) > P(G_P) + \varepsilon/2 \ge P(K_P) - \varepsilon/2 > 1 - \varepsilon,$$

which implies that also $Q(K_\delta) > 1-\varepsilon$. Let now K be the closure of the intersection of all $K_{1/n}$; it is a closed and totally bounded set, hence compact, and we have that $Q(K) > 1 - \varepsilon$ for all $P \in \mathcal{P}$.

7. Weak convergence of probability measures on Hilbert spaces

In this section we firstly review basic fact related to the weak convergence of probability measures on finite dimensional vector spaces. The simple characteristic function technique which is usually applied there, becomes more complex on infinite dimensional Hilbert spaces.

7.1. Weak convergence of probability measures on R^k. On finite dimensional spaces, the notion of weak convergence of probability measures coincides with the notion of convergence of distributions (see [**6**], for example). If F_n and

F are distribution functions of k-dimensional random variables X_n and X respectively, and if $\lim F_n(\boldsymbol{x}) = F(\boldsymbol{x})$ in each point $\boldsymbol{x} \in \boldsymbol{R}^k$ where F is continuous, then we say that the corresponding sequence X_n converges to X in distribution. This occurs if and only if $P_n \implies P$, where P_n and P are probability measures on $\boldsymbol{R}^n$ - distributions of X_n and X respectively.

A useful tool for investigation of weak convergence is the notion of a characteristic function. If P is a probability measure on $\boldsymbol{R}^k$, then its characteristic function is defined by

$$\varphi(\boldsymbol{t}) = \int_{\boldsymbol{R}^k} e^{i\langle \boldsymbol{t}, \boldsymbol{x}\rangle}\, dP(\boldsymbol{x}) \tag{12}$$

where

$$\boldsymbol{x} = (x_1, \dots, x_k), \quad \boldsymbol{t} = (t_1, \dots, t_k), \quad \langle \boldsymbol{t}, \boldsymbol{x}\rangle = \sum_{i=1}^{k} t_i x_i.$$

It is a well known fact (Bochner's theorem) that a function φ defined on $\boldsymbol{R}^k$ is a characteristic function of some probability measure if and only if it is positive definite, continuous at origin and $\varphi(0) = 1$.

It is also a basic fact that $P_n \implies P$ if and only if $\lim_n \varphi_n(\boldsymbol{t}) = \varphi(\boldsymbol{t})$, where φ_n and φ are the corresponding characteristic functions. This is indeed a very strong result, since it says that it suffices to test the condition (1) with only two (classes of) functions, $\boldsymbol{x} \mapsto \cos\langle \boldsymbol{t}, \boldsymbol{x}\rangle$ and $\boldsymbol{x} \mapsto \sin\langle \boldsymbol{t}, \boldsymbol{x}\rangle$.

7.2. Positive definite functions. Let X be any linear vector space. A complex valued function φ defined on X is said to be positive (or non-negative) definite if for any finite $A = (a_1, \dots, a_n) \in \boldsymbol{C}^n$ and $x = (x_1, \dots, x_n) \in X^n$ the following holds:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j \varphi(x_i - x_j) \geq 0.$$

Positive definite functions have some interesting properties, which can be proved directly from the above definition, using an appropriate choice of A and x. We list some of these properties (see **[21]** for proofs):

$$\varphi(0) \geq 0 \tag{i}$$

$$\bar{\varphi}(x) = \varphi(-x) \tag{ii}$$

$$|\varphi(x)| \leq \varphi(0) \tag{iii}$$

$$|\varphi(x) - \varphi(y)|^2 \leq 2\varphi(0)(\varphi(0) - \operatorname{Re}\varphi(x - y)) \tag{iv}$$

$$\varphi(0) - \operatorname{Re}\varphi(2x) \leq 4(\varphi(0) - \operatorname{Re}\varphi(x)) \tag{v}$$

From (iv) it immediately follows that a positive definite function is uniformly continuous on X with respect to any metric topology if and only if its real part is continuous at zero.

7.3. Characteristic functions on Hilbert spaces. Let H be a real separable Hilbert space. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and the norm which defines the topology on H and let $\{e_i\}$ be an orthonormal basis.

Let P be a probability measure on H. The corresponding characteristic function is defined formally in the same way as in finite dimensional spaces:

$$\varphi(x) = \int_H e^{i\langle x,y\rangle}\, dP(y), \qquad x \in H. \tag{13}$$

A characteristic function uniquely determines the corresponding measure [**27**].

It is easy to see that the function defined by (13) is positive definite and continuous at zero (with respect to the given norm). However, these properties are not sufficient for a function to be a characteristic function, as in finite dimensional cases. In order to proceed further, we need some facts about Hilbertian seminorms.

7.4. Hilbertian seminorms. A real valued function p defined on a vector space X is called a Hilbertian seminorm if for all $x, x_1, x_2 \in X$ and $a \in \boldsymbol{R}$:
(i) $p(x) \geq 0$
(ii) $p(ax) = |a|p(x)$
(iii) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$
(iv) $p(x) > 0$ for some $x \in X$
(v) $p^2(x_1 + x_2) + p^2(x_1 - x_2) = 2(p^2(x_1) + p^2(x_2))$

Due to (v), for a Hilbertian seminorm p one can define the corresponding inner product:

$$p(x,y) = \frac{1}{4}(p^2(x_1 + x_2) - p^2(x_1 - x_2)). \tag{14}$$

Let now H be a Hilbert space. Besides its norm, one can define various Hilbertian seminorms on H. One such seminorm is, for example,

$$p_n(x) = \sqrt{\sum_{i=1}^{n} \langle x, e_i\rangle^2}, \qquad n \in \boldsymbol{N}, \tag{15}$$

where $\{e_i\}$ is an orthonormal basis with respect to the original norm. The inner product which corresponds to the seminorm (15) is given by

$$p_n(x,y) = \sum_{i=1}^{n} \langle x, e_i\rangle\langle y, e_i\rangle.$$

Let Π denotes the set of all Hilbertian seminorms p that satisfy

$$p(x) \leq C\|x\| \quad \text{for some } C > 0 \tag{16}$$

$$\sum_{i=1}^{+\infty} p^2(e_i) < +\infty, \tag{17}$$

for an orthonormal (with respect to the original norm) basis $\{e_i\}$. It can be shown that the quantity in (17) does not depend on the choice of an orthonormal basis in H (see [**22**] for the proof and more details). Seminorms p_n defined by (15) belong to Π, since $p_n(x) \le \|x\|$ and $\sum p_n^2(e_i) = n$.

It is easy to see that

$$p \in \Pi \implies cp \in \Pi \quad \text{for any } c > 0 \tag{18}$$

and

$$p_1, \dots, p_n \in \Pi \implies \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2} \in \Pi. \tag{19}$$

Now denote by $\mathcal{I}$ a topology on H defined by the following basis of neighborhoods at zero:

$$\{x \in H \mid p_1(x) < \varepsilon_1, \dots, p_n(x) < \varepsilon_n\},$$

where $p_1, \dots, p_n \in \Pi$, $n \in \boldsymbol{N}$, $\varepsilon_i > 0$. Equivalently, by (18) and (19), a basis of neighborhoods at zero for the $\mathcal{I}$-topology is given by

$$\{x \in H \mid p(x) < \varepsilon\}, \qquad p \in \Pi,\ \varepsilon > 0.$$

Then a sequence $\{x_n\}$ converges in the $\mathcal{I}$ - topology to x if and only if $\lim_n p(x_n - x) = 0$ for any $p \in \Pi$. The $\mathcal{I}$-topology is stronger than the norm topology. If a function φ defined on H is continuous in the $\mathcal{I}$-topology it must be norm continuous, but the converse does not hold.

7.5. Theorem. *Let φ be the characteristic function of a probability measure P on H. Then for any $\varepsilon > 0$ there is a seminorm $p_\varepsilon \in \Pi$ such that for all $x \in H$,*

$$1 - \operatorname{Re}\varphi(x) \le p_\varepsilon^2(x) + \varepsilon \tag{20}$$

and φ is $\mathcal{I}$-continuous on H.

Proof. Since H is a complete separable normed space, by Theorem 6.2 the probability measure P is tight. That is, for a given $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset H$ such that $P(K') \le \varepsilon/2$. So, we have that

$$\begin{aligned} 1 - \operatorname{Re}\varphi(x) = \int (1 - \cos\langle x, y\rangle)\, dP(y) &\le \int_{K_\varepsilon} (1 - \cos\langle x, y\rangle)\, dP(y) + \varepsilon \\ &\le \frac{1}{2} \int_{K_\varepsilon} \langle x, y\rangle^2\, dP(y) + \varepsilon \end{aligned}$$

Since K_ε is compact and $y \mapsto \langle x, y\rangle^2$ is a continuous function, then it is bounded on K_ε and we may define

$$p_\varepsilon = \Big(\frac{1}{2} \int_{K_\varepsilon} \langle x, y\rangle^2\, dP(y)\Big)^{1/2}. \tag{21}$$

It is now easy to show that p_ε is a Hilbertian seminorm which satisfies (16) and (17), hence (20) is proved. From the inequality (iv) in 7.2 and (20) we have that

$$|\varphi(x) - \varphi(y)|^2 \leq 2(p_\varepsilon^2(x-y) + \varepsilon),$$

which implies the uniform continuity of φ in $\mathcal{I}$ topology.

7.6. Example. Consider the function $f(x) = e^{-\|x\|^2/2}$. It is norm continuous and $f(0) = 1$. Using Schoenberg's theorem [5] it can be shown that f is a positive definite function. Suppose that f is $\mathcal{I}$-continuous. Then the norm is also $\mathcal{I}$-continuous, which implies that there is a $p \in \Pi$ and a $\delta > 0$ so that $p(x) < \delta \implies \|x\| < 1/2$. For such a p we have that $\sum p^2(e_i) < +\infty$, hence there is an e_i such that $p(e_i) < \delta$ and so $\|e_i\| < 1/2$, which is a contradiction.

By Theorem 7.5, f is not a characteristic function on H. $\square$

The next theorem is proved by Sazonov [29].

7.7. Theorem. *A function $\varphi : H \mapsto C$ is the characteristic function of a probability measure if and only if it is positive definite, $\mathcal{I}$-continuous at zero and $\varphi(0) = 1$.*

7.8. Weak convergence on H via characteristic functions. Contrary to finite dimensional cases, the convergence of characteristic functions alone is not sufficient for weak convergence of probability measures. Here is where relative compactness of probability measures plays a key role.

Theorem. *Let $\{P_n\}$ be a sequence of probability measures on H and let φ_n be the corresponding characteristic functions. Let P and φ be a probability measure and its characteristic function. If $P_n \implies P$ then $\lim_n \varphi_n(x) = \varphi(x)$ for all $x \in H$.*

Conversely, if a sequence P_n of probability measures on H is relatively compact and $\lim_n \varphi_n(x) = \varphi(x)$ for all $x \in H$, then there exists a probability measure P such that φ is its characteristic function and $P_n \implies P$.

Proof. Since the mapping $x \mapsto e^{i\langle x,y\rangle}$ is norm-continuous, we have that $P_n \implies P$ implies $\varphi_n(x) \to \varphi(x)$ for all $x \in H$. To show the converse, assume that $\{P_n\}$ is relatively compact and that $\lim_n \varphi_n(x) = \varphi(x)$ for all $x \in H$, but $\{P_n\}$ does not converge weakly. Then there are two subsequences $\{P_{n'}\}$ and $\{P_{n''}\}$ with different limits, $P^{(1)}$ and $P^{(2)}$. Then characteristic functions $\varphi_{n'}$ and $\varphi_{n''}$ converge to different limits (i.e., to characteristic functions of $P^{(1)}$ and $P^{(2)}$ respectively), which is a contradiction to the assumption that $\{\varphi_n\}$ is a convergent sequence.

7.9. Example. Let P_n be point masses at e_n. The corresponding characteristic functions are $\varphi_n(x) = e^{ix_n}$, where $x_n = \langle x, e_n\rangle$. Then for every $x \in H$, $\lim_n \varphi_n(x) = 1$ and 1 is the characteristic function of the point mass at zero, P_0. But clearly, $\{P_n\}$ is not a weakly convergent sequence, assuming that H is equipped with the norm topology. To show that exactly, note that if $P_n \implies P$ in the norm topology, then P can only be P_0 because of convergence of characteristic functions. Now since H is a normed space, there is an $f \in C(H)$ such that $f(B_{1/4}) = 1$ and $f(B'_{1/2}) = 0$, where B_r is the ball centered at zero with the radius r. For such a function we have

$$\int f(x)\,dP_n(x) = f(e_n) = 0 \quad \text{and} \quad \int f(x)\,dP_0(x) = f(0) = 0,$$

so P_n does not weakly converge to P.

7.10. Weak convergence on H with respect to strong and weak topology. In this item, we observe H with the strong (norm) topology (H_s) and with the weak topology as defined in 2.2 (H_w). Although, by 2.10, the Borel sets are the same in both cases, there is a difference in the concepts of weak convergence of measures, arising from the fact that $C(H_w)$ is in general a proper subset of $C(H_s)$. Hence, if $P_n \implies P$ in H_w, we need some additional requirements to conclude that $P_n \implies P$ in H_s, unless H is finite dimensional, in which case $H_w = H_s$. In the next two theorems we show that an additional necessary and sufficient condition is "uniform finite dimensional approximation", expressed by (22) below.

7.11. Theorem. *Let $\{P_n\}$ be a sequence of probability measures on H. If $P_n \implies P$ in H_w and for all $\varepsilon > 0$,*

$$\lim_N \sup_n P_n\Big(\sum_{i=N}^{+\infty} \langle x, e_i\rangle^2 \geq \varepsilon\Big) = 0, \tag{22}$$

then $P_n \implies P$ in H_s.

Proof. By Theorem 4.6(ii), it suffices to show that, under the above assumptions,

$$\int f(x)\,dP_n(x) \to \int f(x)\,dP(x), \tag{23}$$

for every uniformly norm continuous and bounded function f. The idea of the proof is to approximate f by a function which is continuous and bounded in H_w. For $x \in H$ let $g_N(x) = \sum_{i=1}^{N-1} \langle x, e_i\rangle e_i$. Then g_N is a linear operator $H \mapsto H$, $\|g_N(x)\| \leq (N-1)\|x\|$ for all $x \in H$ and hence $\|g_N(x) - g_N(y)\| \leq (N-1)\|x - y\|$, so g_N is uniformly continuous. Let now f be any norm continuous real valued function on H_s, with $\|f\| = M_f$. The function $x \mapsto f(g_N(x))$ is continuous in H_w. Consider the difference $d_N(x) = f(x) - f(g_N(x))$ and fix a $\delta > 0$. By uniform continuity of f, there is an $\varepsilon > 0$ so that

$$|d_N(x)| < \delta \quad \text{whenever} \quad \|x - g_N(x)\|^2 = \sum_{i=N}^{+\infty} \langle x, e_i\rangle^2 < \varepsilon.$$

By (22), for such an ε we can find N_0 so that for each $N \geq N_0$ we have

$$P_n(A_N) < \delta \quad \text{for every } n, \quad \text{where } A_N = \Big\{x \in H \;\Big|\; \sum_{i=N}^{+\infty} \langle x, e_i\rangle^2 \geq \varepsilon\Big\}. \tag{24}$$

Now for any $N \geq N_0$ we have

$$\begin{aligned}\left|\int f(x)\,dP_n(x) - \int f(g_N(x))\,dP_n(x)\right| &= \left|\int d_N(x)\,dP_N(x)\right| \\ &\leq \left|\int_{A'_N} d_N(x)\,dP_n(x)\right| + \left|\int_{A_N} d_N(x)\,dP_n(x)\right| \\ &\leq \delta + 2M_f\delta = \delta(1+2M_f).\end{aligned} \tag{25}$$

Further, by continuity of f, the fact that $g_N(x) \to x$ as $N \to \infty$ and the dominated convergence theorem, for $N \geq N_1$ we have

$$\left|\int f(g_N(x))\,dP(x) - \int f(x)\,dP(x)\right| \leq \delta \tag{26}$$

Finally, since $f(g(\cdot)) \in C(H_w)$, we have that

$$\left|\int f(g_N(x))\,dP_N(x) - \int f(g_N(x))\,dP(x)\right| \leq \delta, \tag{27}$$

where $N \geq \max(N_0, N_1)$. The weak convergence of P_n in H_s follows now from (25)-(27).

7.12. Theorem. *Any weak star convergent sequence of probability measures $\{P_n\}$ in H_s satisfies (22).*

Proof. Suppose that $P_n \Longrightarrow P$ in H_s. For an $\varepsilon > 0$, let A_N be defined by (24). Since A_N is closed in the norm topology, by Theorem 4.6(iv) we have that, for any fixed N,

$$\overline{\lim}\, P_n(A_N) \leq P(A_N). \tag{28}$$

Since $\{A_N\}$ is a decreasing sequence of sets with $\bigcap_{N=1}^{+\infty} A_N = \emptyset$, by continuity of probability measures we have that

$$\lim_{N\to+\infty} P(A_N) = 0. \tag{29}$$

Now fix a $\delta > 0$ and choose N_0 large enough so that $P(A_{N_0}) \leq \delta$ for $N \geq N_0$. By (28), there are only finitely many measures, say $P_{n_1}, \ldots, P_{n_k}$ such that $P_{n_i}(A_{N_0}) > \varepsilon$; however, by continuity, there is an integer $N_1 > N_0$ such that $P_{n_i}(A_N) \leq \varepsilon$ for all $N \geq N_1$. Hence, for $N \geq N_1$ we have that

$$\sup_n P_n(A_N) \leq \delta,$$

which is equivalent with (22).

7.13. Theorem. *Let $\mathcal{P}$ be a relatively compact set of probability measures in H_w. Then it is relatively compact in H_s if and only if*

$$\lim_N \sup_{P\in\mathcal{P}} P\Big(\sum_{i=N}^{+\infty}\langle x, e_i\rangle^2 \geq \varepsilon\Big) = 0, \tag{30}$$

for any $\varepsilon > 0$.

Proof. By Theorem 7.11, (30) is a sufficient additional condition for relative compactness in H_s. Conversely, suppose that $\mathcal{P}$ is relatively compact in H_s but (30) does not hold. Then there is an $\varepsilon > 0$ and a $\lambda > 0$ such that for each n there is an $N \geq n$ and $P_n \in \mathcal{P}$ so that

$$P_n\Big(\sum_{i=N}^{+\infty}\langle x, e_i\rangle^2 \geq \varepsilon\Big) > \lambda. \tag{31}$$

By weak compactness in H_s, there is a weakly convergent subsequence of $\{P_n\}$, which together with (31) contradicts Theorem 7.12. Therefore, (30) holds.

7.14. Theorem (Prohorov's theorem in H_w). *For $r = 1, 2, \ldots$, let $B_r = \{x \in H \mid \|x\| \leq r\}$. A set of probability measures $\mathcal{P}$ is relatively compact on H_w if and only if for every $\varepsilon > 0$ there is an integer $r \geq 1$ such that $P(B_r') \leq \varepsilon$ for all $P \in \mathcal{P}$.*

Proof. A key point in the proof is the observation that H_w may be represented as the union of balls $B_r = \{x \in H \mid \|x\| \leq r\}$, $r = 1, 2, \ldots$, which are compact sets in the weak topology (Theorem 2.18) and the weak topology on B_r is metrizable. So, the proof of the "if" part goes in the same way as the proof of Theorem 6.5.

For the converse, it suffices to prove that if P_n is a weakly convergent sequence in H_w then for each $\varepsilon > 0$ there is a ball B_r such that $P_n(B_r') \leq \varepsilon$ for all n. Indeed, assuming that we proved such a claim, suppose that $\mathcal{P}$ is relatively compact and that there is an $\varepsilon > 0$ such that for each positive integer n there is a measure $P_n \in \mathcal{P}$ with $P(B_n') > \varepsilon$. Then there is a subsequence $P_{n'}$ which is weakly convergent in H_w to some probability measure P, and we have that $P_{n'}(B_{n'}') > \varepsilon$. Since $n' \to +\infty$, this is a contradiction with the assumed claim.

So, let $P_n \implies P$ in H_w, where P_n and P are probability measures. Then by continuity of P, for any $\varepsilon > 0$ there is a ball B_r such that $P(B_r) \geq 1 - \varepsilon/2$. Consider now the open sets (in fact, U-sets) in H_w:

$$G_{k,m}^{(r)} = \Big\{x \in H \;\Big|\; \sum_{i=1}^{k}\langle x, e_i\rangle^2 < r^2 + \frac{1}{m}\Big\}, \quad k, m = 1, 2, \ldots$$

Then it is easy to see that the sets $G_{k,m}^{(r)}$ are decreasing as k and m increase. Moreover,

$$B_r = \bigcap_{k=1}^{+\infty}\bigcap_{m=1}^{+\infty} G_{k,m}^{(r)}.$$

By Theorem 4.4, for each k and m we have

$$\underline{\lim} P_n(G^{(r)}_{k,m}) \geq P(G^{(r)}_{k,m}) \geq 1 - \frac{\varepsilon}{2},$$

hence there is an n_0 such that

$$P_n(G^{(r)}_{k,m}) \geq 1 - \varepsilon \quad \text{for all } n \geq n_0.$$

For a fixed n, letting here $k \to +\infty$ and $m \to +\infty$, we get

$$P_n(B_r) \geq 1 - \varepsilon \quad \text{for all } n \geq n_0.$$

Now, for each of measures P_i $(1 \leq i \leq n_0 - 1)$ there is a ball B_{r_i} such that $P_i(B_{r_i}) \geq 1 - \varepsilon$. Let $R = \max\{r, r_1, r_2, \dots, r_{n_0-1}\}$. Then $P_n(B_R) \geq 1 - \varepsilon$ for all $n \geq 1$, which was to be proved.

7.15. Example. Let P_n be unit masses at e_n, as in Example 7.9. We will show that P_n is relatively compact in H_w. Indeed, all P_n are concentrated in B_1, hence by Theorem 7.14, the sequence $\{P_n\}$ is relatively compact. Moreover, since $e_n \to 0$ in the weak topology (Example 2.3), then $f(e_n) \to f(0)$ for any $f \in C(H_w)$. Hence, $\int f(x)\, dP_n(x) = f(e_n) = \int f(x)\, dP_0(x)$, where P_0 is the unit mass at 0. So, $\{P_n\}$ is a weakly convergent sequence in H_w, $P_n \implies P_0$, which is also consistent with Example 7.9.

7.16. Relative compactness via characteristic functions. In the next two theorems, we give conditions for relative compactness in H_w and H_s in terms of characteristic functions. Recall that by Theorem 7.5, to each characteristic function φ and an $\varepsilon > 0$ there corresponds a Hilbertian seminorm p_ε such that (20) holds. Let $\mathcal{P} = \{P_\alpha\}$ be a set of probability, where α belongs to an index set A. A seminorm which corresponds to the characteristic function φ_α of a given P_α with an $\varepsilon > 0$ in the sense of (20), will be denoted by $p_{\alpha,\varepsilon}$. Let us note that $p_{\alpha,\varepsilon}$ are not uniquely determined. One natural choice is given by (21).

In the proofs of the next two theorems, a key role is played by the integration of a Hilbertian seminorm with respect to a finite dimensional Gaussian measure. Let p be a Hilbertian seminorm and $p(\cdot,\cdot)$ a corresponding inner product as in (14). Suppose $\mathcal{G}$ is an N-dimensional Gaussian measure which is concentrated on $\boldsymbol{R}^N$ spanned by $\{e_1, e_2, \dots, e_N\}$, as a product of N coordinate measures $\mathcal{N}(0, \sigma^2)$. Then

$$\begin{aligned}
\int p^2(x)\, d\mathcal{G}(x) &= \int_{\boldsymbol{R}^N} p\Big(\sum_{i=1}^N x_i e_i, \sum_{j=1}^N x_j e_j\Big)\, d\mathcal{G}(x) \\
&= \sum_{i=1}^N \sum_{j=1}^N \int_{\boldsymbol{R}^N} x_i x_j p(e_i, e_j)\, d\mathcal{G}(x) \\
&= \sum_{i=1}^N \int_{\boldsymbol{R}^N} x_i^2 p(e_i, e_i)\, d\mathcal{G}(x) \\
&= \sigma^2 \sum_{i=1}^N p^2(e_i). \qquad (32)
\end{aligned}$$

7.17. Theorem *A set of probability measures $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is relatively compact in H_w if and only if for every $\varepsilon > 0$ there is a set of seminorms $\{p_{\alpha,\varepsilon}\}_{\alpha\in A}$ such that*

$$\sup_{\alpha\in A} \sum_{i=1}^{+\infty} p^2_{\alpha,\varepsilon}(e_i) < +\infty. \tag{33}$$

Proof. By Theorem 7.14, we need to show that the condition (33) is equivalent to the condition that for any $\varepsilon > 0$ there is a ball B_r such that

$$P_\alpha(B_r') \le \varepsilon \qquad \text{for all } P_\alpha \in \mathcal{P}. \tag{34}$$

For a given $\varepsilon > 0$, assume that (34) holds for $\varepsilon/2$ in place of ε and with some B_r. Then, as in the proof of Theorem 7.5, we show that

$$1 - \operatorname{Re}\varphi_\alpha(x) \le p_{\alpha,\varepsilon} + \varepsilon,$$

where φ_α is the characteristic function of P_α and

$$\,^{2}_{\alpha,\varepsilon}(x) = \frac{1}{2}\int_{B_r} \langle x, y\rangle^2 \, dP_\alpha(y). \tag{35}$$

Then we have that

$$\sum_{i=1}^{+\infty} p^2_{\alpha,\varepsilon}(e_i) = \frac{1}{2}\int_{B_r} \|y\| \, dP_\alpha(y) \le r^2,$$

so (33) holds.

Conversely, fix an $\varepsilon > 0$ and assume that (33) holds for some family of seminorms $\{p_{\alpha,\varepsilon}\}$. Let

$$A_{r,N} = \Big\{ y \in H \;\Big|\; \sum_{i=1}^{N} \langle y, e_i\rangle^2 > r^2 \Big\}, \quad N = 1, 2, \ldots$$

Note that $A_{r,1} \subset A_{r,2} \subset \cdots$ and $\bigcup_{N=1}^{+\infty} A_{r,N} = B_r'$, so

$$\lim_{N\to+\infty} P_\alpha(A_{r,N}) = P_\alpha(B_r') \tag{36}$$

For an $y \in A_{r,N}$ we have that

$$1 - \exp\Big(-\frac{1}{2r^2}\sum_{i=1}^{N} \langle y, e_i\rangle^2\Big) > 1 - e^{-1/2} > \frac{1}{3}$$

and so, for every $P_\alpha \in \mathcal{P}$,

$$\frac{1}{3}P_\alpha(A_{r,N}) < \int_{A_{r,N}} \Big(1 - \exp\Big(-\frac{1}{2r^2}\sum_{i=1}^{N} y_i^2\Big)\Big)\, dP_\alpha(y)$$

$$(37) \qquad < 1 - \int_H \exp\Big(-\frac{1}{2r^2}\sum_{i=1}^{N} y_i^2\Big)\, dP_\alpha(y).$$

Let $\mathcal{G}$ be a Gaussian measure on $\boldsymbol{R}^N$, defined as the product of coordinate Gaussian measures $\mathcal{N}(0, 1/r^2)$. Its characteristic function is $y \mapsto \exp\Big(-\sum_{i=1}^{N} y_i^2/2r^2\Big)$ and we have

$$(38) \qquad \int_H \exp\Big(-\frac{1}{2r^2}\sum_{i=1}^{N} y_i^2\Big)\, dP_\alpha(y) = \int_{\boldsymbol{R}^N}\int_H \exp\Big(i\sum_{i=1}^{N} y_i x_i\Big)\, dP_\alpha(y)\, d\mathcal{G}(x).$$

Let

$$\varphi_{\alpha,N}(x) = \int_H \exp\Big(i\sum_{i=1}^{N} y_i x_i\Big)\, dP_\alpha(y).$$

For $x = \sum_{i=1}^{N} x_i e_i$ we have that $\varphi_{\alpha,N}(x) = \varphi_\alpha(x)$ and so $\operatorname{Re}\varphi_{\alpha,N}(x) \ge 1 - p^2_{\alpha,\varepsilon}(x) - \varepsilon$. Then from (38) we get

$$\int \exp\Big(-\frac{1}{2r^2}\sum_{i=1}^{N} y_i^2\Big)\, dP_\alpha(y) \ge 1 - \int_{\boldsymbol{R}^N} p^2_{\alpha,\varepsilon}(x)\, d\mathcal{G}(x) - \varepsilon$$

$$(39) \qquad = 1 - \frac{1}{r^2}\sum_{i=1}^{N} p^2_{\alpha,\varepsilon}(e_i) - \varepsilon.$$

From (37) and (39) we find that

$$P_\alpha(A_{r,N}) < \frac{3}{r^2}\sum_{i=1}^{N} p^2_{\alpha,\varepsilon}(e_i) + 3\varepsilon.$$

Now let $N \to +\infty$ and use (36) to get

$$P_\alpha(B'_r) \le \frac{3}{r^2}\sum_{i=1}^{+\infty} p^2_{\alpha,\varepsilon}(e_i) + 3\varepsilon,$$

which proves that (33) implies (34).

7.18. Theorem. *Suppose that $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is a relatively compact set of probability measures in H_w. Then the following conditions are equivalent:*

(i) For any $\varepsilon > 0$ there is a choice of $\{p_{\alpha,\varepsilon}\}_{\alpha\in A}$ so that

$$\lim_{N\to+\infty}\sup_{\alpha\in A}\sum_{i=N}^{+\infty} p^2_{\alpha,\varepsilon}(e_i) = 0,$$

(ii) For any $\varepsilon > 0$,

$$\lim_{N\to+\infty}\sup_{\alpha\in A} P_\alpha\Big(\sum_{i=N}^{+\infty}\langle x, e_i\rangle^2 > \varepsilon\Big) = 0.$$

Proof. Assume that (i) holds. From (20) it follows that

$$\operatorname{Re}\int \exp(i\langle x, y\rangle)\, dP_\alpha(x) \geq 1 - \varepsilon - p^2_{\alpha,\varepsilon}(y).$$

Put here $y = \sum_{j=N}^{S} a_j e_j$ and integrate with respect to the product of coordinate Gaussian $\mathcal{N}(0,1)$ measures to get

$$\int \exp\Big(-\frac{1}{2}\sum_{j=N}^{S}\langle x, e_j\rangle^2\Big)\, dP_\alpha(x) \geq 1 - \varepsilon - \sum_{j=N}^{S} p^2_{\alpha,\varepsilon}(e_j).$$

Now letting $S \to +\infty$ and using the monotone convergence theorem, we obtain

$$\int \exp\Big(-\frac{1}{2}\sum_{j=N}^{+\infty}\langle x, e_j\rangle^2\Big)\, dP_\alpha(x) \geq 1 - \varepsilon - \sum_{j=N}^{+\infty} p^2_{\alpha,\varepsilon}(e_j). \tag{40}$$

Introduce the notations:

$$\sum_{j=N}^{+\infty} p^2_{\alpha,\varepsilon}(e_j) = S_{\alpha,\varepsilon}(N), \quad \frac{1}{2}\sum_{j=N}^{S}\langle x, e_j\rangle^2 = X(N).$$

Then for any $\lambda > 0$, (40) yields:

$$\begin{aligned}
1 - \varepsilon - S_{\alpha,\varepsilon}(N) &\leq \int \exp(-X(N))\, dP_\alpha(x)\\
&= \int_{X(N)<\lambda} \exp(-X(N))\, dP_\alpha(x) + \int_{X(N)\geq\lambda} \exp(-X(N))\, dP_\alpha(x)\\
&\leq P_\alpha(X(N) < \lambda) + e^{-\lambda} P_\alpha(X(N) \geq \lambda)\\
&= 1 - (1 - e^{-\lambda}) P_\alpha(X(N) \geq \lambda),
\end{aligned}$$

wherefrom we get

$$\sup_\alpha P_\alpha(X(N) \geq \lambda) < \frac{\varepsilon + \sup_\alpha S_{\alpha,\varepsilon}(N)}{1 - e^{-\lambda}}.$$

Letting here $N \to +\infty$ and then $\varepsilon \to 0$ and assuming that (i) holds, we obtain (ii) (with 2λ in place of ε). Note that this part is independent of the assumption that $\mathcal{P}$ is relatively compact in H_w.

To prove the opposite direction, assume that $\mathcal{P}$ is relatively compact in H_w and that (ii) holds. Let $p_{\alpha,\varepsilon}$ be seminorms defined by (35). Then we have

$$2\sum_{i=N}^{+\infty} p_{\alpha,\varepsilon}^2(e_i) = \int_{B_r} \sum_{i=N}^{+\infty} \langle x, e_i\rangle^2 \, dP_\alpha(x) \leq \lambda + r^2 P_\alpha\Big(\sum_{i=N}^{+\infty} \langle x, e_i\rangle^2 \geq \lambda\Big).$$

Taking the supremum with respect to α, letting $N \to +\infty$ and $\lambda \to 0$, we obtain (i).

7.19. Theorem. *A set $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ of probability measures on H_s is relatively compact if and only if for every $\varepsilon > 0$ there is a set of seminorms $\{p_{\alpha,\varepsilon}\}_{\alpha\in A}$, related to P_α as in (20), such that the following two conditions hold:*

(i) For every $\varepsilon > 0$,

$$\sup_{\alpha\in A} \sum_{i=1}^{+\infty} p_{\alpha,\varepsilon}^2(e_i) < +\infty.$$

(ii) For every $\varepsilon > 0$,

$$\lim_{N\to+\infty} \sup_{\alpha\in A} \sum_{i=N}^{+\infty} p_{\alpha,\varepsilon}^2(e_i) = 0.$$

Proof. Directly from theorems 7.13, 7.17 and 7.18.

7.20. Remarks. Let us remark that if the above conditions on seminorms hold for one choice of the family $p_{\alpha,\varepsilon}$, they need not hold for some other choice. For instance, suppose that $\alpha = 1, 2, \ldots$ and let $p_{n,\varepsilon}$ be a family of seminorms related to characteristic functions via (20) and satisfying (33). Then the family of seminorms $q_{n,\varepsilon}$ defined by $q_{n,\varepsilon}^2(x) = np_{n,\varepsilon}^2(x)/\sum_{i=1}^{+\infty} p_{n,\varepsilon}^2(e_i)$ also satisfies (20), but not (33).

Theorem 7.19 is proved in [**27**] by different means. The analysis of a relationship between weak convergence on H_w and H_s is adopted from [**23**]. Separate conditions in H_w may be useful since they are easier to check.

Acknowledgements. My understanding of the material presented in Section 5 is much improved after communications with Sheldon Axler [**3**] and several mathematicians of the Internet based Real Analysis group [**14**]. The presentation in Section 2 is influenced by Sheldon Axler's brilliant lectures of Functional Analysis at Michigan State University [**2**].

References

[1] A.D. Alexandroff, *Additive set functions in abstract spaces I–III*, Mat. Sb. **8(50)** (1940), 307–348,

[2] S. Axler, *Functional analysis*, course notes (1983–1984), Michigan State University, unpublished.

[3] S. Axler, private communication, 1997

[4] R.G. Bartle, N. Dunford, J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305.

[5] C. Berg, J. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Springer-Verlag, New York, 1984.

[6] P. Billingsley, *Convergence of Probability Measures*, Wiley, 1968.

[7] P. Billingsley, *Weak convergence of measures: Applications in Probability*, CBMS-NSF regional conference series in Applied Mathematics, SIAM, Philadelphia, 1971.

[8] G. Birkoff, *Moore-Smith convergence in general topology*, Ann. Math. **38** (1937), 39–56.

[9] N. Bourbaki, *Topologie Généérale*, Hermann, Paris; Russian translation, Nauka, Moskva, 1968.

[10] H. Cartan, *Théorie des Filtres*, C.R. Acad. Sci. Paris **205** (1937), 595–598.

[11] H. Cartan, *Filtres et Ultrafiltres*, C.R. Acad. Sci. Paris **205** (1937), 777–779.

[12] C.C. Chang, H.J. Keisler, *Model Theory*, North-Holland, 1973.

[13] N. Dunford, J.T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York, London, 1958; Russian translation: Moskva, 1962.

[14] Gerald A. Edgar, John Hagood, David Ross, Eric Schechter, Gord Sinnamon, Contributions to Real-Analysis Discussion Group, December 1998.

[15] R. Engelking, *General Topology*, second edition, PWN, Warszaw, 1985.

[16] H. Heyer, *Probability measures on locally compact groups*, Springer-Verlag, 1977.

[17] K. Ito, *Regularization of linear random functionals*, Probability Theory and Mathematical Statistics (Tbilisi), Proceedings, 257–267, Lecture Notes in Mathematics, Vol. 1021, Springer-Verlag, New York–Berlin (1982).

[18] J.L. Kelley, *Convergence in Topology*, Duke Math. J. **17** (1950), 277–283.

[19] A.N. Kolmogorov, *Foundations of the Theory of Probability*, Second English Edition, Chelsea, New York, 1956.

[20] M. Merkle, *Towards a linear limes*, unpublished manuscript, 1997.

[21] M. Merkle, *On positive definite functions defined on vector spaces*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat **1** (1990), 35–40.

[22] M. Merkle, *Multi-Hilbertian spaces and their duals*, Techn. Report No 291, Center for Stochastic Processes, Department of Statistics, University of North Carolina at Chapel Hill, 1990.

[23] M. Merkle, *On weak convergence of measures on Hilbert spaces*, J. Multivariate Analysis **29**:2 (1989), 252–259.

[24] E.H. Moore, *Definition of limit in general integral analysis*, Proc. Nat. Acad. Sci. **1** (1915), 628.

[25] E.H. Moore, E. H., *General Analysis I, part II*, Mem. Amer. Philos. Soc., Philadelphia, 1939.

[26] E.H. Moore, H.L. Smith, *A general theory of limits*, Amer. J. Math. **44** (1922), 102–121.

[27] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, 1967.

[28] Yu.V. Prohorov, *Shodimost slučainih processov i predelynie teoremi teorii veroyatnostei*, Teor. ver. primenen. **1** (1956), 177–238.

[29] V. Sazonov, *A remark on characteristic functionals*, Theory Probab. Appl. **3**:2 (1958), 188–192.

[30] D. Stroock, *Probability Theory, an Analytic View*, Cambridge University Press, 1993.

[31] G.P. Tolstov, *Measure and Integral*, Nauka, Moscow, 1976. (in Russian)

[32] T. Tjur, *Probability Based on Radon Measures*, Wiley, 1980.

[33] J.W. Tukey, *Convergence and Uniformity in Topology*, Ann. of Math. Studies 2, Princeton University Press, 1940.

[34] V.S. Varadarajan, *Meri na topologičeskih prostranstvah*, Matem. sb. **55(97)** (1961), 35–100.

[35] S. Willard, S., *General Topology*, Addison-Wesley, 1970

[36] K. Yosida, *Functional Analysis*, Academic Press, New York 1965.