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PREFACE

These Proceedings contain some of the papers which were submitted or presented at the Symposium "Set Theory. Foundations of Mathematics" held in Belgrade from August 29th to September 2nd 1977, on the occasion of the 70th anniversary of Professor Djuro Kurepa. The Institute of Mathematics in Belgrade organized this Symposium.

Because of the heterogeneity of the submitted reports, the Proceedings are arranged in the alphabetical order of the authors and not according to the subjects. At the end of the Proceedings a panel discussion, held during the Symposium, is attached. All reports are devoted to the work and results of Professor Djuro Kurepa on the occasion of his 70th anniversary.

Professor Kurepa has about 180 published papers in various fields: set theory, algebra, topology, analysis, etc. therefore, we think that only a special publication of his collected papers would give an insight of his fertile scientific work.

The Institute of Mathematics in Belgrade has for the first time decided to organize a symposium in the mentioned fields. After the first result, we hope that such symposiums will become a continuel practice and that they will be held every fourth year either in Belgrade or in some other place in Yugoslavia. We also believe that we could in the future organize them in close cooperation with the Association of Symbolic Logic.

At the end, we would like to use this opportunity to thank, before all, the members of the Scientific Committee for their support and suggestion, as well as the sponsors of the Symposium i.e. The International Union of Mathematician, The Rebublican Association for Science of F.R. of Serbia and The Union of Regional Associations for financial aid.

We also express our thankfulness to all participants of the Symposium, particularly to those who gave their papers for inclusion in these Proceedings.

Secretary
of the Org. Committee
T. Andjelić

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DESCRIPTIVE SET THEORY AND INFINITARY LANGUAGES

John P. BURGESS

Kurepa trees, partitions, Jensen's principles, large cardinals, and other notions from combinatorial set theory play an enormous role in the model theory of generalized-quantifier languages. (See e.g. [29].) Borel and analytic sets, Polish group actions, and notions from descriptive set theory can play almost as large a role in the model theory of certain infinitary languages. (See [31] and [32].) The present paper is a study, by the methods of descriptive set theory, of the class of *strong first-order languages*. These, roughly, are the infinitary languages which are strong enough to express wellfoundedness, at least over countable structures, yet weak enough that the satisfaction relation is Δ_1-definable.

Examples, culled from the literature of exotic model theory, are present in § 1. The set-theoretic machinery for their study is set up in §§ 2-4. §§ 5 and 6 are devoted to an exposition of the properties shared by all strong first-order languages. Most notably: *There is a quasi-constructive complete proof procedure involving rules with* $\aleph_1$ *premisses for any strong first-order language*, and *even the weak version of Beth's Definability Theorem fails for every such language.*

Many of the results in this paper date from the author's days as a student in R.L. Vaught's seminar at Berkeley, 1972-73. At that time I had the benefit of correspondence with Profs. Barwise and Moschovakis, and especially of frequent discussions with Prof. Vaught and D. E. Miller. Most of this work was included in [6], and a few items have appeared in print ([5]; [8], § 2). More recent discussions with Miller led to the discovery of the proof procedure and the counterexample to Beth's Theorem alluded to above, and to the writing of this paper.

§ 1 Some Infinitary Languages

Throughout this paper *structure* means *infinite structure* and *vocabulary* (set of predicates, function symbols, and constants) means *countable vocabulary*. References for some possibly unfamiliar notions such as primitive recursive (*PR*) set functions or Δ_1^{ZFC} definability are recalled at the beginning of § 2.

1.1 *Borel-Game Logic* $L_{\infty B}$

We introduce *codes* for Borel subsets of the power set ω as follows: $\mathcal{E}(0)=\{(0, n): n\in\omega\}$; $\mathcal{E}(\alpha+1)=\mathcal{E}(\alpha)\cup\{(1, e): e\in\mathcal{E}(\alpha)\}$ for α even; $\mathcal{E}(\alpha+1)=\mathcal{E}(a)\cup\{(2, f): f:\omega\rightarrow\mathcal{E}(\alpha)\}$ for α odd; $\mathcal{E}(\lambda)=\cup\{\mathcal{E}(\alpha):\alpha<\lambda\}$ at limits; $\mathcal{E}=\mathcal{E}(\omega_1)$. The Borel set $\mathcal{B}(e)$ coded by $e\in\mathcal{E}$ is determined as follows: $\mathcal{B}((0, n))=\{u\subseteq\omega: n\in u\}$; $\mathcal{B}((1, e))=$ complement of $\mathcal{B}(e)$; $\mathcal{B}((2, f))=\cup\{\mathcal{B}(f(n)): n\in\omega\}$.

The class of formulas of $L_{\infty B}$ in a vocabulary **R** is the smallest class which (i) contains the atomic formulas of **R**; (ii) is closed under negation $\neg$; (iii) is closed under (single) quantification $\forall$, $\exists$; (iv) is closed under conjunction and disjunction $\bigwedge$, $\bigvee$, of arbitrary sets of formulas, so long as the result has only finitely many free variables; and (v) is closed under the following operation: Given $e \in \mathfrak{E}$ and $I \neq \varnothing$ and formulas $\varphi_{i_0 i_1 \ldots i_n}(u_1 \ldots u_k, v_0 \ldots v_n)$ indexed by $I < \omega$ with free variables as shown, we may form the following formula $\varphi(u_1 \ldots u_k)$:

$$(^*B) \qquad \bigwedge_{i_0 \in I} \forall_{v_0} \bigvee_{i_1 \in I} \exists_{v_1} \bigwedge_{i_2 \in I} \forall_{v_2} \bigvee_{i_3 \in I} \exists_{v_3} \ldots$$
$$\ldots \{n : \varphi_{i_0 i_1 \ldots i_n}(u_1 \ldots u_k, v_0 \ldots v_n)\} \in \mathscr{B}(e)$$

The class $L_{\infty B}(\mathbf{R})$ of sentences of $L_{\infty B}$ in vocabulary **R** consists of those formulas without free variables.

Satisfaction for $L_{\infty B}$ is defined as follows: Given an **R**-structure $\mathfrak{A}$ and $b_1 \ldots b_k \in |\mathfrak{A}|$, the formula $\varphi(u_1 \ldots u_k)$ of $(^*B)$ suggests an infinite game for two players PRO and CON. CON opens by picking $i_0 \in I$, $a_0 \in |\mathfrak{A}|$. PRO responds with $i_1 \in I$, $a_1 \in |\mathfrak{A}|$. And so on until infinite sequences $\mathbf{i} = i_0, i_1, i_2 \ldots$ and $\tilde{a} = a_0, a_1, a_1 \ldots$ are generated. PRO wins if $\{n : \mathfrak{A} \models \varphi_{i_0 i_1 \ldots i_n}(b_1 \ldots b_k, a_0 \ldots a_n)\} \in \mathscr{B}(e)$. Since the set of pairs **i**, **a** constituting wins for PRO is a Borel subset of $I^\omega \times |\mathfrak{A}|^\omega$, by Martin's Borel Determinacy Theorem [22], either PRO or else CON has a winning strategy for this game. We define $\mathfrak{A} \models \varphi(b_1 \ldots b_k)$ to hold if PRO has the winning strategy.

If we wish to identify formulas with set-theoretic objects, we can proceed much as is done in [17] for $L_{\omega_1\omega}$. In particular we take nonlogical symbols to be just certain hereditarily countable sets. We can identify the formula of $(^*B)$ with, say, $(e, (\varphi_\sigma : \sigma \in I^{<\omega}))$. It is then not hard to see that sentencehood for $L_{\infty B}$ is a PR notion.

1.2. PROPOSITION Satisfaction for $L_{\infty B}$ is Δ_1^{ZFC}.

PROOF. Any notion defined by a reasonable induction from Δ_1^{ZFC} notions is Δ_1^{ZFC}, so it suffices to show satisfaction for a formula of $L_{\infty B}$ can be defined in a Δ_1^{ZFC} fashion in terms of satisfaction for its subformulas. We consider the case of the formula φ introduced by $(^*B)$. Fix $\mathfrak{A}$ and $b_1 \ldots b_k \in |\mathfrak{A}|$ as in the definition above of satisfaction for φ. Let $f: I^{<\omega} \times |\mathfrak{A}|^{<\omega} \to \{0, 1\}$ code satisfaction for subformulas of φ:

$$f(\sigma, (a_0 \ldots a_n)) = 0 \leftrightarrow \mathfrak{A} \models \varphi_\sigma(b \ldots b_k, a_0 \ldots a_n)$$

A strategy for PRO in the game associated with φ is essentially a pair of functions $\mathscr{S} : I^{<\omega} \times |\mathfrak{A}|^{<\omega} \to I$, $\mathscr{T} : I^{<\omega} \times |\mathfrak{A}|^{<\omega} \to |\mathfrak{A}|$. Applied to sequences $\mathbf{i} = i_0, i_1, i_2, \ldots$ and $\mathbf{a} = a_0, a_1, a_2, \ldots$ $\mathscr{S}$ and $\mathscr{T}$ produce the sequences:

$$(1) \qquad \mathbf{s} = i_0, \mathscr{S}(((i_0), (a_0))), i_1, \mathscr{S}(((i_0, i_1), (a_0, a_1))), i_2, \ldots$$
$$\mathbf{t} = a_0, \mathscr{T}(((i_0), (a_0))), a_1, \mathscr{T}(((i_0, i_1), (a_0, a_1))), a_2, \ldots$$

Let $\sigma_n = \sigma_n(\mathscr{S}, \mathscr{T}, \mathbf{i}, \mathbf{a})$ be the finite sequence of the 0^{th} through n^{th} terms of **s**, and define $\mathscr{T}_n$ similarly from **t**. In this notation, $\mathfrak{A} \models \varphi(b_1 \ldots b_k)$ iff:

$$(2) \qquad \exists \text{ strategies } \mathscr{S}, \mathscr{T}\ \forall\, \mathbf{i} \in I^\omega, \tilde{a} \in |\mathfrak{A}|^\omega\ \{n : f(\sigma_n, \tau_n) = 0\} \in \mathscr{B}(e)$$

Now it is well known that every Borel subset of the power set of ω can be obtained from clopen sets by the fusion operation (1). Indeed the usual proofs of this fact reveal that we can obtain an operation ($\mathcal{A}$) representation of $\mathcal{B}(e)$ in a PR fashion from the code e, i.e. there is a PR function $\mathcal{N}$ from $\mathcal{E}$ to the power set of $2^{<\omega}\times\omega^{<\omega}$ such that for all $x\in 2^{\omega}$:

$$\{n: x(n)=0\}\in\mathcal{B}(e)\leftrightarrow\exists y\in\omega^{\omega}\,\forall n\,(x|n, y|n)\in\mathcal{N}(e)$$

Thus (2) is equivalent to:

(3) $$\exists\mathcal{S},\mathcal{T}\,\forall\mathbf{i},\mathbf{a}\,\forall x\in 2^{\omega}\,\forall y\in\omega^{\omega}\,\exists n$$

$$(x(n)\neq f(\sigma_n,\tau_n))\vee(x|n, y|n)\notin\mathcal{N}((1,e))$$

where, let us recall, $(1, e)$ codes the complement of $\mathcal{B}(e)$.

Now for given strategies $\mathcal{S},\mathcal{T}$ let $\mathcal{Q}=\mathcal{Q}(\mathcal{S},\mathcal{T})$ be the set of all four-tuples $\mathbf{i}_n=(i_0, i_1\ldots i_n)$, $\mathbf{a}_n=(a_0, a_1\ldots a_n)$, $\xi=(x_0, x_1\ldots x_{2n+1})$, $\eta=(y_0, y_1\ldots y_{2n+1})$ such that for all $m\leq 2n+1$, $x_m=f(\sigma_n,\tau_n)$ (where σ_n,τ_n are the obvious initial segments of the sequences in (1)) and $(\xi|m+1, \eta|m+1)\in\mathcal{N}((1,e))$. Partially order $\mathcal{Q}$ by letting one four-tuple p be below another q if every component of p extends the corresponding component of q. Then (3) is equivalent to:

(4) (a) $\exists\mathcal{S},\mathcal{T}$ ($\mathcal{Q}$ is wellfounded)

Moreover, the existence of a winning strategy for PRO is equivalent to the nonexistence of a winning strategy for CON, so (a) is equivalent to:

(4) (b) $\neg\exists\mathcal{S},\mathcal{T}$ ($\mathcal{Q}'$ is wellfounded)

where $\mathcal{Q}'$ is defined dually to $\mathcal{Q}$. Examination of the construction shows $\mathcal{Q}$, $\mathcal{Q}'$ are obtained in a PR fashion from $\mathcal{S},\mathcal{T}$ and the data e, f. Every PR function is Δ_1^{ZFC}, as is the notion of wellfoundedness. Further Martin's Borel Determinacy Theorem, which implies the equivalence of (4) (a) and (b) is provable in *ZFC*. It follows (4) provides a Δ_1^{ZFC} definition of satisfaction for φ in terms of satisfaction for its subformulas φ_σ, as required.

1.3 $L_{\varkappa B}$

For any uncountable cardinal $\varkappa$, the formulas of $L_{\varkappa B}$ are those formulas of $L_{\varkappa B}$ which, as set-theoretic objects, are of hereditary cardinality $<\varkappa$; briefly: $L_{\varkappa B}=L_{\infty B}\cap H(\varkappa)$. Up to a harmless relabelling, these are precisely the formulas with $<\varkappa$ subformulas; and for regular $\varkappa$ constitute the smallest class closed under $\neg,\forall,\exists$; under $\wedge,\vee$ for sets of $<\varkappa$ formulas; and under operation (*B) for index sets I of cardinality $<\varkappa$.

1.4 *Vaught's Closed-Game Logic* $L_{\infty G}$

Let $e\in\mathcal{E}$ be a code for $\{\omega\}$. For this e (*B) can be written more perspicuously:

($_*G$) $$\bigwedge_{i_0\in I}\forall v_0\bigvee_{i_1\in I}\exists v_1\ldots\bigwedge_n\varphi_{i_0\ldots i_n}(u_1\ldots u_k, v_0\ldots v_n)$$

The sublanguage of $L_{\infty B}$ obtained by allowing *only* this special case of ($_*B$) we call $L_{\infty G}$. We also set $L_{\varkappa G}=L_{\infty G}\cap H(\varkappa)$. Vaught [31], [32] has extensively investigated $L_{\omega_1 G}$, and formulas of form $(*G)$ with I countable and the φ_σ quantifier-free formulas of $L_{\omega\omega}$ are called *Vaught formulas.* The game associated with $(*G)$ is closed, and since the determinateness of such games can be proved in ZFC^-, satisfaction for $L_{\infty G}$ is Δ_1^{ZFC-}.

Other fragments of $L_{\infty B}$ can be obtained by restricting the matrix of $(*B)$ to other special forms, e.g. the G_δ-game logic of [6], ch.4C.

1.5 *On Keisler's* $L(\omega)$ *and Related Languages*

We form $L_{\infty QB}$ by restricting the game prefix in $(*B)$ to allow only quantifiers: Given $e \in \mathscr{E}$ and φ_n, $n \in \omega$, we form:

$$(*QB) \qquad \forall v_0 \exists v_1 \forall v_2 \exists v_3 \ldots \{n\colon \varphi_n(u_1 \ldots u_k, v_0 \ldots v_n)\} \in \mathscr{B}(e)$$

which can be regarded as a formula of $L_{\infty B}$ by inserting vacuous propositional operations.

$L_{\omega_1 QB} = L_{\infty QB} \cap HC$ coincides with the restriction to HC of the language Keisler [16] calls $L(\omega)$. This observation justifies our assertion in [5] that satisfaction for $L(\omega) \cap HC$ is Δ_1^{ZFC}.

$L_{\infty QG}$ is obtained by similarly restricting $L_{\infty G}$. Moschovakis and Barwise [2] have studied this language, which (unfortunately) is sometimes called $L_{\infty G}$.

Though obviously (considering propositional logic) $L_{\omega_1 QG} = L_{\infty QG} \cap HC$ is weaker than $L_{\omega_1 G}$, Vaught [32] remarks that *over countable models with some coding built-in* (e.g. models of arithmetic) the expressive power of the two languages coincides.

1.6. *Propositional Game Logic*

We form $L_{\infty PB}$ by restricting the game prefix in (*B) to allow only propositional operations. Thus given $e \in \mathscr{E}$ and $I \neq \varnothing$ and formulas $\varphi_\sigma(u_1 \ldots u_k)$ all in the same free variables, we form:

$$(*PB) \qquad \bigwedge_{i_0 \in I} \bigvee_{i_1 \in I} \bigwedge_{i_2 \in I} \bigvee_{i_3 \in I} \ldots \{n\colon \varphi_{i_0 i_1 \ldots i_n}(u_1 \ldots u_k)\} \in \mathscr{B}(e)$$

This is equivalent to a formula of $L_{\infty\omega}$, viz:

$$(1) \qquad \bigvee_{\mathscr{S}: I^{<\omega} \to I} \bigwedge_{i = i_0, i_1, i_2 \ldots} \bigvee_{y \in \omega^\omega} \bigwedge_{n \in \omega} \bigvee_{\xi \in 2^{n+1}, (\xi, y | n+1) \in \mathscr{N}(e)} \left(\bigwedge_{m \leq n, \xi(m) = 0} \varphi_{\sigma_m} \wedge \bigwedge_{m \leq n, \xi(m) = 1} \neg \varphi_{\sigma_m} \right)$$

where $\mathscr{N}(e)$ is as in § 1.2 and σ_n is the obvious initial segment of:

$$i_0, \mathscr{S}((i_0)), i_1, \mathscr{S}((i_0, i_1)), i_2, \ldots$$

In particular, wellfoundedness cannot be expressed in $L_{\infty PB}$. $L_{\omega_1 PB} = L_{\infty PB} \cap HC$, however, still vastly exceeds $L_{\omega_1\omega}$ in expressive power, since if the formula in $(*PB)$ is in $L_{\omega_1 PB}$, we can only say the equivalent formula (1) is in $L_{\lambda\omega}$ where $\lambda = (2^{\aleph_0})^+$.

$L_{\infty PG}$ and $L_{\omega_1 PG}$ (defined the obvious way) have been studied by Green [10], [11].

1.7 *Solitaire and Souslin-Quantifiers*

We form $L_{\infty SB}$ (resp. $L_{\infty SG}$) by restricting the game prefix (*B) (resp. (*G)) to allow only $\exists$ and $\bigvee$. Formulas of these languages correspond to games in which PRO makes all the moves and CON is a passive spectator. $L_{\infty SB}$ and $L_{\infty SG}$ coincide in expressive power. Indeed we can assign in a PR fashion to every formula of the former an equivalent formula of the latter.
For

(*SB) $\quad \bigvee_{i_0\in I}\exists v_0 \bigvee_{i_1\in I}\exists v_1 \bigvee_{i_2\in I}\exists v_2 \ldots \{n: \varphi_{i_0 i_1\ldots i_n}(u_1\ldots u_k, v_0\ldots v_n)\}\in\mathscr{B}(e)$

is equivalent to:

(1) $$\bigvee_{i_0\in I}\exists v_0 \bigvee_{i_1\in I}\exists v_1 \ldots \bigvee_{y\in\omega^\omega} \bigwedge_{n\in\omega} \bigvee_{\xi\in 2^{n+1},\ (\xi,\, y\restriction n+1)\in\mathscr{N}(e)}$$
$$(\bigvee_{m\leq n,\ \xi(m)=0}\varphi_{i_0\ldots i_m}\wedge \bigwedge_{m\leq n,\ \xi(m)=1}\neg\varphi_{i_0\ldots i_m})$$

and hence to:

(2) $$\bigvee_{i_0\in I}\bigvee_{y_0\in\omega}\exists v_0 \bigvee_{i_1\in I}\bigvee_{y_1\in\omega}\exists v_1 \ldots \bigwedge_{n\in\omega}\bigvee_{\xi\in 2^{n+1},\ (\xi,\,(y_0\ldots y_n))\in\mathscr{N}(e)}$$

etc as in (1).

Distributing $\exists$ through $\bigvee$ and *vice versa,* we also see that any formula of $L_{\infty SG}$ is equivalent to a formula of $L_{\infty\omega_1}$. Malitz has shown that the class of wellorderings of type $\alpha+\alpha$ cannot be defined in $L_{\infty\infty}$, while Takeuti has observed that it is definable in $L_{\omega_1 QG}$.

Further restricting (*SG) to allow only $\exists$ produces $L_{\infty QSG}$. $L_{\omega_1 QSG}=$ $=L_{\infty QSG}\cap HC$ has been studied by Moschovakis and others under the name Souslin-Quantifier Logic. Note that the usual formula expressing wellfoundedness still belongs to this language.

1.8 *Souslin Logic*

Restricting (*SG) to allow only $\bigvee$ produces $L_{\infty PSG}$. Explicitly this language allows:

(*PSG) $$\bigvee_{i_0\in I}\bigvee_{i_1\in I}\bigvee_{i_2\in I}\ldots\bigwedge_{n\in\omega}\varphi_{i_0 i_1\ldots i_n}$$

$L_{\varkappa+PSG}=L_{\infty PSG}\cap H(\varkappa^+)$ has been called $\varkappa$-Souslin Logic, or just Souslin Logic for $\varkappa=\aleph_0$, and has been extensively investigated [9], [10], [11].

Of course (cf. § 1.6) $L_{\infty PSG}$ does not exceed $L_{\infty\omega}$ in expressive power; but Souslin logic vastly exceeds $L_{\omega_1\omega}$. For example, the class of countable wellfounded structures is a PC for Souslin logic, since a countable $\mathfrak{A}=(|\mathfrak{A}|, E^{\mathfrak{A}})$ is wellfounded iff it can be expanded to a model $(|\mathfrak{A}|, E^{\mathfrak{A}}, R^{\mathfrak{A}})$ of:

R linearly orders the universe in order type $\omega\ \wedge$

$$\neg\bigvee_{i_0\in\omega}\bigvee_{i_1\in\omega}\bigvee_{i_2\in\omega}\ldots\bigwedge_n\varphi_n$$

where φ_n expresses that the $i_{n+1}{}^{st}$ element in the R-order stands in the relation E to the $i_n{}^{th}$ element. This means that the wellordering number of Souslin logic is $>\omega_1$, the wellordering number of $L_{\omega_1\omega}$. In fact, it may be as large as ω_2;

see [7]. It is perhaps worth noting (following Vaught) that Souslin logic and $L_{\omega_1 PG}$ coincide in expressive power. For

$$\bigwedge_{i_0\in\omega} \bigvee_{i_1\in\omega} \bigwedge_{i_2\in\omega} \bigvee_{i_3\in\omega} \dots \bigwedge_n \varphi_{i_0 i_1 \dots i_n}$$

is equivalent to

$$\bigvee_{j_0\in\omega} \bigvee_{j_1\in\omega} \bigvee_{j_2\in\omega} \dots \bigwedge_n \psi_{j_0 j_1 \dots j_n}$$

where the ψ_τ are determined as follows: For $\sigma\in\omega^{<\omega}$ let $\#(\sigma)$ be the natural code for σ, $2^{\sigma(0)}\,3^{\sigma(1)}\,5^{\sigma(2)}\dots$ For $\tau=(j_0, j_1, \dots j_n)$ let ψ_τ be the conjunction for all $\sigma=(i_0, i_1, \dots i_m)$ with $\#(\sigma)\leq n$ of $\varphi_{i_0}, \varphi_{i_0 k_0}, \varphi_{i_0 k_0 i_1}, \varphi_{i_0 k_0 i_1 k_1}, \dots$, where $k_0=j_{\#((i_0))}$, $k_1=j_{\#((i_0 i_1))}$, $\dots$.

Green [11] shows that for all $\varkappa$ the wellordering numbers of $\varkappa$-Souslin logic and $L_{\varkappa^+ PG}$ coincide and equal the least ordinal not $H(\varkappa^*)$ recursive in the sense of [4]. Moreover she shows for cf $\varkappa>\omega$, $\varkappa$-Souslin logic and $L_{\varkappa^+\omega}$ coincide in expressive power.

1.9 *Kolmogorov R-Operation Logic*

The formation rules of $L_{\infty R}$ allow us, given formulas indexed by $(I^{<\omega})^{<\omega}$ to form the following horror:

$$(*R)\qquad \begin{array}{l} \bigwedge_{i_{00}} \forall v_{00} \bigvee_{i_{01}} \exists v_{01} \bigwedge_{i_{02}} \forall v_{02} \bigvee_{i_{03}} \exists v_{03} \dots \\ \bigvee_{n_0\in\omega} \bigvee_{i_{10}} \exists v_{10} \bigwedge_{i_{11}} \forall v_{11} \bigvee_{i_{12}} \exists v_{12} \bigwedge_{i_{13}} \forall v_{13} \dots \\ \bigwedge_{n_1\in\omega} \bigwedge_{i_{20}} \forall v_{20} \bigvee_{i_{21}} \exists v_{21} \bigwedge_{i_{22}} \forall v_{22} \bigvee_{i_{23}} \exists v_{23} \dots \\ \dots \bigwedge_r \varphi_{(i_{00}\dots i_{0n_0})\dots(i_{r0}\dots i_{rn_r})} (u_1 \dots u_k,\ v_{00} \dots v_{rn_r}) \end{array}$$

For fixed $\mathfrak{A}$ and $b_1 \dots b_k \in |\mathfrak{A}|$ the obvious game of length ω^2 associated with $(*R)$ is equivalent to the following game of length ω, in the sense that the same player has a winning strategy: CON picks elements of I and $|\mathfrak{A}|$ which we call i_{00} and a_{00}. PRO then has three options: to *challenge* immediately, to pick elements we call i_{01}, a_{01} and then challenge, or to pick such elements without challenging. If PRO does not challenge, CON then picks elements we call i_{02}, a_{02}, and PRO then again has the same three options. If PRO eventually does challenge just after i_{0n_0}, a_{0n_0} have been picked, PRO then also picks elements we call i_{10}, a_{10}. CON then has three options analogous to those PRO had earlier. If CON does not challenge, PRO picks another pair of elements, and CON has the same three options, and so on. If CON eventually challenges after i_{1n_1}, a_{1n_1} have been picked, he also picks elements we call i_{20}, a_{20}, and PRO has three options again, and so on. In the end, PRO wins if either each player challenges infinitely often and the matrix of $(*R)$ comes out true with the a's replacing the v's and the b's the u's, or if at some point it is PRO's option to challenge and he lets infinitely many moves go by without doing so. We leave it to the reader to see that this game really is equivalent to that suggested by $(*R)$. Note that the set of sequences $i\in I^\omega$, $a\in|\mathfrak{A}|^\omega$, which constitute a win for PRO is a Borel (in fact, G_δ) set. This means we can associate to each formula of $L_{\infty R}$, in a PR fashion, an equivalent formula of $L_{\infty B}$, and former language can be regarded as a sublanguage of the latter in a generalized sense.

$L_{\omega_1 R}=L_{\infty R}\cap HC$ was mentioned under the name L^2 in [8], § 2. The languages L^ν, $\nu>\omega_1$, mentioned there are all sublanguages of $L_{\infty B}$ in the same sense that $L_{\infty R}$ is.

§ 2. Some Definability Theory

For any vocabulary **R**, let $\mathfrak{X}(\mathbf{R})$ be the set of all **R**-structures with universe ω. $K \subseteq \mathfrak{X}(\mathbf{R})$ is *invariant* if for all $\mathfrak{A} \in \mathfrak{X}\mathbf{R})$, $\mathfrak{A} \simeq \mathfrak{B} \in K$ implies $\mathfrak{A} \in K$. We will be concerned with four classifications of invariant subsets of $\mathfrak{X}(\mathbf{R})$.

2.1 *Recursion Theory*

Let $X(\mathbf{R})$ be the product of one copy of 2^{ω^n} for each n-ary predicate in **R**, one copy of ω^{ω^n} for each n-ary function symbol, and one copy of for each constant. Any $x \in X(\mathbf{R})$ corresponds in an obvious way to an $\mathfrak{A}_x \in \mathfrak{X}(\mathbf{R})$. E.g. if **R** has just one binary predicate, $x \in 2^{\omega \times \omega}$ corresponds to the structure consisting of universe ω equipped with the binary relation whose characteristic function x is. $K \subseteq X(\mathbf{R})$ is called *invariant* if the corresponding subset of $\mathfrak{X}(\mathbf{R})$ is. This amounts to invariance under a natural action of the group $\omega!$ of permutations of ω on $X(\mathbf{R})$; see [32].

At least for finite **R**, we can classify subsets of $X(\mathbf{R})$ as Σ^0_n, Π^0_n, Δ^0_n, ari-arithmetical, HYP, Σ^1_n, Π^1_n, Δ^1_n, analytical, etc. according to their definability by various types of formulas of second-order arithmetic. For the elements of this theory see [27], ch. 14—16. If we allow parameters to appear in the definitions we obtain the boldface notions $\underset{\sim}{\Sigma}^0_n$, etc. By tedious but routine coding, these boldface notions can be applied even to infinite **R**. We call a subset of $\mathfrak{X}(\mathbf{R})$ $\underset{\sim}{\Sigma}^0_n$, etc., if the corresponding subset of $X(\mathbf{R})$ is.

2.2 *Topology*

Give $2=\{0,1\}$ and ω the discrete topologies. Give each 2^I, ω^I the product topology (making them homeomorphs of the Cantor and of the irrationals, respectively). Give each $X(\mathbf{R})$ the product topology. Finally give $\mathfrak{X}(\mathbf{R})$ the topology that makes $x \to \mathfrak{A}_x$ a homeomorphism. Then each of these spaces is Polish (separable, admitting a complete metric). We may classify subsets as open, closed, F_σ, G_δ, Borel, analytic, co-analytic (CA), PCA, projective, etc. For the elements of this theory see [19].

2.3 *Set Theory*

We assume familiarity with the Levy hierarchy of formulas of the language of set theory. The appendix to [2] contains a useful summary of the needed material. A class K is $\Sigma_n(V)$ (resp. $\underset{\sim}{\Sigma}_n(V)$) if it is definable over the universe V of set theory by a Σ_n formula without parameters (resp. with parameters). $\Pi_n(V)$ is defined similarly; and K is $\Delta_n(V)$ if both $\Sigma_n(V)$ and $\Pi_n(V)$. The boldface notions are defined similarly. K is $\Delta_n{}^T$, where T is a fragment of ZFC, if it is $\Delta_n(V)$ by Σ_n and Π_n definitions whose equivalence is provable in T. K is $\underset{\sim}{\Delta}_n{}^T$, if of form $\{x: (t, x) \in K'\}$ where K' is $\Delta_n{}^T$, and t is a parameter. We are most interested in the cases $T=$KP (Kripke-Platek admissible set theory, with Infinity), ZFC$^-$ (Zermelo-Frankel set theory with Choice and without Power Set), and ZFC.

$HC=H(\aleph_1)$ is the set of hereditarily countable sets. $K\subseteq HC$ is $\Sigma_n(HC)$ (resp. $\underset{\sim}{\Sigma}_n(HC)$) if K is definable over HC by a Σ_n formula without parameters (resp. with parameters from HC). The Π and Δ notions are similarly defined.

Familiarity with the primitive recursive (PR) set functions of [14] is also assumed. These functions include all functions with reasonably simple inductive definitions. They are all Δ_1^{KP}. A class K is PR if its characteristic function is, and is **PR** if of form $\{x:(t,x)\in K'\}$ for some PR K' and some parameter t.

2.4 *Model Theory*

Let L^* be a language. A class K of **R**-structures is an ***elementary class*** for L^*, in symbols $EC(L^*)$, if K is of form $\text{Mod}(\varphi)=\{\mathfrak{A}:\mathfrak{A}\models\varphi\}$ for some $\varphi\in L^*(\mathbf{R})$. K is a *pseudo-elementary* or *projective class* for L^*, in symbols $PC(L^*)$, if for some vocabulary **S** disjoint from **R** and some sentence $\varphi'\in L^*(\mathbf{R}\cup\mathbf{S})$ such that K is the class of all **R**-reducts of models of φ'. Equivalently, K is $PC(L^*)$ if it is of form $\text{Mod}(\exists\,\mathbf{S}\,\varphi')$ for some existential second-order sentence $\exists\,\mathbf{S}\,\varphi'$, $\varphi'\in L^*(\mathbf{R}\cup\mathbf{S})$. By abuse of language, we call $K\subseteq\mathcal{X}(\mathbf{R})$ $EC(L^*)$ or $PC(L^*)$ if it is the restriction of such a class to structures with universe ω.

For the definition of *language* in the abstract see [2] or [3] (where languages are respectively called systems of logics and logics). We call a language L^* *first-order* if:

(1) Sentencehood for L^* is a notion PR, or **PR** in parameters from HC; or the restriction of such a notion to some $H(\varkappa)$.

(2) Satisfaction for L^* is a notion $\Delta_1(V)$, or $\underset{\sim}{\Delta}_1(V)$ in parameters from HC; or the restriction of such a notion to $\varphi\in$ some $H(\varkappa)$.

These conditions correspond roughly to *absoluteness* as in [2] (where the terminology *first-order* is given some intuitive justification). All the languages of § 1 are first-order, as is each $L_{\varkappa\omega}$. We call a first-order language *strong* if:

(3) L^* is closed under $\neg$, $\forall$, $\exists$; under countable $\bigwedge$, $\bigvee$; under substitution of formulas of $L_{\omega\omega}$ for predicates; and the functions corresponding to these closure conditions, e.g. the function $\varphi\mapsto\neg\varphi$, are PR, or **PR** in parameters from HC, or the restriction of such functions to some $H(\varkappa)$.

(4) The class of countable wellfounded structures is $PC(L^*\cap HC)$.

Much of (3) is included in the definition of language in [3] (though not in [2]). These closure conditions guarantee that any $PC(L^*)$ class of **R**-structures is of form $\text{Mod}(\exists\,\mathbf{S}\,\varphi')$ where **S** contains just a single binary predicate not in **R**. (4) corresponds roughly to the notion *not bounded below* ω_1 of [2]. The languages of § 1 are, but $L_{\infty\omega}$ is not, strong.

2.5 *Connections Among the Classifications*

Addison [1] observed that for any of the spaces we have been considering, the class of open sets and the class of $\underset{\sim}{\Sigma}_1^0$ sets coincide, and similarly: $\underset{\sim}{\Pi}_1^0=$ closed, $\underset{\sim}{\Sigma}_2^0=F_\sigma$, $\underset{\sim}{\Pi}_2^0=G_\delta$, $\underset{\sim}{\Delta}_1^1=$ Borel, $\underset{\sim}{\Sigma}_1^1=$ analytic, $\underset{\sim}{\Pi}_1^1=CA$, $\underset{\sim}{\Sigma}_2^1=PCA$.

Ryll-Nardzewski, using Lopez-Escobar's Interpolation Theorem for $L_{\omega_1\omega}$, showed that for *invariant* subsets of $\mathcal{X}(\mathbf{R})$, Borel $= EC(L_{\omega_1\omega})$. Also analytic $=$ $PC(L_{\omega_1\omega})$. See [20].

Kleene [18] in effect showed that for subsets of any of the spaces we have been considering $\Sigma^1_{n+1} = \underset{\sim}{\Sigma}_n(HC)$. (Note that these spaces $\mathcal{X}(\mathbf{R}), X(\mathbf{R})$ are **PR** in parameter **R**, and are subsets of HC.)

Lévy's Teorem (cf. Appendix to [2]) tells us that each $H(\varkappa)$ is an elementary substructure of the universe V with respect to Σ_1 formulas. It follows that for subsets of HC, $\underset{\sim}{\Sigma}_1(HC) = \underset{\sim}{\Sigma}_1(V)$ in parameters from HC.

Barwise [2] in effect shows that for cardinals $\varkappa > \omega$ and for *invariant* classes of structures, **PR** in parameters from $H(\varkappa) = \underset{\sim}{\Delta}_1^{KP}$ in parameters from $H(\varkappa) = EC(L_{\varkappa\omega})$.

Jensen and Karp apparently knew that for subsets of the spaces we have been considering, $\underset{\sim}{\Delta}^1_1 = \mathbf{PR}$ in parameters from HC.

Vaught's work [32] discloses the following: For a fixed Polish space, let $\mathcal{V}(0) =$ Borel sets; $\mathcal{V}(\alpha+1) = \mathcal{V}(\alpha)$ plus complements of sets in $\mathcal{V}(\alpha)$ for α odd; $\mathcal{V}(\alpha+1) =$ sets obtainable from sets in $\mathcal{V}(\alpha)$ by $(\mathcal{A})$ for α even; $\mathcal{V}(\lambda) = \bigcup\{\mathcal{V}(\alpha)\colon \alpha < \lambda\}$ at limits; $\mathcal{V} = \mathcal{V}(\omega_1)$. Where the fusion operation $(\mathcal{A})$ given sets A_σ, $\sigma \in \omega^{>\omega}$, produces $\bigcup_{f\in\omega}\omega \bigcap_{n\in\omega} A_{f|n}$. Classically the sets in $\mathcal{V}$ are known as C-sets, and it is known $\mathcal{V}(1) =$ analytic sets. Then for invariant subsets of $\mathcal{X}(\mathbf{R})$, C-sets $= EC(L_{\omega_1 G})$, and moreover there is a level-by-level correspondence between the $\mathcal{V}$-hierarchy and the complexity of sentences of $L_{\omega_1 G}$, with analytic $= EC$ (Vaught sentences), where the Vaught sentences are, as in § 1.4, the simplest sentences of $L_{\omega_1 G} - L_{\omega_1\omega}$. Moreover Ryll-Nardzewski's equation Borel $= EC(L_{\omega_1\omega})$ can be improved to establish a level-by-level correspondence between the Borel hierarchy and the complexity of sentences of $L_{\omega_1\omega}$.

We extended this work of Vaught's to some other hierarchies in [8], § 2 and [6], ch. 4. The following has been noted with varying degrees of explicitness by several people:

2.6 PROPOSITION. For any strong first-order language L^*, for invariant subsets of $\mathcal{X}(\mathbf{R})$, $\underset{\sim}{\Sigma}_1(V)$ in parameters from $HC = PC(L^* \cap HC)$.

PROOF. That every $PC(L^* \cap HC)$ class is $\Sigma_1(V)$ in parameters from HC is immediate from the fact that satisfaction for $\tilde{L}^*$ is. To prove the converse fix a Σ_1 formula ψ and a parameter $t \in HC$ defining an invariant $K \subseteq \mathcal{X}(\mathbf{R})$.

Let $\in$ be the binary predicate of the language of set theory. The class of countable wellfounded $\in$-structures is $PC(L^* \cap HC)$. Say it is $\mathrm{Mod}(\exists S\, \vartheta)$ where $\vartheta \in L^*(\{\in\} \cup S) \cap HC$. Define inductively for $x \in HC$ a characterizing formula χ_x of $L_{\omega_1\omega}$ by letting $\chi_x(v)$ be:

$$\bigwedge_{y\in x} \exists u \in v\, \chi_y(u) \wedge \forall u \in v \bigvee_{y\in x} \chi_y(u).$$

Let F be a singulary function symbol, and let $\bar{r}, \bar{a}, \bar{t}$ be constants. We assume these symbols and $\in$ and the symbols in $\mathbf{S}$ are all distinct from the symbols of $\mathbf{R}$. Let $\mathbf{T}=\mathbf{R}\cup\mathbf{S}\cup\{F, \bar{r}, \bar{a}, \bar{t}\}$, and let $\varphi\in L(\mathbf{T})\cap HC$ be the conjunction of:

(1) A large enough finite fragment of ZFC.

(2) ϑ

(3) $\chi_{\underset{\sim}{R}}(\bar{r}) \wedge \chi_t(\bar{t})$

(4) $\bar{a}$ is an $\bar{r}$-structure with universe ω

(5) $\psi(\bar{t}, \bar{a})$

(6) F is an injection $\wedge$ range $F=$ universe of $\bar{a}$.

Plus for each n-ary predicate $R\in\mathbf{R}$:

$(7)_R$ $\quad \forall v(\chi_R(v) \to \forall v_1 \dots v_n(R(v_1 \dots v_n) \leftrightarrow (F(v_1) \dots F(v_n)) \in$ the $\bar{a}$-interpretation of the symbol v))

and similarly for function symbols and constants. Here in (4), (6), (7), the definitions of structure, universe, and interpretation are to be written out in terms of $\in$ using the usual set-theoretic definitions.

If $\mathfrak{A}\in K$, then by Lévy's Reflection Principle there is a countable transitive model M of enough of ZFC with t, $\mathfrak{A}\in M$ and $M\models\psi(t, \mathfrak{A})$. Using such an M it is easy to construct a $\mathfrak{L}\in\mathscr{X}(\mathbf{T})$ with $\mathfrak{L}\models\varphi$ and $\mathfrak{L}\,|\,\mathbf{R}=\mathfrak{A}$.

Conversely given $\mathfrak{L}\models\varphi$ with $\mathfrak{L}\,|\,\mathbf{R}=\mathfrak{A}$, (1) and (2) guarantee that $\mathfrak{L}$ is up to isomorphism a transitive set. Then (3), (4), (5) guarantee that the interpretation $\bar{a}^{\mathfrak{L}}$ of $\bar{a}$ in $\mathfrak{L}$ is an $\mathbf{R}$-structure satisfying the definition of K. (We use here the fact that a Σ_1 statement true inside some transitive set is actually true in the universe V.) Finally (5), (6) guarantee that $\mathfrak{A}\cong\bar{a}^{\mathfrak{L}}$, so by invariance of K, $\mathfrak{A}\in K$.

2.7 *Summary*

For any strong first-order language L^*, and for invariant subsets of $\mathscr{X}(\underset{\sim}{R})$, we have:

(a) $\underset{\sim}{\Delta}_1^1=\text{Borel}=\mathbf{PR}$ in parameters from $HC=EC(L_{\omega_1\omega})$,

(b) $\underset{\sim}{\Sigma}_1^1=\text{analytic}=PC(L_{\omega_1\omega})=EC$ (Vaught sentences),

(c) $\underset{\sim}{\Sigma}_2^1=PCA=\underset{\sim}{\Sigma}_1(HC)=\underset{\sim}{\Sigma}_1(V)$ in parameters from $HC=PC(L^*\cap HC)$.

§ 3 A Question of Vaught

3.1 PROPOSITION. For any first-order language L^*, and for invariant subsets of $\mathscr{X}(\mathbf{R})$, we have:

$$EC(L^*\cap HC)\subseteq\Delta_2^1=\Delta_1(HC)$$

PROOF. We only give a sketch since our proof has appeared in [21]. The inclusion and the identity are immediate from 2.7 (*c*). We tacitly assume **R** is nontrivial, i.e. contains at least one binary predicate E. We say $\mathfrak{A}\in\mathcal{X}(\{E\})$ *codes* $x\in HC$ if $\mathfrak{A}\cong(TC(y),\in)$ where $TC(y)=\{y\}\cup y\cup\bigcup y\cup\bigcup\bigcup y\cup\ldots$ is the transitive closure of y. An example to show the inclusion is proper is provided by $\{\mathfrak{A}\in\mathcal{X}(\mathbf{R})\colon \exists\varphi\in L^*(\mathbf{R})\cap HC\ ((|\mathfrak{A}|, E^{\mathfrak{A}})\text{ codes }\varphi\wedge\mathfrak{A}\models\neg\varphi)\}$.

Vaught has asked whether for any invariant Δ^1_2 $K\subseteq\mathcal{X}(\mathbf{R})$ there is some first-order language L^* for which K is $EC(L^*\cap\tilde{H}C)$. We will show this question cannot be answered in ZFC.

3.2 *A Positive Answer*

For any partially ordered set of forcing conditions (PO set) $\mathcal{P}$, let $V^{\mathcal{P}}$ be the corresponding extension of the universe of set theory. (If you will, the Boolean-valued model associated with the complete Boolean algebra of regular open subsets of $\mathcal{P}$) For simplicity let us assume **R** finite. Then we may define $K\subseteq X(\mathbf{R})$ to be *absolutely* Δ^1_2 if there exist Σ^1_2 and Π^1_2 formulas φ, φ in a parameter t from, say, ω^{ω}, defining K, such that for any PO set $\mathcal{P}$:

$$V^{\mathcal{P}}\models\forall x\,(\varphi(t,x)\leftrightarrow\psi(t,x)) \tag{1}$$

Here we are using elements t of V autonymously (writing t rather than t^{v}). We extend this notion in the obvious way to $\mathcal{X}(\mathbf{R})$. Note that if K is invariant, then so is the set defined by φ and t in any $V^{\mathcal{P}}$, since

$$\neg\exists x,y\,(\mathfrak{A}_x\cong\mathfrak{A}_y\wedge\varphi(t,x)\wedge\neg\psi(t,y)) \tag{2}$$

is a true Π^1_2 statement, and Π^1_2 statements are absolute by Shoenfield's Theorem. We show now how, given an invariant absolutely Δ^1_2 $K\subseteq\mathcal{X}(\mathbf{R})$ to constrult a first-order language $L^*\subseteq HC$ for which K is an $E\tilde{C}$. We begin by fixing definitions of the corresponding subset of $X(\mathbf{R})$ satisfying (1) above. To further simplify matters we suppose **R** contains just one binary predicate E.

For an arbitrary **R**-structure $\mathfrak{A}$, let $\mathcal{P}(\mathfrak{A})$ be the PO set of the injective elements of $|\mathfrak{A}|^{<\omega}$, partially ordered by reverse inclusion, i.e. the usual conditions for adjoining a generic bijection between ω and $|\mathfrak{A}|$. Let $\bar{x}(\mathfrak{A})$ be the following term of the forcing language for $\mathcal{P}(\mathfrak{A})$:

$$\{(p,((m,n),i))\colon p\in\mathcal{P}(\mathfrak{A})\wedge m,n\in\operatorname{dom}p\wedge$$
$$(((m,n)\in E^{\mathfrak{A}}\wedge i=0)\vee((m,n)\notin E^{\mathfrak{A}}\wedge i=1))\}$$

i.e. the canonical term for an element x of $X(\mathbf{R})$ with $\mathfrak{A}_x\cong\mathfrak{A}$. By (1) and (2) we have:

$$V^{\mathcal{P}(\mathfrak{A})}\models\varphi(t,\bar{x}(\mathfrak{A}))\leftrightarrow\psi(t,\bar{x}(\mathfrak{A})) \tag{3}$$

$$(4)\qquad V^{\mathcal{P}(\mathfrak{A})} \models \varphi(t, \bar{x}(\mathfrak{A})) \leftrightarrow \exists y \in X(\mathbf{R})(\mathfrak{A}_x \cong \mathfrak{A} \wedge \varphi(t, y))$$
$$\leftrightarrow \forall y \in X(\mathbf{R})(\mathfrak{A}_y \cong \mathfrak{A} \to \varphi(t, y))$$

Any permutation h of ω induces an automorphism H_h of $\mathcal{P}(\mathfrak{A})$ and a permutation $\bar{u} \to \bar{u}^h$ of the terms of the forcing language. For any $p, q \in \mathcal{P}(\mathfrak{A})$ there is an h such that $p, H_h(q)$ are compatible (weak homogeneity). For any h, $\bar{x}(\mathfrak{A})^h$ is still a term for an isomorph of $\mathfrak{A}$. It follows by (4) there cannot exist $p, q \in \mathcal{P}(\mathfrak{A})$ one of which forces $\varphi(t, \bar{x}(\mathfrak{A}))$ and the other of which forces its negation. Thus:

$$(5)\qquad \text{Either } V^{\mathcal{P}(\mathfrak{A})} = \varphi(t, \bar{x}(\mathfrak{A})) \text{ or else } V^{\mathcal{P}(\mathfrak{A})} \models \neg\varphi(t, \bar{x}(\mathfrak{A}))$$

Let $K^+ = \{\mathfrak{A} : V^{\mathcal{P}(\mathfrak{A})} \models \varphi(t, \bar{x}(\mathfrak{A}))\}$.

K^+ is invariant. For if $\mathfrak{B} \cong \mathfrak{A}$, there is an isomorphism $\mathcal{P}(\mathfrak{B}) \cong \mathcal{P}(\mathfrak{A})$ such that the induced map on terms carries $\bar{x}(\mathfrak{B})$ to $\bar{x}(\mathfrak{A})$.

$K^+ \cap \mathcal{X}(\mathbf{R}) = K$. For if $x \in X(\mathbf{R})$ and $\mathfrak{A}_x \in K$, then $\varphi(t, x)$ is true and remains true in $V^{\mathcal{P}(\mathfrak{A}_x)}$ by Shoenfield's Theorem whence by (4) $V^{\mathcal{P}(\mathfrak{A}_x)} \models \varphi(t, \bar{x}(\mathfrak{A}_x))$, i.e. $\mathfrak{A}_x \in K^+$. Conversely, if $\mathfrak{A}_x \notin K$, by (3) and (4) $\mathfrak{A}_x \notin K^+$.

K^+ is $\Delta_1(V)$ in parameter t. For φ is equivalent over all models to some Σ_1 condition $\tilde{\theta}$; and by the general theory of forcing there is a Σ_1 θ' such that for all PO sets $\mathcal{P}$, all $p \in \mathcal{P}$, and all terms $\bar{u}$, $V^{\mathcal{P}} \models \theta(t, \bar{u})$ iff $\theta'(\mathcal{P}, p, t, u)$ holds. Since, $\mathcal{P}(\mathfrak{A})$ and $\bar{x}(\mathfrak{A})$ are PR functions of $\mathfrak{A}$, this implies K^+ is $\underset{\sim}{\Sigma}_1(V)$ in parameter t. Using ψ in place of φ we get Π_1 in place of Σ_1.

Now let L^* be a language with but a single sentence $\rho \in HC$, and $\mathfrak{A} \models \rho$ iff $\mathfrak{A} \in K^+$. L^* is certainly first-order, and we can without difficulty fatten L^* up to a strong language without losing the first-order property. (Cf. [2].) Finally, K is $EC(L^*)$.

The Solovay Absoluteness Theorem, [23], p. 152, implies that if $\forall \varkappa \exists \lambda\, \lambda \to (\varkappa)_2^{<\omega}$, then *every* Δ^1_2 set is absolutely $\underset{\sim}{\Delta}^1_2$. Thus *if enough large cardinals exist, Vaught's question has a positive answer.*

3.3 *A Negative Answer*

It is wellknown that any class K which is $\Sigma_1(V)$ in parameters from HC having $\omega_1 \in K$ contains a closed unbounded ($\widetilde{CUB}$) subset of ω_1. It is also wellknown that if F assigns to each countable ordinal α a wellordering of ω in type α, and for $i = 2^m(2n+1) \in \omega$, $D_i = \{\alpha : m \text{ precedes } n \text{ in } F(\alpha)\}$, then for some i, neither D_i nor $\omega_1 - D_i$ contains a CUB set. Finally it is wellknown that if $\omega_1^L = \omega_1$ then the function F may be taken to be $\Sigma_1(V)$ and hence (since its domain is $OR \cap HC$) $\Delta_1(HC)$. On this assumption, for suitable i, $K = \{\mathfrak{A} \in \mathcal{X}(\{E\}) : \mathfrak{A}$ is a wellordering with order type $\in D_i\}$ is a subset of $\mathcal{X}(\{E\})$ which is invariant (in $\mathcal{X}(\{E\})$) and $\Delta_1(HC)$ hence Δ^1_2, but which cannot be the restriction to $\mathcal{X}(\{E\})$ of any (fully) invariant class which is $\underset{\sim}{\Delta}_1(V)$ in parameters from HC. Thus *if $\omega_1^L = \omega_1$, Vaught's question has a negative answer.*

§ 4 Approximation Theory

Let L^*, L^0 be languages. By an *approximation function* for L^*, L^0 we mean a function $\mathscr{C}$: $\mathrm{OR}\times L^* \to L^0$ which preserves vocabulary; is *PR*, or **PR** in parameters from *HC*, or is the restriction of such a function to some $H(\varkappa)$; and which has the property that for any sentence φ of L^* the following is valid: $\varphi \leftrightarrow \bigwedge_{\alpha\in OR} \mathscr{C}(\alpha, \varphi)$.

4.1 LEMMA. There exists an approximation function for $L_{\infty G}, L_{\infty\omega}$.

PROOF. The basic idea goes back to Moschovakis [25]; see also [31].

We define by induction of subformulas two preliminary functions $\mathscr{A}, \mathscr{S}$: $\mathrm{OR}\times L_{\infty G} \geqq L_{\infty\omega}$. The easy clauses of the induction are:

$$\mathscr{A}(\alpha, \neg\varphi) = \neg\mathscr{A}(\alpha, \varphi) \qquad \mathscr{S}(\alpha, \neg\varphi) = \mathscr{S}(\alpha, \varphi)$$

$$\mathscr{A}(\alpha, \bigwedge\Phi) = \bigvee\{\mathscr{A}(\alpha, \varphi) : \varphi\in\Phi\}$$

$$\mathscr{A}(\alpha, \bigvee\Phi) = \bigvee\{\mathscr{A}(\alpha, \varphi) : \varphi\in\Phi\}$$

$$\mathscr{S}(\alpha, \bigwedge\Phi) = \mathscr{S}(\alpha, \bigvee\Phi) = \bigwedge\{\mathscr{S}(\alpha, \varphi) : \varphi\in\Phi\}$$

$$\mathscr{A}(\alpha, \forall v\varphi) = \forall v\mathscr{A}(\alpha, \varphi) \qquad \mathscr{A}(\alpha, \exists v\varphi) = \exists v\mathscr{A}(\alpha, \varphi)$$

$$\mathscr{S}(\alpha, \forall v\varphi) = \mathscr{S}(\alpha, \exists v\varphi) = \forall v\mathscr{S}(\alpha, \varphi).$$

For φ given by (*G*) of §1.4 the definition is more complex. Fixing α and φ for the moment we define auxiliary functions $\mathscr{A}^*, \mathscr{S}^*$ with domains $\mathrm{OR}\times I^{<\omega}$, OR respectively, by a subinduction:

$$\mathscr{A}^*(0, \sigma) = \bigwedge_{n\leq \mathrm{length}\,\sigma} \mathscr{A}(\alpha, \varphi_\sigma)$$

$$\mathscr{A}^*(\beta+1, \sigma) = \bigvee_{i\in I} \mathscr{A}^*(\beta, \sigma^\frown i)$$

$$\mathscr{A}^*(\lambda, \sigma) = \bigwedge_{\beta<\lambda} \mathscr{A}^*(\beta, \sigma) \text{ at limits}$$

$$\mathscr{S}^*(\beta) = \bigwedge_{n\in\omega} \bigwedge_{\sigma\in I^n} \forall_{v_0}\ldots\forall_{v_n}(\mathscr{A}^*(\beta, \sigma) \to \mathscr{A}^*(\beta+1, \sigma)).$$

We then set:

$$\mathscr{A}(\alpha, \varphi) = \mathscr{A}^*(\alpha, 0)$$

$$\mathscr{S}(\alpha, \varphi) = \mathscr{S}^*(\alpha) \wedge \bigwedge_{n\in\omega} \bigwedge_{\sigma\in I^n} \forall_{v_0}\ldots\forall_{v_n}\mathscr{S}(\alpha, \varphi_\sigma).$$

Readers of [31] should then have no difficulty in verifying that the following are valid:

(1) $\mathscr{S}(\alpha, \varphi) \to \mathscr{S}(\beta, \varphi)$ for $\alpha<\beta$

(2) $\bigvee_{\alpha\in OR} \mathscr{S}(\alpha, \varphi)$

(3) $\mathscr{S}(\alpha, \varphi) \to (\varphi \leftrightarrow \mathscr{A}(\alpha, \varphi))$ for all α

(4) $\varphi \leftrightarrow \bigvee_{\alpha\in OR}(\mathscr{S}(\alpha, \varphi) \wedge \mathscr{A}(\alpha, \varphi))$

(5) $\varphi \leftrightarrow \bigwedge_{\alpha\in OR}(\mathscr{S}(\alpha, \varphi) \to \mathscr{A}(\alpha, \varphi))$.

So it suffices to set $\mathscr{C}(\alpha, \varphi) = (\mathscr{S}(\alpha, \varphi) \to \mathscr{A}(\alpha, \varphi))$.

4.2 APPROXIMATION THEOREM. Let L^* be any first-order language. Then there exists an approximation function for $L^*, L_{\infty\omega}$.

PROOF. By the Lemma it suffices to obtain an approximation function for $L^*, L_{\infty G}$. For simplicity we will consider only the vocabulary $\mathbf{R} = \{E\}$, E a binary predicate, and we will assume satisfaction for L^* is $\Sigma_1(V)$ (no parameters). On these assumptions the approximation function will be PR.

From the Σ_1 definition of satisfaction we obtain a Σ^1_2 formula θ defining $S = \{(x, y) \in X(\mathbf{R})^2 \colon \exists \varphi \in L^*(\mathbf{R}) \cap HC$ (y codes $\varphi \wedge \mathfrak{A}_x \vdash \varphi$)$\}$ and a Σ^1_2 formula θ^- defining the set S^- obtained by replacing φ by $\neg\varphi$ in the definition of S. (Cf. proof of Prop. 3.1.) The statement:

(1) $\neg \exists x, y (\theta(x, y) \wedge \theta^-(x, y) \wedge \mathfrak{A}_x \cong \mathfrak{A}_y)$

is Π^1_2, hence absolute.

From θ we can obtain the index of a recursive functional F such that $(x, y) \in S$ iff:

(2) $\exists z \; F(x, y, z)$ is wellfounded.

By Shoenfield's Theorem, the required z can be found in $J(x, y)$, the class of sets constructible from x, y. Hence (2) is equivalent to the existence of $\alpha < \omega_1$ such that:

(3) $\exists_z \in J_\alpha(x, y)$ $F(x, y, z)$ wellorders ω in order type $< \alpha$ where J_α is the α^{th} level of the constructible hierarchy. From (3) we can readily obtain a Σ^1_1 formula ψ such that the following holds:

(4) $\forall x, y, z, z'$ ($\mathfrak{A}_z$ is embeddable in $\mathfrak{A}_{z'} \rightarrow (\psi(x, y, z) \rightarrow \psi(x, y, z'))$) and for any x, y and for fixed α and z wellordering ω in order type α, (3) is equivalent to $\psi(x, y, z)$. Note that (4) is Π^1_2, hence absolute.

From this ψ we can compute the index of an RE set W such that $\psi(x, y, z)$ is equivalent to:

(5) $$\exists w \in \omega^\omega \, \forall n \in \omega \, (x \| n, y \| n, z \| n, w \| n) \in W$$

where $x \| n$ denotes the restriction of x to $(n+1) \times (n+1)$ for $x \in X(\mathbf{R})$ $(= 2^{\omega \times \omega})$.

Now let φ be a sentence of $L^*(\mathbf{R})$, $\mathfrak{A}$ an arbitrary $\mathbf{R}$-structure. Let $\mathcal{P} = \mathcal{P}(\mathfrak{A})$, $\bar{x} = \bar{x}(\mathfrak{A})$ be as in § 3.2. Let $\mathcal{Q} = \mathcal{Q}(\varphi)$ be the PO set of forcing conditions for making $TC(\varphi)$ countable (i.e. for making $\varphi \in HC$), and let $\bar{y} = \bar{y}(\varphi)$ be the canonical term for an element of $X(\mathbf{R})$ coding φ. Now if $\mathfrak{A} \models \varphi$, then $\mathfrak{A}$, φ satisfy the Σ_1 definition of satisfaction for L^* in V, and will continue to do so in $V^{\mathcal{P} \times \mathcal{Q}}$. Hence in that extension $\bar{x}$ and $\bar{y}$ will satisfy the Σ^1_2 definition θ of S. Conversely, if $\mathfrak{A} \models \neg \varphi$, $\bar{x}$, $\bar{y}$ satisfy θ^- in $V^{\mathcal{P} \times \mathcal{Q}}$ and so by (1) do not satisfy θ. So $\mathfrak{A} \models \varphi$ iff $V^{\mathcal{P} \times \mathcal{Q}} \models \theta(\bar{x}, \bar{y})$. By our detailed analysis of θ above, this condition is equivalent to:

(6) $$V^{\mathcal{P} \times \mathcal{Q}} \models \exists \alpha < \omega_1 \exists z \in X(\mathbf{R}) \; (\mathfrak{A}_z \cong (\alpha, \in) \wedge \psi(\bar{x}, \bar{y}, \bar{z})).$$

For fixed $\alpha \in \mathrm{OR}$, let $\mathcal{R}(\alpha)$ be the *PO* set of forcing conditions for collapsing α, and let $\bar{z}(\alpha)$ be the canonical term for an element of $X(\mathbf{R})$ with $\mathfrak{A}_z \cong (\alpha, \in)$. We claim (6) is equivalent to the existence of α such that:

$$(7) \qquad V^{\mathcal{P}\times\mathcal{Q}\times\mathcal{R}(\alpha)} \models \psi(\bar{x}, \bar{y}, \bar{z}(\alpha)).$$

For if (7) holds for some α, then the Σ_2^1 statement $\exists z (\mathfrak{A}_z$ is a wellordering $\wedge \psi(\bar{x}, \bar{y}, z))$ holds in $V^{\mathcal{P}\times\mathcal{Q}\times\mathcal{R}(\alpha)}$, and hence by Shoenfield's Theorem in $V^{\mathcal{P}\times\mathcal{Q}}$, so (6) holds. Conversely, suppose (6) holds and let $\beta = \mathrm{card}(\mathcal{P}\times\mathcal{Q})^+$, so β is still uncountable in $V^{\mathcal{P}\times\mathcal{Q}}$. For any $p \in \mathcal{P}$, $q \in \mathcal{Q}$, there will exist $p' \leq p$, $q' \leq q$ and $\alpha < \beta$ such that (p', q') forces $\exists z (\mathfrak{A}_z \cong (\alpha, \in) \wedge \psi(\bar{x}, \bar{y}, z))$. It follows (p', q', l_β) forces the same thing, where l_β is the trivial element of $\mathcal{R}(\beta)$. By (4), (p', q', l_β) forces $\exists z (\mathfrak{A}_z \cong (\beta, \in) \wedge \psi(\bar{x}, \bar{y}, z))$, and since p, q were arbitrary, (7) follows.

Now fixing α and $\mathcal{R} = \mathcal{R}(\alpha)$, $\bar{z} = \bar{z}(\alpha)$, (7) is equivalent to:

$$(8) \qquad V^{\mathcal{P}\times\mathcal{Q}\times\mathcal{R}} \models \exists w \in \omega^\omega \, \forall n \in \omega \, (x \| n, y \| n, z \| n, w | n) \in W.$$

For $p \in \mathcal{P}$ with $\mathrm{dom}\, p \geqq n$, define $\xi(n, p)$ to be what p forces $x \| n$ to be. Thus for $i, j < n$, $(\xi(n, p))(i, j)$ is 0 if $(p(i), p(j)) \in E^{\mathfrak{A}}$, and 1 if not. Let η, ζ be similarly defined. Then we claim (8) is equivalent to:

$$(9) \qquad \forall p_0 \in \mathcal{P}, q_0 \in \mathcal{Q}, r_0 \in \mathcal{R} \, \exists p_1 < p_0, q_1 < q_0, r_1 < r_0 \, \exists w_0, w_1 \in \omega$$
$$\forall p_2 < p_1, q_2 < q_1, r_2 < r_1 \, \exists p_3 < p_2, q_3 < q_2, r_3 < r_2 \, \exists w_2, w_3 \in \omega \ldots$$
$$\ldots \forall n (\xi(n, p_n), \eta(n, q_n), \zeta(n, r_n), (w_0 \ldots w_n)) \in W.$$

We will omit the proof of this equivalence, since it is a special case of more general theorems of [15]. Now (9) is equivalent to the following sentence holding in $\mathfrak{A}$:

$$(10) \qquad \bigwedge_{k_0 \in \omega} \forall v_0 \ldots v_{k_0 - 1} \text{ distinct } \bigwedge_{k_0 \in \mathcal{Q}} \bigwedge_{r_0 \in \mathcal{R}}$$
$$\bigvee_{k_1 \in \omega} \exists v_{k_0} \ldots v_{k_0 + k_1 - 1} \text{ distinct } \bigvee_{q_1 < q_0} \bigvee_{r_1 < r_0} \bigvee_{w_0, w_1 \in \omega} \cdots$$
$$\ldots \bigwedge_n \bigvee_\xi \text{ with } (\xi, \eta(n, q_n), \zeta(n, r_n), (w_0 \ldots w_n)) \in W$$
$$\left(\bigwedge_{i, j \leq n, \xi(i, j) = 0} v_i E v_j \wedge \bigwedge_{i, j \leq n, \xi'(i, j) = 1} \neg v_i E v_j \right)$$

where here *distinct* means not merely that $v_{k_0} \ldots$ are distinct from each other, but also that they are distinct from $v_0 \ldots v_{k_0 - 1}$. Tedious but routine coding (cf. Vaught's remarks [32], § 3, on the closure of $L_{\omega_1 G}$ on passage to weak second-order logic) produces a sentence $\mathcal{Q}(\alpha, \varphi)$ equivalent to (10) which belongs to $L_{\infty G}$, and is independent of $\mathfrak{A}$. It suffices to set $\mathcal{C}(\alpha, \varphi) = \neg \mathcal{Q}(\alpha, \neg \varphi)$.

§ 5 The Anti-Beth Theorem

Beth's Definabillity Theorem for a language L^* asserts that for any vocabulary $\mathbf{R}$ and any binary predicate S and constants c, d not in $\mathbf{R}$, that if $\varphi \in L^*(\mathbf{R} \cup \{S\})$ is such that any $\mathbf{R}$-structure $\mathfrak{A}$ has at most one expansion to a model of φ, then there exists $\theta \in L^*(\mathbf{R} \cup \{c,d\})$ such that for any $\mathbf{R}$-structure $\mathfrak{A}$, if $\mathfrak{A}$ has an expansion to a model of φ, then $(\mathfrak{A}, \{(a,b): (\mathfrak{A}, a, b) \models \theta\})$ is that expansion. Replacing "at most one" by "exactly one" produces the weak version of Beth's Theorem.

5.1. ANTI-BETH THEOREM. Let L^* be any strong first-order language. Then even the weak version of Beth's Theorem fails for $L^* \cap HC$.

PROOF. It may help to isolate first the descriptive-set-theoretic content of the construction. Let $X = 2^{\omega \times \omega}$. We think of subsets of X^n, as n-ary relations on X, writing $Z(x_1 \ldots x_n)$ for $(x_1 \ldots x_n) \in Z$. For $x \in X$, $i \in \omega$, define $(x)_1 \in X$ by $(x)_1(j,k) = x(1, 2^j(2k+1))$.

Suppose we are given a family Γ of subsets of and relations on X containing a $T \subseteq X^2$ such that for all x:

(1) $\mathfrak{A}_x$ is wellfounded $\leftrightarrow \exists y\, T(x,y)$

and satisfying:

(2) $\Gamma \subseteq \underset{\sim}{\Delta}^1_2$

(3) All closed sets belong to Γ

(4) Γ is closed under countable —

(5) Γ is closed under taking inverse images under continuous functions

We show how, given an arbitrary $\underset{\sim}{\Delta}^1_2$ set K, to construct a $\underset{\sim}{\Pi}^1_1 H \in \Gamma$ such that:

(6) $\forall x \exists! y\, H(x,y)$

(7) $\forall x, y\, (H(x,y) \to (K(x) \leftrightarrow y(0,1) = 0))$.

To begin with, fix $\underset{\sim}{\Pi}^1_1$ sets P, Q such that:

(8) $\forall x\, (K(x) \leftrightarrow \exists y P(x,y) \leftrightarrow \neg \exists y Q(x,y))$.

Define a $\underset{\sim}{\Pi}^1_1$ set A by:

(9) $A(x,y) \leftrightarrow ((y(0,0) = 0 \wedge P(x,(y)_0)') \vee (y(0,0) = 1 \wedge Q(x,(y)_0))$.

Note:

(10) $\forall x \exists y\, A(x,y)$.

Let B_0 be a $\underset{\sim}{\Pi}^1_1$ set uniformizing A, i.e. $B_0 \subseteq A$ and

(11) $\forall x \exists! y\, B_0(x,y)$.

By the standard analysis of $\underset{\sim}{\Pi}^1_1$ sets there is a continuous $F_0: X^2 \to X$ such that:

(12) $\forall x, y\, (B_0(x,y) \leftrightarrow \mathfrak{A}_{F_0(x,y)}$ is wellfounded).

Define:

(13) $C_0(x,y,z,u) \leftrightarrow z = F_0(x,y) \wedge T(z,u)$.

Note the graph of F_0 is closed so by (3), (4), $C_0 \in \Gamma$. Moreover by (1):

(14) $\forall x, y\,(B_0(x, y) \leftrightarrow \exists z, u\, C_0(x, y, z, u))$.

By (2) C_0 is $\underset{\sim}{\Delta}{}^1_2$, so there exists a $\underset{\sim}{\Pi}{}^1_1$ set $D_0 \subseteq X^5$ such that:

(15) $\forall x, y, z, u\,(C_0(x, y, z, u) \leftrightarrow \exists v\, D_0(x, y, z, v))$.

Let B_1 be a $\underset{\sim}{\Pi}{}^1_1$ set uniformizing D_0, so:

(16) $\forall x\, \exists !\, y, z, u, v\, B_1(x, y, z, u, v)$.

Reviewing the construction, it is clear the same y is involved in (11) and (16).

Now iterate the above steps, picking $F_1 : X^5 \to X$, $C_1 \subseteq X^7$, $D_1 \subseteq X^8$, etc. In the end we define:

(17) $E_n(x, y) \leftrightarrow B_n(x, (y)_0 \dots (y)_{3n+3})$,

(18) $G_n(x, y) \leftrightarrow C_n(x, (y)_0 \dots (y)_{3n+5})$.

Since the maps $y \to ((y)_0 \dots (y)_i)$ are continuous, the E_n will be $\underset{\sim}{\Pi}{}^1_1$ and, by (5), the G_n will belong to Γ. Finally, set:

(19) $H(x, y) \leftrightarrow \forall n\, E_n(x, y)$.

Reviewing the construction, and noting that $(y)_0(0, 0) = y(0, 1)$, we get (6), (7). Moreover:

(20) $\forall x, y\,(H(x, y) \leftrightarrow \forall n\, G_n(x, y))$,

which, with (4), implies $H \in \Gamma$.

Now to apply this construction to model theory. For $n \in \omega$ let $\mathbf{R}^n = \{R_1 \dots R_n\}$ where the R_i are binary predicates, and let $\mathbf{S}^n = \mathbf{R}^n \cup \{\oplus, \otimes\}$, where $\oplus$, $\otimes$ are binary function symbols. Let L^* be a strong first-order language. By the definition of strong, cf. § 2.4, there is a sentence $\tau \in L^*(\mathbf{R}^2) \cap HC$ such that the class of countable wellfounded $\mathbf{R}^1$-structures is $\mathrm{Mod}(\exists \mathbf{R}_2\, \tau)$. Define $T \subseteq X^2$ by:

(21) $T(x, y) \leftrightarrow \mathfrak{A}_{(x, y)} \models \tau$,

and let Γ be the smallest class containing T and closed under (3)—(5) above. It is wellknown that for any Borel $Z \subseteq X^n$ there is a sentence $\zeta \in L_{\omega_1\omega}(\mathbf{S}^n)$ such that for all $x_1 \dots x_n$:

(22) $Z(x_1 \dots x_n) \leftrightarrow (\mathfrak{A}_{(x_1 \dots x_n)}, +, \times) \models \zeta$,

where $+$, $\times$ are the usual arithmetical operations on ω. Now the closure conditions required of Γ correspond to the closure conditions satisfied by strong languages: (3) corresponds to $L_{\omega_1\omega} \subseteq L^*$, (4) to closure of L^* under countable $\wedge$, and (5) to closure under substetution of formulas for predicates. Exploiting this correspondence, for every $Z \in \Gamma$ we can find a ζ in $L^* \cap HC$ satisfying (22). This, with 2.7 (c), implies (2).

Let now a $\Delta^1_2 K \subseteq X$ be given, and suppose K is invariant. Let H be as constructed above from K, and let $\eta \in L^*(\mathbf{S}^2) \cap HC$ correspond to H. Let $\varphi_0 \in L_{\omega_1\omega}(\mathbf{S}^0)$ express that $\oplus$, $\otimes$ are up to isomorphism the usual arithmetical operations on ω. Let $\varphi = (\varphi_0 \wedge \eta) \vee (\neg\varphi_0 \wedge \forall u, v \neg R_2(u, v))$. Then by (6) every $\mathbf{S}^1$-structure $\mathfrak{A}$ has a unique expansion to a model of φ. Suppose $\theta \in L^*(\mathbf{S}^1 \cup \cup\{c, d\})$ is as required by Beth's Theorem. Using the closure properties of L^* we can obtain from θ a $\psi \in L^*(\mathbf{S}^1) \cap HC$ expressing that θ holds of the identity element of $\oplus$ and the identity element of $\otimes$. Then by (7):

$$\forall x\,(K(x) \leftrightarrow (\mathfrak{A}_x, +, \times) \models \psi). \tag{23}$$

It is not hard to see no ψ satisfying (23) can exist if K is the counterexample constructed in the proof of Prop. 3.1. This contradittiction shows Beth's Theorem fails.

§ 6 Some Model Theory

We collect here what is known about first-order languages from §§ 1-5, from Barwise' work [2], and elsewhere.

6.1 DOWNWARD LÖWENHEIM-SKOLEM THEOREM. Let L^* be a first-order language, $\varkappa$ an infinite cardinal, $\varphi \in L^* \cap H(\varkappa^+)$, $\mathfrak{A}$ a model of φ, Z a subset of $|\mathfrak{A}|$ with card $Z \leq \varkappa$. Then there is a substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with $Z \subseteq |\mathfrak{B}|$, card $|\mathfrak{B}| = \varkappa$, and $\mathfrak{B} \vdash \varphi$.

PROOF. This is Prop. 2.1 of [2]. For the languages of § 1, a direct proof using Skolem functions is possible.

If L^* is a language and $\mathfrak{A}$, $\mathfrak{B}$ are structures of the same vocabulary, we say $\mathfrak{A}$ and $\mathfrak{B}$ are L^*-elementarily equivalent, in symbols $\mathfrak{A} \equiv^* \mathfrak{B}$, if they are models of exactly the same sentences of L^*. We say $\mathfrak{A} \approx \mathfrak{B}$ if there exists a family $\mathscr{S}$ of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ with the back-and-forth property $(\forall f \in \mathscr{S}\, \forall a \in |\mathfrak{A}|\, \exists b \in |\mathfrak{B}|\ f \cup \{a, b)\} \in \mathscr{S}$ and *vice versa*).

6.2 KARP PROPERTY. Let L^* be a first-order language. Then for all structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A} \equiv^* \mathfrak{B}$ iff $\mathfrak{A} \approx \mathfrak{B}$.

PROOF. For $\equiv_{\infty\omega}$ this is due to Karp. For the general case it is Prop. 2.5 of [2]. The equivalence of $\equiv^*$ and $\equiv_{\infty\omega}$ is greatly strengthened by the Approximation Theorem 4.2.

We say a sentence φ in vocabulary $\mathbf{R}$ is *compact* if for any vocabulary $\mathbf{S}$ disjoint from $\mathbf{R}$, where we here allow, contrary to our convention everywhere else in this paper, uncountable $\mathbf{S}$, and for any theory $T \subseteq L_{\omega\omega}(\mathbf{R} \cup \mathbf{S})$, if φ is consistent with every finite subtheory of T, then φ is consistent with T.

6.3 GOLD PROPERTY. Let L^* be a first-order language, and φ a sentence of L^* such that both φ and $\neg\varphi$ are compact. Then φ is equivalent to a sentence of $L_{\omega\omega}$ in the same vocabulary.

PROOF. Gold [12] proves this for $L_{\infty\omega}$ but examining her proof one sees it only uses the Karp Property. —

6.4 UPWARD LÖWENHEIM-SKOLEM THEOREMS

(a) Let L^* be a strong first-order language such that the class of *all* wellfounded structures is $PC(L^* \subset HC)$. Then for invariant classes of structures $\underset{\sim}{\Sigma}_1(V)$ in parameters from $HC = PC(L^* \cap HC)$.

(b) Let $L^*, L^{\#}$ be languages satisfying the hypothesis of part (a). Then $L^* \cap HC$, $L^{\#} \cap HC$ have the same Hanf number.

(c) Let η be the common value of the Hanf numbers in part (b), then:

$$\mu\varkappa\,[\varkappa \to (\omega)_2^{<\omega}] < \eta < \mu\varkappa\,[\varkappa \to (\omega_1)_2^{<\omega}]$$

provide these large cardinals exist.

(d) Let L^* be any first-order language. Then the Hanf number of $L^* \cap HC$ is less than $\mu\varkappa\,[\varkappa \to (\omega_1)_2^{<\omega}]$ if it exists.

PROOF. (a) By *invariant* we here mean fully invariant (not just invariant in $\mathfrak{X}(\mathbf{R})$). (a) is then proved just like Prop. 2.6, but we need the stronger hypothesis. Of the languages in § 1, $L_{\omega_1 G}$, for example, satisfies this hypothesis, while Souslin logic does not.

(b) is immediate since the Hanf number depends only on the *PCs*.

(c) These bound were computed by Silver for the language of purely universal sentences of $L_{\omega_1\omega_1}$. Technically this language is not strong, but it is close enough for the arguments for parts (a) and (b) to go through. These bounds apply, for example, to $L_{\omega_1 G_1}$ but not to Souslin logic. For the Hanf number of the latter, see [9], [7], [11].

(d) is now immediate since any first-order language can be fattend up to a strong one. (d) is Prop. 2.4 of [2], and our 2.6 and 3.4 are more explicit formulations of things implicit in Barwise' proof. —

Craig's Interpolation Theorem for a language L^* states that disjoint $PC(L^*)$ classes (in a given fixed vocabulary) can be separated by an $EC(L^*)$. This is equivalent to the conjuction of the *Δ-Interpolation Theorem*, which states that disjoint $PC(L^*)$ classes can be separated by a class which is simultaneously $PC(L^*)$ and co-$PC(L^*)$, with the *Souslin-Kleene Theorem*, which states that any class both $PC(L^*)$ and co-$PC(L^*)$ is $EC(L^*)$. Craig's Theorem implies Beth's, and the Souslin-Kleene Theorem implies the weak version of Beth's Theorem.

6.5 ANTI-CRAIG THOREM. Let L^* be a first-order language containing $L_{\omega_2\omega}$. Then Craig's Theorem fails for L^*.

PROOF. In Prop. 2.11 of [2] Barwise derives this from Malitz' counterexample to Craig's Theorem for $L_{\omega_2\omega}$, which depends on the facts that $(\omega, \in)$ and $(\omega_1, \in)$ can be characterized up to isomorphism in $L_{\omega_2\omega}$, and that any two structures for the empty vocabulary (vocabulary with no nonlogical symbols, just the logical predicate $=$) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy $\mathfrak{A} \approx \mathfrak{B}$.

$A \subseteq HC$ is complete $\underset{\sim}{\Pi}_1(HC)$ if A is $\Pi_1(HC)$ and for any $\underset{\sim}{\Pi}_1(HC)$ B there exist a *PR* function F and a parameter $\tilde{r} \in HC$ such that $B = \{x : F(r,x) \in A\}$. No such set can be $\underset{\sim}{\Sigma}_1(HC)$ or $\underset{\sim}{\Sigma}_1(V)$ in parameters from HC.

6.6 INCOMPLETENESS THEOREM. Let L^* be a strong first-order language. Then the set of logically valid sentences of $L^* \cap HC$ is complete $\underset{\sim}{\Pi}_1(HC)$.

PROOF. Barwise, Prop. 2.15 of [2], shows this set is not $\Sigma_1(HC)$. An obvious simplification of his proof shows it is indeed complete $\Pi_1(\tilde{HC})$. Given any complete proof procedure for $L^* \cap HC$, the set of valid sentences is $\{\varphi : \exists P(P$ is a proof of $\varphi)\}$. Thus 6.6 says there can be no such proof procedure in which proofs are countable objects and being a proof is a property $\underset{\sim}{\Delta}_1$ in parameters from HC.

Let a first-order language L^* be given. We introduce a proof procedure for $L^* \cap HC$ by adjoining then the proof procedure for $L_{\omega_1\omega}$ given in [1] the following rule of inference with $\aleph_1$ premisses:

If $\vdash \mathscr{C}(\alpha, \varphi)$ for all $\alpha < \omega_1$, then $\vdash \varphi$,

where $\mathscr{C}$ is as in the Approximation Theorem 4.2.

6.7 COMPLETENESS THEOREM. Let L^* be a first-order language. The above proof procedure for $L^* \cap HC$ is sound and complete.

PROOF. $\varphi \in L^*$ is *not* valid if $\exists \mathfrak{A} \exists \alpha \neg \mathfrak{A} \vdash \mathscr{C}(\alpha, \varphi)$. This is a Σ_1 statement, and if $\varphi \in HC$, it is true iff it is true in HC, i.e. the ordinal α may be taken $< \omega_1$ and the model $\mathfrak{A}$ may be taken countable. Soundness and completeness are now immediate from the soundness and completeness of the proof procedure in [17].

6.7 shows validity for $L^* \cap HC$ is $\underset{\sim}{\Sigma}_1$ in parameters from HC plus the parameter ω_1. For particular languages from § 1 similar proof procedures have been obtained by Moschovakis (unpublished) and Green [10]. —

In the next four results **R, S, T** are disjoint nontrivial vocabularies.

6.8 DECOMPOSITION THEOREM. Let L^* be a first-order language, $\varphi \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$. Then there exist $\varphi_\alpha \in L_{\omega_1\omega}(\mathbf{R})$, $\alpha < \omega_1$, such that the followingis valid over countable structures:

$$\exists S \varphi \leftrightarrow \bigvee_{\alpha < \omega_1} \varphi_\alpha .$$

6.9 NUMBER OF MODELS. Let L^* be a first-order language. $\varphi \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$. Then up to isomorphism the number of countable models of $\exists \mathbf{S} \varphi$ is either $\leqslant \aleph_1$ or else exactly $2^{\aleph_0}$.

6.10 REDUCTION THEOREM. Let L^* be a strong firstorder language. Then for every $\varphi, \psi \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$ there exist $\varphi_0, \psi_0 \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$ such that the following are valid over countable models:

$$(\exists \mathbf{S} \varphi_0 \to \exists \mathbf{S} \varphi) \wedge (\exists \mathbf{S} \psi_0 \to \exists \mathbf{S} \psi)$$

$$\exists \mathbf{S} (\varphi \vee \psi) \to \exists \mathbf{S} (\varphi_0 \vee \psi_0)$$

$$\neg(\exists \mathbf{S} \varphi_0 \wedge \exists \mathbf{S} \psi_0)$$

6.11 UNIFORMIZATION THEOREM. Assume every real is constructible. Let L^* be a strong first-order language. Then for every $\varphi \in L^*(\mathbf{R} \cup \mathbf{S} \cup \mathbf{T}) \cap HC$ there exists $\psi \in L^*(\mathbf{R} \cup \mathbf{S} \cup \mathbf{T}) \cap HC$ such that the following are valid over conuntable structures:

$$\exists \mathbf{T} \psi \to \exists \mathbf{T} \varphi$$

$$\exists \mathbf{S} \exists \mathbf{T} \varphi \to \exists ! \mathbf{S} \exists \mathbf{T} \psi .$$

PROOF. 6.8—6.10 are the model-theoretic translations of results about invariant $\underset{\sim}{\Sigma}^1_2$ sets in [31]. 6.9 is of course immediate from 6.8 and a theorem of Morley on the number of countable models of a sentence of $L_{\omega_1\omega}$. 6.11 is similarly the model-theoretic translation of an invariant uniformization theorem (see [26], or [8] §1). An (unpublished) example of Silver shows the restriction to countable models cannot be lifted in 6.10. Myers [26] shows 6.11 cannot be proved in ZFC alone. Cf. also [30] for related observations.

6.12 THEOREM. Let L^* be a strong first-order language. Then the following fail for $L^* \cap HC$:

(a) Craig's Interpolation Theorem

(b) The Souslin-Kleene Theorem

(c) The Δ-Interpolation Theorem

(d) Beth's Definability Theorem

(e) Weak Beth's Theorem

PROOF. For (c) this is the model-theoretic translation of the fact that there exist disjoint invariant $\underset{\sim}{\Sigma}^1_2$ sets which cannot be separated by a $\underset{\sim}{\Delta}^1_2$ set. See [2], Prop. 2.13. (e) is Thm. 5.1, and this implies the rest.

One large problem in the model theory of strong first-order languages remains open, which does not lend itself to abstract, descriptive-set-theoretic statement: Can we prove for, say, $L_{\omega_1 G}$, that any sentence preserved under substructure (resp. homomorphic image) is equivalent to a universal (resp. positive) sentence? Harnik [13] has proved preservation theorems for $L_{\omega_1 G}$ for some symmetric relations ($L_{\omega\omega}$-elementary equivalence, the ρ-isomorphism of Scott, isomorphism of direct squares, etc.); his results (by the proofs of [13] or by alternative proofs due to Miller) extend to some of the other languages of § 1.

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ON A FOUNDATION FOR MATHEMATICS — A VIEW OF MATHEMATICS I*

Milan ĐURIĆ

1. Introduction

This paper represents the beginning of our final considerations of an approach to the foundations of mathematics initiated by the paper [5]. We shall deal in it with mathematical activity and its goals. We shall start from the assumption that the primary goal of mathematical activity is the creation of certain entities which will comprise in themselves (all) that activity. Such entities, we shall call mathematical entities. Our next assumption is that any performed mathematical activity creates conditions, i.e., makes a groundwork for a new mathematical activity and hence for the creation of new mathematical entities. These new entities are of a higher level with respect to old ones. If we now assume that all mathematical entities constitute an edifice which we shall call the world of mathematics, then we shall have that this world consists of mathematical entities of various sorts and levels.

In the creation of such a world we accept a symbolic form of presentation. Namely, we assume that there is a collection of symbols which stand for mathematical entities of various sorts and levels. Such a collection will be a symbolic frame of the world of mathematics. We shall denote it by $\mathcal{S}$. If we build up the world on this collection, then we shall say that we have a symbolic form of the world of mathematics and of mathematical entities as its constituents. We shall obtain concrete mathematical entities by naming symbols of such a world according to their creative procedures given in the paper.

If we assume that mathematical entities in question are certain organized wholes, which we call spatial wholes, then we might say that the world of mathematics consists of spatial wholes of various sorts and levels. Together with these entities always go some other entities: connectives between them. In such a way we obtain that the world of mathematics consists of two sorts of entities of various levels. It means that for its creation is enough to start from a subcollection $\mathcal{M}$ of $\mathcal{S}$, consisting of two-sort symbols of various levels. Other symbols of $\mathcal{S}$ are then reserved to stand for properties and other things which are relevent for entities of $\mathcal{M}$.

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The collection $\mathcal{M}$ will serve as a framework for the creation of the world of mathematics. We shall specify the fundamental acts necessary for its creation. The central act occupies of course the creation of spatial wholes — objects of the world. We shall define the concept of a spatial whole and point out its main features. Furthermore, we shall give some examples of spatial wholes and establish the link between the creation of particular kinds of spatial wholes and certain standard mathematical conceptions, like formalism and intuitionism. In such a manner we shall show that these conceptions justify our attitude concerning the goal of mathematical activity. Otherwise, all these investigations will serve as a basis for the process of formalization.

2. Species and spatial wholes

This section is devoted to a general discussion of the mathematical world and to main concepts which arise in the creation of this world. These concepts are species and spatial wholes. We shall see which mathematical activity is comprised in their creation.

Before we begin our consideration of the above concepts we would be shortly concerned with the activity of human beings in general and then within such an activity would find the position of the mathematical activity.

Certainly, in a human activity one can always recognize two things: the goal of activity and means which men have at their disposal to attain the goal. When one is provided with these two things then he has still to decide in which manner to realize the activity. It means that he must have a plan — a scheme for its performing; of course, there also must be criteria for deciding on each of these things.

Before all, the goal of activity has to be determined according to our needs and wishes; these two things are otherwise restricted by certain external moments. Since an activity is always realized within a frame which has its own principles, then we must take into account that it should not violate these principles. If the question is about the organization of a society, then we have principles of various kinds, like social, political and many others. All these particular moments are beyond our interest and therefore we shall not be concerned with them here. However, they will find their place in our global considerations of the organization of $\mathcal{S}$; of course, in a form which we shall be able to set up.

We further have that means for attaining the goal of activity are different and determined according to our wishes to have some, in a certain sense, optimal properties of the goal. Independently of concrete goals people deal with discovering general and always new means for performing their activities and then in concrete situations utilize adequate ones according to desired properties of the goals and considered objects by means of which they build them up. Clearly, we cannot apply any means to each collection of concrete objects. Thus when we specify means we decide on their choice according to the regarded collection of objects. At the end of these general considerations of activities of human beings we shall still be concerned in short with plans for performing activities. The plans are to be given in oral or written form and their purpose is to specify and also to memorize performances of activity.

Mathematics cannot be set apart from these general activities of human beings. It can deal with them only in abstract forms. According to our views, its goals are, in the main, creations of abstract spatial wholes, means for the purpose are certain constructions — operations and plans of activity are symbolic schemata. In the sequel we shall concern all these concepts.

Now we shall begin with a general description of the mathematical world and main activities for its building. We shall start from a large collection consisting of symbols of different sorts and levels. The levels of starting symbols of $\mathcal{S}$ we denote by -1 and the collection containing all these symbols by $\mathcal{S}_0$. We further have different-sort symbols of the levels 0, 1, ... and respective collections $\mathcal{S}_1$, $\mathcal{S}_2, \ldots$ which contain them. If we denote the hierarchy of all levels by $\mathcal{G}$, then we can write the above collection as an indexed collection $\langle \mathcal{S}_i \mid i \in \mathcal{G} \rangle$. This collection is so far without any condition being imposed upon it and its elements.

Since $\mathcal{S}$ contains various symbols in itself we therefore have to carry out some systematizations in it in order to make it capable of suiting our purposes. To do this we shall consider nature around us. Two basic concepts in it are real objects of various levels: electrons, atoms, molecules, actual objects we are surrounded by etc. and forces among them. Forces on a level in nature may be of different characters and sources which we shall not discuss here. These two concepts are quite sufficient for building up the real world; forces are otherwise responsible for its existence as a whole, although they are not sufficient for a complete descirption of it: of all phenomena and events in it. Taking into account the former fact, we shall select in $\mathcal{S}$, by the analogy, a collection $\mathcal{M}$ of two-sort symbols of different levels, which will be sufficient for our purposes: building up the mathematical world. One sort of symbols in it will correspond to objects: natural or abstract and the other to connectives between these symbols. The former symbols we shall call *objects* and the latter, *arrows*. Thus, the collection $\mathcal{M}$ consists of objects and arrows of various levels. Such a collection, which is otherwise quite natural, will serve as a framework for building up the mathematical world. Other symbols of $\mathcal{S}$ will only serve for its description.

In what follows we shall make some further specifications in $\mathcal{M}$. Namely, we shall let the possibility to characterize and hence to differentiate the symbols of $\mathcal{M}$. We can do this by adjoining to each level of $\mathcal{M}$ certain new symbols of $\mathcal{S}$ which will become certain integral parts of the symbols of $\mathcal{M}$, characterizing them. These new symbols we assume to be characteristic properties, which mathematical entities can be supposed to possesss. Since we have in $\mathcal{M}$, on each level, two-sort symbols, then the adjoined symbols, to any level of $\mathcal{M}$, have also to be two-sort: the ones for objects and the others for arrows. We here assume that there are some relationships between the properties of objects and arrows: we assume that arrows have properties of carrying information on objects and their properties; information are otherwise to be specified in each concrete case.

By means of symbols representing properties of objects we can make certain selections in $\mathcal{M}$. These selections are our starting acts. What do we do, in fact? We select (all) objects on a level of $\mathcal{M}$, agreeing in some common (attributes) — characteristic property(ies), in particular collections. Such collections we then call species. We shall specify this somewhat more.

Denote by $\mathfrak{P}$ the collection of (all) possible properties which mathematical entities on a level of $\mathcal{M}$ can be supposed to possess. By applying this collection to the considered level of $\mathcal{M}$ we shall select various collections of objects and arrows on it. Let us see in which way. First, we shall concern the question of the selection of collections of objects on the regarded level of $\mathcal{M}$.

Let us consider a many-valued function

$$S: \mathfrak{P}_{ob} \to \mathcal{M}_{ob'}$$

where $\mathfrak{P}_{ob}$ means the collection of properties which mathematical objects on the considered level of $\mathcal{M}$ can be supposed to possess and $\mathcal{M}_{ob}$ means the collection of objects on that level of $\mathcal{M}$. Such a function we shall call the application of $\mathfrak{P}_{ob}$ to $\mathcal{M}_{ob}$. It assigns, to each property $P \in \mathfrak{P}_{ob}$, a collection of objects of the considered level of $\mathcal{M}$ for each of which one can suppose to possess this property. Such a collection we shall call a species. Thus we define a species as follows:

DEFINITION 1. By a *species* on a level of $\mathcal{M}$ we mean the image of a property P under an application S of $\mathfrak{P}_{ob}$ to $\mathcal{M}_{ob}$, which consists of (all) those objects of $\mathcal{M}_{ob}$ for which one can suppose to possess this property.

When a species $S(P)$ is defined, then any mathematical object which has been or might have been generated before $S(P)$ and which satisfies the condition P, is a member of the species $S(P)$. In the sequel we shall deal with the mode of generation of mathematical objects and in such a way shall contribute to the specification of members of species.

Although the study of species is not our main task in the paper, we shall still deal with certain concepts that concern them. At that, all used signs will have the usual meanings. Otherwise, one can find the definitions of these concepts in [11].

A species $S(P)$ is *empty* if, in the application of S, we cannot select any object of $\mathcal{M}_{ob}$ which satisfies the condition P. If the application S is a single-valued function, then we have the case of a *singleton* species. The size of a species is otherwise to be determined by its relating to the species of natural numbers as it is given in [11].

We further have certain relationships between species. These relations arise from the relationships which exist between the properties. If we have, for instance, that there is a relationship $P \to P'$ between two properties P and P' of $\mathfrak{P}_{ob}$, which means that, if an object has the property P, then it also has the property P', then we shall have that the species $S(P)$ is contained in the species $S(P')$, or that it is a *subspecies* of the species $S(P')$. If the above is also valid conversely, then we shall say that the species $S(P)$ and $S(P')$ are *equal*.

We can now define the concept of splitting up a species. If there is a relation $S(P) = S(P') \cup S(P'')$, where $P' \neq P''$, then we shall say that $S(P)$ is *split up* into species $S(P')$ and $S(P'')$. If $S(P')$ is here a subspecies of the species $S(P)$ and $S(P'')$ the difference $S(P) - S(P')$, then we shall say that $S(P')$ is a *detachable* subspecies of $S(P)$.

One could deal now with further questions concerning species. However, we shall not do this, especially because some of these questions are not essential for this paper and since some of them will arise later in the consideration of species which are endowed with collections of arrows and then with a certain structure. Thus we shall consider that species are specified enough for our further purposes.

Having finished with the selections of collections of objects on particular levels of $\mathcal{M}$, called species, we shall be concerned with the selection of arrows. We shall assume that any species of any level of $\mathcal{M}$ is endowed with a collection of arrows. Let us see in which manner we distribute arrows over species. If we have a species $S(P)$, then we assume that arrows in $S(P)$ are those which naturally belong to it and will do so if they preserve the property P. This property is intrinsic for the objects of species. We shall call the arrows with this property relevant arrows So, their definition is as follows:

DEFINITION 2. By an arrow *relevant* for the species under consideration we understand the arrow which preserves certain intrinsic properties characterizing its objects.

In such a way species of $\mathcal{M}$ are endowed with arrows which carry in themselves information on their objects: their structure and properties. From now on, when we say a species $S(P)$, we shall alwas regard that it is endowed with a collection of relevant arrows. Here $S_{ob}(P)$ will mean the collection of objects of $S(P)$ and $S_{ar}(P)$, the collection of arrows of $S(P)$. Otherwise, if there is no possibility of confusion, we shall denote a species $S(P)$ simply by S, i.e., we shall identify it with the application S.

Now, in order to make the species capable of satisfying our purposes we shall provide them with a certain fundamental structure. We assume here the structure of a (quasi)category*, In the following section we shall explain what this structure means.

Let S be a species on a level of $\mathcal{M}$. Endow it with two unary functions $\mathcal{D}_0, \mathcal{D}_1 : S \to S_{ob}$ and a binary function $\mathcal{C} : S^2 \to S$. In that way we obtain a system $\langle S; \mathcal{D}_0, \mathcal{D}_1, \mathcal{C}\rangle$. We have the following meanings in this system: $\mathcal{D}_0(\alpha)=x$ means that the object x is the source of the arrow α; $\mathcal{D}_1(\alpha)=y$ means that the object y is the target of the arrow α and $\mathcal{C}(\alpha,\beta)=\gamma$, which we shall also write as $\mathcal{C}(\alpha,\beta;\gamma)$, means that the arrow γ is the composition of the arrow α, followed by the arrow β.

If we now involve certain laws to specify the functions in the above system, we shall obtain a desired fundamental structure. First, we have a structure called a quasicategory:

DEFINITION 3. By a *quasicategory* we mean a system $\langle S; \mathcal{D}_0, \mathcal{D}_1, \mathcal{C}\rangle$ for which the following two groups of laws hold:

C1. $$\mathcal{D}_n(\mathcal{D}_m(\alpha)) = \mathcal{D}_m(\alpha), \quad n, m = 0, 1,$$

$$\exists\gamma\,\mathcal{C}(\alpha, \beta; \gamma) \Rightarrow \mathcal{D}_1(\alpha) = \mathcal{D}_0(\beta),$$

$$\mathcal{C}(\alpha, \beta; \gamma) \Rightarrow \mathcal{D}_0(\alpha) = \mathcal{D}_0(\gamma) \wedge \mathcal{D}_1(\gamma) = \mathcal{D}_1(\beta),$$

$$\mathcal{C}(\alpha, \beta; \gamma) \wedge \mathcal{C}(\alpha, \beta; \gamma') \Rightarrow \gamma = \gamma';$$

C2. $$\mathcal{C}(\mathcal{D}_0(\alpha), \alpha; \alpha) \wedge \mathcal{C}(\alpha, \mathcal{D}_1(\alpha); \alpha).$$

The symbols $\wedge$, $\Rightarrow$ and $\exists$ have usual meanings: $\wedge$ (and), $\Rightarrow$ (if ..., then ...) and $\exists$ (there exists). If we do not take differently, these and other logical symbols will have only such meanings throughout the paper.

* In our papers, we have called this term so far a fundamental (quasi)semigroupoid. We think that this term is better because it carries in itself a structural meaning of the concept. Meanwhile, this is only our opinion, And since category theory is a highly developed theory, then there is no reason for changing the names of it and its concepts. Therefore, we accept here the standard name — a (quasi)category.

If we add a new law to the first group of laws

$$\mathcal{D}_1(\alpha)=\mathcal{D}_0(\beta) \Rightarrow \exists \gamma\, \mathcal{C}(\alpha, \beta; \gamma),$$

and also the law

$$\text{C3.} \quad \mathcal{C}(\alpha, \beta; \delta) \wedge \mathcal{C}(\beta, \gamma; \zeta) \wedge \mathcal{C}(\alpha, \zeta; \eta) \wedge \mathcal{C}(\delta, \gamma; \xi) \Rightarrow \eta=\xi,$$

which means the associativity of $\mathcal{C}$, then a quasicategory becomes a category [15]

Furthermore, if we add the law

$$\text{C4.} \quad \forall \alpha \exists \beta (\mathcal{C}(\alpha, \beta; \mathcal{D}_0(\alpha)) \wedge \mathcal{C}(\beta, \alpha; \mathcal{D}_1(\alpha))),$$

then from a quasicategory we obtain a quasisemigroupoid and from a category, a groupoid.

If moreover we take that $\mathcal{D}_0=\mathcal{D}_1$ and that both are constant functions, then a (quasi)semigroupoid is reduced to a (quasi)semigroup and a (quasi)groupoid to a (quasi)group.

Certainly, a morphism between two (quasi) categories is a *functor* [15]; it is a relevant arrow in our sense. We have further morphisms between functors, called *natural transformations,* then morphisms between natural transformations, morphisms between these new morphisms etc. By this process of involving relevant arrows, we could define certain many-valued functors [6] between (quasi)categories possessing various structures as, for instance, the simplicial one, etc.

Since we have specified the basic collections of symbols of $\mathcal{M}$, called species, and have involved certain fundamental structures in them, we shall proceed further to make certain organized wholes from them. From now on we shall fix the fundamental structure on species. We assume it to be a (quasi)category: it means a quasicategory or a category, when it is necessary. A species endowed with such a structure, we shall call a *fundamental world.*

In order to make an organized whole fom a fundamental world in question we must claim that it allows some reasonable creations and other activities in itself. In what follows we shall deal with creations and collections on which they ought to be performed.

The basic purpose of creations on a fundamental world is to give us a possibility to construct new objects from the old ones. We dealt in [6] with certain creations on categories. We created certain concepts having certain geometrical shapes: cylinders, cones, etc. Here we shall be concerned with cones and cocones, since wanted constructions are contained in the creation of certain kinds of these. Thus, here cones and cocones are creative concepts. We shall call them simply *creative concepts.* In what follows we shall explain what they mean.

By a *cone* in a fundamental world W we mean a triple $(U, \Phi, \{v\})$, consisting of a subcollection U of W, a collection Φ of arrows of W and a singleton subcollection $\{v\}$ of W, consisting of an object v of W, called the vertex of the cone, such that for any arrow $\alpha: u' \to u \in U_{ar}$ there are arrows $\varphi: u \to v$ and $\varphi': u' \to v$ of Φ so that $\mathcal{C}(\varphi, \alpha; \varphi')$ holds. In future, when it is obvious from the context what is the basis and vertex of the cone, we shall always identify it with the collection of arrows connecting these.

Let $\mathfrak{C}(U)=\{\Phi_i \mid i\in\mathcal{A}\}$ be the collection of all cones over U, endowed with the collection of cone-arrows. The initial object in this collection we shall call the first cone, abbreviated f.c., and denote it by Φ^c. A Φ^c is defined in the following manner: for each $\Phi\in\mathfrak{C}(U)$ there is an arrow $\gamma : \nu^c\to\nu$, where ν^c is the vertex of Φ^c and ν of Φ, such that $\mathcal{C}(\gamma, \varphi^c; \varphi)$ holds, for $\varphi\in\Phi$ and $\varphi^c\in\Phi^c$. In the opposite direction, we have the concept of a *cocone* with concepts of a cobasis and a covertex. The terminal object in the collection of all cocones over a collection in the world W we shall call the last cocone and abbreviate it as l.c.c.

Vertices of f.c. and l.c.c., we called in [6] a *sequent* and a *presequent*, respectively. If the basis of f.c. and the cobasis of l.c.c. contain only single objects, then their sequent and presequent we called a *successor* and a *predecessor*, respectively.

The above objects: sequents and presequents will be constructive objects in our fundamental world. Such unique objects are well-known in category theory as colimits and limits, respectively. In this case, we shall diverge from standard terminology [15] and accept our terms for these objects.

Since we have specified objects which are to be constructed we must now specify collections of the world, which will allow their construction. Moreover, we must specify certain conditions on the collections, which will determine the character of constructed objects. Thus, our basic task is to specify choices of collections on which we perform constructions and to specify certain requirements on them which will determine the peculiarity of constructed objects.

First, we shall specify the concept of a choice in a fundamental world under consideration.

DEFINITION 4. By a *choice* in a fundamental world W we mean an application σ of a fundamental world J to the world W, i.e., a many-valued functor

$$\sigma : J\to W,$$

which assigns, to each object $i\in J$, a collection $\sigma(i)\subset W$ and to each arrow $i\to i'\in J$, a relevant arrow $\sigma(i)\to\sigma(i')$. The choice σ is *lawlike*, if there is a law or a collection of laws, according to which it is to be performed.

A choice σ, determined by the collection of laws Λ, we shall denote by σ_Λ. Thus, if we have the chosen collection $\sigma_\Lambda(i)$, $i\in J$, on W, then it will mean that it satisfies the conditions of Λ.

In order to ensure the possibility of having various choices for single objects of J, we shall involve the concept of parametrized choice.

DEFINITION 5. By a choice in the fundamental world W, ***parametrized*** by means of a fundamental world $\mathcal{S}$, we mean a collection of choices $\mathfrak{S} = \{\sigma^s \mid s\in\mathcal{S}_{ob}\}$ such that, if there is an arrow $s\to s'\in\mathcal{S}$, then there is also a natural transformation $\sigma^s\to\sigma^{s'}$.

A natural transformaton η between two many-valued functors σ and τ, symbolically $\eta : \sigma\to\tau$, is defined in the same way as the transformation between single-valued ones, but with a difference: instead of an arrow, we now have the concept of a (co)cylinder [6]. Thus, while σ and τ are many-valued functors, $\eta : \sigma\to\tau$ is a (co)cylinder with the lower (co)basis σ and the upper one in τ. If one of these functors is single-valued, then we shall obtain concepts of a cocone and a cone, respectively.

We can represent a parametrized choice $\mathfrak{S}$ as a collection of functors $\sigma:\mathcal{S}\rightarrow \mathrm{Fun}_{mv}(J, W)$ such that σ^s, $s\in\mathcal{S}_{ob}$ are many-valued functors of J to W and σ^f, for an $\mathcal{S}$-arrow f, are natural transformations; it means that, if $f:s\rightarrow s'$ is an $\mathcal{S}$-arrow, then $\sigma^f:\sigma^s\rightarrow\sigma^{s'}$ is a (co)cylinder.

According to the definition, $\mathfrak{S}$ constitutes a fundamental world: its objects are chosen collections of the fundamental world under consideration and arrows are cylinder-arrows [6].

Since our aim is not to have arbitrary choices but choices determined by certain conditions, then we shall impose these upon them. We shall assume that each choice $\sigma^s\in\mathfrak{S}$ satisfies a collection $\Lambda_s, s\in\mathcal{S}_{ob}$, of conditions. Such a choice, we shall denote by $\sigma^s_{\Lambda_s}$. Hence we have that $\mathfrak{S}$ consists of choices $\sigma^s_{\Lambda_s}$, $s\in\mathcal{S}_{ob}$. If Λ means the collection of collections $\Lambda_s, s\in\mathcal{S}_{ob}$, then we shall write $\mathfrak{S}$ by $\mathfrak{S}_{(\Lambda)}$. Thus $\mathfrak{S}_{(\Lambda)}$ will mean a collection of lawlike choices $\sigma^s_{\Lambda_s}$, $s\in\mathcal{S}_{ob}$. We shall call it the *lawlike parametrized choice*. Such a choice will constitute a fundamental world if the existence of an $\mathcal{S}$-arrow $s\rightarrow s'$ implies the existence of a relevant arrow between collections Λ_s and $\Lambda_{s'}$ and a natural transformation $\eta^{s,s'}:\sigma^s_{\Lambda_s}\rightarrow\sigma^{s'}_{\Lambda_{s'}}$. A relevant arrow $\varphi:\Lambda_s\rightarrow\Lambda_{s'}$ together with a natural transformation $\eta^{s,s'}:\sigma^s_{\Lambda_s}\rightarrow\sigma^{s'}_{\Lambda_{s'}}$ will be a relevant arrow between elements of $\mathfrak{S}_{(\Lambda)}$ which we shall simply call a *choice-arrow*.

Certainly, a lawlike parametrized choice $\mathfrak{S}_{(\Lambda)}$ will be specified when we specify its objects and these when we specify the collection $\Lambda=\{\Lambda_s\,|\,s\in\mathcal{S}_{ob}\}$. Thus in order to specify the choice $\mathfrak{S}_{(\Lambda)}$ we have to specify collections Λ_s and their connectives. In what follows we shall be concerned, but only in general, with this question.

There are two moments which we have to differentiate in each collection Λ_s of Λ : effective procedures by means of which we choose subcollections of the world under consideration and conditions which chosen collections have to satisfy, such as size, ordering, constructive properties, etc. First of all, we could specify various algorithms for choosing mentioned collections of objects and arrows of the world in question. Among them, however, we shall accept only those which ensure certain necessary properties of chosen collections and hence wanted peculiarities of constructed objects. Of course, if we want to have peculiarities of the whole choice $\mathfrak{S}_{(\Lambda)}$ we have moreover to specify connectives between its members. We could assume an example in which choices are sequences of objects and arrows between them, chosen by a collection of conditions, and choice-arrows are relevant arrows between these sequences. Throughout the paper, we shall deal with the further specification of the collection Λ. We shall also give some concrete examples.

Besides conditions which are imposed upon objects of $\mathfrak{S}_{(\Lambda)}$, we might also impose crtain conditions upon $\mathfrak{S}_{(\Lambda)}$ as a whole. Namely, we might claim that $\mathfrak{S}_{(\Lambda)}$ as a whole obeys certain conditions: to be directed for instance, to have some creative properties, etc. If Ω is such a collection of conditions on $\mathfrak{S}_{(\Lambda)}$, then we shall emphasize this by writing $\mathfrak{S}^{\Omega}_{(\Lambda)}$ instead of $\mathfrak{S}_{(\Lambda)}$. In the collection Ω, there may be reflected properties of the world $\mathcal{S}$ by means of which the choice $\mathfrak{S}_{(\Lambda)}$ is parametrized. If we suppose that the collections of conditions Λ and Ω are completely specified, then so is the choice $\mathfrak{S}^{\Omega}_{(\Lambda)}$. Otherwise, the collections Λ and Ω may be independent or that the collection Ω contains some further specifications of the collection Λ.

From now on, a choice $\mathfrak{S}_{(\Lambda)}$ on W, parametrized by the world $\mathcal{S}$, we shall regard as a many-valued functor of J into W which assigns, to each object i of J, the fundamental world $\mathfrak{S}^{\Omega}_{(\Lambda)}(i)$, objects of which are collections $\sigma^{s}_{\Lambda_s}(i)$, $s\in\mathcal{S}_{ob}$, of objects and arrows of W determined by rules of Λ_s and relevant arrows of which are choice-arrows $\sigma^{s}_{\Lambda_s}(i)\to\sigma^{s'}_{s'}(i)$, s, $s'\in\mathcal{S}_{ob}$ and which possesses the conditions of Ω, and to each arrow $i\to i'$ of J, a relevant functor $\mathfrak{S}^{\Omega}_{(\Lambda)}(i)\to\mathfrak{S}^{\Omega}_{(\Lambda)}(i')$, i.e., a functor which preserves intrinsic properties of the world $\mathfrak{S}^{\Omega}_{(\Lambda)}$.

Let $\mathfrak{C}(W)=\left\{\mathfrak{S}^{\Omega_\alpha}_{(\Lambda_\beta)}\,|\,\alpha\in\mathcal{A}\wedge\beta\in\mathcal{B}\right\}$ be the collection of lawlike parametrized choices on the fundamental world W; at this we assume that there is a collection $\mathcal{S}=\{\mathcal{S}_\alpha\,|\,\alpha\in\mathcal{A}\}$ of parameter worlds. If $\mathfrak{S}^{\Omega_\alpha}_{(\Lambda_\beta)}$ and $\mathfrak{S}^{\Omega_{\alpha'}}_{(\Lambda_{\beta'})}$ are two members of the collection $\mathfrak{C}(W)$, then we can define a relevant arrow between them. It is a functor $R:\mathfrak{S}^{\Omega_\alpha}_{(\Lambda_\beta)}\to\mathfrak{S}^{\Omega_{\alpha'}}_{(\Lambda_{\beta'})}$ which assigns, to each choice $\sigma^{s_\alpha}_{\Lambda_{s_\alpha}\beta}\in\mathfrak{S}^{\Omega_\alpha}_{(\Lambda_\beta)}$, a choice $R\left(\sigma^{s_\alpha}_{\Lambda_{s_\alpha}\beta}\right)\in\mathfrak{S}^{\Omega_{\alpha'}}_{(\Lambda_{\beta'})}$ and, to each choice-arrow $\varphi\in\mathfrak{S}^{\Omega_\alpha}_{(\Lambda_\beta)}$, a choice-arrow $R(\varphi)\in\mathfrak{S}^{\Omega_{\alpha'}}_{(\Lambda_{\beta'})}$ and moreover preserves the conditions of Ω. Provided with such arrows it becomes a category. We shall return later to some further questions concerning the collection $\mathfrak{C}(W)$ and collections created on it.

Since we have finished with a general consideration of choices on W, we shall be concerned with the concept of spatial whole. We have already said that this concept arise from certain constructive activities on the world under consideration. Since we have done all preparations for such activities, we shall proceed to specify them.

Let $\mathfrak{S}^{\Omega}_{(\Lambda)}$ be a lawlike choice functor of J to the world W, parametrized by the world $\mathcal{S}$. As we have already seen, this functor assigns, to each object $i\in J$, a fundamental world $\mathfrak{S}^{\Omega}_{(\Lambda)}(i)$ on W consisting of choices $\sigma^{s}_{\Lambda_s}(i)$, $s\in\mathcal{S}_{ob}$, and of choice-arrows as connectives between them. By means of this functor is specified the choice activity on the considered fundamental world. Our ultimate aim, however, is not such an activity, but the constructive activity. We shall ensure this if we claim that chosen collections in the world allow some creations; we here decided on cone and cocone creations. In that way, the choice activity on a fundamental world becomes a preparatory activity for the creative activity.

If we have, for instance, a choice $\sigma^{s}_{\Lambda_s}$, $s\in\mathcal{S}_{ob}$ in W and a cone as the creative concept on it, then we can express this as a requirement that there is a single-valued functor $F^s:J\to W$ and a natural transformation $\eta^s:\sigma^{s}_{\Lambda_s}\to F^s$. Certainly, the triple $(\sigma^{s}_{\Lambda_s},\eta^s,F^s)$ is a cone with the vertex F^s; we could say that the functor F^s is a *creative functor* for the choice functor $\sigma^{s}_{\Lambda_s}$. Hence we have that $(\sigma^{s}_{\Lambda_s},\eta^s,F^s)(i)$, $i\in J$, is a cone with the vertex $F^s(i)$ in the world W. We shall denote a cone over

$\sigma^s_{\Lambda_s}$ by ${}_c\sigma^s_{\Lambda_s}$. If all choices $\sigma^s_{\Lambda_s} \in \mathfrak{S}^{\Omega}_{(\Lambda)}$ allow creations of cones, i.e., if for each $s \in \mathcal{S}_{ob}$ there is a single-valued functor F^s together with a natural transformation $\eta^s : \sigma^s_{\Lambda_s} \to F^s$, then we shall emphasize this by ${}_{(c)}\mathfrak{S}^{\Omega}_{(\Lambda)}$. If it is the world about cocones, then we shall accept the denotation ${}_{(cc)}\mathfrak{S}^{\Omega}_{(\Lambda)}$. However, in future, we shall simply write ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ considering that this means that each choice of $\mathfrak{S}^{\Omega}_{(\Lambda)}$ allows a $*$-creation which may be a cone or a cocone, or even both them. These concepts, as we have already said, are *creative concepts* in the paper with a common denotation $*$; if its (co)basis is known, σ for instance, then we shall write it by ${}_*\sigma$. To mention that we could decide on broader kinds of creative concepts such as cylinders and cocylinders. However, we shall only deal with accepted concepts; it means, cones and cocones. Otherwise, these cencepts, as we shall see later, are able to incorporate in themselves logical concepts of production (derivation) with vertices as produced — created objects, peculiarities of which are determined by conditions being imposed on choices. If each choice of $\mathfrak{S}^{\Omega}_{(\Lambda)}$ allows the creation of the concept $*$, then we shall say that $\mathfrak{S}^{\Omega}_{(\Lambda)}$ is *$*$-completed* in itself or outside, depending if the concept $*$ belongs to $\mathfrak{S}^{\Omega}_{(\Lambda)}$ or not; of course, its (co)basis belongs to it. As we know, the number of creative concepts of ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ and connectives between these are determined by means of the world $\mathcal{S}$.

It is clear that for each $s \in \mathcal{S}_{ob}$ there may exist many cones over the same choice $\sigma^s_{\Lambda_s}$ as their basis. We could point this by writing ${}_{*\alpha}\sigma^s_{\Lambda_s}$, if it is the world about the creative concept $*$; here $\alpha \in \mathcal{A}$, where $\mathcal{A}$ means the number of creative concepts over one and the same choice. However, among the possible creative concepts we might decide on the ultimate ones, i.e., on f.c. and l.c.c. concepts: unique or not unique.

Now, if we have a $*$-completed functor ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$, then the following question arises: can we complete the functor ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ as a whole? Certainly, we can do this. Such a functor will be completed if there is a creative concept $\bullet$ which will constitute a cone or a cocone with it; clearly, $\bullet$ is also a many-valued functor having the shape of a creative concept. If the choice functor is $\bullet$-completed, then we shall denote it by ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$. Certainly, the functor ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$ has complete its elements and is completed as a whole.

There is an elementary proposition which establishes the link between constructive objects of ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$: that one of $\bullet$ and those of choices of $\mathfrak{S}^{\Omega}_{(\Lambda)}$.

PROPOSITION 1. *If the concept $\bullet$ in ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$ is its l.c.c. (or f.c.), and if moreover each creative concept ${}_*\sigma^s_{\Lambda_s}$ of ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ is l.c.c. (or f.c.), then the covertex (the vertex) of $\bullet$ is the presequent of presequents (the sequent of sequents) of choices of $\mathfrak{S}^{\Omega}_{(\Lambda)}$.*

PROOF. Denote by $\mathcal{H}$ the collection of all presequents (sequents) of choices $\sigma^s_{\Lambda_s}$, $s \in \mathcal{S}_{ob}$ and by $P(S)$ the covertex (the vertex) of the concept $\bullet$. Clearly, $P(S)$ is the covertex (the vertex) of a cocone (a cone) over $\mathcal{H}$. It is easy to see that such a cocone (cone) is l.c.c. (f.c.). ∎

Hence we have that the constructive object of the concept $*$ is a *construction of constructions* on choices of $\mathfrak{S}^{\Omega}_{(\Lambda)}$. In such a way we have a terminating procedure for the creation of objects within a fundamental world. Such a terminating procedure will be later utilized in the definition of proof.

A $\bullet$-completed functor $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$ will serve for the creation of a spatial whole from the fundamental worlds making its domain and codomain: J and W for instance. This functor assigns, to the world J, a collection $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})\ (J)$ of fundamental worlds and functors chosen on W. We might claim that this functor is such that this collection is also a fundamental world which moreover possesses some properties: to have the first and the last object, to be directed, well-ordered, etc. We might add to it some further requirements which govern the formation of wanted spatial whole as, for instance, separation ones. Denote the collection of all such requirements on the functor by Θ and the functor itself by $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$. By means of such a functor we shall define the concept of spatial whole on a fundamental world.

DEFINITION 6. By an $\mathscr{S}$ J'*-spatial organization* on the fundamental world W of a level of $\mathscr{M}$ we mean a choice functor $_{*}\mathfrak{S}^{\Omega}_{(\Lambda)}: J \to W$, which assigns, to each object $i \in J$, a $*$-completed fundamental world $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}(i)$ of W and, to each arrow $i \to i' \in J$, a relevant functor $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}(i) \to {}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}(i')$, for which there is a functor $\bullet: J' \to W$, where $J' \subset J$, which assigns, to each object $i \in J'$, a creative concept $\bullet\ (i)$ of the same type as those of $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}(i)$, $i \in J'$ and, to each J'-arrow $i \to i'$, a relevant arrow between these concepts, together with a natural transformation $\gamma: \bullet \to {}_{(*)}\mathfrak{S}^{\Omega}_{(\Omega)}$ or $\gamma': {}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)} \to \bullet$ such that the triples $(\bullet, \gamma, {}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})\ (i)$ and $({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}, \gamma', \bullet)\ (i)$, $i \in J'$ are a cocone and a cone respectively and such to satisfy certain conditions given in the collection Θ.

By an $\mathscr{S}$ J'*-spatial whole* we mean the triple $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$ consisting of the worlds J and W and of an $\mathscr{S}J'$-spatial organization $\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$.

According to the definition, a spatial organization $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ on a fundamental world gives a certain creative closeness and in such a way creative possibilities of the world. These possibilities and their peculiarities are, otherwise determined by the collections of conditions Λ, Ω and Θ in which, as we have already said, properties of the worlds $\mathscr{S}$ and J may be included. By means of these collections, we are able to *handle* choices and creations on the world in question. In such a way we enable that certain particularly chosen parts or the whole world allow creative activity and moreover to obtain wanted kind of created objects in it. In what follows we shall be concerned with certain properties and further specifications of spatial organizations on a fundamental world.

We shall get further peculiarities of spatial wholes if we suppose that the functor $\mathfrak{S}^{\Omega}_{(\Lambda)}$ is *transitive*, i.e., if we suppose that $\sigma^{s}_{\Lambda_s} \in \mathfrak{S}^{\Omega}_{(\Lambda)} \Rightarrow \sigma^{s}_{\Lambda_s} \subset \mathfrak{S}^{\Omega}_{(\Lambda)}$ for each $s \in \mathscr{S}_{ob}$. Hence we would have that $\mathfrak{S}^{\Omega}_{(\Lambda)}$ is also a choice on the world in question and that $\sigma^{s}_{\Lambda_s}$, $s \in \mathscr{S}_{ob}$ are its parts.

If we consider now a spatial organization such that the functor $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ is transitive and such that together with $\sigma^{s}_{\Lambda_s}$ it contains the creative concept $_{*}\sigma^{s}_{\Lambda_s}$, then it can possess convenient properties. So, for instance, if we suppose that $\mathscr{S}$ is an ordinal, ω for example, then we can prove the following

PROPOSITION 2. *If the functor* ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ *is transitive and parametrized by the ordinal* ω, *then we can make it to be recursive.* ▮

PROOF. Take any choice $\sigma^{k}_{\Lambda k}\in\mathfrak{S}^{\Omega}_{(\Lambda)}$, $k\in\omega$ and the creative concept ${}_{*}\sigma^{k}_{\Lambda k}$ over it. If we specify the conditions of Ω in such a way that this concept is the choice for the next creation of the same type, i.e., if ${}_{*}\sigma^{k+1}_{\Lambda k+1}={}_{*}({}_{*}\sigma^{k}_{\Lambda k})$ and in the same time specify the choice $\sigma^{0}_{\Lambda_0}$, then ${}_{*}\sigma^{\Omega}_{(\Lambda)}$ will obviously be as required. ▮

We can make it to have some other convenient properties: to be directed or filtered, to have a simplicial form [6]; or, in a special case of this, the form of a tree, etc. If it is the world about filters, then ${}_{\bullet}({}_{(*)}\sigma^{\Omega}_{(\Lambda)})$ will mean a $\bullet$-completed filter; with $\bullet$ as a single-valued functor. They are completed filters in the collection ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ (J) of filters on W. If we have convenient arrows in this collection, then, by means of such arrows, we can complete other filters relating them to the completed ones. This completion is the essence of topological spatial organization (see [7]).

All specifications which we carry into ${}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ determine the peculiarities of the structure of the whole in question. If this functor is completely specified, and will be if we specify mentioned collections of conditions and corresponding creative concepts, then we shall say that we have a specified spatial organization on the fundamental world under consideration and hence specified the ***structural type*** of the whole. Relevant arrows between spatial wholes with specified structural type, called *spatial whole-arrows*, are those ones preserving the type in question. Continuous arrows, in the case of topological spatial wholes, are such arrows [7].

Certainly, in the specifiication of the structural type of a spatial whole, we have to differ wholes with one kind of creations on chosen collections: a cone or a cocone creation and those with both kinds of them; which, of course, can be performed on the same or various collections. The former, we shall call spatial wholes with the *simple* type and latter, spatial wholes with the *mixed* type. If two spatial wholes have the same kind of creations — simple or mixed ones, we shall say that they have the same creative type.

Let us consider, o nce again, the collection **Ch**(W) of choice-functors $\mathfrak{S}^{\Omega_\alpha}_{(\Lambda_\beta)}$ $\alpha\in\mathcal{A}$ and $\beta\in\mathcal{B}$, on a fundamental world W. As we know, any choice functor of **Ch**(W) gives a spatial organization on W. It means that the collection **Ch**(W) serves as a groundwork for the existence a new collection: a collection of spatial organizations on W. We shall denote it by **Sp**(W). We could now deal with this collection: select spatial organizations on W according to their structural type, define relevant arrows between them and accordingly involve a fundamental structure on selected collections, then define concepts of the ***sequent*** and ***presequent spatial organization*** over W, etc. In such a way we make **Sp**(W) itself to carry a certain spatial organization. However, this organization, with respect to those on W, is of the higher level. This fact will be incorporated in our general requirement concerning the existence of spatial wholes of various levels and their vertical connection in $\mathcal{M}$.

Now we shall deal with the question of involvement of new spatial organizations over living ones. If we have a spatial whole on a fundamental world and want to involve another one over it, we have to take into account that creations of the new organization are relevant with respect to the former one, i.e., to preserve it.

In such a way we ensure the compatibility of creations and together with choices the *compatibility* of spatial organizations over one and the same fundamental world; of course, at this, organizations may be of the same or various types.

Now we shall say a few words about the link of spatial organizations in a single organization. If we have two simple and opposite types of spatial organizations, those with cone and cocone creations, then we can combine them in an organization of mixed type assuming that one type of choices and creations is utilized for choice purposes of another type. We have such a situation, for instance, in the case of intuitionistic spatial organizations.

We shall point out one more moment. Namely, we can involve a spatial organization on a fundamental world from an already defined spatial organization on that world by means of certain *well-defined operators*. In this case we have to preserve a part of living spatial organization and to involve a new part; a part which we are going to involve. It means that operators have to be such to enable this. We have this, for instance, in the case of topological spatial wholes [7]: we have involved a topological organization on a fundamental world from an already defined spatial organization: an 1-semigroupoid by means of complementation and closure operators. We shall see later some other examples as for instance Post algebras [13], etc.

Now we shall be concerned with the concept of a subwhole of a spatial whole. This concept, we obtain in the following manner: Let $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$ be a spatial whole. By a *spatial subwhole* of this whole we mean a spatial whole $\langle \tilde{J}, {}_{\bullet}({}_{(*)}\tilde{\mathfrak{S}}^{\Omega}_{(\Lambda)})_{\Theta}, \tilde{W}\rangle$, where $\tilde{J}\subset J$, $\tilde{W}\subset W$ and ${}_{\bullet}({}_{(*)}\tilde{\mathfrak{S}}^{\Omega}_{(\Lambda)})_{\Theta}$ is a subfunctor of ${}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ which imposes the same type structure on $\tilde{W}$ as ${}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ on W.

We can define one more kind of subwhole of a spatial whole $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$ — a choice subwhole. Namely, if $\mathfrak{S}^{ch}_{\Theta}: J\to W$ is a many-valued functor consisting of single-valued functors I^s such that $I^s \in {}_{*}\sigma^{s}_{\Lambda_s}$ for each $s\in\mathscr{S}_{ob}$, which is moreover completed by a single-valued functor I, then the spatial whole $\langle J, {}_{I}\mathfrak{S}^{ch}_{\Theta}, W\rangle$ is a *choice subwhole* of the considered spatial whole. Certainly, it is fully embedded in the whole $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$. We could assume that functors I^s, $s\in\mathscr{S}_{ob}$ are constructive functors for choices σ^s. In that case the functor I for such a choice functor ${}_{I}\mathfrak{S}^{ch}_{\Theta}$ will be the construction of constructions on mentioned choices. Peculiarities of such a construction are determined by means of conditions of Ω.

Now we shall be concerned with certain internal activities in the creation of a spatial whole on a fundamental world W. We shall enable this activity if, besides objects of the world W, we include collections of subobjects of its objects in the creative procedure. These new collections in W will allow certain new creations of spatial wholes within the whole which we create on W and, of course, their inclusion in the creation of the whole itself. In such a way we shall obtain more creative possibilities and hence convenient properties of the whole which we create on W. To do this we must claim that among the choices necessary for the creation of the whole there are also choices which will ensure the creativity of spatial wholes on collections of subobjects. Spatial wholes which allow such an activity we shall call spatial wholes with the local spatial organization. We define them as follows:

DEFINITION 7. We shall say that a spatial whole on the world W will admit a *local spatial organization* if there is a functor $\mathscr{T}: W\to W$ which assigns, to each object $a \in W$, a species $\mathcal{F}(a)$ such that there is an arrow $a \to \mathcal{F}(a)$ with respect to which $\mathcal{F}(a)$ strictly dominates a, and to each arrow $a \to a'$ of W a species-arrow $\mathcal{F}(a) \to \mathcal{F}(a')$.

Certainly, a power-object functor, i.e., a functor which assigns, to each object of W, the species of its subobjects is such a functor.

We now have to make the local spatial organization to be effective in the considered spatial whole $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W \rangle$. We can ensure this by relating the functor $\mathcal{F}$ to a constructive functor being defined on this whole. For that purpose, however, we have to assume that among choice-functors with domains in J, there are also those ones with domains in the world W itself; it means that we assume that W is a subworld of J.

DEFINITION 8. We shall say that a local spatial organization on a spatial whole on the world W is *effective* if there is a choice-functor $\mathfrak{L} : W \to W$ such that the functor $\mathcal{F}$ is naturally equivalent to its sequent functor.

Topoi [10], for instance, are spatial wholes in which the local spatial organization is effective. This is realized by means of the existence of an object which is the representing object for the power-object functor.

Spatial wholes with the above property are very important because they have certain levels-organizations. Namely, on each $\mathcal{F}(a)$, where a is an arbitrary object of the regarded spatial whole, we can involve a spatial organization; these spatial organizations are internal ones. Hence we might say that the functor $\mathcal{F}$ is in fact a spatial whole-functor, i.e., a functor which bears the structure of a certain spatial whole. Spatial wholes on the object a and on $\mathcal{F}(a)$ are spatial organizations on two consecutive levels, within a living spatial whole, the structure on $\mathcal{F}(a)$ is the *hyperspatial structure* with respect to that on the object a.

Thus, spatial wholes with local spatial organizations admit different levels--organizations. With respect to levels of the collection $\mathcal{M}$ as a whole, these organizations are horizontal, i.e., along a fixed level of $\mathcal{M}$.

Since we have finished with considerations of horizontal organization of $\mathcal{M}$, i.e., with the organization of particular levels of $\mathcal{M}$, we shall do this with $\mathcal{M}$ as a whole. It means that we now have to organize $\mathcal{M}$ *vertically*, i.e., to find the link between symbols of various levels of $\mathcal{M}$, in order to obtain a coherent global organization of $\mathcal{M}$. As we have already seen, we organize symbols of a level of $\mathcal{M}$ in certain wholes. Now we assume that symbols of the first higher level with respect to a level of $\mathcal{M}$ which is under consideration represent wholes and arrows between these wholes of the latter level. In that case, we can talk about species of this new level. Its members are clearly symbols which stand for spatial wholes and connectives between these wholes of the first lower level. It means that, if we now want to realize a spatial organization on this species we have to take into account symbols which mean properties of symbols standing for its objects and arrows Namely, creative capabilities of this species are determined by means of structural and other characteristics of its objects and of course of specifications of arrows in it.

Before the creation of spatial wholes on species of the new level of $\mathcal{M}$, we have to make them to be fundamental worlds. It means that we have to specify them in that sense. Since arrows in species have to be relevant, then spatial wholes — objects in them have to be of the same structural type. Hence the notion — structural type is instrinsic for a species. This we could utilize to spefify them. Thus, as a supplement to the specification of species, we would have that *all objects in them are those having the same structural type.* In the same time we have the specification of arrows in $\mathcal{M}$: they are those which preserve structural types in question; we called them relevant arrows.

If we now assume that all we have said above is valid for any two consecutive levels of $\mathcal{M}$, then we could say that the world M is organized completely: horizontally and vertically.

Finally, if we now view the organization of the mathematical world $\mathcal{M}$, we shall notice that it is inductive. Namely, in its organizing we have first to make the organization of a level of $\mathcal{M}$ and after its ending have to pass over to organize the first higher level. What this means? This means that we have to find (all) mathematical entities on species of already created mathematical entities having the particular structure characterized by spatial organizations here given and to continue to create new mathematical entities on, in such a way, created entities. To see which species of mathematical entities will admit a spatial organization we have to know their choice and structural capabilities. Of course, this requires a separate study of spatial wholes and their properties.

In this approach, we assume that there exists a starting level with certain *starting objects* from which we begin the creation of the world; we could assume that these objects are undivisible. Hence we have that all symbols of $\mathcal{M}$, except the starting ones, are created by processes given in the paper: *objects have structural forms of a certain spatial whole and arrows are such to preserve these forms.* From the creative processes arise properties of symbols which stand instead of mathematical entities. Hence we could say that symbols adjoined to symbols of $\mathcal{M}$ to represent their characteristics are also creative and obtained in the process of creation of the world of mathematics.

3. Examples of spatial wholes

We shall deal in this section with certain concrete and typical examples of spatial wholes — wholes with specified structural types. They are topological and intuitionistic spatial whole. These wholes are detailly studied in [7] and [8]. Here we shall only deal with their mode of generation. Afterwards we shall compare these organizations to some standard mathematical conceptions as they are formalism and intuitionism and see what they mean from the standpoint of these organizations.

We obtain a *topological spatial whole* $\langle J, \mathfrak{S}^{top}, W\rangle$ if we assume that J is a discrete fundamental world, i.e., a world consisting of objects and identity arrows such that there is an injection functor I of it to W and that $\mathfrak{S}^{top}$ is a transitive functor which assigns, to each $i \in J$, a filter $\mathfrak{S}^{top}(i)$ and that each such filter allows a cocone creation in W; it means that it is completed in such a way to make a cocone. The collection of all filters on W is endowed with relevant arrows called opposite inclusions. With respect to these arrows the functor $\mathfrak{S}^{top}$ is supposed to obey certain conditions (see [7]). We can see that such a spatial organization has two types of choices and two types of constructions: it allows i) arbitrary f.c. creations and ii) l..c.c. creations on collections with restricted size; it moreover contains the objects o and 1. Otherwise, a topological spatial organization one can involve by means of certain operators as they are the complementation and closure operator (see [7]).

We could also here define the concept of the *pseudotopological spatial whole.* It is enough to take, for this purpose, that the range of choice-functions is in the collection of filters of a fundamental world. It means then that objects of these functors are filters with filter-arrows as connectives. Certainly, we now can impose the spatial structure of topological type on this new fundamental world consisting of filters and filter-arrows.

An *intuitionistic spatial whole* or an intuitionistic topological space $\langle J, \mathfrak{S}^{\mathrm{int}}, W\rangle$ one can obtain if one assumes that each $\sigma^a \in \mathfrak{S}^{\mathrm{int}}$, $a \in W_{\mathrm{ob}}$, is obtained by means of presequent constructions, which every finite collection of W is assumed to admit, in the following manner: each $\sigma^a(i)$, $i \in J_{ob}$ consists of all those objects a' of W for which there is a W-arrow $F(i) \wedge a' \rightarrow a$, where $\wedge$ means the presequent construction and $F: J \rightarrow W$ is a single-valued functor. We assume that for each $a \in W_{ob}$, there is a single-valued functor $S^a: J \rightarrow W$ and a natural transformation $\eta^a: \sigma^a \rightarrow S^a$ such that (σ^a, η^a, S^a) (i), $i \in J_{ob}$ is an l.c.c. in W: it will allow that creation in itself if moreover $S^a(i) \in \sigma^a(i)$. The functor S^a is the creative functor for the functor σ^a, i.e., its *sequent functor.* We still claim that the existence of a connection — an arrow between objects $a, a' \in W$ implies the existence of a natural transformation between sequent functors S^a and $S^{a'}$.

Hence we could say that an intuitionistic topological space has constructively closed parts. However, it has not this property as a whole. To ensure this we shall assume that J and W have strict first objects [6] and that F is such to preserve such an object. If this is fulfilled, then the space as a whole will posses the sequent of all its objects. We shall denote it by 1. This object is, otherwise, equal to $S^o(o')$, where o and o' are strict first objects of W and J, respectively, Such a space has the following properties:

a) it contains the objects o and 1,

b) it is closed with respect to finite p esequents, and

c) it is closed with respect to particular sequents, i.e., sequents of particularly chosen subcollections.

We gave in [8] certain characterizations of intuitionistic topological spaces. Moreover we gave the link between these and topological spaces. We proved the following

PROPOSITION 3. *An $\aleph_0$-topological space is an intuitionistic topological space.* ❙

Now we shall select certain operators on an intuitionistic topological space having the object o. Let $\langle J, \mathfrak{S}^{int}, W\rangle$ be such a space determined by the functor F and parametrized by the world W itself. The object $S^o(i)$, $i \in J$, in it is an W-object satisfying the following condition: the presequent $P(F(i), S^o(i)) = o$. If this object is unique then we might call it a *pseudocomplement* of the object $F(i)$. Furthermore, if $J = W$, then the composition $\mathbf{C}^o = S^o \cdot S^o$ of the sequent functor S^o with itself gives us an operator called the *closure operator.* This operator has the following properties: there is a unique arrow $a \rightarrow \mathbf{C}^o(a)$, $a \in W$; then $\mathbf{C}^o \cdot \mathbf{C}^o \cong \mathbf{C}^o$, $\mathbf{C}^o(o) = 1$, etc. (see [8]).

A particular kind of intuitionistic topological spaces are those which are realized and parametrized by the world W itself, i.e., choices of which are those for which $J = {}_c\mathcal{S} = W$. These spaces, we can involve by means of certain operators: functors possessing certain properties.

Let A be a (quasi) category with defined presequent creations $P(a, a')$, $a, a' \in A_{\text{ob}}$. Denote by δ^a, $a \in A_{ob}$, a relative functor of A to A which assigns, to each object $b \in A$, an object $\delta^a(b)$ and, to each arrow $b \to c \in A_{\text{ar}}$, an arrow $\delta_a(c) \to \delta_a(b)$. If the functor δ_a, where δ_a is a functor such that $\delta_a(b) = \delta^b(a)$, is right adjoint to the functor $P(, a): A \to A$, i.e., if there is a natural isomorphism

$$(P(a', a), b) \cong (a', \delta_a(b)),$$

then we have the following

PROPOSITION 4. *The pair* $\langle A; \delta_a \rangle$ *consisting of a (quasi)category* A *having finite presequents and of a functor* $\delta_a : A \to A$, *which is right adjoint of the presequent functor* $P(, a)$ *is an intuitionistic topological space.*

PROOF. Certainly, the object $\delta_a(b)$ is the unique sequent of all objects a' of A satisfying the above relation. Hence we can define a collection $\mathfrak{S}$ of choice functors, varying objects a and b of A, such that each has the sequent functor and which moreover obeys the connection condition: if there is an arrow $a \to c$, then there is a natural transformation $\delta^a \to \delta^c$. ∎

There is a characterization of the space $\langle A; \delta_a \rangle$, which is specified in the above proposition, given by the following

PROPOSITION 5. *The intuitionistic topological space* $\langle A; \delta_a \rangle$ *is a distributive* $1^{\aleph_0}$*-semigroupoid.*

PROOF. It is an $1^{\aleph_0}$-semigroupoid by the definition: this follows from its bicompleteness. Next we have to show that the distributive law

$$\bigvee_{a' \in A'} (a' \wedge b) \cong \bigvee_{a' \in A'} a' \wedge b,$$

where A' is a subcollection of objects of A and $\vee$ and $\wedge$ are the marks for sequents and presequents, respectively, holds in the space $\langle A; \delta_a \rangle$.

Let A' be the collection of all those a' of A such that $(a', \delta_b(a)) = (a' \wedge b, a)$. Denote by $\mathbf{P}$ the collection of all presequents $a' \wedge b$, $a' \in A$, and by r the last object of $\mathbf{P}$ [6]; it is the unique sequent of all $\mathbf{P}$, i.e., $r = \bigvee_{a' \in A'} (a' \wedge b)$. Hence we have that for every $a \in A$ there is a morphism $a' \wedge a \to r$ and then also a morphism $a' \to \delta_b(r)$. Thus $\delta_b(r)$ is a vertex of a cone over A'. Since $\delta_b(a)$ is the unique sequent of all A', then there is a unique morphism $\delta_b(a) \to \delta_b(r)$ and hence a unique morphism $\delta_b(a) \wedge b \to r$. On the other hand, since $\delta_b(a) \wedge b$ is a vertex of a cone over all $\mathbf{P}$, then there is also a unique morphism $r \to \delta_b(a)$. Hence we have $r \cong \delta_b(a) \wedge b$. Since $\delta_b(a) = \bigvee_{a' \in A'} a'$, then the above relation holds. ∎

If we now claim that the space $\langle A; \delta_a \rangle$ has an effective local spatial organization ensured by the existence of an object which is the representing object for the power-object functor, then we shall obtain the concept of a topos.

We could give many more specifications of the functor $\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ in $\langle J, \bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta'}, W \rangle$ in order to obtain various kinds of spatial wholes and relationships between them. We could obtain various algebras, lattices, topological algebras, numbers: natural and real, equational classes, etc. For instance, pseudo-Boolean algebras, called also Heyting algebras, one can obtain as a special case of an intuitionistic topological space $\langle A; \delta_a \rangle$: it is enough to take that there are unique arrows between objects in it. If we provide this algebra by two collections of operators which have some preserving properties concerning the structure of the algrebra and the collections themselves have some structure and connecting properties, then we could obtain Post algebras. It means that these algebras are certain special cases of spatial wholes involved by means of certain operators. We shall not concern further cases, but shall proceed to consider two standard mathematical views: formalism and intuitionism. We shall see what these views mean from the standpoint of spatial wholes.

A *formal system* or *formalism* can be regarded as a systematic scheme according to which we organize a collection of symbols in a whole with precisely established internal relations: relations between its concepts and rules for the creation of these. We shall show that it creates a kind of spatial whole from such a collection. In what follows we shall sketch such a system given in [9].

Let S be a collection of symbols. As it is well-known, a formal system distinguishes two collections of expressions made from elements of S: the collection of terms $T(S)$ and the collection of formulas $F(S)$. It also gives modes of generation of these collections. The collection $T(S)$ is generated from elements of S by means of certain operations and the collection $F(S)$ from $T(S)$, which is provided with certain relations, by means of logical operations. It is moreover endowed with effective rules for the derivation of formulas from some collections of these, as premises. These rules are known as the rules of inference. According to them, we may take a certain collection of fundamentally valid formulas of axioms and extend it up to a collection of valid formulas or theorems.

Now we shall see what this story means from the standpoint of spatial wholes. Certainly, the collection of symbols S, we can consider as a discrete fundamental world. We are going to specify the kind of spatial whole which a formal system involves on S. According to the above description, the collection $T(S)$ is generated in such a way to contain the collection S and to be closed with respect to finite sequents and presequents. Hence, it is certainly a spatial whole on S.

The next collection of expressions is $F(S)$. Let us see what kind of structure involves the formal system on this collection. To show this we shall first deal with a topological spatial organization on it. Suppose first that $F(S)$ is endowed with certain arrows by means of which it will become a fundamental world; it is enough to take arrows called implications. A topological spatial organization is defined on such a world by means of a many-valued functor ${}_{(*)}\mathfrak{S}$ of S to $F(S)$ which assigns, to each symbol $s \in S$, a filter ${}_{(*)}\mathfrak{S}(s)$ in such a way that there is a single-valued functor $I: S \to F(S)$ and a natural transformation $\eta: I \to {}_{(*)}\mathfrak{S}$. Hence we have that $(I, \eta, {}_{(*)}\mathfrak{S})(s)$, $s \in S$, is a cocone on $F(S)$. With respect to this structure, $F(S)$ becomes closed with respect to arbitrary f.c. and restricted l.c.c. creations. We know [7] that such a structure we can involve by means of certain operators. Hence, we can represent it as a system $\langle F(S); \wedge, \vee, \mathscr{C}, \mathbf{C}\rangle$, where $\wedge$ and $\vee$ are the signs for the presequent and sequent operation respectively, $\mathscr{C}$ is the functor of complementation and $\mathbf{C}$ is the closure functor on $F(S)$. All these functors are defined in [7]. If we now look at the structure which the formal system involves on $F(S)$, we shall notice that it is just such a structure and hence a spatial structure.

It is clear that the structure on $F(S)$ is of the first higher level with respect to that on $T(S)$; objects of $T(S)$ are otherwise included in $F(S)$ through atomic formulas: collections of $T(S)$ selected by certain relations. Thus, the following proposition holds:

PROPOSITION 6. *A formal system involves a two-levels spatial organization on a collection of symbols S.* ▮

Since the structure on $F(S)$ is of topological type, then we could involve certain topological concepts in it, as they are open and closed formulas, separation and compactness conditions, etc. All these concepts one can derive from those for topological spatial wholes. So, for instance, we can see quantifiers as closure and interior operators which we defined in [7]. We shall here mention the definition of the closure operator. A *closure operator* on a fundamental world W is a covariant functor $\mathbf{C}: W \to W$ which fulfils the following conditions:

$C1$: $\mathbf{C}$ is a *successor functor*, i.e., a functor which assigns, to each object $a \in W$, an object $\mathbf{C}(a)$ which is the successor of a with respect to a W-arrow;

$C2$: $\mathbf{C}$ is an *idempotent functor*, i.e., such that $\mathbf{C}\cdot\mathbf{C} \cong \mathbf{C}$ holds;

$C3$: $\mathbf{C}$ is an f.c.$^{<c_\beta}$*-functor*, i.e., a functor which preserves f.c.'s over any $<c_\beta$-subcollection of W, where c_β means its size;

$C4$: $\mathbf{C}$ leaves *fixed* the first object of W.

The interior operator is defined in a similar manner. The complementation operator is defined as a contravariant functor with certain properties. All these functors are not defined in general to be necessarily unique ones.

If objects of the fundamental world are formulas with many variables, then the quantification by variables we can realize by the iteration of these operators along variables, i.e., as a system $I \to \mathbf{C}_{x_1} \to C_{x_1} \cdot C_{x_2} \to \ldots$ of functors and natural transformations, where I is the identity functor and $x_1, x_2, \ldots$ stand for variables in question. By the application of the complementation operator to this system we could obtain the case with the interior operator.

However, beside these concepts, there are other syntactic and semantic concepts which are relevant to various types of formal systems such as proof, consistency, model, etc. Therefore we have to put a general question: in which manner we can find the place of these concepts within those of a spatial whole,

In what follows we shall deal with this question. We shall be concerned with it only in general. We shall first consider the concept of proof.

It is well-known that a *proof* in a system is a procedure by means of which we can deduce (produce) a formula from a collection of formulas using rules which are established in the system which we are concerned with. Since our concept of spatial whole contains in itself various creative procedures, then we can say generally that an object, a formula for instance, is deducible — creative from a collection of objects and arrows if there is a convergent — terminating procedure of applications of creative concepts which starts in this collection and terminates in the desired object; otherwise, a production (derivation) in logical sense can be represented by our creative concept $*$; or, in a more broader case, by the concepts of cylinder and cocylinder. Namely, a production or a derivation is a figure of the form $t_1 t_2 \ldots t_n \rightarrow t$, where $t_1, t_2, \ldots, t_n$ are mathematical objects of a certain kind, terms and formulas for instance, called premises and the object t, the conclusion of the production. We can represent such a figure by our creative concept $*$ in which the vertex will be the conclusion; $t_1 t_2 \ldots t_n$ is its basis. Certainly, in such a case, we can consider $\mathfrak{S}^{\Omega}_{(\Lambda)}$ as a collection of mutually linked produticons. In such a way we could obtain a Post system [12].

Now we have concepts like consistency and model. These concepts are concerned with the characterization of a system, in our case, of a spatial whole. What the *consistency* means. By means of this concept we ensure that the creative procedure of the spatial whole in question cannot produce in it an object which is in a certain sense contestable. Let us see in which manner we determine contestability of an object. The standard way is by selecting certain valuation-fibers. We do this by a relevant arrow — a morphism from the whole in question to the spatial whole consisting of two distinguished and different objects denoted by 0 and 1; in topoi, the representing object for the power-object functor serves for these purposes. Let W be a spatial whole and f a morphism of W to $\{0, 1\}$. By f we select on W two disjoint subcollections called fibers and take them as frames for our purposes: they contain contestable and incontestable objects, respectively and are otherwise bridged over by means of the complementation type functor. Having these frames, we say that an object a is a consequence of a subcollection C of W if $a \in f^{-1}(1)$ for any $f: W \rightarrow \{0, 1\}$ such that $C \subset f^{-1}(1)$, i.e., if a belongs to the same fiber as C does. This fact is known as the *semantic implication* $\models$. This implication we can represent as a certain natural transformation between a functor $I: W \rightarrow W$ having its values in the collection $C \subset W$ and a constant functor $c_a: W \rightarrow W$ having as its values the object a. We can represent this situation as a many-valued functor $\mathbf{S}^C_a = (I, \models, c_a)$ of W to itself.

If there is no f such that $f(C)=1$, then one says that C is *semantic inconsistent*, otherwise it is semantic consistent. If $f(C)=1$, then it is customary to say that f is a *model* for C. Hence we have that a collection C is semantic consistent if it possesses a model. Certainly, models in this approach are certain subcollections which are closed with respect to certain objects; by such a process we can establish if a created object belongs to the fiber or not. Having now models, we could further deal with the concept of spatial structures on collections of them. One could notice that such a situation belongs to our case of spatial wholes with a local spatial organization.

Now we shall deal with the syntactic implication and its connection with the semantic one. A *syntactic implication* $C \vdash a$, from a subcollection $C \subset W$ to an object $a \in W$, as we have already seen, is a proof of a from C. We can represent it as a many-valued functor $\mathbf{P}^C_a$ consisting of inductively connected creative concepts which starts in C and terminates in a. If such a production gives us an object which is contestable, then we shall say that C is *deductively consistent.*

We could now connect these two implications and hence many-valued functors: the semantic $\mathbf{S}_a^C$ and the syntactic one $\mathbf{P}_a^C$ of W to itself. Clearly, we might say that $\vdash$ is a specified form of the implication $\models$; namely, if there is an incontestable and terminating procedure from a collection C, then there is also the implication $\models$. Conversely, it is not always the case. Namely, in a general case of spatial organizations, we do not know always if there is a production which realize this implication.

Now we shall be concerned with *intuitionism*. First we shall deal with the formal part of intuitionistic mathematics. We shall be concerned with the structural type of the intuitionistic propositional logic. We shall show that the system of axioms for this logic involves an intuitionistic topology on the collection of its formulas.

Let us consider the system of axioms for the intuitionistic propositional logic given for instance in [13]. This system we shall write in a form which is more convenient for us at this moment. Namely, we shall write $\delta^b(a)$, or $\delta_a(b)$, instead of $a \to b$, $S(a, b)$ instead of $a \vee b$ and $P(a, b)$ instead of $a \wedge b$, for two formulas a and b. Taking this into account, we shall write the system of axioms in the following form:

A1. $a \to \delta^a(b)$,

A2. $\delta^{b\to c}(a) \to (\delta^b(a) \to \delta^c(a))$,

A3. $a \to S(a, b)$, $b \to S(a, b)$,

A4. $\delta^b(a) \to (\delta^b(c) \to \delta^b(S(a, c))$,

A5. $P(a, b) \to a$, $P(a, b) \to b$,

A6. $\delta^b(a) \to (\delta^C(a) \to \delta^{P(b, c)}(a)$,

A7. $(P(a, b) \to c) \leftrightarrow (a \to \delta^c(b))$,

A8. $P(a, \mathcal{N}(a)) \to b$, $\delta^{\underline{P}(a, \mathcal{N}(a))}(a) \to \mathcal{N}(a)$.

In what follows we shall analyse this system of axioms. We shall see what these axioms mean from the standpoint of spatial whole.

Denote the class of formulas of this logic by $\mathcal{F}$. Elements of this class we shall call objects. This class is certainly provided with a class of unique connectives; there is just one connective between two objects of $\mathcal{F}$. Endowed with such connectives, $\mathcal{F}$ becomes a category. If we have a connective $a \to b$ between objects a and b of $\mathcal{F}$, then the object a is the hypothesis and b, the conclusion of the connective.

Now we shall see what the above axioms specify on the category $\mathcal{F}$. Before all the axioms A4. — A6. specify the category $\mathcal{F}$ to be closed with respect to finite sequent and presequent operations denoted by S and P, respectively; it means that the category $\mathcal{F}$ possesses finite sequents and presequents and hence that it is an $l\aleph_0$-semigroupoid ([6].

Let us consider now a functor $\delta^b : \mathcal{F} \to \mathcal{F}$ which assigns, to each object $a \in \mathcal{F}$, with respect to the chosen object $b \in \mathcal{F}$, an object $\delta^b(a)$. Assume further the object $\delta^b(a)$ to be the unique connective $a \to b$ between objects $a, b, \in \mathcal{F}$. In such a way connectives between objects of $\mathcal{F}$ also become objects of $\mathcal{F}$.

The axioms A1. and A2. specify the functor δ^b, $b \in \mathcal{F}$. According to the axiom A1., there is a connective between object a and the object $\delta^a(b)$ being a connective with respect to the object a. We can express this as the existence of a natural transformation $\eta : I \to \delta_b$, where I is the identity functor of $\mathcal{F}$ to itself and δ_b the functor such that $\delta_b(a) = \delta^a(b)$. According to the axiom A2. we have the existence of connectives between functors. Namely, let $b \to c$ be an object, then, for the functor $b^{b \to c}$, with respect to the connective object $b \to c$, we have the existence of a natural transformation $\eta^{b,c} : \delta^b \to \delta^c$.

The axiom A7. means the adjointness relation of the functor δ_b and the presequent functor $P(\ , b)$. This relation enables us to construct the connectives of $\mathcal{F}$ in an effective manner.

Finally, the axiom A8. specifies a functor $\mathcal{N} : \mathcal{F} \to \mathcal{F}$ which assigns, to each object $a \in \mathcal{F}$, an object $\mathcal{N}(a)$ such that the presequent of a and $\mathcal{N}(a)$ precedes all objects of $\mathcal{F}$; it is certainly the strict first object.

From the above analysis of the axioms for the intuitionistic propositional logic we have that the collection of formulas of this logic has the structure of an intuitionistic space of the form $\langle A; \delta_a \rangle$ which possesses the strict first object; here, A is an $\mathbb{N}_0$-semigroupoid and δ_a a functor on A having mentioned properties; this is the covariant form of the above functor. Hence the following proposition holds:

PROPOSITION 7. *The collection of formulas of the intuitionistic propositional logic has the structure of an intuitionistic topological space.* ▮

Hence we have that the system of axioms of the intuitionistic propositional logic involves a certain kind of spatial structure of intuitionistic type on the collection of its formulas; a spatial structure of another type is contained in building up the collection of terms; this structure is of the first lower level with respect to that on the collection of formulas.

Now we shall deal with certain concepts of nonformalized intuitionistic mathematics. The concepts which we shall concern here are those given in [2], [11] and [16]. First we have the concept of a *species*. This concept is already studied in the paper and therefore we shall not be further concerned with it. Next concept is that of a *spread*. We can obtain this concept by specifying the choice functor $\mathfrak{S}^{\Omega}_{(\Lambda)}$; it means by specifying the collections of conditions Λ and Ω. Let us see in which way.

If we assume that the functor ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ is specified, of course, by specifying Λ and Ω, in such a way to consists of many-valued functors σ^{α}_{k}, $k \in \mathcal{K}$ and $\alpha \in \omega$, such that any $(\alpha+1)$th functor is in fact a cone over a αth one, then we can represent ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ as a collection $\{{}_{*}\sigma^{\alpha}_{k} \mid k \in \mathcal{K} \wedge \alpha \in \omega\}$, where ${}_{*}\sigma^{\alpha}_{k}$ are many-valued functors of the form ${}_{*}\sigma^{\alpha+1}_{k} = ({}_{*}\sigma^{\alpha}_{k'}, \rho^{\alpha}_{k'k}, G^{\alpha}_{k})$ for k', $k \in \mathcal{K}$; here, $\rho^{\alpha}_{k'k}$ are natural transformations, ${}_{*}\sigma^{o}_{k'} = F_{k'}$ is a single-valued functor and G^{α} are constant functors. The determination of the successive concepts ${}_{*}\sigma^{\alpha-1}_{k}$ may be pictured as a process of progressive ramification with simplicial branches: each branch gives a simplicial concept [6]. A spread is a *fan* if the collection $\mathcal{K}$ is finite.

We could deal with other concepts of this logic, as they are choice and lawlike sequences, apartness relation, etc. We shall see for instance what the *apartness relation* means. This relation, usually denoted by #, differs objects in a fundamental world and can serve for choice purposes. Namely, by means of it we can select certain subcollections of the world, objects of which are either identical or in this relation; this relation is otherwise defined to be symmetrical, i.e., such that $\# (a, b) \Leftrightarrow \# (b, a)$, for any two objects a and b. We could deal with this relation as a special arrow, or to express it by means of arrows of the world in question.

According to this, we could say that we could find the position of (all) concepts of (non)formalized intuitionistic mathematics within those of spatial whole. And since we have already said this for the case of formalism, then we might say in general that the creation of spatial wholes contains in itself main parts of mathematical activity.

4. Fundamental acts in creation of the world of mathematics

In this section, we shall formulate, but only in general, fundamental acts which occur in the creation of the world which is intended to contain (all) objects of mathematics and in the creation of which (all) mathematical activity is to be exhausted. Such a world, we have called the world of mathematics. The acts in question are extracted from preceding investigations. All preceding story, we can summarize in five general acts. The first among the acts is the following one:

A1. *Specification of the frame of the world*

We have seen that it is enough to take as a symbolic frame for the creation of the world of mathematics a collection $\mathcal{M}$ consisting of two-sort symbols of various levels. If we adjoin to these symbols some new symbols characterizing these, then we shall arrive at the new act:

A2. *Selection of basic collections*

According to the adjoined symbols, representing properties of symbols of $\mathcal{M}$, all symbols of any level of $\mathcal{M}$ one can select in particular collections called species. Namely, we first select objects on the considered level. which we call species, and afterwards make the distribution of arrows over them. Then we provide such collections of objects and arrows with certain fundamental structure. Hence, the next act is

A3. *Formation of fundamental worlds*

We assume that each species of any level of $\mathcal{M}$, provided with a collection of arrows, bears a fundamental structure — the structure of a (quasi)category. It will possess such a structure if arrows in it are relevant, i.e., if they preserve intrinsic properties of its objects. This structure serves as a groundwork for further purposes contained in the following act:

A4. *Organization of spatial wholes*

Each fundamental world of any level of $\mathcal{M}$, one can organize in a spatial whole. This act is the central one. It contains in itself two activities: choice and creative activity. These activities are comprised in creations of spatial wholes of various levels. We might say that spatial wholes are the main products of mathematical activity and hence objects of an edifice which we have called the world of mathematics. Hence we have that the world of mathematics consists of spatial wholes of various sorts and levels; of course, together with arrows between them. These arrows horizontally connect spatial wholes. Meanwhile, their vertical connection, i.e., the connection between levels is established by the following act:

A5. *Vertical connection of spatial wholes*

Each spatial whole of an arbitrary level of $\mathcal{M}$ is an object of a species of the first higher level with respect to this level. Hence,species of each level of $\mathcal{M}$ consist of spatial wholes, with specified structural type, of the first lower level with respect to their level; they are also endowed with relevant arrows.

The above five acts give us a general procedure for the creation of the world of mathematics. By following them and specifying structural types of spatial wholes we specify the mathematical world. For the complete specification of the world, it is necessary to know all structural types of spatial whole which we can involve on a species of spatial wholes. This problem, however, requires a separate study of various types of spatial wholes and their relationships. Therefore we shall not deal with it.

If we forget the structure of spatial wholes, then the world of mathematics will consist of (quasi)categories of various levels; each (quasi)category of any level of $\mathcal{M}$ has as objects (quasi) categories of the first lower level with respect to its level and functors between them as arrows. If we assume that (quasi) categories are discrete and accept a necessary part of spatial structure we could obtain the frame of the world $\mathcal{U}$ of [5].

Furthermore, as an idealization of the world of mathematics, we could obtain the world of ordinal numbers and also of their cardinal capacities. Going along levels we would have ordinals of various number classes. We could realize this by assuming that choice-functors $\mathfrak{S}^{\Omega}_{(A)}$ are completed transitive functors having the tree structure.

It would be of an interest to find the link between our approach and some other approaches to the foundations of mathematics given for instance in [1]. Moreover one could deal with the connection of some other mathematical worlds, as they are for instance worlds of set theory ([9], [14]), then the world of [3] and others with our world. We shall deal with some of these questions in a separate paper. Moreover we shall apply these investigations to develop some other mathematical theories.

5. Conclusion

We have been concerned in this paper with general aspects of mathematical activity. We have seen that this activity has as its primary goal the creation of certain mathematical entities which we have called spatial wholes and of the world which contain all these entities. This world, we have called the world of mathematics. We have specified certain features of it and its constituents. We have also given fundamental acts for its creation. Certainly, there still remains much work concerning further characterizations of spatial wholes, heredity of their properties along levels, etc.

In the next part of this paper, we shall try to formalize these investigations in a system. We shall give main features of that system and then compare it to some known system. Afterwards we shall return, once again, to the discussion of goals of mathematical activity.

Since we consider that the investigations given in this paper reflect certain features of the real world: its horizontal and vertical evolution and structure, then we shall try to apply them to natural science. Thus, beside the problem of further characterization of the concepts given in the paper and the formalization of this program, we have one more task.

Finally we should say a few words about all what we have done here. The basic idea which has been leading us in this work has been to see mathematical conceptions as various kind procedures by means of which one can create mathematical entities called spatial wholes which have to comprise in themselves mathematical and logical concepts We do not know if we have yet succeded in this.

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CONTINUUM PROBLEM AT MEASURABLE CARDINALS

Aleksandar JOVANOVIĆ

Exposition

Given any set, how to evaluate the cardinal of its power set? The above is known as continuum problem. In ZFC, initial ordinals can be taken to represent cardinals. Thence the problem reads: determine function F, so that for all ordinals α:

(0) $$2^{\omega_\alpha} = \omega_{F(\alpha)}.$$

Cantor has proved that $2^{\omega_\alpha} \geqslant \omega_{\alpha+1}$, for all α. Therefore we can split F so that

(1) $$\omega_{F(\alpha)} = \omega_{\alpha+f(\alpha)}.$$

Putting $f(\alpha)=1$, for $\alpha \in Ord$, we obtain a formulation of generalised continuum hypothesis (GCH).

It is known that

(2) $$\alpha \leqslant \beta \text{ implies } F(\alpha) \leqslant F(\beta)$$

and

(3) $$cf\, \omega_{F(\alpha)} > \omega_\alpha.$$

The (3) is known as König's lemma.

Here we shall first list important recent progress on the matter, assuming the fundamental results of Gödel and Cohen are known.

In [7] Silver has proved the following theorem.

1.1. THEOREM: if ω_α is a singular cardinal of cofinality greater than ω, then:

(4) $$\forall \beta < \alpha\; 2^{\omega_\beta} = \omega_{\beta+1} \quad \text{implies} \quad 2^{\omega_\alpha} = \omega_{\alpha+1}.$$

However, the problem of all singular cardinals is still unsolved. In J. Stern [8] we found the following hypothesis on singular cardinals, for which the consistency and independence are open questions. HCS: let ω_α be a singular cardinal. Then

(4') $$\forall \beta < \alpha\; 2^{\omega_\beta} = \omega_{\beta+1} \quad \text{implies} \quad 2^{\omega_\alpha} = \omega_{\alpha+1}.$$

Jensen in [6] has proved the next theorem.

1.2. THEOREM: if negation of HCS is consistent with ZFC so is the axiom of uncountable measurable cardinals (AM).

For regular cardinals we have the fundamental result of Easton [3]:

1.3. THEOREM: for any function F defined on all ordinals α such that ω_α is a regular cardinal and F satisfies (2) and (3), consistency of ZFC implies the consistency of $ZFC+EA_F$. Here EA_F is the formula

$$\forall \alpha \in D_{om}(F) \ 2^{\omega_\alpha} = \omega_{F(\alpha)}.$$

Here we note that 1.3. theorem, we found in Jech [5], theorem 37, in a somewhat different notation. There presented formulation is adjusted for the following theorem that we have proved. Let F and f be defined by (Ø) and (1). From Chang and Keisler [1], section 4.2. we know that if there is an uncountable measurable cardinal then there is a normal ultrafilter on it.

1.4. THEOREM: let k be an uncountable measurable cardinal and let D be a normal ultrafilter on it. Then

(5) $$\{\beta < K : 2^{|\beta|} = |\beta|^+\} \in D \text{ implies } 2^k = k^+.$$

(6) $$|f(k)| \leqslant |\prod_D f(\beta)|.$$

Above $|X|$ denotes a cardinal of X, $\prod_D$ is ultraproduct modulo normal filter D. (5) says that if continuum hypothensis is true on a set in D, then it is true at measurable cardinal k. Hence it implies that the value 2^k is determined when continuum hypothesis holds on a set in D. (5) is the special case of (6) which can be read as: the number of cardinals α such that $k < \alpha \leqslant 2^k$, is constrained with the value of $|\prod_D f(\beta)|$. Here $f(\beta)$ is a nonempty subset of k, which enumerates the cardinals from ω_β to 2^{ω_β}.

Now it is evident that the axiom of uncountable measurable cardinals contradicts the Easton's result given in 1.3. theorem; to check that, let k and D be as in 1.4. theorem. Define F

$$F(\alpha) = \begin{cases} \alpha+1 \ \textit{iff} \ \alpha \neq k \text{ and } cf\,\omega_\alpha = \omega_\alpha \\ \alpha+2 \ \textit{iff} \ \alpha = k \end{cases}$$

This F satisfies (2) and (3), so by the conclusion of 1.3. theorem we can take as axiom

$$\forall \alpha \in D_{om}(F) \ 2^{\omega_\alpha} = \omega_{F(\alpha)}.$$

But the set of all regular cardinals less then k belongs to D. Hence by (5) $2^k = k^+$, contradicting $F(k) = k+2$ which means that $2^k = k^{++}$. Moreover, since (5) is a special case of (6), similiarly to above we see that if F violates the (6) $ZFC+AM+EA_F$ is inconsistent. What with the opposite question? Taking into account Silver's result that the consistency of $ZFC+AM$ implies the consistency of $ZFC+AM+GCH$, we state the conjecture: let F be defined on all α for which ω_α is regular and let F satisfy (2), (3) and (6). Then the consistency of $ZFC+AM$ implies the consistency of $ZFC+AM+EA_F$.

As we have seen above, the continuum problem was separately treated for singular and regular cardinals. But according to (6), may F be such to prevent the existence of measurable cardinals? Then in $ZFC+EA_F$, HCS would become a theorem.

Proof

First we list two D. Scott's results on normal measure, as we found them in the section 4.2. of Chang-Keisler [1].

DEFINITION. A filter D over a measuralbe cardinal k is said to be normal if:

1. D is an k-complete nonprincipal ultrafilter;
2. in the ultrapower $\prod_D \langle K, < \rangle$, the k-th element is the identity function on k.

2.1. THEOREM: let k be an uncountable measurable cardinal. Then there is a normal ultrafilter over it.

2.2. THEOREM: if k is a measurable cardinal and D a normal ultrafilter on it then

$$\langle R(k+1), \in \rangle \cong \prod_D \langle R(\beta+1), \in \rangle.$$

2.3. COROLLARY: let $\varphi(x)$ be a formula. Then

$$\langle R(k+1), \in \rangle \models \varphi(k) \textit{ iff } \{\beta < k : \langle R(\beta+1), \in \rangle \models \varphi(\beta)\} \in D.$$

As a consequence of the above we note that the set of strongly inaccessible cardinals less than k belongs to D. Also

$$\left| \prod_D R(\beta+1) \right| = 2^k.$$

2.4. THEOREM: let D be an ultrafilter over a cardinal k .Let

$$\mathfrak{A} = \langle A, <_A \rangle = \prod_D \langle k, < \rangle. \text{ If } f \in {}^k k \text{ and } f(\beta) \neq \varnothing$$

when $\beta \in k$, then

$$\left| \prod_D f(\beta) \right| = \left| \{ g_D^{\mathfrak{A}} \in \mathfrak{A} : g_D^{\mathfrak{A}} <_A f_D^{\mathfrak{A}} \} \right|.$$

PROOF: let $g \in \prod_{\beta \in k} f(\beta)$. Then $g \in {}^k k$. Define

1. $g_D = \{ h \in \prod_{\beta \in k} f(\beta) : \{ i < k : g(i) = h(i) \} \in D \}$.
2. $g_D^{\mathfrak{A}} = \{ h \in {}^k k : \{ i < k : g(i) = h(i) \} \in D \}$.

It is clear that $g_D \subset g_D^{\mathfrak{A}}$. Define $\pi : \prod_D f(\beta) \to A$, by $\pi g_D = g_D^{\mathfrak{A}}$. π is $1-1$. For, if $g_D \neq h_D$ and g_D, $h_D \in \prod_D f(\beta)$, then $g_D \cap h_D = \varnothing$. Suppose that $\pi g_D = \pi h_D$. Then $g_D^{\mathfrak{A}} = h_D^{\mathfrak{A}}$, and hence $\{i<k : g(i) = h(i)\} \in D$. It follows that $h_D = g_D$. Contradiction. Put $F = \{g_D^{\mathfrak{A}} \in \mathfrak{A} : g_D^{\mathfrak{A}} <_A f_D^{\mathfrak{A}}\}$. We shall prove that $\pi(\prod_D f(\beta)) = F$. Let $g_D \in \prod_D f(\beta)$. Then $\{\beta<k : g(\beta)<f(\beta)\} = k \in D$. It follows that $g_D^{\mathfrak{A}} <_A f_D^{\mathfrak{A}}$. Hence $g_D^{\mathfrak{A}} \in F$. Let now $g_D^{\mathfrak{A}} \in F$. Then $x = \{\beta<k : g(\beta)<f(\beta)\} \in D$. Let $\bar{g} \in {}^k k$ be such that

$$\bar{g}(\beta) = g(\beta) \quad \text{if} \quad \beta \in x$$

$$\bar{g}(\beta) = 1 \qquad \text{if} \quad \beta \in k \setminus x.$$

Then $\bar{g} \in g_D^{\mathfrak{A}}$. But $\bar{g} \in \prod_{\beta \in k} f(\beta)$ and $\bar{g}_D \in \prod_D f(\beta)$. Therefore $\pi \bar{g}_D = g_D^{\mathfrak{A}}$ and thus π maps $\prod_D f(\beta)$ onto F.

2.5. THEOREM *let k be a measurable cardinal, D a normal ultrafilter over k. Then $\mathfrak{A} = \langle A, <_A \rangle = \Pi_D \langle k, < \rangle$ is well ordered with the relation. $<_A$. Order type of $\mathfrak{A}$ is greater than 2^k.*

PROOF. By lemma 4.2.13. from [1], $<_A$ is a well ordering. Further

$$2^k = \left| \prod_D R(\beta+1) \right| \leqslant \left| \prod_D \langle k, < \rangle \right| \leqslant 2^k.$$

Hence order type $ot\ \mathfrak{A} \geqslant 2^k$ and obviously $ot\ \mathfrak{A} < |2^k|^+$; defining b as $b(\beta) = |R(\beta+1)|$, we see that $b \in {}^k k$ and hence $b_D \in \mathfrak{A}$. The proof then follows from 2.4. theorem and the fact that b_D is not the last element in $\mathfrak{A}$.

2.6. COROLLARY for every $f_D \in \mathfrak{A}$ there is an crdinal γ_f so that f_D is the γ_f-*th* element of $\mathfrak{A}$, and $|\Pi_D f(\beta)| = |\gamma_f|$; for every ordinal $\varkappa < ot\,\mathfrak{A}$ there is an $f^{\varkappa} \in {}^k k$, such that $f_D^{\varkappa}$ is the $\varkappa$-th element in $\mathfrak{A}$.

Now we can give the proof of 1.4. theorem.

Functions F and f are defined by ($\varnothing$) and (1); if $\beta<k$ then $cf|\beta|<k$, $\omega_\beta<k$, $F(\beta)<k$, $2^{\omega_\beta}<k$ and $f(\beta)<k$. Hence the restriction $f\restriction_k \in {}^k k$ and $(f\restriction_k)_D \in \prod_D \langle k, < \rangle$. We define

$$G_f = \{g_D \in \mathfrak{A} : g_D <_A f_D\} \text{ and}$$

$$H = \{h_D \in \mathfrak{A} : \{\beta<k : h(\beta) \in [\omega_\beta, \omega_{\beta+f(\beta)}) \cap c_{ard}\} \in D\}.$$

That is, for $h_D \in H$, $h(\beta)$ is a cardinal and $\omega_\beta \leqslant h_{(\beta)} < \omega_{\beta+f(\beta)}$. Hence, for every $h_D \in H$, there is some $g_D \in G_f$ so that

(*) $\{\beta<k : h(\beta) = \omega_{\beta+g(\beta)}\} \in D$. Define $\pi : H \to G_f$ with

$$\pi h_D = g_D \ \textit{iff}\ (*).$$

It is easy to check that πh_D does not depend on elements of h_D and that π is 1—1. Therefore

$$|H| \leqslant |G_f|.$$

Let $\varkappa$ be a cardinal such that $k \leqslant \varkappa < 2^k$. By the 2.6. corollary there is an $f^\varkappa \in {}^k k$, such that $f_D^\varkappa$ is the $\varkappa$-*th* ordinal in $\mathfrak{A}$, eg. $\gamma_{f^\varkappa} = \varkappa$. From the same corollary

$$\left|\prod_D f^\varkappa(\beta)\right| = |G_{f^\varkappa}| = |\varkappa| = \varkappa.$$

For the function g with the domain k, define the function

$$|g| = \langle |g(\beta)| : \beta < k \rangle.$$

We have

$$\left|\prod_D |f^\varkappa(\beta)|\right| = \left|\prod_D f^\varkappa(\beta)\right| = \varkappa.$$

That implies

$$|G_{|f^\varkappa|}| = \varkappa \text{ and } \gamma_{|f^\varkappa|} \geqslant \varkappa,$$

which means that $|f^\varkappa|$ is at least $\varkappa$-th element in $\mathfrak{A}$. Since $|f^\varkappa|_D \leqslant_A f_D^\varkappa$ $(\{\beta < k : |f^\varkappa(\beta)| \leqslant f^\varkappa(\beta)\} \in D)$, by choice of $f^\varkappa$ must be $f^\varkappa =_D |f^\varkappa|$ and hence

$$X = \{\beta < k : f^\varkappa(\beta) \text{ is a cardinal}\} \in D.$$

Since $\gamma_{f^\varkappa} = \varkappa \geqslant k$ and D is normal, we have

$$\{\beta < k : f^\varkappa(\beta) \geqslant \beta\} \in D.$$

Let $Sinac(k)$ be the set of strongly inaccessible cardinals less than k. As we noticed, $Sinac(k) \in D$. Now we have

$$\text{either } \{\beta < k : f^\varkappa(\beta) \geqslant \omega_{\beta + f(\beta)}\} \in D$$

$$\text{or } \{\beta < k : f^\varkappa(\beta) < \omega_{\beta + f(\beta)}\} \in D.$$

In the first case we would have

$$\{\beta \in k \cap Sinac(k) : f^\varkappa(\beta) \geqslant \omega_{\beta + f(\beta)} = b(\beta)\} \in D,$$

which would imply

$$2^k \leqslant \left|\prod_{D \cap S(Sinac(k))} f^\varkappa(\beta)\right| = \left|\prod_D f^\varkappa(\beta)\right|.$$

Hence $\gamma_{f^\varkappa} \geqslant 2^k$, contradicting assumption for $\varkappa$.
Thus $\{\beta < k : f^\varkappa(\beta) < \omega_{\beta + f(\beta)}\} \in D$.
Since $\varkappa \geqslant k$ and $f^\varkappa =_D |f^\varkappa|$ we have

$$\{\beta < k : f^\varkappa(\beta) \in [\omega_\beta, \omega_{\beta + f(\beta)}) \cap Card\} \in D.$$

It follows that there is some $h_D \in H$, so that $f^\varkappa \in h_D$, or equally $f_D^\varkappa \in H$. Since $\varkappa \neq \varkappa'$ implies $f_D^\varkappa \neq f_D^{\varkappa'}$, we have

$$|[k, 2^k) \cap Card| = |(k, 2^k] \cap Card| = |f(k)| \leqslant |H| \leqslant |G_f| = \left|\prod_D f(\beta)\right|,$$

thus completing the proof of (6). Now let

$$X = \{\beta < k : 2^{|\beta|} = |\beta|^+\} \in D.$$

This means that $f(\beta)=1$, when $\beta\in X$. But from (6) we get

$$|f(k)|\leqslant\left|\prod_{D\cap S(x)} f(\beta)\right|=1. \text{ Hence } 2^k=k^+.$$

NOTE: in the above proof we had $f\restriction_k$ defined on all $\beta<k$; to apply the *Easton's* argument we need $f\restriction_k$ to be defined on $y=\{\beta<k:\omega_\beta \text{ is regular}\}$. Since $y\in D$, such a difficulty can easily be avoided.

From above it follows that actually

$$2^k\leqslant\omega_{k+ot\left(\prod_D\langle f(\beta),<\rangle\right)}.$$

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FROM FOUNDATIONS TO SCIENCE: JUSTIFYING AND UNWINDING PROOFS

Georg KREISEL

Abstract. The first part of this paper recapitulates the general scheme of using techniques developed for discredited foundational aims; specifically, proof theoretic techniques developed for carrying out Hilbert's programme. Since this programme relies on formalization, that is, mechanization, an obvious use is in the mechanical 'handling' of proofs. — The second part of the paper considers three different kinds of 'handling:' finding, checking and unwinding (transforming) proofs. The principal, generally neglected conclusion is that mechanical unwinding presents the most promising application of proof theoretic techniques; particularly where the passage from the informal proof considered to a formalization of its *relevant* features is not particularly problematic. Examples of such cases are proposed.

I. Background

It is a commonplace that the notions and problems (formulated in terms of such notions) which occur to us when we know little about a subject are liable to lose their prominence when we know more. This shift occurs even when, realistically speaking, the notions are quite precise. Here are two examples from so to speak opposite extremes in the case of formulae and proofs.

1. When we know little, *length* of formulae (measured by the number of symbols) will occur to most of us as a subjct of study. It is quite precise for any given notation. But as we go into the subject, we find that length does not determine the mathematical 'behaviour' of formulae at all well; for example, in many decision procedures a bound on the number of quantifier alternations is much more significant. This kind of thing is familiar from the natural sciences: The (mechanical) behaviour of bodies is determined more by their weight and moments of inertia than by their colour or (details of) their shape though colour and shape strike the eye most.

2. When we know little, the first and often almost the only Yes-No question to ask about a proof is whether it is *valid* or, perhaps, whether it uses valid *principles*. Of course this question is meaningful (and often the answer is negative when we have little experience with the subject; for example, a hundred years ago

one applied the power set operation to what Cantor called a *Vielheit*, e.g. the universe). But as we go into the subject we often reach a stage when any analysis or — as one says — justification of the principles is unrewarding in a quite precise sense: any analysis (tacitly: in terms of current concepts) is less convincing than the recognition (=constatation, Konstatierung) of validity. This kind of thing is familiar from experience with children who learn only slowly when it is (intellectually) unrewarding to ask: Why? — The case-study in the Appendix illustrates in detail how experience with the subject matter affects the recognition of validity.

Perhaps the most famous attempt to pursue questions of validity to the bitter end is Hilbert's programme. Fairly recently, I have set out what I believe we have learned from work on this progamme [4]. The idea was to justify abstractly valid principles by the following kind of reduction. If an elementary statement has a proof π by such principles then it has also a proof π_e by elementary means. And if the principles are *formalized*, the reduction is, in turn, expressed by an elementary statement (for details, see [4]). The latter should be proved by elementary means, once and for all; cf. Hilbert's famous 'final solution' ([4], pp. 111—112).

As is well-known, the most striking so to speak legalistic defect of Hilbert's programme is established by Gödel's incompleteness theorem; naturally modulo second thoughts about abstract validity. A far more *specific*, and *therefore more convincing* defect is established by looking at particular abstract principles which have been reduced according to Hilbert's aim, and to see what is gained or lost by the reduction; cf. [4] pp. 116—117. Indeed, quite generally, defects of reductions are most easily seen in cases where they have been carried out, where Ockham's razor has been applied. Otherwise there is always a lingering doubt that we shall see something new and marvellous when 'unnecessary' growth has been removed.

Be that as it may, it is quite clear that the 'reductions' involve transformations of proofs: $\pi \to \pi_e$. And even if one has no doubts about (the validity of) π or less doubts about π than about π_e (for example, because π_e is more involved than π and so has a higher chance of containing copying errors), there remains the possibility that π_e *tells us something we want to know that* π *doesn't*. Finding that 'something' becomes a principal problem: it may need more imagination than the step from π to π_e.

Remarks. (a) The problem above, of exploiting work done for the sake of discredited aims, is familiar in the philosophy of science under the somewhat grandiose heading: *Logik der Forschung* (logic of scientific discovery). It is very popular among scientists working on cosmology or theories of evolution where such problems are the order of the day. (b) In particular, what were principal notions or principal results for the discredited aims turn into lemmas, of interest only when reformulated, and combined with other constructions. A good example is provided by so-called consistency proofs using ε_0-induction, reformulated as a formal equivalence between the logical principle of soundness (=reflection) and the mathematical principle of ε_0-induction ([4], p. 121, l. 7—8). This has recently been combined with combinatorial arguments by Paris and Harrington [7], who established an equivalence to a 'more' mathematical principle, namely their version of Ramsey's theorem, to which we return in the Appendix.

II. Mechanical Handling of Proofs

For familiar foundational reasons which were recalled above, *formalization* of the principles studied (of course not: of the metamathematical methods used) is needed for Hilbert's programme. Others tried to connect formalization with mathematical rigour, which requires metamathematical arguments to be formalized too. However far-fetched all this may be for the phenomena of mathematical reasoning itself, formalization or, equivalently, mechanization is an obviously essential element in the use of digital computers. since they operate only on formal data.[1] We consider here three kinds of uses: *finding* proofs, say for a given conjecture; *checking* proofs, of a given assertion; and *transforming* proofs, for example, a prima facie non-constructive proof of an existential theorem into a realization, an analytic proof of an algebraic theorem into an algebraic one, and the like.

1. *Past experience*: *computation and highbrow mathematics.* Of course, the huge bulk of computer uses in pure or applied mathematies concerns computations or, more generally, classes of assertions A, for example, equations $t=t'$, for which decision methods are known that can be realistically implemented by a computer. So formulated, the uses *can* be regarded as examples of finding or checking proofs; for example, if we think we have an argument for A, but are not sure[2]. However, the only feature of the argument which is relevant to this use is the conclusion A itself. The computer checks the *result* of the argument, and does not look at its details. Put differently, given the result, the computer makes a fresh start. As a corollary, the third type of use mentioned above, the transformation of proofs, does not occur here at all.

In high-brow mathematics the situation is different. Finding and checking proofs are, at least generally, done without using mechanical rules. This is a commonplace as far as discovery is concerned. But also checking is rarely done mechanically, for example, by careful comparison with some given set of formal rules (mathematicians make logical inferences, but seldom remember rules of predicate calculus even after having seen them). By far the most efficient checking is done by comparing or confronting intermediate steps with what is known already, possibly in superficially quite different parts of mathematics. This is of course related to discovery where results from different areas of knowledge are combined. In short, for the phenomena of mathematical reasoning just mentioned the business of formalization seems quite far-fetched.

In contrast there is another part of high-brow mathematical activity which does have a mechanical look, namely the analysis or unwinding of proofs; it is mechanical, once one has decided *what* to read off the proof. As a matter of empirical fact (cf. p. 113—116 of [5]), though mechanical, this unwinding occasionally makes one's head spin, and one gets lost — as in computations with large numbers. From this point of view it is promising to use computers for such unwinding. And, as suggested by Part I, methods developed in traditional proof theory turn out to be relevant here.

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[1] Many instruments which are called 'computers' are here thought of as combining a (central) digital computer and a (peripheral) analogue device; the latter may operate on, say, continuous data, and then supplies the computer with discrete formal data.

[2] An 'essential element' and not necessarily the sum total; for example, if we are interested in a conjecture A, *one* type of use of a computer is to present not a formal proof of A, but of $P_A \rightarrow A$ with an invitation to the user to consider if P_A is valid.

Reminders. To avoid a general air of unreality, it is as well to recall at this point a few simple facts. (a) Naturally, even if the programme of unwinding works out, it cannot be expected to be of comparable importance to, say, high speed computation. This is a particular case of the truism that the use of computers *within* mathematics is a very minor part of the total picture. (b) Conversely, inasmuch as the programme is useful, it cannot be expected that *clever* mathematics will often play an essential role. This is a particular case of the fact of experience, say in operational research, that one rarely gets a startling gain in efficiency by some new mathematical device for solving a (decision) problem. Far more often does one get an improvement by spotting constraints on the problem (as originally stated): one finds that in practice only few of the assertions occur which were thought to be relevant at first blush. Actually, this point applies to some extent within mathematics too when there are high, say exponential bounds for deciding all formulae of the class C; the practical conclusion is that one had better look for a more amenable class, say a subclass of C. (c) But also one should remember that there are *occasional* exceptions to the general features of present day high-brow mathematics emphasized above. The proof of the four-colour-conjecture by Haken and Appel (explained in [1] with the benefit of advice from professional scientific journalists) was certainly *discovered* by a high-brow use of computers. At our present stage of experience it is as reasonable to look for a *check* without the use of computers as it would have been a hundred years ago to look for a finitist proof of a theorem discovered non-constructively.

2. *The passage from informal to formal proofs: the alleged spanner in the works*. When one speaks of (mechanically) unwinding or, generally, transforming proofs, one has to have a proof to start with! So naively, it seems we need machinery to pass from some given informal proof π to corresponding formal data π' and perhaps (b) that, for a *mechanical* transformation, π' has to be built up by *formal* rules. Both these ideas are quite naive. The first neglects general experience in the application of theories, the second specific experience in proof theory.

(a) What is needed is a formal representation of those features of π which are *relevant* to the transformation. Sure, one *can* ask: How do you know what is relevant, (as a child asks: Why?) But, before one imposes unrealistic demands on uses of proof theory, it is much more profitable to remember how mathematical theories are applied elsewhere. If physical theory is to be applied to some phenomenon, say the motion of the planets, it is left to the physicist to discover the physically significant features of the phenomenon. There is no 'machinery' for deciding whether chemical composition or cosmic radiation is significant — and if there were, the application of the machinery might take so long that the more significant features (position and velocity) are already out of date. The physicist uses a certain familiarity with the phenomena to spot the significant features.

And physical theory *is* of use whenever the effort involved in the passage from the raw phenomenon to the choice of data is not out of all proportion to the effort of applying the theory to those data.

(b) For the kind of unwinding mentioned in §1, most details of a proof are *not* relevant; for example, none of the details involved in proving so-called identities, that is, $\prod_1^0$-axioms, and if the latter are true then the transformed proof will again use only true ($\prod_1^0$) axioms.

As a corollary, when we have the job of unwinding a proof π, we shall look for chunks of the proof that are used only for proving $\prod_1^0$-theorems, and suppress them altogether from the representation π' to which the proof theoretical transformation is applied; cf.: a physicist who is given data including the spectral lines of the light coming from a planet, will ignore this optical information if his job is to determine the motion of the planet.

The fact that proof theoretic methods are occasionally of use, is not in doubt. As documented in [5], pp. 113—116, even without a computer they have been applied to unwind proofs, and to extract information which the discoverers of those proofs wanted to know and did not find by themselves. Spotting relevant features of those proofs was not a major obstacle.

NB. Of course there is intrinsic logical and above all aesthetic interest in giving a closer analysis of the passage from informal proofs to (relevant) formal representations. But under ordinary circumstances the use of such a scheme is more likely to hamper than to help the effective application of computers in the unwinding of proofs. — The reader should compare here cases of mechanizing the choice of relevant features in natural science. This was necessary, for example, when sending a *robot* to Mars to look for life, since only a limited number of types of measurement (of supposedly relevant data) could be incorporated. The robot was surely much better than a scientifically untrained or thoughtless observer. But perceptive scientists on the spot would surely have done better than the robot by *not* restricting themselves to a prescribed repertoire.

3. *New examples of candidates for mechanical unwinding.* The 'new' examples are here regarded as a continuation of those discussed in [5], pp. 113—116 (where also some loose ends are pointed out which can probably be tied up by use of a computer). The 'old' examples concerned questions raised by distinguished mathematicians about their own proofs, and so it was reasonable to take the interest of the questions for granted. The interest of the new questions will be discussed briefly at the end of (a), respectively (b) below; 'briefly' because, as always, only the general interest of an open problem can be decided, the exact interest depending on the specific solution.

Warning. To fix ideas the unwinding considered below is done by normalization or cut elimination (so that one ends up with a cut free proof). This is fine for realizations of existential theorems. It is not good for finding, say, a first order proof which corresponds to a higher order proof (of say, a logical theorem). Giving a better unwinding, which in general associates a (first order) proof *with* cut to higher order proofs, is certainly a *principal open problem.*

(a) Milnor [6] showed by use of topological arguments that the only (possibly non-associative) division algebras over a real closed field have dimenmensions 1, 2, 4, 8. So, for each integer $n \neq 1, 2, 4, 8$ there is a purely logical proof of the non-existence of a division algebra of dimension n from the axioms of real closed fields, since the property of being the multiplication table for such an algebra is expressed by a first order formula.

Problem. What do the (purely) logical first order proofs look like, which are obtained by unwinding Milnor's proof (say, for $n=16, 64, 256$)?

Reminder (from §2). Naturally, one will not formalize many details of Milnor's proof, but only those steps which are relevant to the unwinding procedure.

It is known that Milnor's result does not extend to all (ordered) fields. A standard counter example is the following commutative and associative division algebra of dimension 3 over the rationals:

The elements are of the form

$$a+b\sqrt[3]{2}+c\sqrt[3]{4},$$

where a, b, c are rational.

Sums and products are defined as usual, that is, for the field of rationals extended by $\sqrt[3]{2}$.

The irrationals $\sqrt[3]{2}$ and $\sqrt[3]{4}$ satisfy cubic equations. This is optimal since inspection of standard methods yields the following:

Corollary: For all odd n Milnor's result holds for ground fields in which every polynomial of degree n has a zero (The fields need not be real closed).

Discussion. Mathematically speaking, the problem of unwinding presents a *risk*; specifically, when more is lost than gained. (A — conscious or unconscious — attraction of finitist foundations consisted in apparently removing this risk by the claim that the unwinding was needed for *justifying* Milnor's proof). The corollary above indicates *one* kind of possible gain, incidentally in terms of conventional concepts. The unwound proof will exhibit the particular (finite subset of) axioms for real closed fields that are needed for the conclusion, and may thus suggest a neat *generalization* of Milnor's result (to a larger class of fields). — On the other hand, foundationally or pedagogically speaking there is no risk. There are sufficiently many people with foundational *convictions* that unwinding is either always or never informative, that somebody is bound to learn something from the unwinding.

(b) When — in contrast to (a) above — both mathematical and logical proofs of some (logical) formula are actually available, unwinding is used to *compare* the proofs. For example, suppose DO are (first order) axioms for dense orderings without first or last element, and that F is a formula in the language of DO with the single free variable x. Then $\mathrm{DO} \rightarrow \forall x \forall x' (F \Leftrightarrow F')$ where F' is $F[x/x']$. For each such F, the implication can be proved by elimination of quantifiers, but also (mathematically) by use of the categoricity of DO for countable models and their automorphisms. The mathematical proof can be formalized in type theory, and unwound by normalization: but we really have no idea what the resulting (logical) proof looks like.

Discussion. One, very familiar way of *expressing* the malaise produced by the existence of such spectacularly different proofs is to doubt the validity of the set-theoretic notions used in the mathematical proof. But note that there are also non-ideological doubts about — the concepts needed to state — structural relations between those proofs.

Appendix : a case study

The purpose of this appendix is to expand the general discussion in Section I by reference to a specific case. Plenty of familiar material could be used for this purpose, for example, the discovery and recognition of any of the basic current axiom systems. But I (and the readers likely to profit from this article) find hackneyed examples distasteful, and so a very interesting recent discovery by Paris, already mentioned in the text, will be used instead. Besides, there is no analysis in print which puts this discovery into a broad context.

Paris discovered a striking variant RT_A('A' for arithmetical) of Ramsey's own finite version RT_F of his theorem RT on partitions of the set of pairs of a countable infinite set; for exact statements, see [7]. According to the title of [7], the most remarkable property of RT_A is that it is a 'mathematical' theorem which can be stated but not proved in first order arithmetic. As already indicated at the end of Part I, this alone is not particularly convincing since ε_0-induction (for, say, the complete $\prod_1^0$ predicate) is hardly any more meta-mathematical than RT_A, and has been known for more than 40 years to have the same remarkable property. This is made precise at the end of (b) below.

(a) As for background, it has been known since the work of Jockusch [2] that RT itself cannot be proved in most 'usual' conservative extensions of first order arithmetic with full induction; more specifically, any finite subset of axioms of those extensions is satisfied by some (finite) segment of the arithmetic hierarchy, and RT is not. On the other hand, RT_F itself can be comfortably proved in first order arithmetic, in fact, bounds for the corresponding Ramsey functions lie in E_4—E_3 of Grzegorczyk's hierarchy of the primitive recursive functions; cf. [8] p.140, Lemma 6.

(b) *Validity of* RT_A. Far and away the simplest proof of RT_A uses a deduction (by compactness) from RT itself. The same applies to RT_F.

Corollary. Taken in their literal sense, as $\prod_2^0$ theorems, a separation between RT_A and RT_F (so to speak, on the ground of a different 'kind' of validity) is suspect. — More precisely, as cannot be repeated too often, it is an *assumption* that the classification of theorems according to formal derivability in any particular (incomplete) system is significant. The discovery that RT_F and RT_A are separated by this classification, using first order arithmetic, casts doubt on the assumption

The situation changes if interest shifts from the literal sense to ***bounds*** for Ramsey functions, specifically *upper* bounds. NB. It is a striking discovery that, in contrast to the bulk of elementary mathematics, this shift is significant here: usually bounds are read off quite directly from a proof of a $\prod_2^0$ theorem.

Proofs via RT supply α-recursive bounds for some $\alpha<\varepsilon_{\varepsilon_0}$, by [3] inasmuch as the most obvious formalization of the proof of RT uses $\prod_1^0$—analysis (which is formally identical to the theory of the first level of ramified analysis [9]). As always this can be improved by bounding the complexity of the induction schema used in the proof of RT. — Evidently, these bounds are far beyond E_4 which, by above, bounds the original Ramsey functions (of RT_F).

The proof of RT_A in [7], via the so called $\sum_1^0$ - reflection (or soundness) principle for first order arithmetic, supplies an ε_0-recursive bound. This follows from — one direction of — the well-known equivalence, for example, in 1. 7—8 on p 121 of [5], between ε_0-induction and the reflection principle. For reference below: the bound in question is *primitive recursive* in f_{ε_0}, the particular ε_0-recursive function of Wainer's hierarchy (used in [10]).

Discussion. Realistically speaking, this proof, though very agreeable to a logician, is unsatisfactory for those who really want to know bounds for Ramsey functions. (After all, for a logician, consistency is a much more interesting assertion than RT_A!) The proof requires the verification that a number of arguments *can* be formalized in first order arithmetic; evidently, a delicate matter (for a novice) in a context where there are also arguments which cannot be so formalized, specifically, proofs of RT_A!

The best upper bound for RT_A so far obtained is $f_{\varepsilon_0}(n+4)$, in [10]. The proof uses a careful proof-theoretic analysis of subsystems of first-order arithmetic in terms of Wainer's hierarchy.

It seems plausible that the machinery of [10] can be developed to give this bound by means familiar to the *principal consumer*, the combinatorial mathematician interested in RT_A. Specifically, one would use an ordering (of type ε_0) of finite partitions, called 'algebras of sets' in [10], and one would apply induction on that ordering to a combinatorial property of such partitions. In contrast, the unwinding of the proof of RT_A in [7] together with the deduction of ($\sum_1^0-$) reflection from ($\sum_1^0-$) ε_0-induction uses orderings of infinite cut-free proof trees and unfamiliar (derivability) properties of formulae at the nodes of those trees.

(c) *Formal underivability of RT_A : lower bounds.* Once again, a number of proofs are available. First of all, there are more or less familiar constructions of models, originally by Paris, later by Kochen-Kripke (unpublished), in which RT_A fails. By itself, this does not establish any lower bounds at all because, after all, even a (numerically) true $\prod_1^0$ statement can be formally underivable. The device used here is to have models in which all true $\prod_1^0$-statements hold, and appeal to the fact that, for $\alpha<\varepsilon_0$, all α-recursive functions are provably recursive. If a $\prod_2^0$-statement $\forall x \exists y A(x, y)$ has an α-recursive bound, defined by a Gödel-number e_α, then the $\prod_1^0$-statement

$$(*) \qquad \forall x \forall z \{T(e_\alpha, x, z) \rightarrow \exists y [y < U(z) \wedge A(x, y)]\}$$

is true, $\forall x \exists z\, T(e_\alpha, x, z)$ provable, and so $\forall x \exists y\, A(x, y)$ is derivable from (*).

Corollary (for people interested in the formal independence of $\sum_1^0-\varepsilon_0$-induction). Once one has (i) a model in which all theorems and all true $\prod_1^0$-sentences of arithmetic do, but RT_A does not hold, and (ii) any ε_0-recursive *upper* bound for RT_A (as in (b) above), it is immediate that f_{ε_0} is not provably recursive.

Secondly, there is the proof in [7] which derives the $\sum_1^0$-reflection principle (in primitive recursive arithmetic) from RT_A. Appealing again to the proof theoretic equivalence mentioned in (b), we find that any Ramsey function *enumerates* all α-recursive functions for $\alpha<\varepsilon_0$, and so cannot be equal to any such function. Trivially, as in (*) above, no Ramsey function could be dominated by any such function either. In terms of Wainer's hierarchy (in [10]) : f_{ε_0} is *primitive recursive* in any bound for RT_A. This is done by unwinding the proofs of (i) $(\sum_1^0-)$ reflection from RT_A in [7] and (ii) of $(\sum_1^0-)$ ε_0-induction from $(\sum_1^0-)$ reflection.

Again, neither of the proofs mentioned can be satisfactory to the principal consumer because it involves the passage from provably recursive to $<\varepsilon_0$-recursive functions. Only the latter are so defined that the property in question, rapid growth, is evident.

The third proof, by Solovay [10], shows, in terms familiar to the combinatorial mathematician, except for the notion

$$\alpha\text{ - recursive function: } \alpha<\varepsilon_0,$$

that all such functions are almost everywhere *lower bounds* for Ramsey functions of RT_A. In fact, by [10], $f_{\varepsilon_0}(n-4)$ is a lower bound.

Discussion. There is, I believe, a useful parallel between Solovay's proof and Higman's well-known characterization of subgroups of finitely presented groups (ignoring for the moment the relative interest of this part of group theory and of the partition calculus resp.). Higman discovered that a few *notions* of recursion theory combined with a good deal of group theory permit a satisfactory answer to the question:

Which finitely generated groups can be embedded in finitely presented groups?

Solovay succeeds in using a notion first thrown up in proof theory to answer the question:

How fast do Ramsey functions of RT_A grow?

Certainly, no bounds anywhere in combinatorial (or other ordinary) mathematics, have ever come near the (lower) bounds for RT_A. A critical view of traditional proof theory, specifically of the consistency programme was of some help (as claimed at the end of Part I) because — on the traditional view — the emphasis on extensional properties of provably recursive functions is quite trivial compared to the metamathematical methods used in the consistency proof.

Remark. Just as the discovery (in [8], 16.4, based on section 14 about infinite cardinals) of the original lower bounds for RT_F, Solovay's argument obviously involves the fruits of experience with infinitary partition calculus. This is a counterpart to Jensen's successful use in (infinitary) set theory of some developments in proof theory of Bachmann's ideas for defining fundamental sequences. Certainly, not everything is the same as everything else (unless viewed very superficially). But the *particular* traditional distinctions between 'the' finite and 'the' infinite are not all that important as far as proofs are concerned; certainly less than appears to the inexperienced.

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PS (added March 1978). Since this paper was written, Part (b) of the Appendix has been improved. (i) On the formal side several of us noticed that ε_0-induction (applied to arithmetic predicates) axiomatizes the arithmetic theorems, and hence

$$\sum_1^0 -\alpha-\text{induction: } \alpha<\varepsilon_1$$

the $\prod_2^0$—theorems which follow from RT in Δ_1^0—analysis with induction restricted to arithmetic predicates (with parameters). (ii) More interestingly, J. KETONEN established the conjecture at the end of (b), proving RT_A by induction on (a predicate involving) the membership relation in H^α, for $\alpha\leq\varepsilon_0$, where H^α is his hierarchy of so called α-large, finite sets of natural numbers. (The relation is coded arithmetically). His proof uses a general scheme for weakening *suitable* definitions D of familiar closure conditions on ordinals $\varkappa$ (Mahlo, weakly compact, n-subtle); roughly speaking, by rewriting (set-theoretic) D in combinatorial language D_c where the variables for ordinals used as indices are separated from those used as elements of sets. As a result it makes sense to let the set quantifiers in D_c range over $\varkappa$-large sets of natural numbers in place of arbitrary subsets of ordinals $<\varkappa$. Ketonen's proof of RT_A shows that ε_0 is the least ordinal which satisfies (the latter, arithmetic interpretation of) $\omega-S_c$ where ω-S is the appropriate definition of *n-subtle for all n* as a partition property. -Ketonen's scheme gives further substance to the *Remark* on p. 121 at the end of this paper.

Incidentally, the formal work in (i) is sometimes useful for (ii), for example, to check bounds for (the least ordinal satisfying) D_c.

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LE NOUVEL ESPRIT MATHEMATIQUE

Maurice LOI

Le sujet de cette conférence tient d'abord à une question d'opportunité: je prépare un ouvrage dont le titre sera d'ailleurs celui de la conférence d'aujourd'hui, qui rappelle tout ce qu'elle devra à l'oeuvre de Gaston Bachelard en épistémologie, et il me sera particulièrement utile d'avoir votre opinion sur les quelques idées que exposer devant vous.

Ces idées sont à l'origine du Séminaire de philosophie et mathématiques. Elles ont leur source dans les travaux d'Albert Lautman, qui viennent d'être réédités par les Editions 10/18 à Paris, et dans la constatation que l'enseignement, la philosophie et la culture méconnaissent le véritable esprit des mathématiques contemporaines, dont je voudrais montrer la liaison avec l'heuristique. Même lorsque les programmes ont été modifiés dans les écoles, même quand les notions nouvelles y ont été introduites, l'ancien epsrit dogmatique et sclérosé règne toujours, oubliant la dynamique de la science. La vie des concepts est ignorée et pour donner du mouvement à des notions mortes on a recours à l'agitation enfantine, à du bricolage, des exercices et des problèmes destinés à faire "sécher" les élèves parce que les outiles nécessaires à leur résolution ne leur sont pas toujours donnés, ce qui leurs impose des complications stupides. A l'opposé on encombre l'apprentissage de notions simples et nécessaires de considérations pédantes et inutiles à ce niveau. C'est une conception dépassée des mathématiques qui domine cette pédagogie ou rigueur et créativité sont trop souvent opposées, alors que l'accroissement de la rigueur mathématique et les recherches logiques ont permis d'augmenter de façon considérable les moyens d'invention de l'esprit humain dans tous les domaines, comme je m'efforcerai de le montrer au cours de cette conférence. Et c'est une autre raison de cette conférence; le mépris contemporain de trop de personnes pour la déduction et la rigueur, le "déductivisme" disent-ils d'un ton méprisant. Or les conquêtes essentielles de la science ont été obtenues avec l'intervention dominante de la déduction. L'impossibilité d'y arriver par le moyen de simples inductions basées sur l'observation directe avait été reconnue par Galilée lui-même comme je le rappellerai tout à l'heure. Or l'a priori est devenu signe da l'arbitraire, du conventionnel et Poincaré n'a pas peu contribué à répandre cette idée. La science expérimentale équivaut alors à la science objective. Bien sûr, il y a eu un usage abusif de la déduction au Moyen-Age,e aboutissant à des théories mystiques ou fantastiques, mais les alchimistes ont bien usé aussi de la méthode expérimentale. Ces médisances contre la raison humaine ne doivent pas nous faire oublier la fécondité de la déduction, qui n'est stérile qu'entre les mains des ignares. En fait, elle constitue souvent un moyen bien plus efficace et plus sûr de recherche que l'expérience ou l'observation directe. Mieux: sans elle la science ne peut pas se constituer. Même dans l'étude des phénomènes sociaux, les plus hardis inventeurs et constructeurs de plans de réformes et les critiques les plus impitoyables des théories justificatrices des institutions et des règles sociales effectivement existantes, sont précisément ceux qui se distinguent par une plus grande tendance à l'usage de la déduction, par exemple Rousseau et Marx. Or ce mépris et cette ignorance de la déduction ne permettent pas de juger correctement les mathématiques, qu'on réduit à quelques recettes ou procédés de calcul sans portée au delà des exercices et des problèmes d'examens et de concours. Il est vrai que la société de consommation les utilise comme moyen de sélection pour recruter ses cadres. Mais les mathématiques ont quand-même un autre intérêt et c'est la société qu'il faut changer.

I. L'essor des Mathématiques et leur valeur Invéntive

Les mathématiques, malgré leur ancienneté, connaissent de nos jours un essor impétueux et accéléré dont il est possible à un profane d'apprécier l'ampleur en consultant la circulaire mensuelle de la *SMF* donnant le programme des séminaires, colloques, congrès, en visitant une bibliothèque spécialisée; en feuilletant un numéro de *Mathematical Review*, de *Current mathematical Publications* ou encore de *Zentralblatt für Mathematik und ihre Grenzgebiete*. On sera impressionné par le nombre de problèmes résolus et la variété des résultats obtenus, par la floraison de théories audacieuses, par la quantité de livres et de publications divers. Jean Dieudonné a pu écrire: "On peut dire sans exagération qu'il y a eu plus de problèmes mathématiques fondamentaux résolus depuis 1940 que de Thalès à 1940"* L'âge d'or, qui a commencé pour les mathématiques au début du XIXe siècle, n'est pas prêt de finir. Cette première constatation prouve l'activité créatrice de l'esprit humain en mathématique pour élaborer des méthodes et des théories nouvelles permettant de résoudre des problèmes posés depuis longtemps, théories nouvelles qui font naître inversement l'idée de problèmes nouveaux, lesquels ne pouvaient être formulés abstraitement auparavant. Le degré d'abstraction de plus en plus poussé des mathématiques ne les empêche pas — au contraire, pourrait-on dire — d'être utilisées dans des secteurs les plus divers, ou ce sont parfois les théories et les idées les plus récentes et les plus élaborées qui se révèlent les plus nécessaires. Ainsi Einstein eut besoin au début du siècle de la théorie des groupes, de la géométrie riemanienne et du calcul tensoriel pour élaborer sa théorie de la Relativité. Ce faisant il ne procédait pas du tout à la façon dont l'imaginent encore en 1977 trop de personnes, utilisant un langage, un simple moyen d'expression pour une idée déjà existante. Grâce aux mathématiques formelles les plus élaborées et les plus éloignées de l'expérience, des notions aussi fondamentales que l'espace et le temps, dont Kant avait fait des absolus, furent bouleversés. Des conclusions étonnantes, telles que celles de l'équivalence de la masse et de l'énergie ont été obtenues comme des conséquences mathématiques du principe d'invariance par les transformations de Lorentz de toutes les équations gouvernant les phénomènes physiques. Le point de vue du mathématicien triomphe de celui des empiristes.

* Es *ai sur l'unité des mathématiques*, par Albert Lautman, p.20 note 2.

Conséquence ce qui n'a pas assez retenu l'attention des philosophes, ni medifié notre culture colporteuse d'idées surrannées.

Aussi, trop souvent, n'estime-t-on pas à sa juste valeur le rôle des mathématiques dans la pensée scientifique. Or désormais elle est tout entière présente dans son effort mathématique, ou, pour mieux dire, c'est l'effort mathématique qui forme l'axe de la découverte. C'est l'expression algébrique qui seule, souvent, permet de penser le phénomène, comme si l'esprit acquerrait des facultés nouvelles en la maniant, rendant possible le mouvement spirituel de découverte. Je pourrais citer d'autres exemples analogues de la physique: algèbre stellaire, géométrie symplectique, théorie des groupes etc, ou des sciences biologiques ou humaines, qui montreraient le rôle heuristique des mathématiques dans l'oeuvre de théorisation, de réflexion et de définition des concepts. C'est le premier aspect essentiel du Nouvel esprit mathématique d'être une source d'idées qui permettent la compréhension et la maîtrise des phénomènes comme l'avait rêvé Descartes dans sa philosophie pratique et conquérante : saisir l'intelligence des choses à partir de leurs vrais principes qui donnent la lumière intellectuelle, telle est la véritable mathématique, où il n'opposait pas induction et déduction comme le font certains de nos contemporains, qui voient dans l'induction la source unique des inventions et considèrent la "sèche" déduction comme un simple moyen de preuve et d'exposition de résultats déjà trouvés. Or la conquête de vérités importantes ne peut être effectuée par la simple observation passive, mais exige l'exercice d'activités mentales bien plus élevées et compliquées. Dans la plupart des cas les expériences sont de simples vérifications de conclusions auxquelles les expérimentateurs sont déjà arrivées indépendamment d'elles: "Je fus d'abord persuadé par la raison avant d'être assuré par les sens" écrivait Galilée (*Dialogue* des grands systèmes, seconde journée). Pasteur, deux siècles plus tard, a justement défini l'expérimentation comme une observation guidée par des idées préconçues, c'est-à-dire, en d'autres termes, une observation précédée et accompagnée de procédés déductifs.

Un précurseur : Descartes.

Descartes avait le souci d'une logique féconde qui serve non seulement à exposer mais à découvrir. Les mathématiques l'ont justement séduit par l'évidence de leurs raisons et l'enchaînement de leurs conclusions. Elles lui ont donné ses idées-clés : toute vérité est un degré, auquel on accède en partant du précédent et qui lui donne lui-même un accès du suivant. Aux touts perçus par l'intuition il faut désormais substituer des composés artificiels, fabriqués par nous et dont par conséquent la structure et tous les éléments nous sont exactement connus. Ainsi la science, au lieu d'être, comme le croyaient les anciens une contemplation d'objets idéaux, se présentera désormais comme une création de l'esprit, une composition synthétique. La tâche essentielle du savant sera, par conséquent, non pas d'apporter une nombreuse collection de résultats, mais de mettre sur pied de bons instruments de combinaison, de constituer une méthode puissante et efficace. Les voies de la synthèse algébrique sont ouvertes. Tel est précisément le but que Descartes se propose avant toute chose. La physionomie nouvelle que va prendre la science, c'est la géométrie qui la définit, qui la commente, et en donne en même temps une vision concrète : par l'algèbre, une algèbre nouvelle il est vrai, clarifiée et perfectionnée, il est possible de résoudre les problèmes relatifs aux grandeurs et auf figures en suivant une voie sûre et régulière. La sûreté, la régularité de la méthode ; voilà ce qui est essentiel aux yeus de Descartes, voilà ce qui doit distinguer la science moderne de la géométrie ancienne, ce champ clos ou les virtuoses de la démonstration pouvaient seul se mouvoir et accomplir leurs prouesses. C'est en ce sens que Descartes est un précurseur du Nouvel esprit mathématique et pour ainsi dire de Bourbaki: l'algèbre pour lui n'est pas un recueil de résultats, c'est une technique, c'est une méthode de combinaison et de construction. Par le simple jeu du mécanisme algébrique nous faisons surgir un monde géométrique illimité que ne nous aurait jamais révélé l'intuition directe de la figure. En réhabilitant le calcul délaissé par les Grecs au profit de la géométrie, Descartes prépare la route pour la mathématique formelle. Tous les scrupules des géomètres grecs touchant la définition des courbes s'évanouissent, et les détours qu'ils employaient pour y échapper perdent leur raison d'être. La théorie de la construction géométrique devient inutile ainsi remplacées par cette synthèse créatrice, autrement féconde qu'elle.

II. Le rôle central du concept de fonction.

Sous le vêtement de la notion de courbe apparaît (bien sûr il faudra attendre la fin du XVIIe siècle pour que le mot apparaisse et que l'idée soit précisée) la notion générale de fonction grosse de toutes les questions qui bientôt surgiront à sa suite. Cette notion n'a pas seulement constitué un perfectionnement des mathématiques, elle a marqué un changement radical dans leur orientation, qui n'est pas toujours apprécié comme il convient malgré ses nombreuses conséquences et applications pratiques.

L'intérêt philosophique de cette découverte a été apprécié par la suite par quelques philosophes tels que Hegel ,Marx et Engels: le passage de la pensée de Parménide à la pensée d'Héraclite. Hegel nota dans la *Phénoménologie de l'esprit* que la tâche pédagogique moderne est en quelque sorte inverse de la tâche pédagogique antique, qu'il faut maintenant rendre fluides ces déterminabilités. Telle est la tâche qu'il se proposera dans sa *Logique*. Selon Parménide tout être intelligible par la raison doit être considéré comme invariable tandis que selon Héraclite c'est le changement qui est la loi dominante de l'univers. La constitution de la mathématique grecque marqua le triomphe de Parménide: la philosophie d'Héraclite ne laissant place à aucune fixité, elle aurait abouti à nier la valeur de la mathématique et empêché le développement de la science. Bien sûr la pensée grecque est bien plus complexe que cette schématisation peut le laisser croire. Platon, par exemple, fut tout autant fasciné par Héraclite que par Parménide, mais il appelle dialectique ce qui sera appelé plus tard métaphysique. Car Platon avait déjà une conception riche et souple de la raison, qui savait s'inspirer des découvertes de la science. Aussi n'est-ce pas un hasard si Albert Lautman fit si souvent référence à Platon lorsque vers les années trente du XXe siècle il voulut élaborer une philosophie mathématique. Tel fut aussi l'effort de Brunschwig et de Bachelard. Mais au XVIIe siècle Pascal opposait encore esprit de géométrie et esprit de finesse alors que le mathématicien moderne use autant de l'un et de l'autre. Une constatation doit être soulignée: le dogmatisme fut d'abord surmonté dans la science. Voilà une leçon dont Bachelard sut tenir compte mais que bien des philosophes contemporains devraient méditer.

Quelle fut l'importance de ce tournant dans la pensée mathématique?

Pour la science grecque tout problème se ramenait à la recherche d'un ou plusieurs nombres, déterminées d'une manière complète quoique implicite, par les données de la question. Manifeste en ce qui concerne les problèmes d'arithmétique, cela n'était pas moins certain dans le domaine géométrique, puisque les figures considérées pas les Anciens (points, droites, plans, cercles etc.) dépendaient chacune d'un nombre fini et même peu élevé de paramètres. Etudier les relations entre certains nombres laissées invariables dans tout le cours du raisonnement ainsi que la manière d'utiliser ces relations pour calculer quelques-uns d'entre eux, les autres érant supposés donnés, voilà ce que se sont proposé mathématiciens jusqu'au XVIIe siècle.

D'Eudoxe et Archimède à Leibniz et Newton.

Eudoxe et Archimède furent des exceptions et n'eurent pas de successeurs directs. Le cadre de la géométrie antique ne fut réellement dépassé et une arme nouvelle donnée à la science que lorsqu'on considéra la variation continue de certains éléments numériques ou géométriques — ce qui revient au même. — liés les uns aux autres et ainsi furent jetées les bases de l'édifice que devaient achever Newton et Leibniz.

Mais ce stade devait être bientôt dépassé. Il ne consituait que le début d'une évolution qui n'a cessé par la suite de se poursuivre dans le même sens et elle se continue encore à l'heure actuelle. Lorsque les notions nouvelles déduites de celles de fonction furent appliquées à la physique et eurent montré la légitimité de ce nouveau point de vue, que le calcul infinitésimal permettait pour la première fois d'aborder: il n'était plus possible à la science de le laisser de côté. Dès que l'on commença à s'attaquer au mouvement, à capter l'invisible c'est-à-dire le changement — ce qui n'avait pas été possible avant qu'on disposât des instruments mathématiques adéquats — et à mettre ses lois à la base de la physique, il apparut que dans l'étude de la nature on ne pouvait continuer à considérer comme seule individualité, comme seul objet de recherches, le nombre déterminé ou ses équivalents géométriques (point, droite, cercle etc.) L'être mathématique, en un mot, ne fut plus le nombre: ce fut la loi de variation, la fonction, qui devint le centre autour duquel s'organise la science. La mathématique n'était pas seulement enrichie de nouvelles méthodes, elle était transformée dans son objet et dans ses fondements.

La transformation ne fut pas totale du premier coup. L'Analyse ne fit pas d'un seul coup le saut qu'elle allait être obligée de faire et garda un pied sur la rive qu'elle devait quitter. C'est seulement au XIXe siècle avec Fourier, Dirichlet, Cauchy, Riemann, que la notion de fonction prit son sens moderne et toute sa portée: une fonction $y=f(x)$ ne s'obtient plus nécessairement par un certain nombre d'opérations prises dans une liste déterminée quelle qu'elle soit. C'est une correspondance quelconque établie entre chaque valeur attribuée à x et une valeur y, supposée seulement déterminée dès que la première est donnée, mais sans qu'on s'astreigne à employer pour cela tels ou tels modes de détermination plutôt que d'autres.

La nouvelle tendance dialectique de la science et l'unité des mathématiques.

Cette fois la novelle tendance de la science ne pouvait pas manquer de prendre conscience d'elle-même. Définir une fonction arbitraire, c'est définir sa valeur pour chaque valeur de x; si cette fonction est supposée représentés par une ligne, cette ligne est, elle aussi, quelconque, et n'est déterminée que lorsqu'on connaît chaque point. La connaissance de la fonction ou de la courbe équivaut donc non plus à celle de certains nombres mais à celle d'une infinité de nombres. Et c'est encore sous cette forme que se posaient les nouveaux problèmes, où aucune image simple ne s'offrait plus à l'esprit. L'intuition géométrique ne pouvait plus rien nous apprendre. Pour remédier à cette ignorance, la raison ne pouvait le faire qu'analytiquement: il fallait créer et développer la Théorie des ensembles. Bien sûr il faudrait, dans le même ordre d'idées, parler du calcul des variations, des équations différentielles et intégrales, du calcul fonctionnel, de la théorie du potentiel et de bien d'autres choses pour montrer pourquoi le concept de fonction marque bien le début d'une ère nouvelle et qu'il en est le noeud essentiel. Si je me suis attardé quelque peu sur cette notion fondamentale qui a ouvert des portes nouvelles à la pensée, c'est parce que dans l'enseignement, en France du moins, elle a été quelque peu obscurcie par un engoument exagéré et naïf en faveur du concept d'ensemble ou de relation. Bien sûr on a dit et répété: "la mathématique moderne est la science des relations" en oubliant de préciser que la relation fondamentale de base de l'édifice, reste la fonction. Russell l'a bien vu, qui lui fait jouer le rôle essentiel dans *The Principles of Mathematics*, ou un chapitre est consacré aussi à la notion de variable, une des notions essentielles de la nouvelle mathématique. Mais comme l'écrivit Hermann Weyl en 1949: "Nul ne peut dire ce qu'est une variable"*. Elle n'atteint quelque précision qu'avec le développement de la théorie des ensembles et des mathématiqnes. Birkoff et Mac Lane proclament, eux, le mot d'ordre: "tout est fonction" dans leur traité d'algèbre. Il s'agit alors du concept pris dans toute son ampleur qui est omniprésent en science et non sous sa forme la plus pauvre, comme dans trop de manuels d'enseignement.

Non seulement le concept de fonction fut à l'origine des travaux de Cantor et il devient le véritable objet du calcul fonctionnel exactement au même titre qu'un point ou un nombre, mais il peut être pris comme notion fondamentale et primitive pour exprimer les propriétés de certains ensembles sans faire appel aux éléments. C'est lui qui sous des noms divers: application, homomorphisme, homéomorphisme morphisme, isomorphisme, transformation, correspondance interprétation, représentation, opérateur, foncteur etc. est si souvent utilisé. Ces divers synonymes suggèrent une activité féconde tissant l'unité profonde des mathématiques, parce qu'elle a pour but de révéler des rapports qui illuminent les données. Elle est devenue la clé de voûte de l'évidence et avait déjà retenu l'attention au XIXe siècle de mathématiciens comme Lejeune-Dirichlet et Dedekind. Le premier écrivait: "Il arrive très souvent en mathématiques ou dans les autres sciences que si un système d'objets ou d'éléments ω est donné, chanque élément ω détérminé soit remplacé d'après une certaine loi par un élément déterminé ω' correspondant à ω". On a l'habitude d'appeler substitution un tel acte et on dit que ω' est le transformé de ω par cette substitution et Ω, lequel est constitué par les ω' le transformé de Ω. Il est encore plus commode de dire, comme nous le ferons, que cette substitution est une application de Ω, que ω' est l'image de ω et Ω' l'image de Ω'. Dedekind ajoute en note: "C'est dans cette capacité de l'esprit de comparer un object ω avec un objet ω', ou de mettre ω et ω' en relation, ou de faire correspondre à ω un ω', capacité sans laquelle il n'y aurait tout simplement pas de pensée, que repose aussi, comme je le montrerai ailleurs, toute l'arithmétique." L'idée de la définition de l'application dans 'Zahlen' remonte en effet à Dirichlet: "Par une application f d'un ensemble S j'entends une loi qui attache à chaque élément déterminé s de S un objet déterminé qui s'appellera l'image de s"**

* Philosophy of Mathematics and Natural Science

** Zahlentheorie hrsg. von R. Dedekind, 1879 — 163 pp 469—70 cité par J. Largeault: Logique et philosophie de Frege p. 418.

Clifford à son tour attira l'attention sur le rôle crucial du concept de fonction: "La mise sur pied d'une correspondance entre deux ensembles et la recherche des propriétés qui se conservent au cours de cette correspondance, peut être considérée comme l'idée centrale des mathématiques modernes: on la retrouve à travers toute la science pure et ses applications."*

III. La fin du dogmatisme et les limites de Descartes

Après avoir souligné les grands mérites de l'épistémologie cartésienne, il est nécessaire d'en tracer les limites précises et de souligner ce qu'elle peut contenir de périmé en 1977. A ce sujet je peux suivre presque à la lettre le dernier chapitre du *Nouvel Esprit scientifique* de Gaston Bachelard.

Tout d'abord le dogmatisme de la méthode, qui fut notons-le quelquefois le fait des cartésiens plutôt que de Descartes lui-même, devient un frein pour la connaissance. Les complications inutiles qui se rencontrent dans beaucoup de résultats classiques sont justement dues à l'emploi de méthodes qui n'ont rien à voir avec le résultat escompté, de méthodes n'admettant pas, en général, le même groupe de transformations que le résultat. L'importance accordée à l'intuition, au simple: à l'évidence et aux idées innées ne convient plus du tout à la science moderne où même des notions comme celles d'espace et de temps sont bouleversése, pas plus que le hautain mépris pour la logique formalle. Leibniz serait un meilleur guide, comme l'a noté Bourbaki.

L'intuition cartésienne, certes, est l'intuition intellectuelle, l'aperception du rapport logique de principe à conséquence, tandis que Kant n'admettra plus d'autre intuition que l'intuition sensible et repoussera avec force l'intuition intellectuelle qui est pour lui le vice fondamental de toutes les métaphysiques antérieures, y compris la métaphysique cartésienne. Sortir du sommeil dogmatique était certainement indispensable, comme l'a écrit Kant, mais il ne fallait pas oublier que sans stabilité il n'y a plus de science: "Donne-moi un endroit où se tenir ferme et j'ébranlerai le monde" notait déjà Aristote. La pensée scientifique détermine dans l'univers changeant les points fixes, les pôles inamovibles et s'en sert comme de repères. Une des premières démarches de l'esprit humain fut de découvrir sous le devenir, ou au-dessus, des permanènces. De là sont nés les problèmes de la substance, de l'essence, de la forme, de l'être, de l'existence, de la vérité, sur lesquels méditèrent les métaphysiciens mais qui furent aussi au centre de l'activité mathématique, activité d'où surgirent de nouvelles manières de penser et l'esprit acquit des capacités insoupçonnées. En grec le terme même d'"épistêmê" est étymologiquement dérivé d'une racine signifiant "fermeté" et "stabilité". Ainsi le changement a-t-il été d'abord considéré comme une dégradation et non pas comme un progrès. La méthode scientifique conduit à un équilibre stable, à la stabilisation et à la consolidation du monde des perceptions et des pensées, sans lesquels le changement ne peut être maîtrisé. Le cas des mathématiques est exemplaire: la géométrie est l'étude des propriétés invariantes dans un déplacement ou quelquefois dans une similitude. Depuis Klein et Sophus Lie une géométrie est désormais l'étude des propriétés invariantes d'un groupe de transformations, la topologie est une géométrie dont le groupe est celui des homéomorphismes etc. Klein a, en effet, montré avec beaucoup de force que le plus important pour une géométrie n'est pas la nature

* Cité par Jean-Claude Pont dans la Topologie algébrique p. 121 (Mathematical papers pp 334-5).

des points qu'elle étudie, mais la structure du groupes de transformations qui y définit l'égalité des deux figures. Il faudrait citer aussi la théorie des invariants algébriques dont l'intérêt retient encore l'attention des mathématiciens, la théorie de la relativité en physique théorique où l'essentiel ce sont les absolus, les invariants.

Tout changement d'ailleurs n'est pas forcément un progrès, mais l'esprit a besoin d'une certaine tension pour progresser. Une féconde bipolarité lui est indispensable. Et c'est un mathématicien qui le note, Jean Dieudonné, dans son avantpropos à l'oeuvre d'Albert Lautman (p.17): Tant il est vrai que le grand laboratoire des idées, c'est désormais au sein de la science qu'il se trouve. On peut dire que les vrais savants sont à la pointe de la culture et de l'innovation. Malheureusement la philosophie contemporaine non seulement n'est plus l'antichambre de la science, mais elle ignore la science contemporaine dont elle se fait une conception dogmatique, et Bachelard est une exception. Pourtaint les Grecs avaient déjà très bien saisi la nécessité de cette tension de l'esprit. Lorsqu'un problème était résolu Platon "tenait la blessure ouverte" et se refusait à "cacher derrière un mot la difficulté du concept". Aristote affirmait que la science commence avec l'étonnement. Mais la mode en 1977, où tout un chacun se réclame pourtant de la science, n'est plus à l'étonnement. Tout est présenté comme allant de soi, naturel, facile, à l'aide d'une philosophie paresseuse qui est la négation de la véritable culture. Celle-ci ne peut ignorer l'extraordinaire essor des mathématiques, où nous voyons à l' oeuvre l'effort de la raison et le triomphe de l'intelligence. Il n'est plus possible d'immobiliser la perspective de la clarté intellectuelle, d'imaginer que le plan des pensées les plus claires se présente toujours le premier, que ce plan doit rester le plan de référence et que toutes les autres recherches s'ordonnent à partir du plan de la clarté primitive. Le simple est une conquête et non plus une donnée ou un point de départ.

L'ideal de complexité.

Le temps cartésien des natures simples et absolues est révolu. On pourrait dire que c'est un idéal de complexité qui anime la science contemporaine, ou plutôt il s'est établi un véritable chassé-croisé du simple au complexe et inversement. "Il n'y a pas de route royale pour la science" disait le mathématicien grec Ménechme, l'un des précepteurs d'Alexandre le Grand, qui remplaça l'incomparable Eudoxe précurseur des mathématiques modernes. Les mathématiques sont abstruses et difficiles et toute assertion qu'elles sont simples n'est vraie que pour les initiés ou les pseudo-pédagogues à la suite de Piaget. Mais on paye cher cette facilité, cette confiance dans l'acquis et le spontané, ce repos dans les idées reçues.

Tout le problème de l'intuition se trouve bouleversé. Des concepts aussi primitifs que "point", "droite", "plan", "espace", "nombre", etc ont été enrichis à tel point qu'ils présentent maintenant de multiples facettes. Ils se sont complexifiés en s'enrichissant. Une telle variété d'aspects exige qu'on en finisse avec la stupide raideur dont font preuve trop d'enseignants ou de formateurs d'enseignants, qui soutiennent encore qu'un concept doit être noté d'une seule et unique façon partout et toujours sous peine d'ambiguïté Ils ne voient pas que c'est précisément le choix du bon formalisme, du langage adéquat au but poursuivi, à la solution d'un problème out tout simplement à l'énoncé précis et rigoureux de ce problème qui est devenu la caractéristique de la pensée mathématique contemporaine, de son intelligence et de sa souplesse. On saisit mieux pourquoi les mathématiciens accordent tant d'importance non seulement au résultat mais aussi au style et à l'élégance, pourquoi la "beauté", c'est-à-dire l'exacte concordance entre les moyens mis en oeuvre et les fins à atteindre, occupe une telle place dans les motivations profondes des mathématiciens. Si les rapports entre la pensée et le langage mathématique étaient aussi rigides que les ignorants le prétendent, tout le monde ferait et écrirait des mathématiques de la même façon uniforme. Ce n'est heureusement pas le cas!

La conscience claire du sens axiomatique des principes mathématiques doit être acquise pour bien dessiner le simple après une étude approfondie du complexe. La liste des axiomes dans la géométrie plane axiomatique de Hilbert n'est pas seulement plus complète que celle d'Euclide: ils correspondent désormais à un point de vue diamétralement opposé au point de vue constructif. En effet, au lieu de définir les points, droites etc. à partir d'autres notions pour en déduire ensuite leurs propriétés, elle laisse la nature de ces objets complètement indéterminées se contentant d'énoncer leurs propriétés fondamentales, qualifiées 'axiomes'. Et l'exemple de l'axiomatique de Hilbert ne devait pas resté isolé. En particulier l'Algèbre allait de cette façon se constituer d'une manière autonome. Le style des écrits mathématiques en fut profondément modifié comme l'a noté Claude Chevalley dans un article de la *Revue de Métaphysique et de Morale* en 1935: "Ce souci d'exacte adéquation des méthodes remet en honneur, tout en lui donnant un sens précis, la recherche de l'élégance des démonstrations, quelque peu négligée par les géomètres de l'école précédente" (p.382).

Pour être utile l'intuition doit être savante et rationnelle, sinon elle est 'un obstacle épistémologique', comme aimait à dire Bachelard, et non plus une aide. En particulier la suprématie de la géométrie euclidienne ne saurait être plus légitime que la suprématie du groupe des déplacements. En fait ce groupe est relativement pauvre; il a cédé la place à des groupes plus riches, plus aptes à décrire rationnellement l'expérience fine. On comprend alors l'abandon total de l'opinon de Poincaré relative à la commodité suprême de la géométrie euclidienne. Cette opinion est plus qu'une erreur partielle et l'on trouve à méditer plus qu'un conseil de prudence dans les prévisions du destin de la raison humaine. En la rectifiant on aboutit à une véritable révolution dans le domaine rationnel et l'on apprécie mieux le rôle créateur de l'esprit mathématique. L'idée est communément admise en génétique aujourd'hui que l'évolution biologique dans l'espèce humaine s'est considérablement ralentie et a été relayée par une évolution culturelle*. Dans la formation de l'intelligence, les mathématiques ont certainement occupé une place centrale pour en former la charpente. Valéry dit quelque part dans Eupalinos: "Les nombres ont été les premiers mots."

Mathématiques et philosophie

Mais les philosophes en 1977 s'occupent de tout: politique, linguistique, histoire, sociologie, économie, psychologie, psychanalyse, archéologie du sexe, arts, statut de la philosphie etc, mais ils ignorent souvent les mathématiques, riches pourtant d'idées philosophiques. Il est vrai que les mathématiciens le leur rendent bien en méprisant la philosophie comme une vaine spéculation sans intérêt, qui a perdu sa source principale et le terrain privilégié où naissent les problèmes es-

* Voir à ce sujet Atlan N (1975) *Variabilité des cultures et riabilité génétique. Ann. genet.* 18, n. 3 149—152.

sentiels de la connaissance. Car la mathématique et la philosophie sont nées ensemble en Grèce: Thalès est considéré comme le créateur des mathématiques, du moins au sens où nous l'entendons c'est-à-dire dans leur rigueur démonstrative, et les historiens de la philosophie voient en lui l'initiateur de la spéculation rationnelle.

C'est Kant qui a établi entre la métaphysique et la mathématique une opposition tranchée et on peut dater de cette époque la scission entre la science et la philosophie. Il a insisté sur leur hétérogénéité absolue, sans doute préccupé d'établir la valeur objective de la science et de ruiner au contraire celle de la métaphysique comme connaissance spéculative et transcendante. Mais c'est aussi au point de vue historique parce qu'il veut réagir contre la philosophie de Leibniz et et de Wolff. Il affirme que les jugements mathématiques sont synthétiques à priori et surtout qu'ils sont nécessairement et exclusivement fondés sur l'intuition, alors que Leibniz les considérait comme analytiques et reposant sur le principe d'identité. La mathématique et la logique modernes donnères raison à Leibniz contre Kant, comme l'a si bien noté Bourbaki. Kant croyait que la logique n'avait pas fait un pas depuis Aristote et n'en ferait plus aucun; la logique moderne a donné à cette assertion le plus éclatant démenti. D'autre part il concevait la mathématique comme la science du nombre et de la grandeur et croyait que la méthode mathématique n'est applicable qu'à ces objets spéciaux. Or la mathématique moderne a rompu le cadre où la tradition l'enfermait et vérifié cette parole de Boole: "Il n'est pas de l'essence des mathématiques de s'occuper exclusivement des idées de nombre et de grandeur."* Boole en inventant le calcul logique et Grassmann en inventant le calcul géométrique n'ont fait que ressusciter des idées de Leibniz et réaliser au XIXe siècle la Caractéristique universelle.

Leibniz plus moderne que Kant

En ce sens on peut dire que Leibniz est plus moderne que Kant. La fusion de la logique et de la mathématique, que Leibniz avait entrevue est aujourd'hui réalisée, mais le développement de la science a montré l'erreur de Kant d'avoir considéré l'espace et le temps comme des absolus éternels de notre sensibilité. Son dogmatisme sur ces problèmes influença bien des savants et des philosophes, comme par exemple Henri Poincaré, qu'il empêcha de découvrir la Relativité, alors qu'il disposait de tout l'outilage technique nécessaire à la constitution de la théorie. Or à cette époque, c'est-à-dire dans les premières années du siècle, c'est le moment où, en Françe, sous la conduite du même Henri Poincaré, de Borel, de Baire et Lebesgue, les notions nouvelles de Cantor sont introduites dans la théorie des fonctions de variable réelle. Elles en bouleversent les principes et les conceptions classiques. La logique traditionnelle montre immédiatement sont insuffisance, car des paradoxes sont inventés, dont la réfutation est malaisée et reste douteuse, des raisonnements dont les faiblesses ne peuvent être démontrées mènent à des conclusions incertaines ou difficiles à admettre. Cette crise atteint sa plus grande acuité exactement en 1904 l'année du centenaire de la mort de Kant, lorsque Zermelo publie son fameux théorème.

Une révision de concepts les plus fondamentaux de l'Analyse paraît alors nécessaire. Qu'est-ce que définir en mathématiques? Une existence ne peut-elle pas être purement nominale et nullement réelle? Un ensemble peut-il être considéré comme défini sans que chaque élément le soit aussi? Qu'est-ce qu'un concept

* *Laws of Thought* p. 12

mathématique véritablement pensé? Hadamard et Denjoy se refusèrent alors à borner la vérité mathématique aux lisières de ce que les hommes sont capables d'exprimer immédiatement par leurs conventions de langage. Dénoncer alors certaines conception de Kant comme le fit, seul, le philosophe Louis Couturat, exigeait un courage certain. Car les idées de Kant régnaient sans partage dans les milieux mathématiques et philosophiques. Deux grands mathématiciens comme Poincaré et Hilbert s'en réclamaient ouvertement. Or ces idées kantiennes étaient devenues une entrave à l'essor de l'esprirt scientifique, dont Kant avait pourtant vu toute la puissance. Mais après avoir fait sa révolution copernicienne il ne sut pas en tirer toutes les conséquences et fut trop préoccupé de mettre des bornes à la raison. Ce sont les savants qui ont fait ce travail, tout particulièrement en mathématiques. Ils n'ont pu le faire qu'en se libérant des oeillères d'une culture dépassée.

IV. Nature des mathématiques

Arrivé à ce point, je voudrais examiner la question de la nature des mathématiques, qui est sous-jacente à mon propos et à toute présentation des mathématiques, donc de leur enseignement. Traditionnellement deux thèses se sont affrontées dans l'histoire. La première consiste à supposer l'existence d'un monde idéal et complet d'objets mathématiques que les mathématiciens doivent découvrir. Cette première conception est appelée platonicienne par référence aux monde des Idées de Platon, encore que ce dernier, malgré le rôle essentiel qu'il accordait aux mathématiques, les considérât comme intermédiaires entre les Idées et la réalité. Cette conception fut et est encore celle de nombreux mathématiciens ou philosophes rationnalistes. Frege et Hermite s'en réclamèrent ouvertement. L'imagination n'a alors aucun rôle, le savant découvre ce qui existe déjà en dehors de lui "tout comme le géographe" aimait à dire Frege, lequel refusa violemment le nouveau point de vue de Hilbert sur la géométrie dans la mesure où il lui semblait compromettre l'objectivité de la science.

La deuxième thèse consiste à considérer que les notions mathématiques s'obtiennent par abstraction à partir des objets sensibles du monde réel. Cette deuxième conception fut avancée par Aristote, "le chef des empiristes" disait Kant, et elle fut effectivement la leur au cours de l'histoire tout comme à notre époque. Le critère de la vérité mathématique réside alors essentiellement dans les applications pratiques, la rigueur semble négligeable et même, un raffinement inutile. L'observation et l'expérimentation sont les sources fondamentales des innovations. Alors le bricolage, le tâtonnement, l'à peu près jouercnt un rôle essentiel dans l'enseignement. Dans cette conception l'accord avec le monde réel ne pose aucun problème et va de soi, la physique et la technique sont les sources fécondes dont le mathématicien ne doit pas s'écarter sous peine de stérilité.

Un épistémologue contemporain, Jean-Toussaint Desanti a résumé le problème en écrivant: "Les mathématiques sont elles du ciel, sont-elles de la terre?

Une création humaine

En fait une troisième conception existe, bien plus intéressante mais souvent méconnue: les mathématiques sont une création humaine. Une telle solution donne à l'imagination une importance fondamentale. Elle fut adoptée dans l'histoire par certains mathématiciens et philosophes mais curieusement n'a pas retenu l'attention. Elle rapproche l'activité du mathématicien de celle du poète, de l'artiste.

Elle rend compte des préoccupations d'harmonie et d'esthétique qui animent souvent les mathématicien, Certains d'entre eux les considèrent même comme essentielles et caractéristiques de leur activité. Les nombres sont pour Dedekind comme pour Hankel des créations de l'esprit humain: „Je conseillerais plutôt, écrit-il à Weber de ne pas entendre par nombre la classe même, mais quelque chose de nouveau... que l'esprit engendre. Nous sommes de race divine et possédons . . . le pouvoir de créer." A la même époque Cantor proclamait: "L'essence des mathématiques, c'est la liberté!" et Weierstrass renchérissait: "Le véritable mathématicien est un poète". Wittgenstein indiqua: "Le mathématicien est un inventeur, non un découvreur."*. Plus près de nous Albert Lautman, Jean Cavaillès et Gaston Bachelard conçurent les mathématiques de cette manière. Léon Brunschwig insista, lui aussi sur la dynamique de l'intelligence mathématique. Cette conception des mathématiques les libère de l'escalavage du réel des empiristes dogmatiques et des liens du rationnalisme classique.

Cette révélation de la véritable nature des mathématiques, l'idée d'une nouvelle orientation philosphique est contemporaine de la géométrie non euclidienne, qui prouva la capacité de l'esprit à créer de toutes plèces un domaine de pensée dont la contradiction avec les "vérités intuitives" était flagrante. La Théorie de la Relativité exigea aussi une nouvelle philosophie de l'espace et du temps qui ne pouvait plus être une philosophie du donné, où l'intuition est fondamentale. La raison devait se mettre à l'école des mathématiques les plus modernes et les plus éloignées de la culture traditionnelle: les tenseurs, les différents sortes d'algèbres et de géométries devenaient les instruments habituels du physicien. Le formalisme le plus abstrait se révélait indispensable pour l'investigation la plus concrète. La métaphore célèbre de Kant dans sa préface à la *Critique de la raison pure* sur l'erreur de la colombe platonicienne devait être renversée: le vide du formalisme est indispensable pour atteindre les profondeurs de l'objet. L'esprit doit prendre de l'altitude pour mieux dominer sa proie. Trop près du but la vue manque de perspective pour élaborer la théorie nécessaire. L'immédiateté de la capture n'est pas le propre de l'homme. C'est par la pensée et l'effort qu'il est devenu un géant.

Le rôle de l'imagination et de la philosophie

Dans une telle conception des mathématiques l'imagination a toute sa place, qu'Hilbert a soulignées. A la question "Comment un homme qui était mathématicien peut-il écrire des romans?" — "Mais c'est tout simple, répond Hilbert, il n'avait pas assez d'imagination pour les mathématiques, mais il en avait assez pour les romans."* C'est une autre caractéristique du Nouvel Esprit mathématique que de donner ce rôle essentiel en mathématiques à l'imagination, tout à fait à l'opposé de la conception dominante du XVIIe siècle. Ce n'est plus "la folle" du logis", responsable des divagations de l'esprit, mais ce qui donne sa forme, sa couleur et son relief à une pensée nouvelle.

Pour Descartes l'erreur s'introduit par l'intervention intempestive d'une puissance qu'il exorcise: l'imagination. Pascal est encore plus net dans ses *Pensées***: "C'est cette partie décevante dans l'homme, cette maîtresse d'erreur et de fausseté et d'autant plus fourbe qu'elle ne l'est pas toujors; car elle serait règle infaillible de vérité si elle l'était du mensonge. Mais étant le plus souvent fausse,

* Constance Reid, *Hilbert*, Springer Verlag 1970 p. 175.

** Edition Brunswicg Hachette 1945 pp 362—363—367 et passim.

elle ne donne aucune marque de sa qualité, marquant du même caractère le vrai et le faux." — "Je ne parle pas des fous, je parle des plus sages et c'est parmi eux que l'imagination a le grand don de persuader les Hommes. La raison a beau crier, elle ne peut mettra le prix aux choses."

Au XVIIe siècle l'invention est avant tout l'oeuvre de la raison. Leibniz occupe peut-être une place à part avec le sens très vif qu'il a eu du changement, de l'activité essentielle à toute réalité. Il dépasse le mécanisme cartésien et prélude à l'énergétisme et au transformisme modernes. La Caractéristique et la Logique se confondent pour lui avec la combinatoire, l'art de penser et surtout l'art d'inventer, qui n'est autre que la Mathématique. Car Leibniz a trop conscience de l'unité de l'esprit humain et de l'unité de la science pour séparer synthèse et analyse. Ce sont les logiciens empiristes qui opposent les sciences déductives et les sciences inductives, comme s'il y avait deux méthodes distinctes et contraires pour découvrir et démontrer la vérité. La mathématique formelle et abstraite est la véritable logique des autres sciences et l'on peut dire sans paradoxe que la seule méthode expérimentale est la déduction.

L'imagination créatrice est à l'oeuvre en mathématiques et remet en cause la doctrine traditionnelle d'une raison absolue et immuable, philosophie dogmatique périmée. L'esprit doit se plier aux conditions du savoir, se mettre à l'école des mathématiques, cette invention humaine qui avec quelques autres comme le langage, la poésie, la musique etc, ont créé l'homme tel qu'il est et lui ont permis de se rendré maître et possesseur de la nature. Une telle conception dynamique et vivante pose en termes essentiellement nouveaux le problème de la vérité, de l'objectivité, de la subjectivité, de la nécessité et de la rigueur, des rapports des mathématiques avec le réel. Les solutions du rationalisme dogmatique ou de l'empirisme opportuniste, en fait tout aussi dogmatique, sinon plus, ne peuvent plus être adoptées. Elle souligne avec force l'importance des définitions, car on observe et on décrit ce qui existe, mais on doit définir ce que l'esprit crée et qui n'est pas donné. On comprend mieux aussi le rôle fondamental des théorèmes d'existence et de la cohérence en mathématiques. Ce sont les notions de base. Avec ces théorèmes d'existence les mathématiciens cherchent un critère très large applicable à une multitude de problèmes différents pour savoir si une solution existe ou non. Une fois trouvé le caractère garantissant l'existence d'une solution, nous pouvons chercher à la découvrir avec l'assurance que cette recherche ne sera pas vaine. L'importance de ces théorèmes d'existence est garantie par la pratique des mathématiciens. Les étudiants et les pédagogues sont souvent sceptiques à leur sujet car il existe une grande différence entre les preuves de l'existence d'une solution et les méthodes utilisées pour trouver ces solutions. Un théorème d'existence doit s'appliquer dans tous les cas: sa détermination est souvent difficile et son application effective peur être compliquée et fastidieuse. Un exmple moderne communiqué par J. Dieudonné suffira à le montrer. La démonstration d'un tel théorème de la théorie des groupes, démontré par l'absurde en 1963 par Walter Feit et John G. Thompson occupe 258 grandes pages du *Pacific Journal of Mathematics*. Son énoncé est pourtaint relativement simple et court: tous les groupes finis d'ordre impair sont résolubles. Il est vrai que la plupart des exemples présentés aux étudiants sont simples et l'existence peut être démontrée par des méthodes plus simples et en général constructives. Aussi pensent-ils souvent à la métaphysique quand on esquisse devant eux la notion d'existence de solutions. Pourtant c'est une question fondamentale liée à la solution des problèmes plus traditionnels. Songeons à la fameuse question de la trisection de l'angle avec la règle et le compas, ou à celle de la résolution des équations algébriques. Quand le problème de l'existence fut clairement posé, on sut y répondre. Dans la recherche moderne les questions d'existence sont posées d'abord et les réponses sont absolument vitales afin que les théories reposent sur de saines fondations. Pour s'on rendre compte il suffit de consulter le *Traité d'Analyse* de Goursat, ou celui, plus récent, de Dieudonné. Il y a là une exigence profonde de l'esprit humain qui ne peut être négligée. L'imagination intervient aussi dans l'élaboration de nouveaux formalismes, de nouveaux automatisme. Ceux-ci déchargent l'esprit, certes, de certaines opérations fastidieuses, mais ne dispensent pas, contrairement à ce que certains prétendent, de penser. Bien plutôt, grâce à eux, l'esprit acquiert de nouvelles capacités, il apprend à penser avec des flèches, avec des diagrammes, avec de nouveaux languages qui sont autant d'instruments décuplant ses possibilités. D'autre part les grands mathématiciens, c'est-à-dire ceux qui trouvent une nouvelle façon d'envisager une question, une nouvelle méthode pour résoudre un problème jusque là insoluble, ne se contentent jamais d'utiliser mécaniquement les procédés classiques. Ils poussent d'abord aussi loin que possible l'exploration des sources des automatismes employés et savent restituer ainsi à la pensée son autonomie, grâce à quoi elle pourra prendre un nouvel essor par delà les frontières où elle s'était d'abord crue prisonnière. Souvent la recherche conduit à une nouvelle théorie ou à un renouvellement complet de la problématique traditionnelle. Ce travail d'investigation, qui est la véritable vie de l'esprit, une preuve de sa liberté, ne devrait pas laisser indifférents les philosophes ni les hommes de culture. Il devrait être, comme par le passé au coeur de leurs préoccupations et permettre de réhabiliter des auteurs injustement oubliés, qui avaient compris, eux, la richesse spirituelle des mathématiques, tels Louis Couturat et Albert Lautman, par exemple, qui virent en elle une des plus hautes manifestations de la puissance productrice de l'intelligence.

Malheureusement le dogmatisme, s'il n'est plus soutenable en sciences est toujours présent dans la philosophie et la pédagogie qui suivent les modes les plus contestables, les prétendus novateurs en pédagogie étant souvent les plus fermés à l'opinion des autres, qu'ils refusent d'examiner, J'en connais qui vous traitent en ennemis si vous ne partagez pas leur foi. C'est pourquoi vous ne trouverez pas le Nouvel Esprit Mathématique dans les manuels ou les instructions officielles. Il faut, pour le connaître, vous adresser aux mathématiciens. Il faut entrer en contact avec l'oeuvre d'un maître. Abel (1802—1829) à qui l'on domandait comment il avait fait pour produire des résultats aussi remarquables en six ou sept ans répondit: "En étudiant les maîtres et non pas leurs disciples."

C'est la science en train de se faire qui nous montre le chemin d'une philosophie et d'une culture adéquates aux innovations scientifiques, face à toutes les démissions de l'esprit.

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Symposium: Set Theory. Foundations of Mathematics, Beograd, 29.08. — 2.09. 1977.

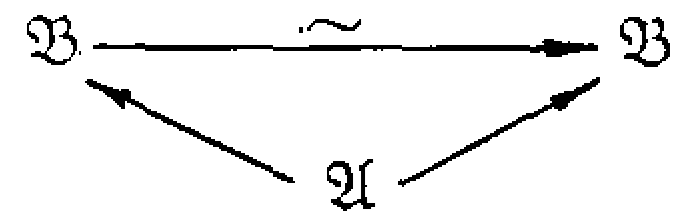
HOMOGENEOUS-UNIVERSAL MODELS OF THEORIES WHICH HAVE MODEL COMPLETIONS

Žarko MIJAJLOVIĆ

1. Introduction

In the present work our attention is turned to those Jónnson classes of models which are classes of models of theories which have model completions. Main reason for that lies in the fact that the class of models of a theory which has the model completion is almost a Jónnson class, therefore that part of model theory which concern model completions may be applied in full power. In such sense this paper is closly related to the works of others as of M. Yasuhara [6], Comfort-Negreponties [3] etc. (relatively complete list of references on the subject can be found in the works just cited). The terminology that is used in this paper is mostly according to [2] and [5], however we repeat some of it, since it is not uniquely determined in general, and also some assumptions and conventions are introduced.

A language is denoted by L, the language of a theory T by $L(T)$ and of a model $\mathfrak{A}$ by $L(\mathfrak{A})$. It is assumed throughout that $L(T)$ is countable and that T has infinite models. Universes of models $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ are denoted by A, B, C respectively, and the cardinal number of A by $|A|$. By $\mathfrak{M}(T)$ is denoted the class of all models of T. As usual $\mathfrak{A} < \mathfrak{B}$ means that $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$ and $\mathfrak{A} <_1 \mathfrak{B}$ states the fact that $\mathfrak{B}$ is an existential extension of $\mathfrak{A}$ (i.e. $\mathfrak{A} \subseteq \mathfrak{B}$ and for every existential formula ψ and valuation **a** in A $\mathfrak{A} \models \psi$ [**a**]) iff $\mathfrak{B} \models \psi$ [**a**]. Symbol $\vec{a}$ stands for a sequence $a_1, a_2, \ldots, a_n$ if the subscript n is of no importance in the consideration. So if f is a function, then $f\vec{a}$ stands for $fa_1, fa_2, \ldots, fa_n$. The arrow in a diagram $\mathfrak{A} \to \mathfrak{B}$ represents an unnamed embedding f: $\mathfrak{A} \to \mathfrak{B}$ and similiarly $\overset{\sim}{\to}$, $\overset{<}{\to}$ represent an (unnamed) isomorphism and an elementary embedding respectively. If an arrow has more then one occurence in a diagram, then each occurrence of the arrow may represent a different embedding. A name of an element $a \in A$ is denoted by $\underline{a}$. A model $\mathfrak{A}$ is an universal model of T if it is a model of T and if for every $\mathfrak{B} \models T, |B| \leqslant |A|$, $\mathfrak{B}$ is embeddable into $\mathfrak{A}$. A model $\mathfrak{B}$ of T is a homogeneous model of T if for every $\mathfrak{A} \models T$, $|A| < |B|$, the diagram $\mathfrak{B} \leftarrow \mathfrak{A} \to \mathfrak{B}$ can be



completed to the shown commutative diagram.

A model $\mathfrak{A}$ is an elementary universal model of T if $\mathfrak{A} \models T$ and for every model $\mathfrak{B}$, $\mathfrak{B} \equiv \mathfrak{A}$ and $|A| \leqslant |B|$ implies $\mathfrak{B} \overset{<}{\to} \mathfrak{A}$. A model $\mathfrak{A}$ is an elementary homogeneous model of T if $\mathfrak{A} \models T$ and for every set $X \subseteq A$, $|X| < |A|$, and any map $p: X \to A$ $(\mathfrak{A}, x)_{x \in X} \equiv (\mathfrak{A}, px)_{x \in X}$ implies the existence of an automorphism $f: \mathfrak{A} \overset{\sim}{\to} \mathfrak{A}$ such that $f \restriction X = p$. With $T_\forall$, $T_{\forall\exists}$ are denoted respectively the sets of universal, universal-existential sentences which are consequences of T. We state well known basic facts which relate theories $T_\forall$, $T_{\forall\exists}$ to the threory T.

THEOREM 1.1. 1° $\mathfrak{A} \models T_\forall$ *iff there is* $\mathfrak{B} \models T$ *such that* $\mathfrak{A} \subseteq \mathfrak{B}$.

2° $\mathfrak{A} \models T_{\forall\exists}$ *iff there is* $\mathfrak{B} \models T$ *such that* $\mathfrak{A} <_1 \mathfrak{B}$. $\dashv$

In connection with this theorem, we remark that in general the following holds: $\mathfrak{A} \models T_{\Pi_{n+1}}$ iff there is $\mathfrak{B} \models T$ such that $\mathfrak{A} <_n \mathfrak{B}$, where T_{Π_n} is the set of all Π_n^0 consequences of T and $\mathfrak{A} <_n \mathfrak{B}$ means that $\mathfrak{A} \subseteq \mathfrak{B}$ and for every Π_n^0 formula ψ and assignment $\boldsymbol{a}$ in A $\mathfrak{A} \models \psi[\boldsymbol{a}]$ iff $\mathfrak{B} \models \psi[\boldsymbol{a}]$.

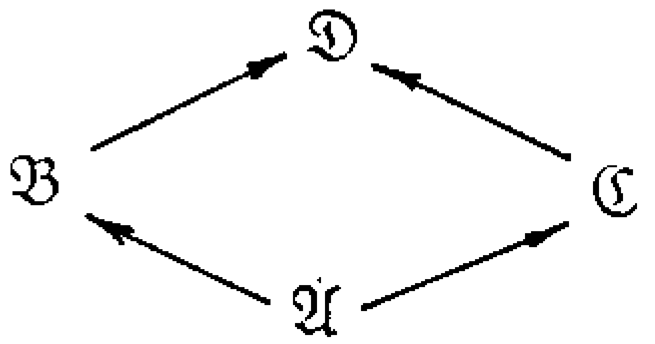
For convenience we repeat the definition of a notion of Jónnson class of models (for basic properties of Jónnson classes see for example [1] and [3]). A class K of models of a language L is a Jónnson class if K satisfies the following conditions:

1° K contains models of arbitrarily large cardinals.

2° K is closed under isomorphic images.

3° K has the joint embedding property (JE): For any $\mathfrak{A}, \mathfrak{B} \in K$ there is $\mathfrak{C} \in K$ such that $\mathfrak{A} \to \mathfrak{C} \leftarrow \mathfrak{B}$.

4° K has the amalgamation property (AP): For any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$ diagram $\mathfrak{B} \leftarrow \mathfrak{A} \to \mathfrak{C}$ can be amalgamated to the commutative diagram. In the terminology of M. Yasuhara [6] every $\mathfrak{A}$ is amalgamative in K.



5° K is closed under union of chains of models.

6_k° For any $\mathfrak{A} \in K$ and $X \subseteq A$, $|X| < k$, there is $\mathfrak{B} \subseteq A$, $\mathfrak{B} \in K$, $|B| < k$ such that $X \subseteq B$ (k is an infinite cardinal).

Under cited conditions, as B. Jónnson has shown (1960), if $k = k^{\underline{k}}$ then K contains an universal-homogeneous model for K. In this paper it is assumed that K is an elementary class i.e. $K = \mathfrak{M}(T)$ for some T. If $\mathfrak{M}(T)$ is a Jónnson class we say simply that T is a Jónnson theory (similar convention is applied to any property P which concern the class $\mathfrak{M}(T)$ By LST (Löwenheim-Skolem-Tarski) theorem, T satisfies 1° and 6° for $k \geqslant \omega_1$. By Chang-Los-Suzko preservation theorem T has property 5° iff T has universal-existential axiomatization. Hence, the really problem that may occur is "Does T have JE and AP?".

The property JE can be syntatically described.

PROPOSITION 1.2. A theory T has *JE* iff the following holds: If θ, ψ are basic formulas (i.e. conjunctions of atomic and negations of atomic formulas) then the consistency of theories $T+\exists\vec{x}\,\theta$, $T+\exists\vec{y}\,\psi$ implies the consistency of of $T+\exists\vec{x}\,\theta+\exists\vec{y}\,\psi$.

PROOF ($\Rightarrow$) Let $\mathfrak{A},\mathfrak{B}\models T$ such that $\mathfrak{A}\models\exists\vec{x}\,\theta$, $\mathfrak{B}\models\exists\vec{y}\,\psi$ where θ,ψ are basic formulas. By *JE* there is $\mathfrak{C}\models T$ such that $\mathfrak{A}\to\mathfrak{C}\leftarrow\mathfrak{B}$, hence $\mathfrak{C}\models T+\exists\vec{x}\,\theta+\exists\vec{y}\,\psi$.

($\Leftarrow$) Let $\mathfrak{A},\mathfrak{B}\models T$ and assume that there is no $\mathfrak{C}\models T$ such that $\mathfrak{A}\to\mathfrak{C}\leftarrow\mathfrak{B}$. Then the theory $\Gamma=T+\Delta(\mathfrak{A})+\Delta(\mathfrak{B})$ ($\Delta(\mathfrak{A})$ is the diagram of $\mathfrak{A}$), is inconsistent, hence there are basic formulas $\theta(\vec{x})$, $\theta(\vec{y})$ and $\vec{a}\in A$, $\vec{b}\in B$ such that $\theta(\vec{a})\in\Delta(\mathfrak{A})$ and $\psi(\vec{b})\in\Delta(\mathfrak{B})$ so that $T+\theta(\vec{a})+\psi(\vec{b})$ is inconsistent. Hence $T\vdash\theta(\vec{a})\Rightarrow\neg\psi(\vec{b})$ so $T\vdash\forall\vec{x}\,\forall\vec{y}\,\neg(\theta(\vec{x})\wedge\psi(\vec{y}))$, $\{x_1,\ldots,x_n\}\cap\{y_1,\ldots,y_m\}=\varnothing$. Therefore $T\vdash\neg(\exists\vec{x}\,\theta(\vec{x})\wedge\exists\vec{y}\,\psi(\vec{y}))$ and $\mathfrak{A}\models\exists\vec{x}\,\theta$, $\mathfrak{B}\models\exists\vec{y}\,\psi$, but this contradicts our hypothesis. $\dashv$

COROLLARY 1.3. Assume that any two countable models of T can be embedded into a model of T. Then T has *JE*.

PROOF Let θ, ψ be basic formulas and assume that $T+\exists\vec{x}\,\theta$, $T+\exists\vec{y}\,\psi$ are consistent theories. By *LST* theorem there are countable models $\mathfrak{A}$, $\mathfrak{B}$ of T such that $\mathfrak{A}\models\exists\vec{x}\,\theta$, $\mathfrak{B}\models\exists\vec{y}\,\psi$. By *JE* for countable models there is $\mathfrak{C}\models T$ so that $\mathfrak{A}\to\mathfrak{C}\leftarrow\mathfrak{B}$. Then $\mathfrak{C}\models\exists\vec{x}\,\theta\wedge\exists\vec{y}\,\psi$ so $T+\exists\vec{x}\,\theta+\exists\vec{y}\,\psi$ is a consistent theory. $\dashv$

In some cases properties *JE* and *AP* are transferred from one theory to another. Let us see some examples of such kind.

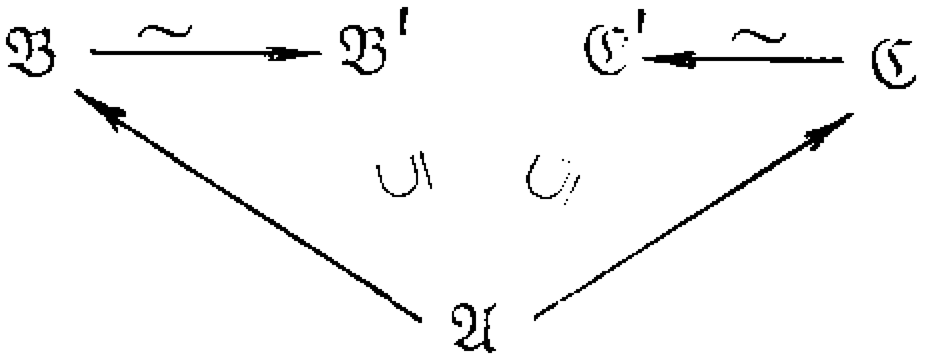
PROPOSITION 1.4. 1° T has *JE* iff $T_\forall$ has *JE*.
2° (M. Yasuhara, [6]) T has *AP* iff $T_{\forall\exists}$ has *AP*.

PROOF 1° ($\Rightarrow$) Assume that T has *JE*, and let $\mathfrak{A},\mathfrak{B}\models T_\forall$. Then there are $\mathfrak{A}',\mathfrak{B}'\models T$ such that $\mathfrak{A}\subseteq\mathfrak{A}'$, $\mathfrak{B}\subseteq\mathfrak{B}'$. T has *JE* so $\mathfrak{A}',\mathfrak{B}'$ can be embedded into a model $\mathfrak{C}\models T$. Since $\mathfrak{C}\models T_\forall$, $T_\forall$ has *JE*. ($\Leftarrow$) Proof is trivial.

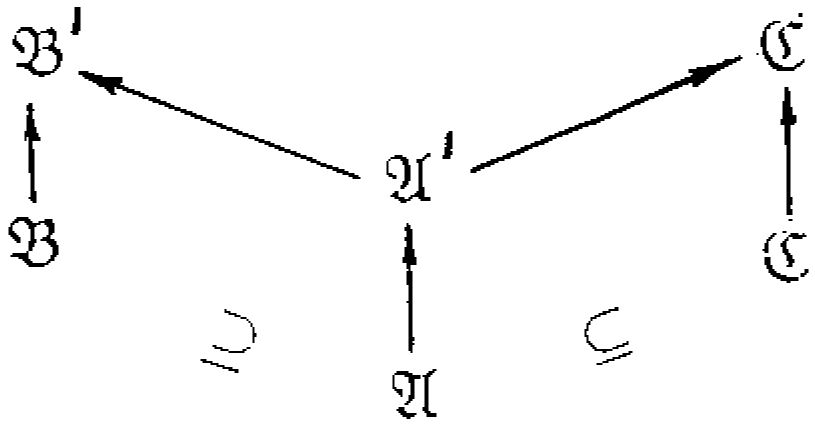
2° Assume that T has *AP*. Let $\mathfrak{A},\mathfrak{B},\mathfrak{C}$ be models of $T_{\forall\exists}$ and

(1) $$\mathfrak{B}\supseteq\mathfrak{A}\subseteq\mathfrak{C}.$$

Remark It is sufficient to amalgamate diagrams of the form (1) since every diagram of the sort $\mathfrak{B}\leftarrow\mathfrak{A}\to\mathfrak{C}$ can be completed to the commutative diagram (2).



(2)



(3)

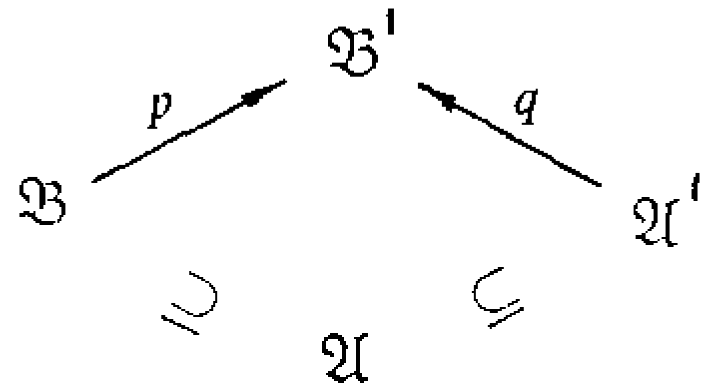
We want to transfer the diagram (1) to T i.e. to construct a commutative diagram (3).

The existence of the model $\mathfrak{A}'$ is provided by $T.1.1$, moreover it may be taken $\mathfrak{A} <_1 \mathfrak{A}'$. Now we prove that any diagram of the form

$$\mathfrak{B} \supseteq \mathfrak{A} <_1 \mathfrak{A}' \tag{4}$$

can be amalgamated. Consider the theory $\Gamma = T + \Delta(\tilde{\mathfrak{B}}) + \Delta(\tilde{\mathfrak{A}}')$ where $\tilde{\mathfrak{B}} = (\mathfrak{B}, b, a)_{b \in B, a \in A}$, $\tilde{\mathfrak{A}}' = (\mathfrak{A}, a', a)_{a' \in A', a \in A}$. Γ is consistent theory. Assume it is not. In such a case there are basic formulas $\theta(\vec{z}, \vec{x})$, $\psi(\vec{y}, \vec{x})$ so that $\theta(\vec{b}, \vec{a}) \in \Delta(\mathfrak{B})$ and $\psi(\vec{a}'\, \vec{a}) \in \Delta(\mathfrak{A})$ for some $\vec{a} \in A$, $\vec{a}' \in A'$, $\vec{b} \in B$ and $T + \theta(\vec{b}, \vec{a}) + \psi(\vec{a}', \vec{a})$ is inconsistent. Hence $T \vdash \forall \vec{x}\vec{y}\vec{z}\, (\theta(\vec{z}, \vec{x}) \Rightarrow \neg \psi(\vec{y}, \vec{x}))$, so since the formula $\forall xyz\, (\theta(\vec{z}, \vec{x}) \Rightarrow \neg \psi(\vec{y}, \vec{x}))$ is universal, $\mathfrak{B} \models \theta(\underline{b}, \underline{a}) \Rightarrow \forall \vec{y}\, \neg \psi(\vec{y}, \underline{a})$, and thus $\mathfrak{B} \models \forall \vec{y}\, \neg \psi(\vec{y}, \vec{a})$. But $\mathfrak{A} <_1 \mathfrak{A}'$ so $\mathfrak{A}' \models \forall \vec{y}\, \neg \psi(\vec{y}, \underline{a})$ so $\mathfrak{A}' \models \neg \psi(\underline{a}', \underline{a})$, what is contradiction. Hence Γ has a model $\mathfrak{B}' = (\mathfrak{B}, c_b, c_{a'}, c_a)_{a \in A, b \in B, a' \in A'}$ and (4) is amalgamated to the diagram (5) where $p(b) = c_b$, $q(a') = c_{a'}$.

(5)



In similar way a model $\mathfrak{C}'$ is obtained with the required property and therefore the diagram (3). T has AP so $\mathfrak{B}' \leftarrow \mathfrak{A}' \rightarrow \mathfrak{C}'$ can be amalgamated and therefore $\mathfrak{B} \subseteq \mathfrak{A} \supseteq \mathfrak{C}$ can too.

($\Leftarrow$) Trivally holds. $\dashv$

COROLLARY 1.5. If T has JE and AP then $T_{\forall\exists}$ is a Jónnson theory. $\dashv$

If T has AP, it is not necessarily that T has too. For example, this case occur whenever T is model complete, but not submodel complete.

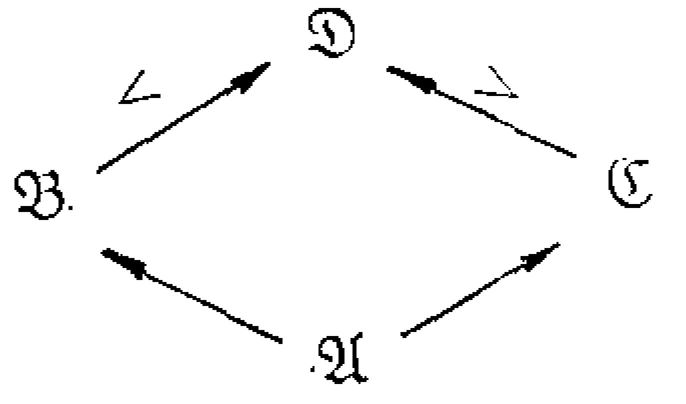
2. Full models

Now we consider those theories T which have model completion T^*. Hence, it is assumed (here and throughout) that T has a model completion. For convenience we repeat the definition of the notion of model completion (it was introduced by A. Robinskon, see [4], [5]). A theory T^* is model completion of T if the following holds:

1° Every model of T^* is a model of T.

2° Every model of T is a submodel of a model of T^*.

3° Any diagram $\mathfrak{B} \leftarrow \mathfrak{A} \rightarrow \mathfrak{C}$, $\mathfrak{A} \models T$, $\mathfrak{B}, \mathfrak{C} \models T^*$ can be amalgamated to the commutative diagram:



Some of basic properties of this notion are: If T has a model completion, then it is unique (up to logical equivalence). T^* is model complete and has universal-existential axiomatization.

It turns out that T and T^* have in common properties JE and AP.

THEOREM 2.1. 1° *T has JE iff T^* has JE.*

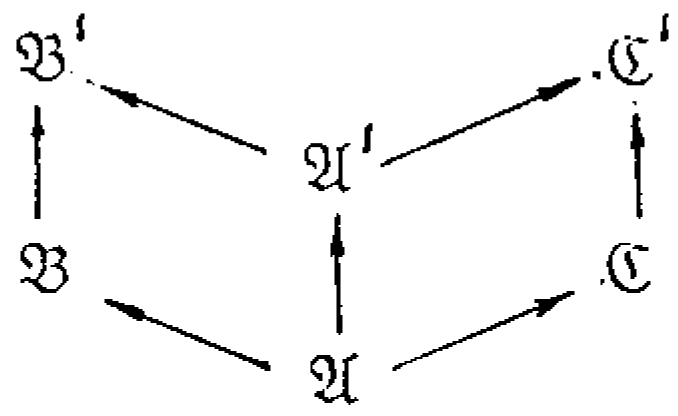
2° *T and T^* have AP.*

PROOF 1° ($\Rightarrow$) Let $\mathfrak{A}, \mathfrak{B} \models T^*$ and assume that T has JE. Since $T^* \models T$ it follows $\mathfrak{A}, \mathfrak{B} \models T$ so there is $\mathfrak{C} \models T$ such that $\mathfrak{A} \to \mathfrak{C} \leftarrow \mathfrak{B}$. Since T^* is model completion of T, it can be chosen $\mathfrak{C} \models T^*$.

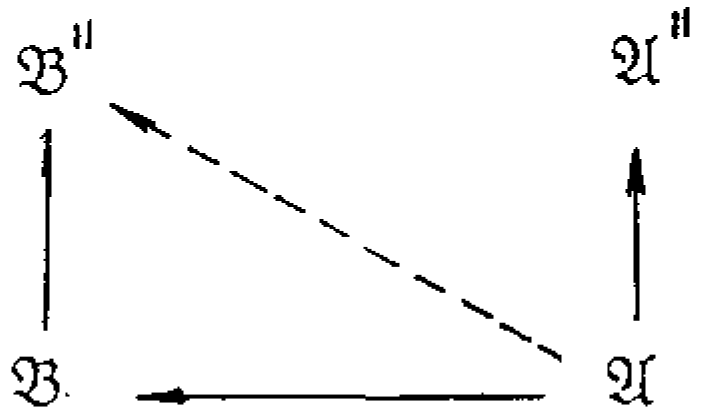
($\Leftarrow$) It follows immediately since every model of T is a submodel of T^*.

2° According to the property 3° of model completion and since every model of T^* is a model of T, it follows that T^* has AP.

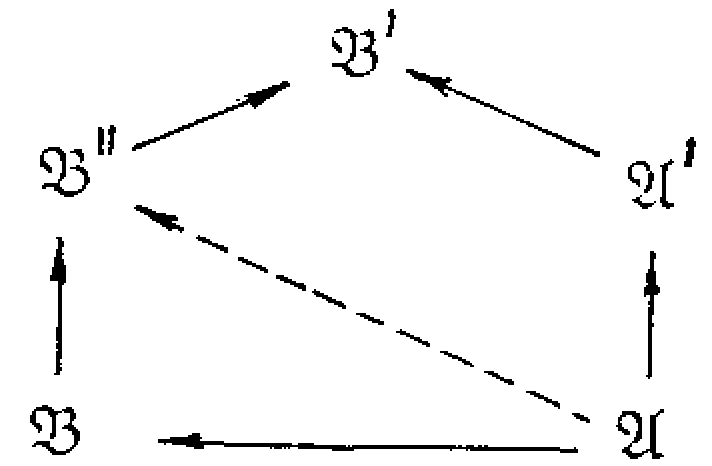
Now, let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be models of T and assume that $\mathfrak{B} \leftarrow \mathfrak{A} \to \mathfrak{C}$. This diagram can be transferred into a diagram in T^* i.e. there is a commutative diagram (1). Existence of $\mathfrak{A}'$ is provided by property 2° of model completion. Further, there is a model $\mathfrak{B}''$ of T^* such that $\mathfrak{B} \to \mathfrak{B}''$. According to the property 3° diagram (2) can be amalgamated to the commutative diagram (3). In the similar way the model $\mathfrak{C}'$ is obtained, and so the diagram (1) exists. The diagram $\mathfrak{B}' \leftarrow \mathfrak{A}' \to \mathfrak{C}'$ can be amalgamated, so we have obtained commutative diagrams (4) and (5). ⊣



(1) $\mathfrak{A}', \mathfrak{B}', \mathfrak{C}' \models T^*$



(2)



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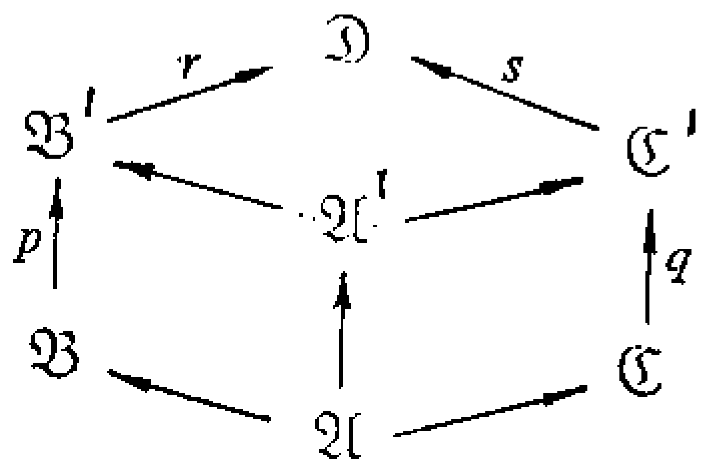
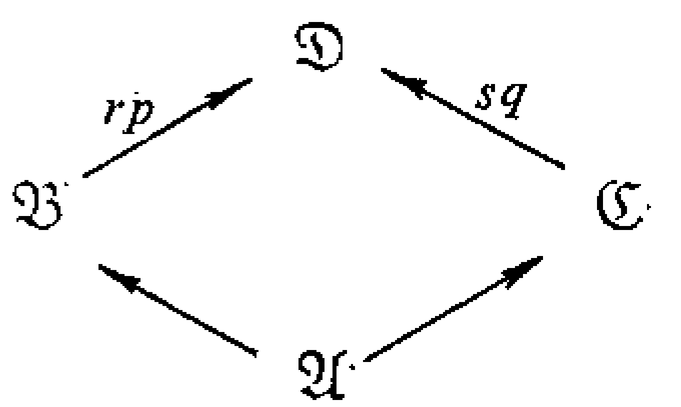


(4)



$\mathfrak{D} \models T^*$

(5)

COROLLARY 2.2. T^* is the model completion of $T_{\forall\exists}$. ⊣

It should be remarked that T^* in general is not a model completion of $T_{\forall}$ (but it is the model companion of $T_{\forall}$).

COROLLARY 2.3. (Test for a class to be a Jónnson class). Assume that a theory T has a model completion T^*, universal-existential axiomatization and a prime model. Then T is a Jónnson theory.

PROOF Closure of T under union of chains of models of T follows from universal-existential axiomatization and AP from the previous theorem. Since T has a prime model $\mathfrak{A}$ (i.e. $\mathfrak{A}$ is embeddable into every model of T), for any $\mathfrak{B}, \mathfrak{C} \models T$ a diagram $\mathfrak{B} \leftarrow \mathfrak{A} \rightarrow \mathfrak{C}$ exists and by AP it can be amalgamated, so T has JE. $\dashv$

Since T^* has universal-existential axiomatization and AP, it may lack only JE in order to be a Jónnson theory. Model complete theory T is model completion of itself, so if T has a prime model, then it is a Jónnson theory. We have assumed that T has model completion T^*, so AP is provided for T but JE is not in general. However the question of JE can be removed if the following relation $\simeq_T$ is introduced in $\mathfrak{M}(T)$.

DEFINITION 2.4. Models $\mathfrak{A}$, $\mathfrak{B}$ of T are *compatible* in T, $\mathfrak{A} \simeq_T \mathfrak{B}$, iff there is a model $\mathfrak{C}$ of T such that $\mathfrak{A} \rightarrow \mathfrak{C} \leftarrow \mathfrak{B}$. (Often the subscript T will be omitted in $\simeq_T$).

A model $\mathfrak{A}$ of T is a *semiuniversal* model of T if for any model $\mathfrak{B} \models T$ $\mathfrak{A} \simeq \mathfrak{B}$ and $|B| \leqslant |A|$ implies $\mathfrak{B} \rightarrow \mathfrak{A}$, that is, $\mathfrak{A}$ is an universal model in the class of all models compatible with $\mathfrak{A}$. A model $\mathfrak{A}$ of T is a *full* model of T if $\mathfrak{A}$ is semiuniversal and homogeneous model of T. A model $\mathfrak{A}$ of T is a *semiprime* model of T if it is prime in the class of all models of T compatible with $\mathfrak{A}$.

EXAMPLE 2.5. If T is the theory of fields, then the Galois field Z_p is semiprime model of T. Every algebraically closed field F of infinite transcedental degree over Z_p is semiuniversal and in fact a full model of T.

In the following proposition the basic properties of the relation $\simeq$ are given.

PROPOSITION 2.6. 1° $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \simeq_T \mathfrak{B}$ for any theory T which has $\mathfrak{A}$, $\mathfrak{B}$ as models.

2° $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\mathfrak{A} \simeq \mathfrak{B}$.

3° The relation $\simeq$ is an equvalence relation in $\mathfrak{M}(T)$.

4° Assume that $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \models T$. If $\mathfrak{A} \equiv \mathfrak{B}$ and $\mathfrak{B} \simeq \mathfrak{C}$ then $\mathfrak{A} \simeq \mathfrak{C}$.

5° Let $\mathfrak{A}, \mathfrak{B}$ be models of T^*. Then $\mathfrak{A} \equiv \mathfrak{B}$ is equivalent to $\mathfrak{A} \simeq \mathfrak{B}$.

6° If T has a prime model, then every semiprime model of T is prime and every semiuniversal model is universal.

Proofs of these assertions are simple so they are omitted.

THEOREM 2.7. *Let $\mathfrak{A}$ be a model of T and $C(\mathfrak{A})$ the class of all models of T compatible with $\mathfrak{A}$. Then $C(\mathfrak{A})$ is an elementary class of models with JE and AP. If T has an universal-existential axiomatization, then $T_{\mathfrak{A}} = Th\,(C\,(\mathfrak{A}))$ is a Jónnson theory. If $C^*(\mathfrak{A})$ is the class of all models of T^* in which models of $C\,(\mathfrak{A})$ are embeddable, then $T^*_{\mathfrak{A}} = Th(C^*(\mathfrak{A}))$ is a complete theory and the model completion of $T_{\mathfrak{A}}$. Also, $C^*(\mathfrak{A})$ is a class of equivalence under $\simeq_{T^*}$.*

PROOF In order to prove that $C(\mathfrak{A})$ is an elementary class we use the theorem (Frayne, Morel, Scott) which states that a class of models is elementary if it is closed under elementary equivalence and ultraproducts. So let $\mathfrak{B}\in C(\mathfrak{A})$ and $\mathfrak{C}\equiv\mathfrak{B}$. Then $\mathfrak{C}\models T$ and $\mathfrak{C}\simeq\mathfrak{B}$ so $\mathfrak{C}\in C(\mathfrak{A})$. Further, assume that $\mathfrak{A}_i\in C(\mathfrak{A})$, $i\in I$. Hence there are models $\mathfrak{B}_i$ so that $\mathfrak{A}_i\rightarrow\mathfrak{B}_i\leftarrow\mathfrak{A}$. Let U be an ultrafilter over I. Then $\mathfrak{A}$ and $\mathfrak{A}'=\prod_{i\in C}\mathfrak{A}_i/U$ are embedded into $\prod_{i\in I}\mathfrak{B}_i/U$. Since $\mathfrak{A}'$ is a model of T, it follows that $\mathfrak{A}'\simeq_T\mathfrak{A}$, and therefore $C(\mathfrak{A})$ is an elementary class. That $T_{\mathfrak{A}}$ satisfies JE and AP it is obvious. So assume that T is closed under union of chains of models and let us prove that $T_{\mathfrak{A}}$ is too. Since $\mathfrak{M}(T_{\mathfrak{A}})$ is an elementary class it suffices to prove that $T_{\mathfrak{A}}$ is closed under countable chains of models. Therefore let $\mathfrak{A}_1\subseteq\mathfrak{A}_2\subseteq\ldots$ where $\mathfrak{A}_n\in C(\mathfrak{A})$, $n\in\omega$, and $\mathfrak{A}'=\bigcup_n\mathfrak{A}_n$. Then $\mathfrak{A}'\models T$. Further consider models $\tilde{\mathfrak{A}}_1=(\mathfrak{A}_1,a^1)_{a^1\in A_1}$, $\tilde{\mathfrak{A}}_2=(\mathfrak{A}_2,a^1,a^2)_{a^1\in A_1,a^2\in A_2},\ldots$ and $\Gamma=T_{\mathfrak{A}}+\Delta(\mathfrak{A}_1)+\Delta(\mathfrak{A}_2)+\cdots$. The theory Γ is finitelly consistent, hence there is a model $\tilde{\mathfrak{B}}\models\tilde{\Gamma}$ i.e. $\mathfrak{B}\models T$ and $\mathfrak{A}'\rightarrow\mathfrak{B}$. Thus $\mathfrak{B}\simeq\mathfrak{A}$ and $\mathfrak{A}'\simeq\mathfrak{B}$, so $\mathfrak{A}'\simeq\mathfrak{A}$. Now we prove that $T^*_{\mathfrak{A}}$ is a complete theory and model completion of $T_{\mathfrak{A}}$. That $C^*(\mathfrak{A})$ is an elementary class it can be proved as it was done for $C(\mathfrak{A})$. Assume that $\mathfrak{B},\mathfrak{C}\in C^*(\mathfrak{A})$. Then there are $\mathfrak{B}',\mathfrak{C}'\in C(\mathfrak{A})$ so that $\mathfrak{B}'\subseteq\mathfrak{B}$ and $\mathfrak{C}'\subseteq\mathfrak{C}$. Since $\mathfrak{B}'\simeq\mathfrak{C}'$ it follows $\mathfrak{B}\simeq\mathfrak{C}$ and therefore $\mathfrak{B}\equiv\mathfrak{C}$ because $\mathfrak{B}$, $\mathfrak{C}$ are models of T^*. Hence, $T^*_{\mathfrak{A}}$ is a complete theory. The last two statements are easy to prove. ⊣

Now we proceed to description of saturated models of T^*.

THEOREM 2.8. 1° *If $\mathfrak{C}$ is an infinite saturated model of T^* then $\mathfrak{C}$ is a full model of T.*

2° If $\mathfrak{C}$ is a full model of T of cardinality $\alpha\geqslant\omega_1$ then $\mathfrak{C}$ is a saturated model of T^*.

PROOF. During this proof we shall use the theorem which states that a model $\mathfrak{C}$ is saturated iff it is elementary universal and elementary homogeneous.

1° Assume that $\mathfrak{C}$ is a saturated model of T^*.

CLAIM. *$\mathfrak{C}$ is a semiuniversal model of T.* For that let $\mathfrak{A}\simeq\mathfrak{C}$ and $|A|\leqslant|C|$. Further, there is $\mathfrak{B}\models T^*$ such that $\mathfrak{A}\rightarrow\mathfrak{B}$ and by LST theorem it may be assumed that $|B|=\max(|A|,\omega)$. Then $\mathfrak{B}\simeq\mathfrak{C}$, so $\mathfrak{B}\equiv\mathfrak{C}$ and by universality of $\mathfrak{C}$ it follows $\mathfrak{B}\rightarrow\mathfrak{C}$ and therefore $\mathfrak{A}\rightarrow\mathfrak{C}$.

CLAIM. *$\mathfrak{C}$ is a homogeneous model of T.* Let $\mathfrak{C}\overset{f}{\leftarrow}\mathfrak{A}\overset{g}{\rightarrow}\mathfrak{C}$ and $|A|<|C|$. Define a partial isomorphism p on $\mathfrak{C}$ by $pfa=ga$, $a\in A$. Since T^* is the model completion of T, it follows $(\mathfrak{C},fa)_{a\in A}\equiv(\mathfrak{C},pfa)_{a\in A}$ so there is an automorphism $h:\mathfrak{C}\overset{\sim}{\rightarrow}\mathfrak{C}$ such that $p\subseteq h$.

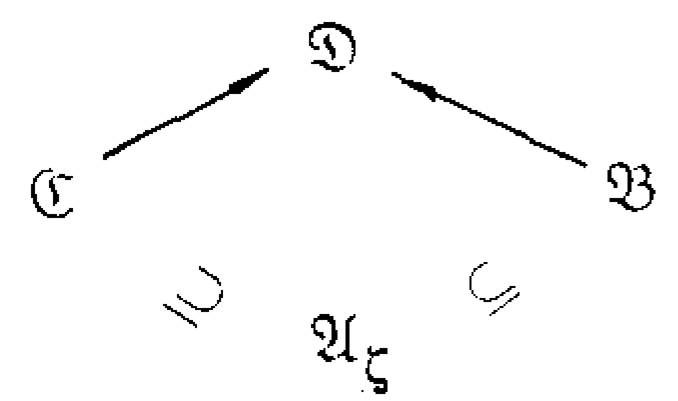
2° Assume that $\mathfrak{C}$ is a full model of T.

CLAIM. *$\mathfrak{C}$ is a model of T^**. For that let $\beta = cf\alpha$. Then there is a sequence of sets X_ξ, $\xi<\beta$ so that (1) if $\xi<\zeta$ then $X_\xi \subseteq X_\zeta$, (2) $|X_\xi|<\alpha$, (3) $C=\bigcup_{\xi<\beta} X_\xi$. By transfinite induction we define a sequence of models $\mathfrak{A}_\xi$, $\mathfrak{B}_\xi$, $\xi<\beta$ so that the following hold: (4) For all $\zeta<\xi$ $\mathfrak{A}_\zeta \subseteq \mathfrak{B}_\xi$ (5) If $\xi \geqslant 1$ then $\mathfrak{B}_\xi \subseteq \mathfrak{A}_\xi$ (6) $X_\xi \subseteq A_\xi$, (7) $|A_\xi|$, $|B_\xi|<\alpha$ and (8) $\mathfrak{B}_\xi \models T^*$.

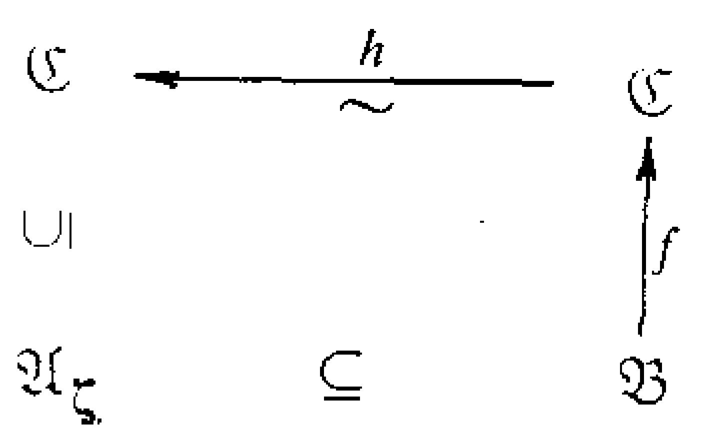
Let $\mathfrak{A}_0$ be such that $\mathfrak{A}_0 < \mathfrak{C}$, $X_0 \subseteq A_0$ and $|A_0| \leqslant \omega$, its existence is provided by *LST* theorem. Assume $\xi \geqslant 1$ and $\mathfrak{B}_\xi$ has been defined. By the inductive hypothesis $|B_\xi|<\alpha$. Therefore, since $|X_\xi|<\alpha$, $|B_\xi \cup X_\xi|<\alpha$. Hence by *LST* theorem there is $\mathfrak{A}_\xi < \mathfrak{C}$ so that $B_\xi \cup X_\xi \subseteq A_\xi$ and $|A_\xi| = |B_\xi \cup X_\xi|$. Thus $|A_\xi|<\alpha$, $\mathfrak{B}_\xi \subseteq \mathfrak{A}_\xi$, and $X_\xi \subseteq A_\xi$.

Models $\mathfrak{B}_\xi$ are defined in the following way.

If $\xi<\alpha$ is a limit ordinal, $\xi \neq 0$, then $\mathfrak{B}_\xi = \bigcup_{\zeta<\xi} \mathfrak{B}_\zeta$. The theory T^* is closed under union of chains of models, hence $\mathfrak{B}_\xi \models T^*$. Now assume that $\xi=\zeta+1$. By the induction hypothesis $\mathfrak{A}_\zeta \subseteq \mathfrak{C}$, $|A_\zeta|<\alpha$. Further, there is $\mathfrak{B} \models T^*$ so that $\mathfrak{A}_\zeta \subseteq \mathfrak{B}$ and by *LST* theorem it may be taken $|B|=|A_\zeta|$ i.e. $|B|<\alpha$. T^* is amalgamative, therefore the diagram $\mathfrak{C} \supseteq \mathfrak{A}_\zeta \subseteq \mathfrak{B}$ is completed to the amalgam (9)

(9) 

Hence $\mathfrak{B} \simeq \mathfrak{C}$. Since $\mathfrak{C}$ is a semiuniversal model, there is $f\colon \mathfrak{B} \to \mathfrak{C}$. Also, $\mathfrak{C}$ is α-homogeneous model, so there is an automorphism h of $\mathfrak{C}$ so that the diagram (10) commutes.

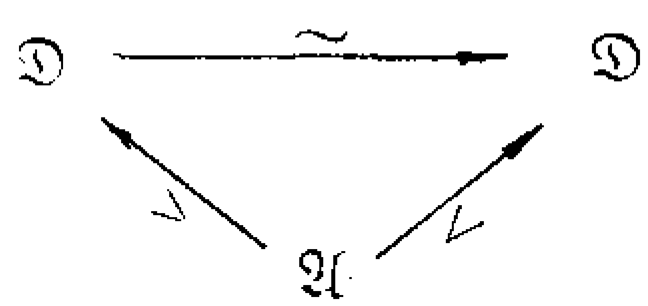
(10) 

Let $\mathfrak{B}_\xi = hf(\mathfrak{B})$. Then $\mathfrak{B}_\xi \models T^*$, $\mathfrak{A}_\zeta \subseteq \mathfrak{B}_\xi$ and $|B_\xi|<\alpha$. At the end we set $\mathfrak{B}_0 = \mathfrak{B}_1$. It should be observed that β is a limit ordinal, so for all $\xi<\beta$ $\mathfrak{B}_{\xi+1}$ is defined, hence $\mathfrak{A}_\xi \subseteq \mathfrak{B}_{\xi+1}$ and $X_\xi \subseteq B_{\xi+1}$. Therefore $\mathfrak{C} = \bigcup_{\xi<\beta} \mathfrak{B}_\xi$. Since T^* is closed under union of chains of models, it follows that $\mathfrak{C} \models T^*$.

CLAIM. $\mathfrak{C}$ *is an elementary universal model of* T^*. For that, let $\mathfrak{B}$ be a model of T^*, $|B|\leqslant\alpha$ and $\mathfrak{B}\equiv\mathfrak{C}$. Then $\mathfrak{B}\simeq\mathfrak{C}$ and $\mathfrak{B}\models T$ so there is $f\colon\mathfrak{B}\to\mathfrak{C}$. Since T^* is model complete, f is elementary in fact.

CLAIM. $\mathfrak{C}$ *is an elementary homogeneous model.* In order to prove this assertion we need the following...

DEFINITION 2.9. *A model* $\mathfrak{D}$ *is a weak homogeneous model if every diagram of the sort* $\mathfrak{D}\overset{\geq}{\leftarrow}\mathfrak{A}\overset{\leq}{\rightarrow}\mathfrak{D}$, $|A|<|D|$, *can be completed in the commutative diagram:*



(The following question can be stated: Does the weak homogeneity implies elementary homogeneity?)

It is obvious that $\mathfrak{C}$ is a weak homogeneous model. That $\mathfrak{C}$ is elementary homogeneous follows directly from the previous claim and the following...

LEMMA (Morly-Vaught) If $\mathfrak{C}$ is an elementary universal model then $\mathfrak{C}$ is weak homogenous iff it is elementary homogeneous.

For the proof see [3; 11.14]. $\dashv$

There are several results similar to the previous theorem. We would like to mention two theorems of such kind. One is in [3; 11.19] and it is connected with the notion of conservative enlargement L of a class of models K. This theorem aserts that α homogenous-universal models of K and L coincide. However, in this theorem uniformity in assigment of models of class L to models of class K is assumed, what is not the case in our theorem. The second one is the theorem of H. Simmons (6; 3.4.1) which states that if a given theory has the model companion, then all its k-objective (in the sense of M. Yasuhara [6]) models are k-saturated.

3. Full models of a theory with a dense ordering

In some cases it is possible to say exactly in which cardinals a theory T has full models, and according to the theorem 2.8., its model completion has saturated models.

THEOREM 3.1. *Let* $\mathfrak{A}$ *be a saturated model of cardinality* α *and assume that it (or its definable expansion) contains a nontrivial dense partial ordering, i.e. in* $\mathfrak{A}$ *holds* $\forall xy\exists z(x<y\Rightarrow x<z<y)$. *Then an* η_α *set can be embedded into* $\mathfrak{A}$ *and therefore* $\alpha=\alpha^{\underline{\alpha}}$.

PROOF Let g be a maximal chain without endpoints and X, $Y \subseteq g$ so that $X < Y$ (i.e. for all $u \in X$, all $v \in Y$, $u < v$), $|X \cup Y| < \alpha$. The set $\Sigma(x) = \{u < x \mid u \in X\} \cup \{x < v \mid v \in Y\}$ is finitely consistent with $Th(\mathfrak{A}_{X \cup Y})$, hence $\Sigma(x)$ is realized in $\mathfrak{A}$, i.e. there is $a \in A$ so that $\mathfrak{A} \models \Sigma(a)$. Therefore $X < a < Y$. Assume that $a \notin g$. Let $b \in g$. Then there are the following possibilities:

1° For some $u \in X$ $b \leqslant u$, so $b \leqslant a$.

2° For some $v \in Y$ $v \leqslant b$, so $a \leqslant b$.

3° $X < b < Y$.

If 3° does not hold for any $b \in g$, then by 1° and 2° $g \cup \{a\}$ is linearly ordered, so by maximality of g $a \in g$, but this contradicts to our assumption. Hence $a \in g$ or there is $b \in g$ so that $X < b < Y$, in any case there is $c \in g$ so that $X < c < Y$. Thus, g is an η_α set so $|g| \geqslant \alpha$. But $g \subseteq A$, hence $|g| = \alpha$. Hence g is an η_α set of cardinality α so (Gillman, cf. [3]) $\alpha = \alpha^{\underset{\smile}{\alpha}}$. $\dashv$

Assume that T is a Jónnson theory. According to the theory of Jónnson classes, if $\alpha > \omega$ and $\alpha = \alpha^{\underset{\smile}{\alpha}}$ then there is a homogeneous-universal model of T of cardinality α. By the previous theorem we have the following ...

COROLLARY 3.2. Assume that T contains a nontrivial partial dense ordering, and let α be a cardinal, $\alpha > \omega$. Then T has a full model and T^* has a saturated model of cardinality α iff $\alpha = \alpha^{\underset{\smile}{\alpha}}$. $\dashv$

We list several examples of theories with ordering on which previous theorems can be applied.

T	T^*
1. Theory of linear ordering.	Theory of linear dense ordering without endpoints.
2. Theory of linearly ordered Abelian groups.	Theory of linearly ordered Abelian divisible groups.
3. Theory of Boolean algebras.	Theory of atomless Boolean algebras
4. Theory of distributive lattices with endpoints.	Theory of distributive, complementary, dense lattices with endpoints
5. Theory of ordered fields.	Theory of o.dered real closed fields.

Depending on a theory several names are connected with the theory in two sense: 1° In proof that an appropriate theory T^* is a model completion of T, 2° That the class of models of T is a Jónnson class. For informations of that kind one may consult [2], [3] and [4].

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REPRESENTATION THEOREM FOR MINIMAL σ-ALGEBRAS

Kanji NAMBA

The purpose of this paper is to state some properties of minimal separating σ-algebras and of σ-compact topological spaces. Original motivation of this work is to consider the problem of existence of a minimal separating σ-algebra without any singleton. This fine problem, comes from a problem of statistics, is proposed by H. Morimoto who communicated me the following elementary but fundamental example of such a σ-algebra which appears in [18]:

Let X be an uncountable set and x be an element of X, then the σ-algebra consisting of subsets A of X with the property that "$x \in A$ and A is co-countable or $x \in A$ and A is countable" is minimal separating and does not contain $\{x\}$.

In statistics, various σ-fields are considered as mathematical expressions of statistical experiments. In some special cases, one of the properties of the σ-fields with statistical relevance called "pairwise sufficiency" reduces to their separating property.

Existence of minimal pairwise sufficient σ-fields is of interest and the σ-field given at the outset of this paper is one such example. It naturally leads to the question as to whether any more examples exists and, further, how they are characterized, and these are exactly the problem treated here.

Considering the structure of the above example, it is natural to imagine that there are many other types of such σ-algebras, and this is realized by considering a natural correspondence between the notions of minimality of σ-algebras and σ-compactness of related topological spaces, and that of σ-complete 2-valued measures and limit points of σ-topological spaces.

The author wishes to express his thanks to Prof. H. Morimoto for his generous support and encouragement.

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1. Definitions, notions and elementary properties

We begin with the notions and definitions of concepts needed for the descriptions and discussions below. A cardinal number k is called regular if it is not a λ sum of smaller cardinals for all $\lambda<k$. A set B consisting of subsets of X is called a k-algebra over X provided that it is closed under complementation and λ-union for all $\lambda<k$. ω_1-algebra is usually called a σ-algebra, that is, closed under complementation and countable union. k-algebra B is called a separating algebra if for any distinct elements x, y of X, there is a set A of B such that $x\in A$ but $y\notin A$, in other words if

$$\forall A\in B(x\in A\equiv y\in A)\rightarrow x=y.$$

k-algebra B is called minimal if it is minimal in the sense of set inclusion. A subset $\{G_i:i\in I\}$ of k-algebra B is called a generator of B if it is the smallest k-algebra containing the subset. For a k-algebra B the following two properties are equivalent:

(a) B is minimal separating,

(b) $\{G_i:i\in I\}$ is a generator of B if and only if it separates the points of X.

Let $\{G_i:i\in I\}$ be a generator of separating k-algebra B over X, and put

$$G_{i0}=G_i \text{ and } G_{i1}=X-G_i.$$

Then there is a natural correspondence j between X and a subset Y of 2^I which consists of functions with domain I and values in $2=\{0, 1\}$ in such a way that

$$j(x)(i)=k\equiv x\in G_{ik}.$$

By this correspondence j, the set X may be considered as a subset Y of 2^I and B may be considered as a k-algebra over Y with the generators

$$Y_{ik}=\{p\in Y:p(i)=k\},$$

because of the property

$$G_{ik}=j^{-1}(Y_{ik})$$

and inverse image keeps complementation and union. Of course such B is always a separating algebra.

Let a be a subset of I with the cardinality less than k, that is, $\#a<k$, by a neighbourhood of $p\in Y$ of 2^I we mean the set

$$U(p;\ a)=\{q\in Y:\forall i\in a(p(i)=q(i))\}.$$

The k-topology of Y is introduced by the system of neighbourhoods

$$U_p=\{U(p;\ a):a\subset I,\#a<k\}.$$

ω and ω_1-topology are usually called weak and σ-topology, respectively. A k-topological space Y, i.e. the subspace Y of 2^I with k-topology, is called λ-compact if for any function which associates p with its neighbourhood $U(p;\ a_p)$, there is a subset b of Y with $\#b<\lambda$ such that

$$Y\subset\bigcup_{p\in b}U(p;\ a_p).$$

It is well-known that this property is characterized by the following properties:

(a) Let $\{O_j; j\in J\}$ be an open covering of Y, then there is a subset b of J with $\# b<\lambda$ such that

$$Y\subset \bigcup_{j\in b} O_j.$$

A dual form of this expression is:

(b) Let $\{C_j : j\in J\}$ be a family of closed subsets of Y with less than λ intersection property, that is

$$\# b<\lambda \rightarrow \bigcap_{j\in b} C_j\neq \varnothing ,$$

then their intersection is not empty, namely

$$\bigcap_{j\in J} C_j\neq \varnothing .$$

Let X and Y be k-topological spaces of 2^I and 2^J, respectively. Then a function $f: X\rightarrow Y$ is called uniformely continuous if there is a function

$$g: P_k(J)\rightarrow P_k(I)$$

where $P_k(I)=\{a\subset I: \# a<k\}$ such that for all $b\in P_k(J)$ and $p\in X$

$$U(p; g(b))\subset U(f(p); b).$$

Let A be a subset of k-topological space Y of 2^I. Then a subset a of with $\# a<k$ is called a support of A if for every $p, q\in Y$,

$$\forall i\in a\,(p(i)=q(i))\rightarrow p\in A\equiv q\in A.$$

A subset A with support is closed and open, i.e. a clopen set of Y. Let B^* be the set of all such subsets of Y. Then B^* is a k-algebra provided that k is a regular cardinal. It is also clear that B^* is a separating k-algebra including the k-algebra generated by its basic open sets.

Let Y be a k-topological space of 2^I. Then it is called a k-space if for any subset a of I with $\# a<k$, there is a subset b of Y with $\# b<k$ such that

$$Y\subset \bigcup_{p\in b} U(p; a).$$

By this definition, we have that k-compact k-topological space is a k-space.

2. k-compactness and minimality of k-algebras

We begin with an easy property of k-spaces.

LEMMA 1. In k-space Y of 2^I, the k-algebra B^* of sets with support of cardinality less than k coincides with the k-algebra B generated by the basic open sets of Y.

PROOF. Let A be an element of B^*, then there is a subset a of I with $\# a<k$ such that

$$A=\bigcup_{p\in A} U(p; a).$$

Since Y is a k-space, there is a subset b of Y with $\# b<k$ such that

$$\bigcup_{p\in A} U(p; a)=\bigcup_{p\in b} U(p; a)=\bigcup_{p\in b}\bigcap_{i\in a} Y_{ip(i)}.$$

By the definition of B, it is closed under less than k union and intersection. Therefore we have

$$A=\bigcup_{p\in b}\bigcap_{i\in a} Y_{ip(i)}\in B.$$

This means that $B=B^*$ by the inclusion mentioned above.

Next lemma reveals a property of minimality of k-algebras.

LEMMA 2. Suppose that the k-algebra B generated by all the basic open sets of k-topological space Y of 2^I ia a minimal separating k-algebra. Then Y is k-compact and hence a k-space.

PROOF. Suppose Y is not k-compact, then there is a function

$$p\rightarrow U(p; a_p)$$

such that for any subset b of Y with $\# b<k$, we have

$$Y-\bigcup_{p\in b} U(p; a_p)\neq\varnothing .$$

Let B_1 be the set of all A in B with the following property:

(1) There is a subset b of Y with $\# b<k$ such that for any $q, r\in Y$

$$q, r\notin \bigcup_{p\in b} U(p; a_p)\rightarrow q\in A\equiv r\in A.$$

Since k is a regular cardinal, we have that B_1 is a k-algebra. Now we shall show that B_1 is separating. Suppose $p\neq q$, then since B is separating, there is a set A of B such that $p\in A$ but $q\notin A$, so we have

$$p\in A\cap U(p; a_p) \text{ and } q\notin A\cap U(p; a_p).$$

By the definition of B_1, we have $A\cap U(p; a_p)\in B_1$ and so it is separating. By the minimality of B, we have $B=B_1$.

Since the basic open set Y_{i0} belongs to B for every i, we have a subset b_i of Y with $\# b_i<k$ such that

$$q, r\notin \bigcup_{p\in b_i} U(p; a_p)\rightarrow (q\in Y_{i0}\equiv r\in Y_{i0}).$$

Hence there is a function $s: I\rightarrow 2$ such that

$$Y-\bigcup_{p\in b_i} U(p; a_p)\subset Y_{is(i)}.$$

We consider a neighbourhood of s in k-topological space 2^I,

$$W(s, a)=\{p\in 2^I:\ i\in a\,(p(i)=s(i))\}.$$

For any subset a of I with $\#a<k$, we put

$$b=\bigcup_{i\in a} b_i,$$

then we have $\#b<k$ and

$$\varnothing\neq Y-\bigcup_{p\in b} U(p;\, a_p)\subset\bigcap_{i\in a} Y_{is(i)}=Y\cap W(s;\, a).$$

Now we shall show that s is an element of Y.
So suppose $s\notin Y$ and let p^* be a fixed element of Y. Let B_2 be the set of all A in B with the following property:

(2) There is a subset a of I with $\#a<k$ such that any for $q\in Y$

$$q\in W(s;\, a)\to(q\in A\equiv p^*\in A).$$

By the relation

$$W(s;\, \bigcup_{i\in c} a_i)=\bigcap_{i\in c} W(s;\, a_i),$$

we see that B_2 is a k-algebra. Now we shall show that B_2 is separating. let p, q be two elements of Y such that $p\neq q$. Then we have $p\neq p^*$ or $q\neq p^*$, so we may assume $p\neq p^*$. Since $p^*, q, s\neq p$, there is a subset a of I with $\#a<k$ such that

$$q,\, p^*\not\in U(p;\, a),\;\; U(p;\, a)\cap W(s;\, a)=\varnothing.$$

This means that for any $r\in Y$, we have

$$r\in W(s;\, a)\to(r\in U(p;\, a)\equiv p^*\in U(p;\, a)).$$

By the definition of B_2, we have $U(p;\, a)\in B_2$ and $q\notin U(p;a)$. This means that B_2 is separating and so by minimality of B, we have $B=B_2$. By $p^*\neq s$, there is a subset a of I with $\#a<k$ such that

$$U(p^*;\, a)\cap W(s;\, a)=\varnothing.$$

Since $U(p^*;\, a)\in B_2$, there is a subset a_1 of I with $\#a_1<k$ such that

$$r\in W(s,\, a_1)\to(r\in U(p^*;\, a)\equiv p^*\in U(p^*;\, a)).$$

We consider a point

$$q^*\in Y-\bigcup_{p\in c^*} U(p;\, a_p)\subset Y\cap W(s;\, a\cup a_1)$$

where $c^*=\bigcup_{i\in a\cup a_1} b_i$, then by $q^*\in W(s;\, a\cup a_1)\subset W(s;\, a)$, we have

$$q^*\in U(p^*;\, a).$$

This contradicts with

$$U(p^*;\, a)\cap W(s;\, a)=\varnothing.$$

This contradiction shows that $s\in Y$, that is Y is close.

Now we consider the neighbourhood $U(s;\, a_1)$. Then by putting

$$b^*=\bigcup_{i\in a_s} b_i$$

we have $\#b^*<k$ and

$$Y-\bigcup_{p\in b^*}U(p;\,a_p)\subset U(s;\,a_s)=Y\cap W(s;\,a_s).$$

This means that

$$Y\subset\bigcup_{p\in b^*}U(p;\,a_p)\cup U(s;\,a_s)$$

which contradicts to the choice of $U(p,\,a_p)$. Hence Y is k-compact.

LEMMA 3. Let X and Y be k-topological spaces of 2^I and 2^J. If X is k-compact and $f\colon X\to Y$ is continuous, then f is uniformly continuous and the image $f(X)$ is k-compact.

PROOF. Let $f\colon X\to Y$ be continuous and a be a subset of J with $\#a<k$. By the continuity of f, there is a subset a_p of I such that

$$f(U(p;\,a_p))\subset U(f(p);\,a).$$

By the k-compactness of X, we have a subset $b(a)$ of X with $\#b(a)<k$ such that

$$X\subset\bigcup_{p\in b(a)}U(p;\,a_p).$$

Now we put

$$a^*=\bigcup_{p\in b(a)}a_p.$$

For $q\in X$, there is $p\in b(a)$ such that $q\in U(p;\,a_p)$, so we have

$$r\in U(q;\,a^*)\subset U(p;\,a_p)\to f(r)\in U(f(p);\,a).$$

This means that

$$f(U(q;\,a^*))\subset U(f(p);\,a)=U(f(q);\,a),$$

so f is uniformly continuous.

To each q in $f(X)$, let there correspond b_q, any subset of J with $\#b_q<k$, and consider the function

$$q\to V(q;\,b_q)$$

defined on $f(X)$. By the continuity of f, there is a similar

$$p\to U(p;\,a_p)$$

on X such that

$$f(U(p;\,a_p))\subset V(f(p);\,b_{f(p)}).$$

Since X is k-compact, we have a subset c of X with $\#c<k$ such that

$$X\subset\bigcup_{p\in c}U(p;\,a_p).$$

Hence we have

$$f(X)\subset\bigcup_{p\in c}f(U(p;\,a_p))\subset\bigcup_{p\in c}V(f(p);\,b_{f(p)}).$$

This means that $f(X)$ is k-compact.

LEMMA 4. Let X and Y be k-topological spaces of 2^I and 2^J. If X is k-compact and $f: X \to Y$ is a 1—1 onto continuous function, then $f^{-1}: Y \to X$ is uniformly continuous.

PROOF. Let A be a closed subset of X. Then A is k-compact as a closed subset of X, so $f(A)$ is k-compact and so a closed subset of Y. This means that the image of a closed set is closed, and since f is 1—1 onto, the image of open set is open. By the relation $f(U)=(f^{-1})^{-1}(U)$, we have that the inverse image of an open set U by f^{-1} is open. This means that the function f^{-1} is contionuous. Since $Y=f(X)$ is k-compact, f^{-1} is uniformly continuous.

LEMMA 5. Let X be a k-compact subset of 2^I with k-topology. Then the k-algebra $\underline{B}$ generated by the basic open sets of X is a minimal k-algebra.

PROOF. Let $\{G_j: j \in J\}$ be a separating subset of B. By B^* we denote the k-algebra generated by $\{G_j: j \in J)$. Now we define a function $f: X \to 2^J$ by the relation

$$f(p)(j)=k \equiv p \in G_{jk}$$

where $G_{j0}=G_j$ and $G_{j1}=X-G_j$. Since X is k-compact and so a k-space, we have that every element of B has a support. This means that the above function f is continuous. Since $\{G_j : j \in J\}$ is separating, f is 1—1. Let Y be $f(X)$, then

$$f: X \to Y$$

is a 1—1 onto continuous function. Hence its inverse

$$f^{-1}: Y \to X$$

is uniformly continuous. This means that for any i of I, there is a subset b_i of J with $\# b_i < k$ such that

$$f^{-1}(V(f(p); b_i)) \subset U(p; \{i\}).$$

By the compactness of Y, there is a subset c of Y with $\# c < k$ such that

$$Y = \bigcup_{q \in c} V(q; b_i).$$

Hence by the relation

$$p \in f^{-1}(V(q; \{k\})) \equiv f(p)(k) = q(k).$$

we have

$$f^{-1}(V(f(p); b_i)) = \bigcap_{k \in b_i} f^{-1}(V(f(p); \{k\})) = \bigcap_{k \in b_i} G_{k f(p)(k)}.$$

Hence by the definition of B^*, we have

$$U(p; \{i\}) = \bigcup_{\substack{f(q) \in c \\ q(i)=p(i)}} \bigcap_{k \in b_i} G_{k f(q)(k)} \in B^*.$$

Since $\{U(p; \{i\} : i \in I\}$ is a generator of B, we have $B=B^*$. This means that every separating subsets of B is a generator of B, hence B is a minimal separating k-algebra.

Combining these lemmas, we have the following

THEOREM. Let X be a subset of 2^I with k-topology. Then the k-algebra generated by the basic open sets of X is minimal separating if and only if X is k-compact.

3. Examples and remarks

Let X be a totally disconnected k-complete topological space, that is, any distinct points of X are separated by a clopen set and the intersection of less than k open sets is again an open set. Let $\{G_i : i \in I\}$ be a separating clopen basis of X. Then X can be considered as a subspace of 2^I with k-topology. In the k-topological space 2^I, the element of $\bar{X}$, the closure of X, means a k-additive 2-valused measure on k-algebra B determined by the basic open sets of X. The cannonical relation of point p of $\bar{X}$ and measure μ_p is

$$p \in \bar{A} \equiv \mu_p(A) = 1.$$

Since in the k-topological space X, we have the relation

$$\overline{(\bigcup_{\nu \in a} A_\nu)} = \bigcup_{\nu \in a} \bar{A}_\nu$$

for every a with $\# a < k$, the additivity condition follows.. And if $A \in B$, then it has a support a with $\# a < k$ and so

$$\bar{A} \cap \overline{X - A} = \varnothing .$$

Conversely any k-additive 2-valued measure $\mu : B \to 2$ determines an element of 2^I by $p(i) = 1 - \mu(G_i)$ which belongs to the closure of X in 2^I. Hence the closure $\bar{X}$ is just the set of all k-additive 2-valued measures on B. An element of X is called a principal or a point measure and an element of $\bar{X} - X$ is a non-principal measure. The notion of k-additive 2-valued measure and k-complete maximal filter or ideal are considered as alternating expressions of the same concept by considering the element of $2 = \{0,1\}$ as quantity 0,1 or as truth value 0=falsity, 1 = truth.

Next, we shall give some examples of k-compact sets by showing the following lemma

LEMMA 6. Let λ be a cardinal number. Then the set

$$X_\lambda = \{f \in 2^I : \# \{i \in I : f(i) = 1\} \leqslant \lambda\}$$

is k-compact in the k-topological space 2^I if and only if

$$\forall \eta < k \, (\eta^\lambda < k).$$

PROOF. First, if there is some η which satisfies

$$\lambda \leqslant \eta < k \leqslant \eta^\lambda$$

then X_λ is not k-compact. Because we can take a subset a of I with $\# a = \eta$, then we have

$$\# \{f \in 2^a : \# \{i \in a : f(i) = 1\} \leqslant \lambda\} = \eta^\lambda \geqslant k.$$

We associate a neighbourhood $U(p; a)$ for every p of X_λ, then no less than k union cover X_λ and so it is not k-compact. The case $k \leqslant \lambda$ is trivial.

Next, we shall show that $\forall \lambda < k (\eta^\lambda < k)$ implies the k-compactness of X_λ. Let $U(p; a_p)$ be a given neighbourhood of p in X_λ. We shall show X_λ can be covered by less than k union of $U(p; a_p)$'s. Let f be a function with the domain in I and values in 2, we denote by f^* the function $f^* : I \to 2$ defined by

$$f^*(i) = \begin{cases} f(i) & \text{if } i \in \mathrm{dom}(f) \\ 0 & \text{if } i \in I - \mathrm{dom}(f) \end{cases}$$

By the induction on ν, we define a subset a_ν of I as follows

$$a_0 = a_{\varnothing^*}$$

where $\varnothing$ is the empty function. For a successor ordinal, we put

$$a_{\nu+1} = \bigcup_{f \in X_\lambda} a_{(f|a_\nu)^*}$$

where $f|a$ is the restriction of f to a. For a limit ordinal, we put

$$a_\nu = \bigcup_{\tau < \nu} a_\tau.$$

We shall show that $\# a_\nu < k$ for all $\nu < \lambda^+$, the smallest cardinal greater than λ. Since the case that ν is a limit ordinal is clear by the regularity of k and $\lambda^+ \leqslant 2^\lambda < k$, we shall show that $\# a_\nu < k$ implies $\# a_{\nu+1} < k$. So we consider the set

$$d_\nu = \{f | a_\nu : f \in X_\lambda\},$$

then by the assumption of $\# a$ and by the property of λ, we have $\# d_\nu \leqslant \# a_\nu^\lambda < k$, so using $\# a_{(f|a_\nu)^*} < k$ for each $f \in X_\lambda$, we have

$$\# a_{\nu+1} \leqslant \sum_{f \in d_\nu} \# a_{f^*} < k.$$

Next, we consider two cases

(1) $$a_{\nu+1} - a_\nu = \varnothing \text{ for some } \nu < \lambda^+,$$

(2) $$a_{\nu+1} - a_\nu = \varnothing \text{ for all } \nu < \lambda^+.$$

The case (1): For every $f \in X_\lambda$, we have

$$a_{(f|a_\nu)^*} \subset a_{\nu+1} = a_\nu.$$

For any $f \in X_\lambda$, consider $(f|a_\nu)^* \in X_\lambda$, then by (1), we have

$$f \in U((f|a_\nu)^*; a_{(f|a_\nu)^*}),$$

so we obtain that

$$X_\lambda \subset \bigcup_{f \in d_\nu} U(f^*; a_{f^*}).$$

This means that X_λ can be covered by less than k union of given neighbourhoods.

The case (2): We put

$$A_\nu=\{f\in X_\lambda : a_{(f|a_\nu)^*}-a_\nu\neq\Phi\},$$

and assume that

$$f\in \bigcap_{\nu>\lambda+} A_\nu.$$

Then we have for all $\nu<\lambda^+$,

$$a_{(f|a_{\nu+1})^*}-a_{\nu+1}\neq\Phi.$$

Suppose that $f(i)=0$ for all $i\in a_{\nu+1}-a_\nu$, then

$$(f|a_{\nu+1})^*=(f|a_\nu)^*.$$

Hence we have

$$a_{(f|a_{\nu+1})^*}=a_{(f|a_\nu)^*}\subset a_{\nu+1}.$$

This contradiction shows that for all $\nu<\lambda^+$, there exists an $i\in a_{\nu+1}-a_\nu$ such that $f(i)=1$. This means that

$$\#\{i\in I: f(i)=1\}>\lambda,$$

which contradicts the assumption $f\in X_\lambda$, and so we have

$$X_\lambda=X_\lambda-\bigcap_{\nu<\lambda+} A_\nu=\bigcup_{\nu<\lambda+}(X_\lambda-A_\nu).$$

Let $f\in X_\lambda-A_\nu$. Then we have that $a_{(f|a_\nu)^*}\subset a_\nu$ and so

$$X_\lambda-A_\nu\subset\bigcup_{f\in d_\nu} U(f^*;\, a_{f^*}).$$

Hence we have

$$X_\lambda\subset\bigcup_{f\in d^*} U(f^*;\, a_{f^*})$$

where $d^*=\bigcup_{\nu<\lambda+} d_\nu$ and the condition $\# d^*<k$ follows from $\lambda^+\leqslant 2^\lambda<k$. Therefore X_λ can be covered by less than k unicn of given neighbourhccds. Any way, the space X_λ is k-compact.

By the proof of above lemma, we have that if a family D of subsets of I satisfies the condition

(1) $$a\in D,\ b\subset a\rightarrow b\in D,$$

(2) $$\# a<k\rightarrow\#\{b\in D: b\subset a\}<k,$$

then the set of all representing functions of the sets in D is k-compact. For example, if D satisfies the conditions and a partial ordering $\leqslant$ is defined on I, then the set D' of elements of D which is well-ordered or linearly ordered by $\leqslant$, satisfies this condition. Hence if $k=(2^\omega)^+$, then the set of all well-ordered countable subsets of I is k-compact. But of course this set is not ω_1-compact, namely not σ-compact, if I includes a countable increasing sequence.

By using Lemma 6 and the property that the continuous image of a k-compact set is k-compact, we have that

$$X_{f,\lambda}=\{q\in 2^I: \#\{i\in I: q(i)\neq f(i)\}\leqslant\lambda\}$$

is k-compact for any $f: I \to 2$ and λ such that $\forall \eta < k (\eta^\lambda < k)$. Hence any closed subset of less than k union of such $X_{f,\lambda}$ is also k-compact.

Now, we consider the case $k > \omega$, for example $k = \omega_1$, then for any $\eta < k$ and $n < \omega$, we have $\eta^n = \eta < k$. By this, the set

$$X^*_\omega = \{f \in 2^I : \#\{i \in I : f(i) = 1\} < \omega\},$$

being a union of countable k-compact sets, is k-compact. Since each point of this set is not an isolated point, the k-algebra determined by this- k-compact set is an example of a minimal separating k-algebra without a singleton. Since there is no restriction on the cardinality of index set I, each cardinality determines at least one non isomorphic minimal separating k-algebra without a singleton. One may consider, may be pathological, the k-compact space consisting of all finite sets, in which case the minimal separating k-algebra consists of elements which are not sets but classes.

Now we consider, for example, the space ω^I with k-topology. Since each natural number n of ω can be considered as an element of 2^ω by usual binary expansion, we may consider

$$\omega^I \subset (2^\omega)^I = 2^{\omega \times I}$$

So we have that the set

$$X^\#_\omega = \{f \in \omega^I : \#\{i \in I : f(i) \neq 0\} < \omega\}$$

is k-compact, and the k-algebra generated by its basic open sets

$$\{f \in X^\#_\omega : f(i) = n\}$$

is minimal separating and have no singleton.

One intuitive example of minimal separating σ-algebra would be as follows: Suppose there are at most countably many particles and their states, the family X of all positions and states of finite particles in, for example, n-dimensional Euclidean space forms a σ-compact set, and the σ-algebra determined by this topological space is minimal separating σ-algebra without singleton.

We consider the property

(*) $$\forall \eta < k (\eta^\lambda < k).$$

If $\lambda < k$ and k is regular, then (*) implies

$$k^\lambda = \sum_{\eta < k} \eta^\lambda < k^2 = k < k^+$$

hence k^+ also satisfies the property (*). On the other hand, if $cf(k)$, the cofinality of k, satisfies $cf(k) \leq \lambda$, then $cf(k^\lambda) > \lambda$, by König's theorem, so we have $k^+ \leq k^\lambda$ and so k^+ does not satisfy the property (*). The least cardinal greater than η_0 satisfying (*) is defined by $k = (\eta_0{}^\lambda)^+$, because $(\eta_0{}^\lambda)^\lambda = \eta_0{}^\lambda < k$.

If for example the continuum hypothesis $2^\omega = \omega_1$ is true, then

$$\{f \in 2^I : \#\{i \in I : f(i) = 1\} \leq \omega\}$$

is k-compact for $k = \omega_2, \omega_3, \ldots$ but not for $k = \omega_1, \omega_{\omega+1}, \ldots$.

Interesting problem concerning this is the problem of implication

$$\forall n<\omega\,(2^{\omega}=\omega_{n+1})\rightarrow 2^{\omega_\omega}=\omega_{\omega+1}$$

which is proposed by R. M. Solovay, and is called the singular cardinals problem And this problem is equivalent to $\omega_{\omega+2}$-compactness of above set X_ω under the assumption of

$$\forall n<\omega\,(2^{\omega_n}=\omega_{n+1})$$

Another interesting problem is explicit characterization of ω_1 or σ-compact sets, for example the existence of σ-compact set which is not included in the continuous image of the set of the form $X_\omega^{\#}$. And the characterization of the structure of complete Boolean algebra determined by closed subsets of 2^I divided by the ideal of k-compact sets.

The case $k=\omega$ is well-known case of weak topology, by Tihonov theorem the topological space 2^I is compact, hence a subset is compact if and only if it is closed. This means that the ω-algebra (Boolean algebra of clopen sets) B generated by basic open sets of X is minimal separating if and only if X is closed. There is natural correspondence between the closure $\overline{X}$ of X and the set B^* of all maximal filters (or ideals) of B.

We have already mentioned that every k-compact k-topological space X of 2^I is a k-space. Now we consider the problem of converse implication. That is, whether every closed k-space in k-topological space in 2^I is k-compact or not.

When $I=k$, this property is known as tree property. To explain about this, we define the notion of binary tree, here we say simply a tree. A subset T of $P=\bigcup_{\nu<k}2^\nu$ is called a k-tree if the following conditions are satisfied:

(a) $$f\in T,\ g\in P,\ g\subset f\rightarrow g\in T,$$

(b) $$0<\#(T\,|\,\nu)<k \text{ where } T\,|\,\nu=\{f\in T:\mathrm{dom}(f)=\nu\} \text{ and } \nu<k.$$

A function $f:k\rightarrow 2$ is called a total branch of T if

$$\forall\nu<k\,(f\,|\,\nu\in T).$$

We say that a cardinal k have the tree property if every k-tree has a total branch.

LEMMA 7. k has tree property if and only if every closed k-space in the k-topological space 2^k is k-compact.

PROOF. Let T be a k-tree without any total branch. For any $f\in T$, we associate a function $f^*:k\rightarrow 3$ defined by

$$f^*(\nu)=\begin{cases} f(\nu) & \text{if } \nu\in\mathrm{dom}(f)\\ 2 & \text{if } \nu\in k\text{-}\mathrm{dom}(f).\end{cases}$$

Then by the inclusion $3\subset 2^2$, we may consider f^* as an element of 2^k by $3^k\subset(2^2)^k=2^{2\times k}=2^k$. Now we consider a subset T^* of 2^k defined by

$$T^*=\{f^*\in 2^k:f\in T\}.$$

Then T^* is a k-space. Since T has no total branch, T^* is a closed subset of 2^k and each point of which is an isolated point. Let $U(p;a_p)$ be a neighbourhood of p in T^* with

$$U(p;\ a_p)\cap T^*=\{p\}.$$

Then by $\#T^*=k$, we have that T^* cannot be covered by less than k union of such neighbourhoods. This means that T^* is a closed k-space which is not k-compact.

Next, suppose k has the tree property and X be a closed k-space which is not k-compact. Let $\{U(p;\nu_p):p\in X\}$ be a covering of X by which X cannot be covered by less than k union of the sets. Let T be the set of functions defined by

$T=\{f|\nu : f\in X,\ T$ cannot be covered by $<k$ of $U(p;\ \nu_p)$'s$\}$. Since X is a k-space which is not k-compact, T is a k-tree. Hence, by tree property of k, T has a total branch $f: k\to 2$. But since X is closed, we have $f\in X$. This means that $f\in U(f;\ \nu_f)$ and so $f|\nu_f\notin T$, which is a contradiction.

Followings are known examples about this notion:

(1) ω has tree property. This is known as König's infinity lemma or Brower's fan theorem and is a special case of Tihonov's compactness theorem.

(2) ω_1 does not have tree property. Such an example is known as Aronszajn tree

(3) (Specker) if a regular cardinal k satisfies $\forall\nu<k(2^\nu\leqslant k)$, then k^+ does not have the tree property.

(4) (J. Silver) if k is a real valued measurable then k has tree property.

It is known, by R. M. Solovay, that the consistency of existence of 2-valued measurable cardinal and that of real-valued measurable cardinal are equivalent under ZFC, Zermelo-Fraenkel set theory with axiom of choice. And every real--valued measurable cardinal is weakly inaccessible cardinal less than or equal to 2^ω, every 2-valued measurable cardinal is strongly inaccessible, that is, k is regular and $\forall\,\nu<k\,(2^\nu<k)$.

In the case k is strongly inaccessible, every subset X of k-topological space 2^I is always a k-space, and the property

$$\forall\,\nu<k\,(\#(T\,|\,\nu)<k)$$

is always satisfied. In this case k is called weakly compact. That, is, a cardinal k is weakly compact if

2^k with k-topology is k-compact.

Followings are known about this notion:

(1) the first strongly inaccessible, the first Mahlo cardinal is not weakly compact. More generally the first cardinal satisfying $\pi_1{}^1$ property is not weakly compact.

(2) every measurable cardinal is weakly compact and it is a limit of weakly compact cardinals.

J. Silver proved that the consistency of existence of weakly compact cardinal implies the consistency of

"ω_2 as tree property"

with the exioms of set theory ZFC.

More general case is considered and it is called strongly compact, or simply compact, cardinal if for any set I

$$2^I \text{ with } k\text{-topology is } k\text{-compact.}$$

This notion is also described by using tree like structures. A subset T of

$$P=\{f|a: a\subset I,\ \#a<k\}$$

is called a k-function tree if the following conditions are satisfied:

(a) $$f\in T,\ g\in P,\ g\subset f\to g\in T,$$

(b) $$0<\#\{f\in T: \mathrm{dom}(f)=a\}<k \text{ for } \nu<k \text{ and } \#a<k.$$

A function $f: I\to 2$ is called a total function of T if

$$\forall a\subset I(\#a<k\to f|a\in T).$$

We say that a cardinal k has the k-function tree property if every k-function tree has a total function. For strongly inaccessible cardinals, strong compactness is equivalent to function tree property. For example, we know the followings;

(1) every strongly compact cardinal is measurable.

(2) (Vopenka-Hrbacek) if strongly compact cardinal exists then $V\neq L(a)$ for every set a.

(3) (R. Solovay) $2^\lambda=\lambda^+$ for every singular strong limit cardinal greater than a compact cardinal.

(4) if there exists a strongly compact cardinal, then the first strongly compact cardinal can be the first measurable cardinal.

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PRESENTATION OF NATURAL DEDUCTION

R. P. NEDERPELT*

Introduction

The merits of a system of natural deduction are not only determined by its value as a logical system in itself. Since it formalizes deductions in a manner close to intuitive reasoning, natural deduction can also be used as a (logical) framework for mathematical argumentation. One may say that many mathematical texts are tacitly based on a form of natural deduction, as regards the logical part of the deductive patterns.

Jáskowski and Gentzen constructed the first systems of natural deduction in the early thirties (see Prawitz [7, appendix C]). Many suggestions have been made since with a view to formalizing the natural deduction structure present in usual mathematical reasonings.

Text-books concerned with logic on this basis are, for instance, Quine [9], Suppes [10] and Kalish-Montague [5]. The incorporation of a natural deduction system in the common mathematical practice can be very useful, in particular for didactical purposes.

In section I of this paper we shall propose another system of natural deduction, resembling that of Kalish and Montague, which can be used for the logical part of mathematics. The system to be described is quite satisfactory in practice, as became apparent when applying it to undergraduate mathematics tuition.

A natural way of reasoning in mathematics has, however, more aspects than the logical ones. These other, non-logical aspects were isolated by N.G. de Bruijn. His investigations led to a system called "the mathematical language Automath" (see [1]), which may serve as a formal notational system for rendering mathematics in a natural manner. The system is founded on typed lambda-calculus, not on axiomatic set theory.

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In section II of this paper we shall describe, in coherence with section I, the major principles of such a "system of natural reasoning". The description will be rather informal and incomplete.

It will also be shown how the rules of natural deduction can be expressed within the system, so that an important part of natural reasoning finds a formalized counterpart. In systems like this, a large part of everyday mathematics can actually be expressed, as is shown in e.g. Jutting [4] and Zucker [11].

The structure underlying a system like de Bruijn's can be made clearer by uniformations, leading to a system which is a typed lambda-calculus, the types themselves having a lambda structure. This uniform system will be described in section III of this paper. It does not have the natural aspects of the other systems. It has, however, a relatively simple and transparent structure and is therefore very useful for theoretical investigations into "systems of natural reasoning", e.g. with respect to (strong) normalizability. We shall give a precise description of this system and summarize some of its properties.

I. A practical system of natural deduction

With the aim of obtaining a practical system for natural deduction, directly applicable in everyday mathematics, we reformulate the introduction and elimination rules for $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\forall$ and $\exists$ (see e.g. Prawitz [7] or [8]), with modifications to be described below.

Basic units in the systems we shall call *sentences*, written in a sequential (not a tree-like) order, one sentence below the other. A sentence can express something like an axiom, a theorem, a definition, an assumption or a derived statement. If desired, one may add comments, e.g. containing justification for a derived statement. Such justifications may be based on logical rules (like the introduction and elimination rules), on premisses, valid assumptions and previous results, but also on mathematical arguments; this part of the reasoning is not formalized in the present system.

As primitive symbols we have the logical constants $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\forall$, $\exists$ and contradiction. We do not consider the logical constant $\Leftrightarrow$ primitive; it can be defined in the usual manner in terms of $\wedge$ and $\Rightarrow$.

We note that in mathematical practice the following observation is often used: if "F implies G" is a derived rule, then a proof of F suffices as a proof of G. (Thus a proof of b is also a proof of $a \Rightarrow b$, and so on.) We embody this meta-rule in the present system, for practical reasons.

A. *Introduction rules*

Assumptions play an essential role in natural deduction. They are generally used with the purpose of simplifying in a natural manner the statement which has to be proved: a particular related statement is temporarily taken as an added datum, another statement, simpler than the original one, is the new object for proving. As soon as this aim is achieved, the assumption is "discharged", and the original statement has been proved.

This way of dealing with assumptions will be expressed in the notations used: sentences which are assumptions will be specially marked by a box; the range of validity of an assumption will be marked by a vertical line starting from the left end of the box.

Thus doing it becomes apparent how the outer structure of a statement to be proved is reflected in its proof, as is often the case. For example, a proof of $\forall_{x \in A}[P(x) \Rightarrow Q(x)]$ will usually have the following shape:

Let $x \in A$

Assume $P(x)$
    . . . .
    $Q(x)$

For presenting an outer proof structure in this manner it is desirable to organize a proof in such a way that validity ranges of assumptions are disjoint or nested: one should arrange these validity ranges in a block configuration as is known from programming languages.

From this example it may be seen that the sentence "Let $x \in A$" will appear as an assumption in our $\forall$-introduction rule. Our preference for assumptions rather than parameters in this rule is prompted by mathematical practice: in a proof of $\forall_{x \in A}[P(x)]$, the natural first step is: "Let $x \in A$".

It will be clear that the latter sentence is not an assumption in the proper sense, as it also introduces the variable x. There is, however, a strong analogy with "normal" assumptions of the kind "Assume p", notably with respect to validity and use. Therefore we shall all the same call "Let $x \in A$" an assumption, distinguishing this kind of assumption from the other by using the word "let" instead of "assume".

Our $\vee$-introduction rule deviates from the usual rules. Our argument for this is that the two "natural" proofs for $a \vee b$ look like a proof of an implication; for example: start with : "Assume $\neg a$" and derive a proof of b. Because our system is based on classical logic (see subsection C), the usual $\vee$-introduction rules are derived rules. (We confine ourselves to one rule for $\vee$-introduction and one for $\wedge$-introduction, the symmetry of $\vee$ and $\wedge$ being presupposed.)

Thus we propose the following standard proof schemes for introduction of $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\forall$ and $\exists$, respectively:

1. . . . a 2. . . . b	Assume $\neg a$ . . . b	Assume a . . . b	Assume a . . . contradiction
concl.: $a \wedge b$	concl.: $a \vee b$	concl.: $a \Rightarrow b$	concl.: $\neg a$

Let $x \in A$ . . . $P(x)$	. . . $t \in A$ $P(t)$
concl.: $\forall_{x \in A}[P(x)]$	concl.: $\exists_{x \in A}[P(x)]$

In applying any of these schemes one should insert (from above downwards) a sequence of sentences in the place of the dots, each sentence being justifiable in the sense explained before.

B. *Elimination rules*

We denote the legitimacy of a deduction of G from F by: $F \mathbin{.\cdot.} G$. As elimination rules for $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\forall$ and $\exists$ we propose:

$a \wedge b \mathbin{.\cdot.} a$

$a \vee b,\ a \Rightarrow c,\ b \Rightarrow c \mathbin{.\cdot.} c$

$a \Rightarrow b,\ a \mathbin{.\cdot.} b$

$\neg a,\ a \mathbin{.\cdot.}$ contradiction

$\forall_{x \in A}[P(x)],\ t \in A \mathbin{.\cdot.} P(t)$

$\exists_{x \in A}[P(x)],\ \forall_{x \in A}[P(x) \Rightarrow Q] \mathbin{.\cdot.} Q$

In the $\exists$-elimination rule Q must not depend on x.

In a way, each elimination rule is the inverse of the corresponding introduction rule (cf. Prawitz [7, p.33]). There is, however, an essential difference in use between the two kinds of rules which, in our opinion, disturbs the symmetry: in principle, introduction rules give the general structure of a proof (cf. what has been mentioned in subsection A), elimination rules, however, are used for proceeding stepwise in the body of the proof. The difference in the notation of introduction and elimination rules, as shown above, reflects this asymmetry.

C. *Double negation rule*

Because logic, as it is generally used, is classical, we add the double negation rule:

$\neg\neg a \mathbin{.\cdot.} a$

The absurdity rule (contradiction $\mathbin{.\cdot.}$ a) is now a derived rule.

The main difference between the logical system described above and the usual Gentzen-type systems are found in the ∀- and ∨-introduction rules. The block structure for validity ranges of assumptions is also present in the system of Kalish and Montague ([5]). The latter system employs an existential instantiation rule instead of the usual ∃-elimination rule (cf. the comment by Prawitz on this subject in Appendix C of [7]).

The system outlined above is suitable for tuition purposes. It has the advantages of natural deduction, both in setting up proofs and in understanding them. It also agrees closely with usual patterns of reasoning: a proof written with the aid of this system scarcely deviates from usual proofs, the differences being hardly more than boxes and validity lines.

On the other hand, as stated in the beginning of this section, formalization in the above system is not pushed very far. There are no formal devices for frequently occurring mathematical routines, like applying a theorem or a definition, justifying a deduction step, and so on. In the next section we shall describe how these sides of mathematical reasoning can be effectively formalized.

II. A system for natural reasoning

We shall describe a system with a wide range of applications and a high level of formalization. The system now to be proposed is natural in the sense that it is closely related to the usual way of reasoning and proving in mathematics. In the first instance, the system refers mainly to the nonlogical part of mathematics. However, rules of logic can be expressed and applied in the system. One may choose natural deduction as a basic for logic, in the manner of the previous section (as we do in II F), thus preserving the "natural" character of the system.

The system is directly derived from the "mathematical language Automath", designed by N.G. de Bruijn for rendering mathematical texts in a formal way (see [1]). Various versions of this language have been developed by de Bruijn, in cooperation with, among others, D.T. van Daalen, L.S. Jutting and J. Zucker (see [2]). Most of the features of these various versions will be present in the "system for natural reasoning", which we shall describe in this section.

A text formalized in such a system consists of a sequence of sentences, constructed one by one in accordance with the rules of the system, the "syntax". We shall not discuss the syntax rules in detail. For this we refer to the precise definitions of a few Automath systems in [2] or [3].

A mathematical text selected for being formalized in a system like the one at issue must not show any omission in its chain of reasoning; if necessary, it must be made complete. An appropriate "translation" of that text, i.e. a formalization in the system, will be complete as well, in the sense that every sentence can be mechanically verified as to correctness according to the syntax. The latter property obviously implies that one may attach a high degree of mathematical cogency to a text, which has been translated and verified in such a system.

A number of mathematical texts have actually been formalized in systems of this kind. For example: Jutting has translated a well-known mathematical text (see [4]), and the formalized text has been verified by means of a computer programme; Zucker formalized a part of classical real analysis (cf. [11]).

A. *Typing*

In mathematics one usually attaches types to objects; one says: x is a natural number, p is a proposition. In our system we incorporate a relation "s has type t" in a formal way, denoted as $s : t$.

We fix two basic terms: π and τ, representing the type of all propositions and the type of all "classes", respectively. For example: $(p \Rightarrow q) : \pi$ and $\mathbf{N} : \tau$. (Here $\mathbf{N}$ denotes the class of all natural numbers). A class like $\mathbf{N}$ can be the type of some term of lower rank, as, for instance, in the sentence $x : \mathbf{N}$. But a proposition like $p \Rightarrow q$ can likewise be the type of some term, viz. its proof, as we shall now explain.

It is common in mathematics to deal with proofs of propositions only at a meta-level. Contrary to this, we shall incorporate proofs as terms in our system, denoted and manipulated just as the other terms in the system. This idea is well-known (for references: see [11, p. 135]). It is based on the observation that a proof of a proposition results from a kind of "construction".

As type of a proof we take the proposition it proves; if t is a proof for $p \Rightarrow q$, we write $t : (p \Rightarrow q)$. Conversely, if $r : \pi$ and $t : r$ then (proof) t asserts (proposition) r.

By the above agreements concerning typing we obtain a hierarchical relation between terms of the system. Terms π and τ are (the only) representatives of the highest level of abstraction, to be assigned *degree* 0. Terms like $p \Rightarrow q$ and $\mathbf{N}$ belong to a lower level (degree 1); terms like x and t belong to the lowest level (degree 2). In the present system we restrict ourselves to these three levels.

There is a notable contrast between our relation : ("has type") and the set-theoretical relation $\in$ ("is element of"). In set-theory, an element may belong to different classes: $x \in \mathbf{N}$ implies $x \in \mathbf{R}$, since $\mathbf{N} \subset \mathbf{R}$. As to relation :, however, we impose *uniqueness of type*: each term of degree 1 or 2 has a fixed type. (For a remark on this uniqueness: see the following subsection.) Typing thereby becomes an unambiguous, effective procedure; this facilitates mechanical checking.

Thus, in the case that $\mathbf{N}$ and $\mathbf{R}$ have been introduced independently, "natural number x" cannot be considered as a real number by a direct embedding of $\mathbf{N}$ into $\mathbf{R}$. This has obvious disadvantages, like the necessity of some non-trivial embedding device; on the other hand, obscuring identifications are absent.

B. *Conversions*

We note the complicating circumstance that a term in the system may have different manifestations, being interchangeable by means of *conversions*. There are three kinds of basic conversions. The first results from the application of a definition to (part of) a term; this is called *definitional conversion* (for an example: see subsection D). The second concerns the application of a function to an argument; it is called *functional conversion* or β-conversion (see subsection E). The third is caused by the renaming of a certain variable in a few occurrences in a term, without disturbing the pattern of binding in the term; it will be called *renaming conversion* or α-conversion.

Conversions change a term in appearance, without changing its nature. Different appearances of one term, related by conversions, will be called *equivalent*.

The above implies that the "uniqueness of type", discussed in subsection A, should be understood modulo conversion.

C. *Assumptions*

In the natural deduction system of § I, assumptions appeared in two shapes: either in the simple version "Assume p", or in the more complex version "Let $x \in A$" (cf. I A). Since we regard proofs as terms, we may replace the former version by "Let t be a proof of proposition p". The latter version becomes "Let x be a term which has type A", in correspondence with our view upon typing. Formally, both versions of assumptions can be denoted, quite similarly, by the sentences $\boxed{t : p}$ and $\boxed{x : A}$, respectively, t and x being variables, p and A being terms. (An arbitrary assumption $\boxed{u : v}$ can be correctly interpreted by regarding the type of v.)

D. *Axioms, definitions, theorems*

We shall now describe how axioms, definitions and theorems can be incorporated.

Axioms (including basic notions) will be denoted by means of a double box. For example, in regarding N as a basic notion we obtain the sentence: $\boxed{\boxed{\mathbf{N} : \tau}}$.

Then Peano's first axiom will be rendered by: $\boxed{\boxed{one : \mathbf{N}}}$. Axioms may contain one or more assumptions, like in Peano's second axiom, postulating a successor to every natural number; we may express this axiom by means of two sentences: $\boxed{x : \mathbf{N}}$ $\boxed{\boxed{s(x) : \mathbf{N}}}$. Here the assumption variable x returns in the latter sentence.

In such cases, when a sentence depends on an assumption variable, one may *instantiate*, i.e. (simultaneously) substitute a term for each occurrence of this variable in the sentence. It is then a natural requirement that the substituted term has an "appropriate" type. For example, from the last axiom one may infer that $s(one)$ has type N. Analogous rules hold in the case in which a sentence depends on more than one variable.

Definitions will be written as in the following examples:

$two := s(one) : \mathbf{N}, \qquad three := s(two) : \mathbf{N},$

$\boxed{y : \mathbf{N}}$ $plustwo\,(y) := s(s(y)) : \mathbf{N}.$

In the last example the definition consists of two sentences, the latter depending on the former.

The three above examples concern definitions of terms of degree 2. It is also possible to write a sentence containing a definition of a term of degree 1; such a term defines a "class", a proposition or a predicate.

For proofs of theorems we use the same notation as for definitions that concern terms of degree 2. We justify this policy with the following remark: a proof of a theorem *th* (where $th:\pi$) fixes a term p with $p : th$, while a definition of an "object" belonging to a "class" *cl* (where $cl:\pi$) fixes a term b with $b : cl$.

We shall show in an example how a theorem can be expressed (and proved). Suppose that the relation equality for natural numbers ($=_N$) is given as a basic notion by $\boxed{x:\mathrm{N}}$ $\boxed{y:\mathrm{N}}$ $\boxed{\boxed{(x=_N y):\pi}}$. Let reflexivity of $=_N$ be given by axiom:

$$\boxed{x:\mathrm{N}}\ \boxed{\boxed{refis(x):(x=_N x)}}.$$

Now a proof of the theorem *plustwo* (*one*) $=_N$ *three* can be expressed by:

$$proof\ 1 := refis\,(s\,(s\,(one))):(plustwo\,(one)=_N three).$$

At first sight this seems incorrect, because the axiom for reflexivity yields the relation

$$refis\,(s\,(s\,(one))):(s\,(s\,(one))=_N s\,(s\,(one))),$$

by substituting $s\,(s\,(one))$ for x. But by means of definitional conversion (see subsection B) and instantiation we may change $(plustwo\,(one)=_N three)$ into $(s\,(s\,(one))=_N (s\,(s\,(one)))$, by applying the definitions given above of *plustwo* (*one*), *three* and *two*.

As some of our examples showed, axioms and definitions may consist of more than one sentence, all but the last being assumptions. This may also be the case with (proved) theorems. Such an initial sequence of assumptions is called a *context* for the axiom, definition or theorem at issue; the assumption variables of a context may occur in the final sentence. The interdependence may even be stronger: each assumption variable in a context may occur in "type-parts" of assumptions which follow in that context. See, for instance, the axiom for the double negation rule, given in subsection F.

E. *Functions*

Functional abstraction and application form part of the system. For functional abstraction we use an adapted lambda-calculus notation, demonstrated by the following definition: $idfun := [\lambda x:\mathrm{N}]x : [\lambda x:\mathrm{N}]\,\mathrm{N}$. Here $[\lambda x:\mathrm{N}]x$ is the identity function for natural numbers; the type of this function, N^{N}, is denoted by the "type-valued function" $[\lambda x:\mathrm{N}]\mathrm{N}$. Application of funtcion f to argument x is denoted by $\{x\}f$. A motivation of the unusual order of function and argument is given in [6, p. 11—12].

A natural requirement regarding functions is that an argument of a function must have a type equivalent (in the sense explained in subsection B) to the domain of that function.

For functions and arguments the laws of *functional conversion* (also called β-conversion) hold, allowing for example the conversion of $\{A\}[\lambda x : B]C$ into $\overset{x}{_A}C$, i.e. the result of substituting term A for all free occurrences of x in term C. (We gave a general description of conversions in subsection B; q.v.)

Example: application of *idfun* to *two* yields $\{two\}\,idfun$, which is equivalent to $\{two\}[\lambda x : \mathrm{N}]x$ by definitional conversion. Then by functional conversion we may change the latter into *two*. Hence, $\{two\}\,idfun$ and *two* are equivalent: they are both "appearances of the same term".

F. *Deduction rules*

As stated before, logical rules are not primitive in the present system: one may choose one's own logical basis. We shall show how one may incorporate rules for natural deduction by means of axioms and definitions. In this respect the formal correspondences between $\Rightarrow$ and $\forall$ on one hand and functional abstraction on the other, can be successfully exploited.

For example, the "meaning" of $p \Rightarrow q$ is that for every proof of proposition p we may produce a proof of proposition q. This is a functional relation. Hence it seems natural to define $p \Rightarrow q$ as type-valued function $[\lambda x : p]q$. Then application of modus ponens can be simply effectuated by functional application (and a few conversions):

$$\boxed{s:p}\ \ \boxed{t:(p\Rightarrow q)}\ \ modponapp(s,t) := \{s\}\,t : q\,.$$

The "meaning" of $\forall_{x\in A}[P(x)]$ is that to every x in A, a proof of $P(x)$ can be attached. This is again a functional relation. So one may define $\forall_{x\in A}[P(x)]$ as the type-valued function $[\lambda x : A]P(x)$. The role of the $\forall$-elimination rule will again be taken over by the rule of functional application.

Contradiction may be introduced as basic notion:

$\boxed{\boxed{contradiction : \pi}}$. Then $\neg\, p$ can be defined as $p \Rightarrow contradiction$. Now $\neg$-elimination becomes a special case of modus ponens.

The double negation rule has to be expressed by means of an axiom as follows:

$$\boxed{p:\pi}\ \ \boxed{n:\neg(\neg p)}\ \ \boxed{\boxed{doubneg(p,n):p}}\,.$$

The logical constants $\wedge$, $\vee$, $\Leftrightarrow$ and $\exists$ can be defined in terms of $\Rightarrow$, $\neg$ and $\forall$, in the usual way. The introduction and elimination rules for $\wedge$, $\vee$ and $\exists$ can subsequently be derived as theorems.

III. *A uniform system*

In the system of § II there exists a strong correspondence between contexts ("sequences of assumptions", see II D) and functional abstractions. For example, the axiom $\boxed{x:\mathrm{N}}\ \ \boxed{\boxed{s(x):\mathrm{N}}}$, using a context consisting of a single assumption, could be replaced by $\boxed{S : [\lambda x : \mathrm{N}]\, \mathrm{N}}$, using functional abstraction. The role of "instantiation", substitution of a term for an assumption variable (cf. II D), will be taken over by functional application: $s(two)$ becomes $\{two\}S$.

We shall now propose a uniform system, developed by de Bruijn and Nederpelt (see [6]), wherein, to begin with, all assumptions are written as abstractions of the form $[\lambda k : L]$. We denote axioms and basic notions as abstractions, too, because one may regard these as being assumptions with unlimited validity range. For example, the above axiom will be written as $[\lambda S : [\lambda x : \mathrm{N}]\mathrm{N}]$.

Definitions obtain a uniformized shape as well, because instead of, e.g., $z := A : B$, A and B being terms, we write $\{A\}[\lambda z : B]$. Here variable z is defined as being functional application term A, both having type B. The role of definitional conversion (replacement of z by A) is taken over by functional conversion: $\{A\}[\lambda z : B]C$ is equivalent to $\Sigma^z_A C$.

It this system we write theorems together with their proofs, in a manner similar to that in which definitions are written. For example: $\{D\}[\lambda z : E]$ may express theorem E and its proof D, z being a name for the proof.

In the case in which a definition or a proved theorem depends on a non-empty context, the formulation in the present system is somewhat more complicated than suggested above.

By means of uniformation, such as above, we obtain a simplified system, which is a typed lambda-calculus with lambda-structured types and two constants: π and τ. This typed lambda-calculus, which we call Λ, can be regarded as a model for "systems of natural reasoning" like that described in § II, in the sense that it gives a simple and uniform framework for such systems.

As an example we give the reformulation of the theorem $plustwo(one) =_N three$ discussed in § II. In Λ this theorem becomes a single line, containing all needed information:

$$[\lambda \mathrm{N} : \tau]\, [\lambda S : [\lambda x : \mathrm{N}]\, \mathrm{N}]\, [\lambda\, ONE : \mathrm{N}]\, \{\{ONE\}\, S\}\, [\lambda\, TWO : \mathrm{N}]$$

$$\{\{TWO\}\, S\}\, [\lambda\, THREE : \mathrm{N}]\, \{[\lambda y : \mathrm{N}]\, \{\{y\}\, S\}\, S\}\, [PLUSTWO,$$

$$[\lambda y : \mathrm{N}]\, \mathrm{N}]\, [\lambda\, ISN : [\lambda x : N]\, [\lambda y : \mathrm{N}]\, \pi]\, [\lambda\, REFIS : [\lambda x : \mathrm{N}]$$

$$\{x\}\, \{x\}\, ISN]\, \{\{ONE\}\, PLUSTWO\}\, \{THREE\}\, ISN.$$

The proof of this theorem looks similar, but for the last part

$\{\{ONE\}\, PLUSTWO\}\, \{THREE\}\, ISN$, which reads:

$\{\{\{ONE\}\, S\}\, S\}\, REFIS$.

We do not uniformize π and τ into one constant, as is done in [6], since we wish to prevent assertions concerning propositions from having consequences for "classes", and vice versa. The double negation rule, for example, would in that case imply some form of the axiom of choice (cf. [11, p. 141]).

We shall now give a precise definition of Λ as being a class of terms in a typed lambda-calculus.

The *alphabet* under consideration consists of *constants* π and τ, an infinite number of *variables*: $x, y, \ldots$ and the *improper symbols* [,], {, }, λ and : .

Terms are recursively defined by:

(1) π and τ are terms; each variable x is a term.

(2) If A and B are terms and if x is a variable, then $[\lambda x : A]B$ and $\{A\}B$ are terms.

The relation: K is a *subterm* of L is the reflexive and transitive relation generated by:
A and B are subterms of $[\lambda x : A]B$ and of $\{A\}B$.

Type$_K$ is a partial function from the set of subterms occurring in term K to the set of all terms, which function is recursively defined by:

(1) If variable x occurs in K as a subterm and x is bound by $[\lambda x : A]$ in K, then $\text{Type}_K(x)=A$.

(2) (monotony:) If $[\lambda y : A]B$ is a subterm of K, if $\text{Type}_K(B)$ is defined and if $\text{Type}_K(B) = C$, then $\text{Type}_K([\lambda y : A]B) = [\lambda y : A]C$. Under analogous conditions: $\text{Type}_K(\{A\}B) = \{A\}C$.

Degree$_K$ is a partial function from the set of subterms occurring in term K to the set of the non-negative integers, which function is recursively defined by:

(1) If subterm A of K ends in τ or π, then $\text{degree}_K(A) = 0$.

(2) If subterm A of K ends in variable x, bound by $[\lambda x : B]$, and if $\text{degree}_K(B)$ is defined, then $\text{degree}_K(A) = \text{degree}_K(B)+1$.

(In Λ there is no upper bound for the values of the degree function.)

Bound terms are terms without free variables.

(In bound terms all subterms have a degree and all subterms not ending in τ or π have a type.)

α-reduction, denoted $>_\alpha$, is the monotonous relation generated by $[\lambda x : B]\, C >_\alpha [\lambda y : B]_y^x C$, with the usual restriction that the pattern of binding may not be disturbed.

β-reduction, denoted $>_\beta$, is the monotonous relation generated by $\{A\}[\lambda x : B]C >_\beta {}_A^x C$. (In substituting A for x, variables must be renamed in the usual way, in order to prevent "clash of variables".)

Reduction, denoted $>$, is the reflexive and transitive closure of both α- and β-reduction. If $K>L$, then L is called a *reduct* of K. (One may consider reduction as "one-way conversion"; cf. § II B and E.)

Legitimate terms are bound terms K with the following property: For each subterm of K of the form $\{A\}B$ there exist C and D with the properties that $\text{Type}_K^\gamma(B) > [y, C]D$ and that $\text{Type}_K(A)$ and C have a common reduct; here $\gamma = \text{degree}\ (B)$ and Type_K^γ is Type_K iterated γ times, which iteration is defined in the natural way.

Now Λ is defined as the set of all legitimate terms.

The limitation to legitimate instead of bound terms has two reasons. The first is of a intuitive nature: it is a natural requirement for a system, close to mathematical practice, that arguments A may only be related to terms B with an appropriate functional character. That is to say, B must, in a sense, be a function with a certain domain C. Moreover, argument A must be an object belonging to this domain C.

A second reason is that function applications (β-reductions) to bound terms may bring about a non-terminating process, just as in ordinary λ-calculus. In restricting oneself to legitimate terms this is impossible (see following theorem (3)).

We conclude with four theorems valuable for theoretical purposes:

(1) *Church-Rosser property* or diamond property: If $A > B$ and $A > C$, then B and C have a common reduct.

(2) *Normalization*: Every term in Λ has a normal form (i.e. a reduct to which no β-reduction can be applied), which is effectively computable; this normal form is unique but for α-reduction.

(3) *Strong normalization*: For no A in Λ is there an infinite reduction sequence $A >_\beta A_1 >_\beta A_2 >_\beta \dots$.

(4) *Closure*: If A is in Λ and $A > B$, then B is in Λ.

For proofs of these theorems: see [6] and [3].

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ON THE QUANTIFIER OF LIMITING REALIZABILITY

N. A. SHANIN

Logical connectives intermediate between $\exists$ and $\underset{\cdot}{\exists}$ as well as between $\vee$ and $\underset{\cdot}{\vee}$ [$\underset{\cdot}{\exists} x F$ denotes $\neg \forall x \neg F$ and $(F_1 \underset{\cdot}{\vee} F_2)$ denotes $\neg(\neg F_1 \,\&\, \neg F_2)$] are sometimes useful in searching for interesting "in contents" constructive analogs of theoremes of classical mathematics. We introduce two such logical connectives prompted by the theory of limiting computable (in other terms semicomputable) functions, namely the quantifier of limiting realizability $\underset{\rightarrow}{\exists}$ and the limiting disjunction $\underset{\rightarrow}{\vee}$. They are defined in terms of the basic connectives of constructive logic as follows:

$$\underset{\rightarrow}{\exists} z\, F \leftrightharpoons \exists y\, ((y \,\mathrm{stab}) \,\&\, \forall z\, ((z \lim\cdot \mathrm{val}\, y) \rightarrow F)),$$

$$(F_1 \underset{\rightarrow}{\vee} F_2) \leftrightharpoons \underset{\rightarrow}{\exists} x\, ((x=0 \rightarrow F_1) \,\&\, (x \neq 0 \rightarrow F_2)).$$

The expression (y stab) stands for the condition ≪y is a gödelnumber of a stabilizing unary total recursive function≫ (this means: a gödel number of such a total unary recursive function f for which a value x_0 of the argument quasi-exists ($\neg\neg\exists$) such that $f(x_0+x)=f(x_0)$ for any x). ($z \lim\cdot \mathrm{val}\, y$) stands for the condition ≪z is the limit value of the unary recursive function with gödelnumber y≫. Several properties of $\underset{\rightarrow}{\exists}$, $\underset{\rightarrow}{\vee}$ have been presented in the report. A detailed exposition can be found in [1].

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НЕСКОЛЬКО КОМБИНАТОРНЫХ ПРОБЛЕМ

Б. С. СТЕЧКИН

Здесь приводится несколько довольно общих задач, к которым я обращался в последнее время. По некоторым из них даются частные или прикидочные результаты. Большинство из этих проблем настолько широки, что безосновательно ожидать скорого и исчерпывающего их решения, однако и менее общие продвижения по этим задачам представили бы несомненный интерес. Вот перечень обсуждаемых здесь тематик:

I Критерий гамильтоновости.

II Закон факторизации.

III Реализуемость валентностей.

IV Структурные константы.

V Структурно-векторные покрытия.

Выражаю свою искреннюю признательность профессору Дж. К. Роте и доктору К. Баклавскому, которые во многом способствовали написанию этой работы.

I. Критерий гамильтоновости.

Согласно теореме Менгера, см. [I], граф d — связен тогда и только тогда, когда любая пара его вершин соединена по крайней мере d вершинно-непересекающимися путями. Будем говорить, что граф d-покрывающе-связен, если всякая пара его вершин соединена по крайней мере d вершинно--непересекающимися путями такими, что пути эти покрывают всех вершины графа, т.е. множество всех вершин всех этих путей есть все множество вершин графа. Граф будем именоваь четно-покрывающе-связным, если для всякой пары его вершин существует в этом графе система из четного числа вершинно-непересекающихся и покрывающих путей, соединяющих эти вершины.

ГИПОТЕЗА. *Граф гамильтонов тогда и только тогда, когда он четно-покрывающе связен.*

Необходимость очевидна, поскольку всякая пара вершин гамильтонового цикла соединима в точности двумя вершинно-непересекающимися и покрывающими путями. Критерий становится тавтологичным, если в графе имеется вершина, степень которой не превосходит трех.

Пусть G-плоский граф, который четно-покрывающе-связен, тогда либо найдется пара вершин соединимая двумя вершинно-непересекающимися и покрывающими путями, и значит G-гамильтонов, либо всякая пара его вершин соединима по крайней мере четырьмя вершинно-непересекающимися и покывающими путями, но последнее, в частности, влечет тот факт, что G является четырехсвязным графом, а согласно теореме Татта [2] плоский и четырехсвязный граф является гамильтоновым. Таким образом имеет место

ТЕОРЕМА. *Плоский граф гамильтонов тогда и только тогда, когда он четно-покрывающе-связен.*

Имеющиеся достаточные условия гамильтоновости, см. [I], редуцируют задачу к случаю наличия пары вершин соединимой не слишком большим четным числом вершинно-непересекающихся и покрывающих путей в неплоском графе. Наблюдается, наконец, и алгоритмическое „равновесие" этой гипотезы.

Выражаю благодарность доктору Ю. В. Матиясевичу, беседы с которым помогли более точно сформулировать эту гипотезу.

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II. Закон факторизации.

Через $(P, \leqslant)$ будем обозначать локально конечное частично-упорядоченное P с отношением частичного порядка $\leqslant$ на нем, см. [1, 2]. Посредством F будем обозначать факторизацию, т. е. разбиение, множества P, так что фактормножество $F(P)=\{f_i\}$ есть множество непересекающихся классов эквивалентности $f_i \subseteq P$ объединение которых дает все P. Причем будем предполагать, что факторизация F каким-то образом „наследует" порядок $\leqslant$, т. е. порядок $\leqslant$ на P посредством факторизации индуцирует некоторый новый порядок $\lessdot$ на фактормножестве $F(P)$, например, по правилу:

$$f_1 \lessdot f_2 \Leftrightarrow \exists p_i \in f_i \, (i=1, 2): p_1 \leqslant p_2 .$$

Итак, пусть имеется множество $(P, \leqslant)$ и его фактормножество $(F, \lessdot) = (F(P, \leqslant), \lessdot)$.

(α) *Когда множество $(F, \lessdot)$ является частично-упорядоченным?*

(β) *Как связаны между собой Мебиус-функции μ_P и μ_P?*

К сожалению в рамках ротовских алгебр инциденций $AI(P)$ и $AI(F)$ ответ на основной вопрос (β) сильно зависит от ответа на первый вопрос. Оказалось возможным преодолеть это неудобство, см. [2]; был построен класс алгебр инциденций $AIK(P)$ с ядром K, которые определены и для „плохих" порядков, например, при отсуствии транзитивности. Стало быть резонно говорить и о наличии чисто технической, формальной связи между μ_P и μ_F.

Сейчас знание закона факторизации ограничено случаем существования между $(P \leqslant)$ и $(F, \leqslant)$ сильно алгебраических связей типа связи Галуа, см. [3].

Вот конкретная задача на эту тему: Пусть $(B(S_n), \subseteq)$ — беллиан, т.е. частично упорядоченное множество всех неупорядоченных разбиений множества $S_n = \{a_1, a_2, \ldots, a_n\}$ упорядоченное по склейке блоков. Пусть F-факторизация беллиана по размерам блоков, тогда $(F(B(S_n), \subseteq), \leqslant) = (F, \leqslant) = (P_{(n)}, \leqslant)$— есть частично упорядоченное множество партиций, т. е. неупорядоченный разбиений числа n на натуральные слагаемые. Мебиус-функция партиций неизвестна.

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III. Реализуемость валентностей.

При решении экстремальных задач о гиперграфах выявляются весьма важные численные характеристики гиперграфов — их валентности, см. [1]. Пусть $S_n = \{a_1, a_2, \ldots, a_n\}$ — неупорядоченное n-элементное множество (вершин), и пусть $C^k(S_n) = \{S \subseteq S_n : |S| = k\}$ — множество всех k-подмножеств S_n, или иначе — полный k-граф; положим

$$\mathcal{P}(S_n) = \sum_{k=0}^{n} C^k(S_n).$$

Рассматриваются гиперграфы $G = \{e_i\} \subseteq \mathcal{P}(S_n)$ и k-графы $G^k \subseteq C^k(S_n)$. Валентность $\nu(S, q; G)$ от системы вершин $S \subseteq S_n$, числа $q \in \mathbf{N} + \{0\}$ и гиперграфа $G \subseteq \mathcal{P}(S_n)$ определяется как число

$$\nu(S, q; G) = |\{e \in G : |e \cap S| = q\}|. \tag{1}$$

Ясно, что при $|S| = q = 1$ это есть обычная степень. Если есть два гиперграфа $G, F \subseteq \mathcal{P}(S_n)$, то

$$\sum_{S \in F} \nu(S, q; G) = \sum_{e \in G} \nu(e, q; F), \tag{2}$$

и в частности

$$\sum_{S_p \subseteq S_n} \nu(S_p, q; G) = \sum_j \binom{j}{q} \binom{n-j}{p-q} \nu(S_n, j; G). \tag{3}$$

Кроме того

$$\nu(S_{p,q}; G) = \sum_{i \geq 0} (-1)^i \binom{q+i}{q} \sum_{S_{q+i} \subseteq S_p} \nu(S_{q+i}, q+i; G). \tag{4}$$

Естественен вопрос о реализуемости числовых последовательностей валентностями некоторого гиперграфа.

(α) *Каким условиям должна удовлетворять числовая последовательность* $\{v_i\}\ 1\leqslant i\leqslant\binom{n}{p}$ *для того чтобы существовал гиперграф* $G\in\mathcal{L}$ *(из некоторого априорного класса гиперграфов* $\mathcal{L}$*) такой, что*

$$\{v_i\}_{1\leqslant i\leqslant\binom{n}{p}}=\{v(S_p,\ q;\ G)\}_{S_p\subseteq S_n}\ ?$$

Эрдеш и Галлаи, см. [1] в п. 1, решили эту задачу для обычных степеней $p=q=1$ и обычных графов. Здесь мы „естественным" образом выведем их ограничения на $\{v_i\}$. Из (4) при $G=G^2\subseteq C^2(S_n)$ и $q=1$ находим

$$v(S_p,\ 1;\ G^2)=\sum_{S_1\subseteq S_p}v(S_1,\ 1;\ G^2)-2\sum_{S_2\subseteq S_p}v(S_2,\ 2;\ G),$$

или

$$\sum_{S_1\subseteq S_p}v(S_1,\ 1;\ G^2)=v(S_p,\ 1;\ G^2)+2\sum_{S_2\subseteq S_p}v(S_2,\ 2;\ G^2)=$$

$$=v(S_p,\ 1;\ G^2)+2v(S_p,\ 2;\ G^2)=v(S_n-S_p,\ 1;\ G^2)+2v(S_p,\ 2;\ G^2)\leqslant$$

$$\leqslant v(S_n-S_p,\ 1;\ G^2)+p(p-1)\leqslant p(p-1)+\sum_{S_1\subseteq S_n-S_p}v(S_1,\ 1;\ G^2)\wedge p,$$

(здесь и везде далее $a\wedge p=\min\{a,\ p\}$), что и дает известное неравенство Эрдеша — Галлаи

$$\sum_{i=1}^{p}v_i\leqslant p(p-1)+\sum_{i=p+1}^{n}v_i\wedge p.$$

Условие четности суммы Σv_i следует из (3). Однако задача не решена даже для обычных степеней в гиперграфском случае. Девдней [2] имеет для этого случая рекурсивные ограничения на $\{v_i\}$. Поэтому можно поинтересоваться и более частным вопросом: действительно ли вся информация о связях между валентностями заключена в (3) и (4)? Здесь мы обратим внимание лишь на один простой факт.

ТЕОРЕМА. *Пусть* $\{v_i\}\ 1\leqslant i\leqslant\binom{n}{p}$ *— последовательность целых неотрицательных чисел. Для того чтобы последовательность* $\{v_i\}$ *реализовывалась валентностями* $\{v_i(S_p,\ 1;\ G^1)\}$ *некоторого 1-графа* $G^1\subseteq C^1(S_n)$ *необходимо и достаточно, чтобы*

$$\sum_{i=1}^{\binom{n}{p}}v_i\equiv 0\left(\mathrm{mod}\binom{n-1}{p-1}\right),\tag{5}$$

$$\{v_i\}_{1\leqslant i\leqslant\binom{n}{p}}=\sum_i C(i)\binom{m}{i}\binom{n-m}{p-i}\tag{6}$$

где $$m=\sum_{i=1}^{\binom{n}{p}}v_i\Big/\binom{n-1}{p-1}.$$

Формула (6) означает, что число o должно впоследовательности $\{v_i\}$ наличествовать ровно $\binom{m}{0}\binom{n-m}{p}$ раз, число 1-ровно $\binom{m}{1}\binom{n-m}{p-1}$ раз и т. д., подробнее обозначения см. [1]. Примечатеьно, что в этом случае существует и конструктивный критерий, именно (6) можно заменить на условие

$$\{v_i\}=\{\,|\,S_m\cap S_p\,|\,\}_{S_p\subseteq S_n,\,(S_m\subseteq S_n)},\tag{7}$$

поскольку правые части (6) и (7) в точности совпадают. В частности из (7) и немедленно следует достаточность теоремы.

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IV. Структурные константы.

В последний период своей жизни Паль Туран большое внимание уделял комбинаторно-геометрическим вопросам, вопросам качественного использования экстремальных комбинаторных задач, [1—6]. В частности, это привело к одной чисто геометрической задаче, которую мы здесь попытаемся изложить с наибольшей полнотой.

Пусть X — линейное нормированное пространство, через $\sigma_k=\{\alpha_1,\ldots,\alpha_k\}$ будем обозначать совокупности из k точек этого пространства, причем для простоты примем, что $\|\alpha_i\|=1$, $i=1,\ldots,k$, хотя это ограничение в большинстве случаев можно заменить и более слабым. Пусть $k\geqslant l\geqslant 1$ — целые числа, положим

$$\delta(l,k;X)=\min_{\sigma_k\subseteq X}\max_{\sigma_l\subseteq\sigma_k}\Big\|\sum_{\alpha\in\sigma_l}\alpha\Big\|,$$

константы эти будем называть структурными геометрическими константами. Задача заключается в их вычислении для данного пространства, или же класса пространств данного типа;

$$\underline{\delta}(l,k)=\inf_X\delta(l,k;X),\quad\overline{\delta}(l,k)=\sup_X\delta(l,k:X),$$

где inf и sup берется по всем пространствам данного типа.

Принято обосабливать тот специальный случай, когда $l=2$, это связано с тем, что в этом случае максимизация длинной диагонали параллелограмма эквивалентна минимизации его короткой диагонали, точнее, пусть

$$d(2,k;X)=\max_{\sigma_k\subseteq X}\min_{\alpha_1,\alpha_2\in\sigma_k}\|\alpha_1-\alpha_2\|,$$

так называемые упаковочные константы. Тогда если в X выполняется правило параллелограмма, то

$$\delta^2(2,k;X)+d^2(2,k;X)=4.$$

Так что задачи вычисления структурных и упаковочных констант в этом случае эквивалентны до тех пор, пока действует правило параллелограмма, при его нарушении задачи „расслаиваются".

Структурные константы можно понимать и как локальную характеристику заполнения всего пространства единичными шарами.

Имеется очень глубокая связь структурных констант с числами Турана. Пусть $n \geqslant k \geqslant l \geqslant 1$ — целые числа, пусть $T(n, k, l)$ обозначает то наименьшее m, для которого существует m-членное семейство $F=\{S_l^{(i)}\}$, $1 \leqslant i \leqslant m$ из l-подмножеств $S_l \subseteq S_n$ (множества $S_n=\{a_1, a_2, \ldots, a_n\}$ из n элементов) такое, что

$$\forall S_k \subseteq S_n \exists S_l \subseteq S_k : \{S_l\} \in F.$$

Вычисление $T(n, k, l)$ есть комбинаторная проблема Турана, см. [7]. Оказывается имеет место следующая

ТЕОРЕМА. *Пусть $n \geqslant k \geqslant l \geqslant 1$, а X — линейное нормированное пространство, тогда для всякого $\sigma_n \subseteq X$ найдется по крайней мере $T(n, k, l)$ подмножества $\sigma_l \subseteq \sigma_n$ таких, что*

$$\Big\| \sum_{\alpha \in \sigma_l} \alpha \Big\| \geqslant \delta(l, k; X).$$

ДОКАЗАТЕЛЬСТВО. На множество точек σ_n как на вершинах построим l-граф $G^l \subseteq C^l(\sigma_n)$ по правилу: l точек $\sigma_l \subseteq \sigma_n$ считаем за одно l-ребро тогда и только тогда, когда $\| \sum_{\alpha \in \sigma_l} \alpha \| \geqslant \delta(l, k: X)$. Рассмотрим построенный таким образом l-граф G^l, он содержит по крайней мере $T(n, k, l)$ ребер, поскольку в противном случае, согласно определению $T(n, k, l)$

$$\exists \sigma_k^* \subseteq \sigma_n : \forall \sigma_l \subseteq \sigma_k^* \Big\| \sum_{\alpha \in \sigma_l} \alpha \Big\| < \delta(l, k; X),$$

или, что одно и тоже,

$$\exists \sigma_k^* \subseteq \sigma_n : \max_{\sigma_l \subseteq \sigma_k^*} \Big\| \sum_{\alpha \in \sigma_l} \alpha \Big\| < \delta(l, k; X),$$

но тогда

$$\min_{\sigma_k \subseteq X} \max_{\sigma_l \subseteq \sigma_k} \Big\| \sum_{\alpha \in \sigma_l} \alpha \Big\| \leqslant \max_{\sigma_l \subseteq \sigma_k^*} \Big\| \sum_{\alpha \in \sigma_l} \alpha \Big\| < \delta(l, k; X),$$

стало быть

$$\min_{\sigma_k \subseteq X} \max_{\sigma_l \subseteq \sigma_k} \Big\| \sum_{\alpha \in \sigma_l} \alpha \Big\| < \delta(l, k; X),$$

но это противоречит определению $\delta(l, k; X)$. ч т. д.

Некоторые частные случаи этой теоремы были известны и раньше. Но именно данная общность позволила перенести приложения в банахово пространство. Приведем один результат который принадлежит В. Арестову и В. Бердышеву, (приводится с согласия авторов).

ТЕОРЕМА. *Пусть B банахово пространство, тогда* $\underline{\delta}(2, 3) = \inf_B \delta(2, 3; B) = 2/3$, *где* inf *берется по всем банаховым пространствам.*

ДОКАЗАТЕЛЬСТВО. Покажем сперва, что для всякого банахова пространства $\delta(2, 3; B) \geqslant 2/3$; положим $h = \max_{\sigma_2 \subseteq \sigma_3} \| \sum_{\alpha \in \sigma_2} \alpha \|$, и пусть $\sigma_3 = \{\alpha_1, \alpha_2, \alpha_3\}$. Тогда если $\alpha_1 + \alpha_2 = z_1$, $\alpha_1 + \alpha_3 = z_2$, $\alpha_2 + \alpha_3 = z_3$, то $\|z_i\| \leqslant h$ в силу определения h, но тогда $\alpha_1 = \frac{1}{2}(z_1 + z_2 - z_3)$, значит

$$1 = \|\alpha_1\| \leqslant \frac{1}{2}(\|z_1\| + \|z_2\| + \|z_3\|) \leqslant \frac{3}{2} h,$$

следовательно

$$\forall \sigma_3 \subseteq B \max_{\sigma_2 \subseteq \sigma_3} \| \sum_{\alpha \in \sigma_2} \alpha \| = h \geqslant 2/3,$$

значит $\delta(2, 3; B) \geqslant 2/3$.

Докажем теперь обратное неравенство, расмотрим для этого трехмерное пространство l_1 и три вектора из него

$$\alpha_1 = \{1/3, 1/3, 1/3\}, \alpha_2 = \{-1/3, -1/3, 1/3\}, \alpha_3 = \{1/3, -1/3, -1/3\},$$

поскольку $\|x + y\|_{l_1} = \sum_i |x_i + y_i|$, то легко видеть что в данном случае $\|\alpha_1 + \alpha_2\| = \|\alpha_2 + \alpha_3\| = \|\alpha_2 + \alpha_3\| = 2/3$. и т. д.

Аналогичный результат для случая гильбертова пространства был получен Д. Катоной [3], который показал, что $\delta(2; 3; H) = 1$. Это в частности позволяет сравнить вероятностные приложения последних двух теорем в случае гильбертова

$$\mathcal{P}\{\|\xi + \eta\| \geqslant x\} \geqslant 1/2 \mathcal{P}\{\|\xi\| \geqslant x\}^2$$

и банахова

$$\mathcal{P}\left\{\|\xi + \eta\| \geqslant \frac{2}{3} x\right\} \geqslant \frac{1}{2} \mathcal{P}\{\|\xi\| \geqslant x\}^2$$

пространств; здесь ξ и η — независимые и одинаково распределенные в X случайные векторы.

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V. Структурно-векторные покрытия.

Пусть $\mathcal{A}=\{a_1, a_2, \ldots, a_n, \ldots\}$ — некоторое (конечное) множество, а $\leqslant$ — бинарное отношение на нем. Пусть V_t — обозначает множество всех t — мерных векторов $\bar{v}=(v_1, v_2, \ldots, v_t)$ таких, что $v_i \in \mathcal{A}$, $i=1, 2, \ldots t$. Введем на этом множестве векторов V_t бинарное отношение $\leqslant$ по правилу

$$\forall \bar{u}, \bar{v} \in V_t \quad \bar{u} \leqslant \bar{v} \Leftrightarrow \bar{u}_i \leqslant v_i \forall_i, \ 1 \leqslant i \leqslant t.$$

Рассматривается проблема покрытий, общая постановка которой такова: Пусть $Q, P \subseteq V_t$, каково то наименьшее $Q' \subseteq Q$, для которого

$$\forall \bar{p} \in P \exists \bar{q} \in Q' : \bar{q} \leqslant \bar{p}.$$

Принято, однако, её формулировать для некоторых специальных подмножеств множества V_t, именно, если на V_t ввести некоторое отношение эквивалентности, порождающее разбиение V_t на непересекающиеся классы эквивалентности $V_t^{(i)}$, так что $\sum\limits_i V_t^{(i)} \equiv V_t$, то в качестве Q и P обычно рассматривают $V_t^{(l)}$ и $V_t^{(k)}$ — какие-то из этих классов. В ряде случаев принадлежности вектора классу эквивалентности удается идентифицировать со значениями некоторой „весовой" функции. Рассмотрим несколько конкретных задач.

Пусть $(\mathcal{A}, \leqslant) = (\mathcal{P}(S_n), \subseteq)$ есть булеан (или n-мерный гиперкуб), так что любая компонента v_i это есть некоторое подмножество $v_i \subseteq S_n$ множества S_n из n элементов; запись $|v_i|$ обозначает число членов этого подмножества, так что $0 \leqslant |v_i| \leqslant n$. Рассмотрим несколько способов за дания отношения эквивалентности на множестве векторов V_t для этого случая

$$\forall \bar{u}, \bar{v} \in V_t \bar{u} \sim \bar{v} \Leftrightarrow (|\bar{u}_1|, |\mathcal{U}_2|, \ldots, |\mathcal{U}_t|) = (|v_1|, |v_2|, \ldots, |v_t|), \qquad (\sim)$$

в этом случае всякий класс эквивалентности однозначно определяется „весовым" вектором $\bar{r}=(r_1, r_2, \ldots r_t)$ где $0 \leqslant r_i \leqslant n$, $i=1, 2, \ldots, t$. Пусть $\bar{k}$ и $\bar{l}$ — два таких вектора, причем $k_i \geqslant l_i$, $i=1, 2, \ldots, t$, вопрос заключается в нахождении наименьшего покрытия класса $V_t^{(\bar{k})}$ классом $V_t^{(\bar{l})}$, в этом случае очевидна следующая

ТЕОРЕМА. *Число векторов из $V_t^{(\bar{l})}$ в наименьшем покрытии класса $V_t^{(\bar{k})}$ классом $V_t^{(\bar{l})}$ равно*

$$\prod_{i=1}^{t} T(n, k_i, l_i),$$

где T число Турана.

Пусть G_t-обозначает некоторую группу t-подстановок, положим тогда

$$\forall \bar{u}, \bar{v} \in V_t \bar{u} \approx \bar{v} \Leftrightarrow \exists \sigma \in G_t : \sigma(|\bar{u}_1|, \ldots, |\mathcal{U}_t|) = (|v_1|, \ldots, |v_t|), \qquad (\approx)$$

так что эквивалентность $(\sim)$ отвечает тому случаю, когда G_t состоит только из тождественной подстановки. Если G_t-симметрическая группа всех t-подстановок, то „весовой" функцией служат неупорядоченные системы состоящие из t чисел.

И в этом часном случае задача не решена.

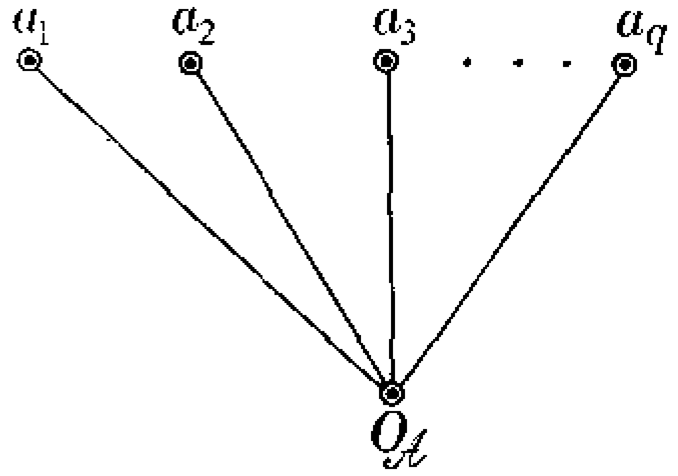
Пусть теперь отношение эквивалентости задается по правилу:

$$\forall \bar{u}, \bar{v} \in V_t \bar{u} \approxeq \bar{v} \Leftrightarrow \sum_{i=1}^{t} |\bar{u}_i| = \sum_{i=1}^{t} |v_i| \qquad (\approxeq)$$

Здесь всякий класс характеризуется одним „весовым" числом; естественно распространить этот случай на равенство каких-то функционалов определенных на V_t.

Примером иного $(\mathcal{A}, \leqslant)$ может служить линейно-упорядоченное множество, в этом случае всякий вектор $\bar{v} \in V_t$ можно интерпретировать как некоторое мультимножество, в котором i-ый элемент повторен v_i раз. Невелик прогресс и в этом случае, см [1].

Наконец, геометрические интерпретации приводят к исследованию случая когда $(\mathcal{A}, \leqslant)$ имеет вид



Приведенные выше примеры задаваемых на V_t отношений эквивалентности имеют смысл и для всякого $(\mathcal{A}, \leqslant)$ в случае когда $\leqslant$ частичный порядок, поскольку всякое частично упорядоченное множество вложимо в некоторый гиперкуб.

Имеется несколько конкретных реализаций этой проблемы, так если $t = 1$, то имеем проблему покрытий в упрядоченном множестве $(\mathcal{A}, \leqslant)$, если при этом $(\mathcal{A}, \leqslant) = (\mathcal{P}(S_n), \subseteq)$, а экивалентность задается любым из приведенных выше способов, то получаем проблему Турана; если же $(\mathcal{A}, \leqslant) = (\mathcal{P}(S_1), \subseteq)$ $t = n$, то для случая $(\approx)$ где G_t-симметрическая группа или для случая $(\approxeq)$ опять-таки получаем проблему Турана. Здесь я выражаю свою благодарность Н. Н. Кузюрину ознакомившему меня с некоторыми специальными случаями этой проблематики.

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MATHEMATICS AND PHILOSOPHY

Panel discussion*

Dedicated to Professor Đuro KUREPA on the Occasion of His 70th Birthday

Speech by Kajetan ŠEPER, Zagreb

Ladies and Gentlemen,

In the first place I wish to thank the Organizing Committee of the Symposium for having accepted our proposal to hold this panel discussion.

This discussion being dedicated to Professor Đuro Kurepa on the occasion of his 70th birthday, I am taking this opportunity to say a few words about Professor Kurepa. Please excuse me for the digressions I shall make.

When I was attending the high school at Osijek, somewhere in 1951 or 1952, I came accross Professor Kurepa's work "Teorija skupova", a first text-book on sets in our country. By that time I had read the well-known Moritz Cantor's "Vorlesungen über Geschichte der Mathematik", and a bit of philosophical logic and ordinary mathematics which I found in our libraries. No wonder that the sets were a refreshment for me. Even now I remember the footnote of the text on the null set and the all set. At that time the theory was attractive to me. However, I have never been fully satisfied with it: at the beginning I thought I did not understand what the theory was about, and later on I realized that I had to *accept* the theory in order to be able to *understand* what it was about.

As an undergraduate at the Department of Mathematics of the Faculty of Natural Sciences and Mathematics of the University of Zagreb, I met Professor Kurepa personally, in 1953 or 1954, studied with him and passed through a number of courses and seminars. Mathematical logic did not exist in Zagreb at all, neither did any foundational studies, with the exception of the traditional course in the foundations of geometry, but Professor Kurepa announced a list of various themes, among them the propositional calculus, the predicate calculus, axiomatics of real numbers, and the like. That was crucial for the whole further development of mathematical logic and foundations of mathematics in Zagreb, in Croatia, and perhaps in Yugoslavia, too.

* This panel discussion was organized by the Zagreb section of the Seminar for constructive mathematics and model theory Zagreb—Beograd (of the Mathematical Department of the Faculty of Natural Sciences and Mathematics, Zagreb, and the Mathematical Institute, Beograd).

Once I tried to sketch Prof. Kurepa's influence concerning mathematical logic in Zagreb. Regardless of the interinfluential laterals, I obtained a four-rank tree. I called it Kurepa "small tree". Of course, this tree should be enlarged by taking into the account his influence concerning other mathematical theories — set theory, topology etc., together with his influence in other or bigger regions — Belgrade, Yugoslavia etc.

It is not my intention to give here any account of Professor Kurepa's work, his activity, influence and importance — although I should perhaps apologize for that — but to say — and I feel obliged to do so — that Professor Kurepa has not been just a professional mathematician, a teacher and a pedagogue, but a real scientist and a philosopher, a humanist, and a human in the best sense of the word. He was the father, the originator and the pioneer of mathematical logic and foundational studies in Croatia, and of modern mathematical theories in Croatia and Yugoslavia. Generally speaking, he was catalizer, and initiator, a bringer and a transferer of knowledge.

As a student of his, and an admirer of his personality, with all of its virtues and individualities, qualities and peculiarities, temperament and character, I fullheartedly thank Professor Kurepa, in my own name and in the name of all of my colleagues, for everything he has done both as a scientist and as a man. Happy anniversary and many happy returns of the day!

CONSTRUCTIVE PROCESSES IN MATHEMATICS

Mathlmatical and Philosophical Aspect

SOME THESES CONCERNING THE DEVELOPMENT OF MATHEMATICS

Kajetan ŠEPER, Zagreb

1. Our **introductory general thesis** is that *constructive mathematics, in a broad sense, is a measure for determining the value of mathematics as a positive science in all epochs, and especially at present.* In our current opinion, the development of mathematics can be compared with a two-side balance: one side carries the practical, numerical, computer-computational, concrete, constructive mathematics, and the other — the theoretic, conceptual, abstract, non-constructive, platonistic mathematics. Although this balance has never been balanced, one yet clearly observes in each epoch an overloading of one of its sides. Its balancing by the new, the progressive and the necessary is the golden transition period; this period is the most valuable time interval in the historical development of mathematics both for its fruits and for its influence.

1. At the very beginning of civilization the scales did not actually exist. All mathematics was concrete, practical, inductive; in other words, if we use the comparison mentioned above, the constructive side of the balance overweighed.

2. It was the scientific and philosophical genius of the ancient Greeks that created the balance, i.e. the other side, the abstract, the theoretic, the deductive one.

3. This theoretic side already overweighed at the time of the ancient Greeks, and such a state was transmitted to and prevailed through the Middle Ages.

4. The European spirit, commercial and early-industrial, rebalanced the scales, and

5. raised the overloaded side by putting heavy weights onto the neglected side — the infinitesimal calculus has by no means been called a calculus at random, and mathematics and natural sciences became undiscernible.

6. The europeanized Greek genius again loaded the research with axioms and deductions, the actual infinity, and the absolute, and

7. created the Cantorian intellecto-universe. Thus the abstract theoretic side prevailed and its closed empire of ideas got its name: PLATONISM.

8. The force of history, however, is stronger than the ideas; science, and production, and society develop and so does the need for an equilibrium and also the requirement for a new open system, for a constructive universe, for CONSTRUCTIVISM.

9. Perspectives:

a) We conjecture "Periodicity". It should be mentioned that this conjecture concerns the immediate future; otherwise, we do not conjecture anything.

b_1) Goodman and Myhill conjecture "Compatibility and Interaction".

Cf. [1], p.83:

"One can distinguish two traditions in the study of the foundations of mathematics. The non-constructive tradition, represented today by set theory and category theory, . . . (and) the constructive tradition (which) is represented today by intuitionism and much of proof theory. These two tendencies in foundational studies are not incompatible. Rather, it is the interaction between them that is likely to lead to the most fruitful development of foundations as a whole. Current examples include the use of infinite proof-figures in proof theory and the use of elementary, rather than higher order, theories in studying categories. Our subject here is a recent development in constructivity which promises to open new avenues for such interaction."

Cf. [1], p. 94:

"Thus one may hope that the ultimate bastion of classical idealism, set theory, can be made to give way piecemeal to the insights which, in particular cases, it gives into the structure of its own objects."

b_2) Trostnikov conjectures "Quantitative Gnoseology".

Cf. [2], p. 252:

"Возможно, в будущем произойдет следующее: метаматематика вступит в более тесную, чем ныне, связ с определенними разделами материалистической философии и психологии и так образуется область, которую можно назвать "*количественная гносеология*", предметом которой будет проблема согласования различных "языков (каждый из которых опирается на свою специфическую структуру сознания), с помощю которых му конструируем, верифицируем и переконструируем объекты нашего" научного сознания, все полнее и глубже проникая в тайны материи."

2. From this observation it seems to us that *balancing is historically necessary in order for mathematics to be able to enter a new epoch*, and that *preponderance of one scale is a characteristic feature of each epoch.* Therefore it seems to us *that we*

are living in the transition period of balancing by means of historically heavier weights of the constructive, the numerical, the discrete ,the finite. This is our **first conclusive general thesis.**

3. *The process perceived clearly parallels the socio-economic systems in the evolution from the primitive society,* through slavery, feudalism, and early capitalism, *up to the contemporary systems* (highly developed capitalism and socialism). This correspodence suggests to us and substantiates our opinion that *Constructive Mathematics is a Socio-Economo-Political Problem, and not just a Philosophical One,* as it is widely accepted, spread and debated. This is our **second conclusive general thesis.**

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[Discussed by N.A. Shanin (Leningrad), S.R. Zervos (Athens), M. Krasner (Paris), Th. Stavropoulos (Athens), S. Panayiotis (Athens), J. Pelant (Prague), D. Rosenzweig (Zagreb).]

CONTRIBUTION TO THE DISCUSSION

M. KRASNER, Paris

KRASNER: Prof. Shanin said that the constructivism, in caracterizing certain mathematical objects by means of some information (which he compared to the macroscopical information of quantum mechanics), is considering only such objects and reasons only in passing from information to information. Even in supposing such "informational" point of view admitted, I don't believe that the information used by constructivists is the only possible and that the constructivistic way of using it is exhausting.

From another side, Prof. Shanin believes that constructive mathematical objects are more able to imitate (or "model") that of experimental sciences, that do that of classical mathematics and he considers this circumstance as a decisive advantage of the constructivistic point of view. If even it was so, I think that the mathematics, as any other adult science, has its own internal logic, and the existence and the interest if its objects are not determined by their ability of imitation of objects of other sciences or of material world. In particular, many highly interesting objects of algebra and of number theory have, until now, no relations with that of experimental or human sciences, even when they can be described constructivistically.

Let us remind the discussion between Borel, Hadamard and Lebesgue. It is clear that the constructivism is a development (and accomplishment) of Borel's ideas, and that usual naive and axiomatic set theory as basis of all classical mathematics derives from Hadamard's point of view (with some Hilbertian aftertaste). But, there exists a point of view inspired by Lebesgue's ideas, the "definitionism", where only the objects having a definition exist (clearly, the word "definition" has not so a narrow sense as for Lebesgue: in particular, there may exist definitionistic systems, where the definitions may not be finite)' The definitionism uses a wider information than constructivism, and in a wider way, although constructivistic objects are among the definitionistic ones, and the "relative'.

study of only constructivistic objects has (rather mathematical, than logical or philosophica) interest for a definitionist (and ,even, for a platonistic or axiomatic mathematician).

[In translating in Russian, I added: "Prof. Shanin gave the impression, by what he said, that every problem about constructivistic objects is soluble. That is certainly wrong."].

SHANIN: But, when we prove the existence of a solution of some problems, this proof gives, in the same time, a construction of some such solutions.

KRASNER: Yes, but there are constructivistically formulable problems, for which the constructivism, exactly as the ordinary mathematics, can give no answer, for example that of the validity of the Fermat's last theorem.

So, I recognize the interest of the constructivistic point of view, but I consider it as too narrow for me.

SHANIN: How too narrow? And all the hierarchy of the constructivistic types? For every part of Analysis a constructivistic analogue could be built.

KRASNER: For example, in constructivism do not exist the property of being an object or, also, properties opposite in absolute sense (I must say that they, also, don't really exist in the naive and in ZF-axiomatic set theory).

SHANIN: The arguments of Prof. Krasner about the autonomy of mathematics in respect to other sciences are a typical example of what happens when a constructivist and a classical mathematician meet ..... etc, ...

KRASNER: But I am not a classical mathematician from point of view of Foundations.

DIOPHANTINE EQUATIONS AND CONSISTENCY OF FORMAL THEORIES*

Mirko MIHALJINEC, Zagreb

For any recursively-enumerably axiomatizable first order formal theory, the set of Gödel numbers of its theorems is recursively enumerable. Of this kind are for instance the theory P (formalized Peano's arithmetics, see [1], pp.43, 300—301, it might be better to speak about Peano-arithmetics because the axiom of induction is expressed for formulas with one free variable in the language of the signature $\langle O, S, +, \cdot, < \rangle$), the theory S (formalized second order arithmetics, [1], pp.334—335), the theory ZFC (formalized set theory with the axiom of choice, [2], pp.507—508). If f is a recursive function which enumerates such a set of Gödel numbers, and if a is the Gödel number of a false formula (e.g. $0=s(0)$ in the language of P), then the consistency of the theory can be expressed in the following way: $\neg(\exists x)f(x)=a$. As the set of values of f (range, codomain of f) is recursively enumerable, according to the Matijasevič's theorem it is diophantine (see [4]), and there is a polynomial p (see [8]) in 14 variables with integral coefficients such that consistency of the theory in question is equivalent to the formula: $\neg(\exists x_1)\ldots(\exists x_{13})\, p(a, x_1, \ldots, x_{13})=0$ (the coefficients of that polynomial can be effectively calculated as soon as the theory is specified, although it is practically impossible because of the size of the numbers involved). Even more, in order to check whether a formula of the language of such a theory is a theorem, one should calculate its Gödel number b and check if the equation $p(b, x_1, \ldots, x_{13})=0$ has a solution in nonnegative integers (although the corresponding algorithm, for instance for above mentioned theories, does not exist — that is connected with the

* Translated from the Serbo-Croatian by D. Rosenweig.

negative answer to the Hilbert's tenth problem). An important consequence is that provability of a statement can be reduced to solvability of a completely specified diophantine equation. From the viewpoint that the theory ZFC contains (almost) all of contemporary mathematics, it might be said that all mathematical problems can be reduced to solvability of corresponding diophantine equations. A word of caution is however necessary in this place, as such a view about ZFC certainly is exaggerated, because a formal system, however rich, cannot contain all of mathematics. Clear interpretability of a system is as important as its consistency.

If we compare the Gödel's second theorem about unprovability of consistency of a formal system P within the system itself ([1], pp.307—315) with the Gentzen's proof (which is finitary-constructive) of consistency of P ([1], pp.315—327), we can see that it has undoubtfully been proved in an arithmetically clear way that the equation $p(a, x_1, \ldots, x_{13})=0$ has no solution in nonnegative integers $x_1, \ldots, x_{13}$ and that this statement is not provable in the system P.

It is hence an enrichment of Peano-arithmetics and the theory of diophantine equations. Although consistency of P can be proved in the system S ([1], pp.338—339) that proof is (unlike the Getzen's one) not finitary-constructive, as such a proof for consistency of S is not known ([1], p.342) even after the results of Spector and Tait (see [7], p.7), and the possibility of such a proof is highly doubtful. This certainly holds for ZFC too, so we cannot be convinced about unsolvability of the diophantine equation derived from the statement "ZFC is consistent".

Solvability of diophantine equations has been object of research for a long time ([9], [10], [6], [3], pp.176—195, [14]). The methods of contemporary algebraic geometry and model theory do enrich our knowlegde about diophantine equations ([11], [12], [13], [15]). The question is, are the results so obtained provable as theorems in P, are there among them some theorems which are provable in a finitary-constructive way and which are not theorems of P? Is there a statement about unsolvability of some diophantine equation which is provable in a finitary-constructive way and which is not a theorem of S (may be even not of ZFC)?

"The study of diophantine equations, that is the solution of equations in integers, or, alternatively, in rationals, is as old as mathematics itself. It has exercised a fascination throughout the centuries and the number of isolated results is immense [as it is witnessed, for example, by Dickson's three tomes]. Some more-or-less general techniques and theories have been developed and there are some grandiose conjectures, but the body of knowledge is less systematic than that in more recently established branches of mathematics because here we are concerned with the most basic and intractable mathematical material: the rational integers." ([13], pp.193—194).

"I wish to note expressly that Theorem XI (and the corresponding results for M and A) do not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that c a n n o t be expressed in the formalism of P (or M or A)." (K. Gödel, [5], p.106.)

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[Discussed by A.N. Šanin (Leningrad), S.P. Zervos (Athens), Ž. Mijajlović (Belgrade).]

ON MARKOV'S PRINCIPLE*

N.A. SHANIN, Leningrad

Professor Shanin kindly conformed to the request of the organizer of the panel discussion to give a special lecture on Markov's principle, to speak especially in behalf of it, and to present the related point of view of those constructivists, primarily of Markov and of Shanin himself, who express their opinion about the *consistency with the idealizations and intuitive notions accepted in constructive mathematics* of that principle.

During the discussion we brought out our objections to the application and the plausibility of the principle in constructive mathematics.

At the end of the discussion we came to a *terminological* agreement only: according to the term 'constructive' in (the algorithmic foundations of) 'constructive mathematics' one has to distinguish *at least* two levels of abstraction and security. Markov's principle is concerned with the higher level i.e. with constructive mathematics in a *wide* (or *wider*) *sense*.

[Discussed by K. Šeper (Zagreb), D. Rosenzweig (Zagreb), M. Mihaljinec (Zagreb).]

* Summarized by K. Šeper

1. Contribution to the Discussion of Markov's Principle.

Kajetan ŠEPER, Zagreb

Constructive mathematics (CM) is the science of *constructive processes* (CP's) and *constructive objects* (CO's) — the results of such processes in case the processes terminate, and of our abilities of realizing these processes. More precisely, CP's are defined in terms of *algorithms* of various kind and CO's in terms of *words* in specific alphabets. The *abstraction of potential realizability* (APR), and the related idea of *potential infinity* based on it, is a characteristic feature of CM. Constructive mathematical logic is formed on the basis of CM, and depends upon CM; it models one's intuitive constructive thinking formally by means of syntactical and semantical systems. Our discussion is carried through on an intuitive ground, and is concerned with the phrase 'the process of applying an algorithm to an admissible input value *terminates*', or synonimously, 'the algorithmic process terminates in a *finite* number of steps'. Our initial attitude is that this phrase should be 'immediately clear' by our constructive point of view; in other words, that the phrase means that "we are able to indicate, *actually* or *potentially under* APR, the *number of steps* needed for terminating the application *of the process*, or equivalently, *one of its upper bounds*'. That number will be called here the *halting characteristic* of the process. During the discussion *Markov's principle* (MP) will be mentioned frequently.

We are discussing here the following problem: *Is the acceptance and use of MP in CM legitimate i.e. consistent with the idealizations and intuitive notions accepted in CM, and with APR, especially*?

We have objections to the acceptance and use of MP in CM. One applies it only when one does not have such a good insight in the algorithmic process under consideration that allows him to infer termination of the process, or, we hope seldom, if one does not care about it. In such a case, however, one is very often able to infer 'the impossibility of nontermination of the process' ('A') i.e. the impossibility of continuation of the development of the process after each step. Then, by use of MP, one is allowed to infer 'termination of the process' ('B*'), and, as a consequence, to treat the result of the process as being a CO. Of course, in order to find the result actually one is allowed to develop the process as long as he wants. Such a procedure is just suggested by the constructivists who accept MP and who believe that the process will finally stop. If one succeeds to compute the result, the application of MP becomes superfluous. Otherwise, generally one is in essentially the same position as if he did not have the information A — it does not indicate anything about termination, and it is left to one's decision of how long will he compute. So, we consider B* as not established by A, but rather as an *open* problem.

MARKOV himself, in his papers written before 1967, clarifies the principle by saying that he does not see any reason of *knowing* in advance exactly the halting characteristic of the process as being a necessary condition for asserting termination of the process. As a matter of fact, a number of great theorems in all branches of CM are obtained by using MP. In ŠANIN's well-known papers on constructive mathematical logic, MP is accepted and widely incorporated in the whole body of his semantical anlysis of the propositions of current CM i.e. CM+MP. (We wish to notice here that we got a feeling, after reading MARKOV's papers published after 1967, that even MARKOV would not treat the principle in such a generality any more.)

In our opinion, however, the acceptance of MP alters the intuitive notions of our constructive universe, and the entire motivation for CM, radically. The notion of 'finiteness' (*effective*, static, determined, bounded, actual or obtaining possibly under APR), which is essential and primary in our understanding of CP's, CO's and the idea of potential infinity, becomes altered into another weakened notion of '*floating* finiteness' (noneffective, dynamic, nondetermined, nonbounded, obtaining via potential infinity), or '*potential noninfinity*' i.e. *non*-' potential infinity'. So, the notions of 'termination' and 'CO' become altered, too; they become a kind of '*floating* termination' and '*floating* CO', respectively. The difference between the two notions of 'finiteness', or 'termination', or 'CO', the one being 'effective' and the other 'floating', can be characterized in the following way. The former is persistently *actual* or *potential under* APR, and so *based on* APR *directly*, and is defined by an *existential quantifier* (which in turn has to be interpreted by means of some contensive arguments), The latter, however, is unsteadily *procedure-like*, and *based on* the idea of *potential infinity*, and so *based on* APR, too, but *indirectly*, and is defined by a *negated universal quantifier*. In CM the primary notion is that of effective finiteness (effective termination, effective CO), directly established under APR, and that of potential infinity being secondary and defined by it. In CM+MP the primary notion is that of potential infinity, based on APR, and that of floating finiteness (floating termination, floating CO) being secondary and defined by it.

We do not say that MP is inconsistent with APR and the like. In any way, APR does not imply the idea of floating finiteness (floating termination, floating CO). We just say that this idea is based on the idea of potential infinity, and so on APR indirectly.

We do not say that MP is an *additional idealization* to APR and the like, either. (Cf. also ROSENZWEIG's discussion in this symposium.) Although we could say so, *if we have in mind our understanding of constructiveness* i.e. the essential and primary notions of finiteness, termination, CO etc., and, in addition, if we have in mind that, if we are working in CM+MP, we indeed *abstract from our actual knowing* of termination and argue as if such knowing is present, we yet avoid to say so. By saying that the acceptance of MP introduces an additional idealization into the body of CM, we could not abstain from saying that the acceptance *extends* the limited computational and combinatorial power of *homo sapiens from outside*, and, consequently, — we are firmly convinced — that it *extends* the class of constructively true propositions, too, and so, that it is *not consistent* with APR and the like, and that it *contradicts* to the essential and primary constructiveness in its whole, as well.

Exactly in the same sense as BROUWER abstracts from *laws* determining the components of sequences one after the other, and introduces in this way so-called 'choice sequences' (or synonimously, 'infinitely proceeding sequences'), so does MARKOV abstract from *halting characteristics* determining terminations and the corresponding results of algorithmic processes, and introduces in this way what we are calling here, 'floating termination' and 'floating CO'.

According to HEYTING ([1], p.71), "the only essential feature" of the components of a choice sequence "is that it does not matter by which means they are determined one after the other", and so, choice sequences "are not constructible objects in the strict sense".

How could the components of a sequence (termination of an algorithmic process and the corresponding result) be *determined*, if not by a law (halting characteristic)?

How could we *know* they are *determined*, if not by *knowing* a law (halting characteristic, respectively)?

We do say, however, that CM+MP, in relation to CM, deals with another weaker conceptual subject, and that it does not treat the fundamental constructive notions, such as finiteness, termination, CO etc., *adequately*. We consider CM+MP as the science of *floating CO's*. In such a theory CO's get mixed among all the weaker and weaker floating CO's. We do not feel any scientific, or philosophical, or practical reason to accept such a weak form of CO's, and in the same time not to accept for instance infinitely proceeding sequences or the like. We do not feel any need for a 'closure' of 'all' the — wider and wider classes of — *total* functions i.e. *total* algorithms, which in a definite sense MP implies. (We mean by that, that MP eliminates the known troubles with the existential quantifier in the definition of these functions i.e. the *circulus vitiosus* in the definition. See also *addenda* below.) We consider MP just as an ***approximate guiding principle, heuristic*** in nature, and its application as an ***ephemeral quasi-constructive guiding argumentation.*** Theoretically the subject is much more interesting, and the relation of CM to CM+MP should be examined formally in more detail. Nevertheless, we prefer any modeling of the notions of the wider and weaker theory CM+MP in the frame of the (narrower and stronger) theory CM i.e. in the frame of the strict CM. For instance, instead of having the notions of total functions, decidable set etc. in CM+MP, we prefer to manage them in CM by the notions of *weakly total* function, *weakly decidable* set etc., respectively. In fact, we feel and believe that *ultraconstructivistic tendency* in one or another form — we are considering complexity theory as one of the various aspects of the tendency — will play a role *sine qua non* in the development of mathematics in the future.

Addenda. Now, we wish to give here some quotations and comments.

After HEYTING's and PÉTER's clarification of the point (see the quotations given below), it is generally supposed that everybody working in this area is familiar with what the problem is about.

"There ought to be distinguished between

a) theories of the constructible;
b) constructive theories." ([1], p.69.)

"The notion of a constructible object must be a primitive notion in this sense that must be clear what it means that a given operation is the construction of a certain object. It has been explained by Miss Péter in her conference in this colloquium that any attempt to define the notion of a constructive theory leads to a vicious circle, because the definition always contains an existential quantifier, which in its turn must be interpreted constructively." ([1], p.70.)

"Als eine Zusammenfassung und Verallgemeinerung der durch diesen speziellen Rekursionsarten definierten Funktionen ist der HERBRAND-GÖDEL-KLEENEsche Begriff der allgemein-rekursiven Funktion entstanden [4]. Das ist ein sehr nützlicher Begriff, da er die einheitliche Behandlung sämtlicher speziellen rekursiven Funktionsarten ermöglicht; bisher ist aber keine allgemein-rekursive Funktion bekannt, die für irgendeine mathematische Unetrsuchung wichtig ist, und nicht unter eine der bekannten speziellen rekursiven Funktionsarten eingereiht werden könnte. Aber der Hauptziel bei der Einführung dieses Begriffes war eben die exakte Fassung des Konstruktivitätsbegriffes. Die sogenannte Churchsche Thesis identifiziert den Begriff der berechenbaren Funktion mit diesem Begriff. Hier möchte ich nicht darauf eingehen, worüber Kalmár sprechen wird, nämlich ob tatsächlich alle berechenbaren Funktionen allgemein-rekursiv sind; ich möchte gerade die entgegengesetzte Frage aufwerfen: können die allgemein-rekursiven Funktionen sämtlich mit Recht "effektiv-berechenbar", d.h. "konstruktiv" genannt werden?

Eine allgemein-rekursive Funktion wird durch ein Gleichungssystem angegeben, wobei vorausgesetzt wird, dass es zu jeder Stelle ein endliches Berechnungsverfahren gibt, welche aus Einsetzungen von Zahlen für Variablen und Ersetzungen von Gleichem durch Gleiches besteht, und den Wert der betrachteten Funktion an der angegebenen Stelle eindeutig liefert. Nun ist aber dieses "es gibt" etwas unsicheres, wie darauf schon der sprachliche Ausdruck hinweist, und zwar in den meisten Sprachen. "Es gibt" — wer denn? "Il y a" d.h. "er hat da" — wer und wo? "There is" d.h. "da ist" — wo denn? Kleene meint, wer das in dieser Allgemeinheit nicht annimmt, mag dieses "es gibt" konstruktiv auffassen. Das ist leicht zu sagen, gerade da bisher keine echt-allgemein--rekursive Funktion bekannt ist, und so kann man nicht wissen, was mit einer solchen Einschränkung verloren geht. So werden eigentlich zwei Begriffe der allgemein-rekursiven Funktion definiert: einer mit klassisch aufgefasstem, und einer mit intuitionistisch aufgefasstem "es gibt". Es wäre interessant durch ein Beispiel zu zeigen, inwiefern der letztere Begriff enger ist, nämlich durch eine

Funktion, welche klassisch allgemein-rekursiv ist und intuitionistisch nicht; das ist aber kaum zu hoffen, da in den bisherigen Betrachtungen noch überhaupt kein Beispiel für eine allgemein — und nicht speziell-rekursive Funktion vorgekommen ist. Nun, der klassische Begriff der allgemein--rekursiven Funktion ist nicht konstruktiv, und die intuitionistische (Definition) enthält ein Circulus vitiosus: hier soll das in der Definition aufretende "es gibt" konstruktiv sein — man wollte aber gerade mit dieser Definition der Allgemein-Rekursivität die Konstruktivität exakt definieren.

Derselbe Circulus vitiosus taucht überall auf, wie man ihn auch umgehen mag." ([2], pp.227 and 228.)

"Es hat den Anschein, dass sich der Konstruktivitätsbegriff überhaupt nicht zirkelfrei erfassen lässt." ([2], p.233.)

However, even after HEYTING's and PÉTER's papers, MENDELSON argues as if he did not know what is the subject about, and moreover, he gives a misleading statement of the subject of PÉTER's discussion. We quote here sec.2 of his paper entirely.

"2. According to the precise mathematical definition, a function $f(x_1, \ldots, x_n)$ is general recursive if there exists a system of equations **E** for computing f, i.e. for any $x_1, \ldots, x_n$, there exists a computation from E of the value of $f(x_1, \ldots, x_n)$ (Kleene [5]). Both occurrences of the exitential quantifier "there exists" are meant here in the non-constructive classical sense. To this, Péter ([2], p.229) makes the following objections: (i) The existential quantifier must be interpreted constructively; otherwise, the functions defined in this way cannot be considered constructive. (ii) If the existential quatifiers are meant in the construtive sense, and if the notion of "constructive" is defined in terms of general recursive functions, then this procedure contains a vicious circle.

Both objections seem to be without foundation. " (6. I am assuming that Péter intends "constructive" to have the same meaning as "effectively computable".) In the case of (i), the general recursive functions defined using the non-constructive existential quantifiers are certainly effectively computable in the sense in which this expression is used in Church's Thesis; no bound is set in advance on the number of steps required for computing the value of an effectively computable function, and it is not demanded that the computer know in advance how many steps will be needed. In addition, for a function to be computable by a system of equations it is not necessary that human beings ever know this fact, just as it is not necessary for human beings to prove a given function continuous in order that the function be continuous. Since objection (i) is thus seen to be unjustified, there is no need to assume, as is done in (ii), that the existential quatifiers are interpreted constructively. However, there is another error in (ii); "constructive" (or "effectively computable") is not *defined* in terms of general recursive functions. Church's Thesis is not a definition; rather it states that the class of general recursive functions has the same extension as the class of effectively computable functions; and the latter class has its own independent intuitive meaning. Thus, there is no vicious circle implicit in Church's Thesis". ([3], pp.202 and 203.)

MENDELSON's objections to PÉTER's objections to the definition of general (i.e. total) recursive functions are seen immediately to be without foundation and unjustified. His discussion is carried through in the non-constructive classical sense, in another universe, in a universe of speechifying, and so it has nothing to do with PÉTER's criticism. The discussion failed to hit the point. PÉTER *intends* "constructive" to have the same meaning as "effectively computable"; it *is* demanded that the computer *knows* in advance how many steps will be needed for computing the value of an effectively computable function; and, in addition, for a function to be computable by a system of equations it *is* necessary that human beings *know* this fact, just as it *is* necessary for human beings to *prove* a given function continuous in order that the function be continuous. Otherwise, *human beings* will try to solve all these *open* problems. PÉTER's initial question is:

Can all the general (i.e. total) recursive functions properly be called "effectively computable" i.e. "constructive", ([2], p.228.) Nowhere in PÉTER's paper one can find any mentioning that Church's Thesis is a definition, or that there is a vicious circle implicit in Church's Thesis, but that it seems that the notion of constructiveness (or finite-computability, or constructive theory, or effectively computable total function) cannot be made precise by a net mathematical definition that would be free of a vicious circle ([2], p.233). It seems as MENDELSON did re-discover in his paper that the class of effectively computable functions has its own independent intuitive meaning ([3], p.203).

KLEENE explains the situation very carefully and restrainedly. We give here a quotation of a passage from the fourth paragraph of footnote 171 in his text-book.

"We have been assuming without close examination the CONVERSE OF CHURCH'S THESIS: *If a function is Turing computable (or general recursive, or λ-definable), then it is intuitively computable (or effectively calculable).* In defending this implication to an intuitionist, or to any other kind of constructivist who considers an algorithm to exist only when it is proved by his standards that it always works, we only ask him to accept the following: if the hypothesis that a function is Turing computable holds by his standards, so does the conclusion. Put thus, it is hard to see how it can be questioned. Only if one allows a nonconstructive interpretation of the hypothesis, and yet insists on a constructive interpretation of the conclusion, is the converse of Church's thesis in doubt. "([6], p.241.)

Unfortunately, KLEENE does not discuss the meaning of 'it is proved by *a constructivist's standards* that an algorithm always works (i.e. terminates)'.

Undoubtly, ŠANIN's new 1973 paper is fully influenced by HEYTING's and PÉTER's papers, or at leasts by the facts they discuss. According to ŠANIN ([7], pp.217, 218, 222, and 223), let us consider some propositions with their clarifications, and some definitions.

Let A be any alphabet, A any algorithm over the alphabet A, and P any A-word (i.e. word in A).

(C_1) [(C_9), (C_{10})] The process of applying algorithm A to P terminates [is potentially infinite, is not potentially infinite].

(C_2) [(C^-_{11})] For any A-word X, the process of applying A to X terminates [is not potentially infinite].

(C^-_{12}) For any natural number n, $\neg W_A(P, n)$.

Here $W_A(X, n)$ stands for the condition "The process of applying algorithm A to word X terminates after not more than n steps". Obviously, this condition is testable by means of an algorithm applicable to (i.e. total with respect to) every word of the form X, n.

The notion of '*total algorithm with respect to words of a certain type*' is defined in this case by (C^*_2) ([7], p.218), an obvious generalization of (C_2). The sign — in (C^-_{11}) indicates an inessential for our discussion modification of (C_{11}). (C^-_{12}) slightly differs symbolically from (C_{12}).

(C_1) [(C_2)] is said to be *true* if it has a potentially realizable contensive demonstration.

An algorithm A over the alphabet A is said to be *total* (with respect to all A-words) if proposition (C_2) is true.

(C_9) is clarified (or 'deciphered') by (C^-_{12}).

(C_{10}) and (C^-_{11}) are correspondingly clarified.

We cannot imagine any such potentially realizable contensive demonstration of (C_1) which would not indicate the halting characteristic. On the other hand, if we accept MP, as a contensively conclusive argumentation, as ŠANIN does in the paper, then we do not see why ***termination*** i.e. (C_1) is not *clarified or defined by* (C_{10}) i.e. *by* $\neg \forall n \neg W_A(P, n)$, and *totalness* i.e. ($C_2$) *by* ($C^-_{11}$) i.e. *by* $\forall X \neg \forall n \neg W_A(X, n)$, where the prefixed quantifier — connective combinations are to be interpreted contensively as usual. By the acceptance of MP, a *pure* contensive demonstration of (C_1), i.e. such that does not make use of MP, get mixed among contensive demonstrations of (C_1) that make use of it.

According to PÉTER ([2], p.228), there are indeed two notions of general (i.e. total) recursive function that depends on the interpretation of 'there is' in the definition. The one is the *classical* notion and the other the *intuitionistical* (i.e. *constructive*) notion. However, the former is not a constructive notion, and the definition of the latter contains always a vicious circle. Nowadays 'there is' is interpreted contensively, or, in other words, is considered as a primitive notion, and is not defined by a net mathematical definition; hence, there is no vicious circle in the'definition' of the constructive notion of total recursive function. If 'there is' is interpreted in the sense of general applicability of MP, one more notion of total recursive function (call it *MP — constructive*, or *floating-constructive*) is introduced that has an intermediate status between those before mentioned.

References

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2. Contribution to the Discussion of Markov's Principle

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I see Markov's principle (MP) as a way around the difficulties arising in constructive interpretation of the quantifiers occurring in the definition of a total recursive function. A constructivist mathematician could live very well just with and open hierarchy of known total functions, e. g. of Péter-recursions. If one however insists on a closed, general definition, then such a directed application of *reductio ad absurdum* is the only known way to secure it. An attempt to interprete $\forall x \exists y\ T(a, x, y)$ just like any other sentence of the same form falls into an endless loop, while leaving such an interpretation to unspecified intutive arguments makes the demarcation between constructivism and intuitionism seem quite arbitrary and unmotivated.

So I understand MP as an additional idealization, *consistent with* but certainly *not derivede from* the abstraction of potential realizability and constructive interpretation of logical connectives and quantifiers.

Results proved by means of "constructive mathematics in the narrow sense", i.e. without MP, could de described as computations with an *a priori* upper bound on computational complexity. In such a case a programmer could say: "I could compute this if only I had computing apparatus of such and such speed, "where "such and such,, means a previously known function. On the other hand, results proved in "extended constructive mathematics", i.e. by MP, represent computations with no *a priori* complexity bound. No real or imaginary programmer, however powerful a computer he had, could risk an uncontrolled run of such a program.

These arguments are of course highly theoretical, as in any presently conceivable situation only first three or four levels of the Grzegorczyk hierarchy are effectively computable (compulable in the sense of German *berechenbar*; more complex functions are effectively *rechenbar* but not *berechenbar* by humans in this time).

Nevertheless, such considerations guide me to distinguish between constructive mathematics without and with MP as different degrees of idealization, hence to try to eliminate MP where possible and to isolate results for which I don't know how to eliminate it.

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