The Mediterranean International Conference of Pure&Applied Mathematics and Related Areas (MICOPAM 2018)

Antalya, Turkey

October 26th–29th, 2018

Proceedings Book

of

MICOPAM2018

Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary

Editor

Yilmaz SIMSEK

Associate Editors

Miloljub ALBIJANIĆ, Mustafa ALKAN and Irem KUCUKOGLU

NASLOV
Proceedings Book of MICOPAM 2018

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IZDAVAČ
Školsi servis Gajić DOO, Beograd

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IZDANJE I GODINA
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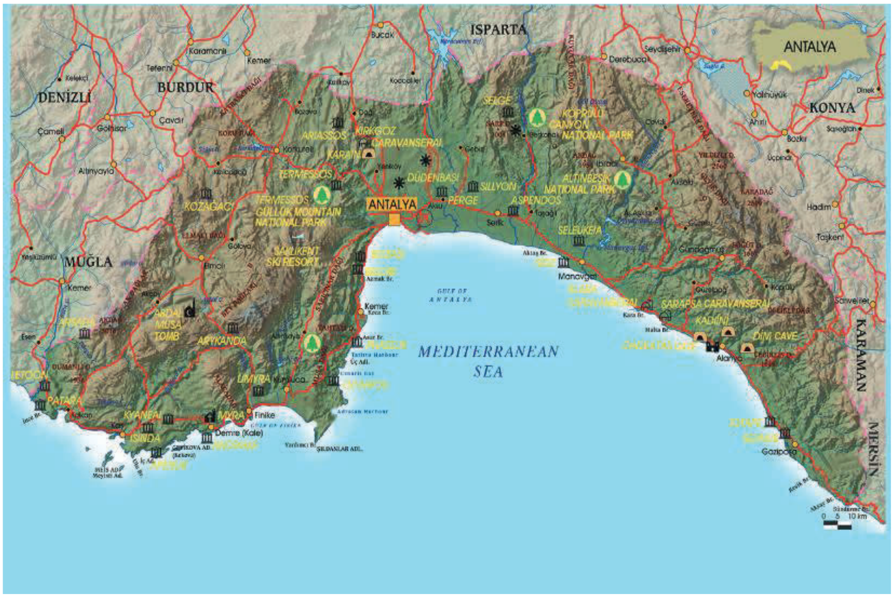
FOREWORD

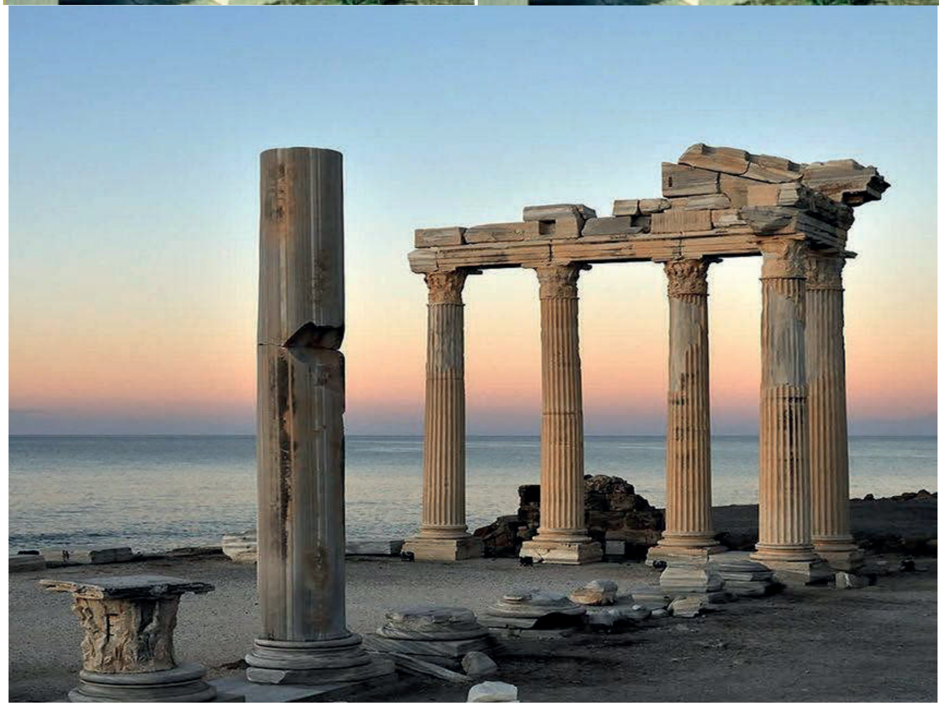
Dear distinguished participants of the Mediterranean International Conference of Pure&Applied Mathematics and Related Areas (MICOPAM 2018) held in Antalya, Turkey, on October 26–29, 2018. On behalf of the Scientific and Organizing Committees, Welcome to Turkey's pretty and historical Mediterranean resort town Antalya, which hosts our conference MICOPAM2018 conference which dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary.

By the way Antalya, which is one of our historical cities, has been a source of inspiration for many empires and civilizations. I hope you will visit some part of this pretty and historical city of Turkey. In order to show some of the historical sites of this beautiful city, we have included an excursion program to our conference.

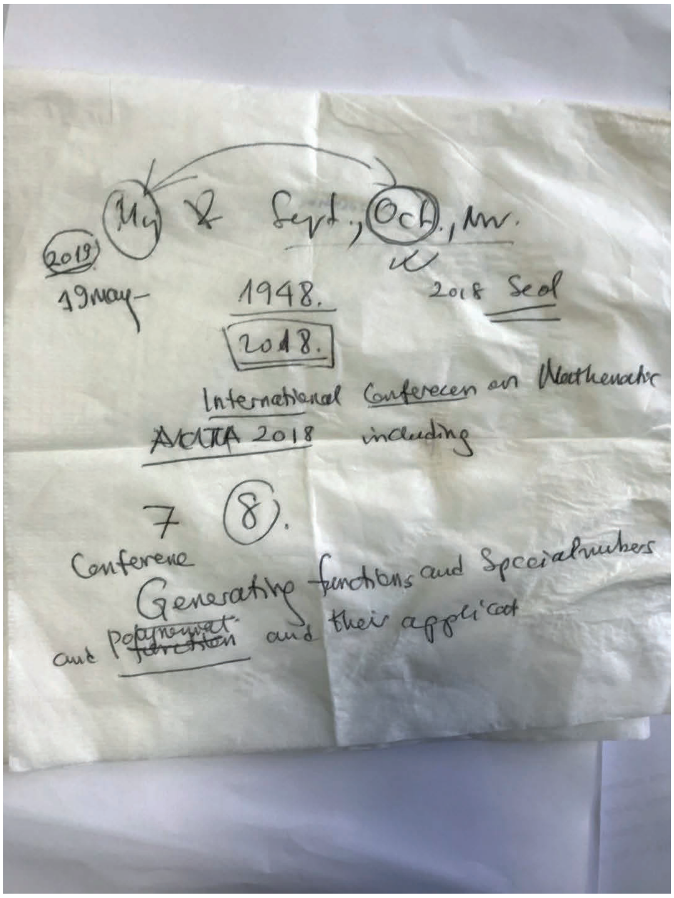
This excursion includes a trip to Campus of Akdeniz University, Antalya Kaleici (Old Town), Perge Ancient City (where the mathematician Apollonius lived), Aspendos Ancient Theatre, Side Ancient City (where you see the splendid Agora, Theatre and Temples built in the 17th century B.C.).

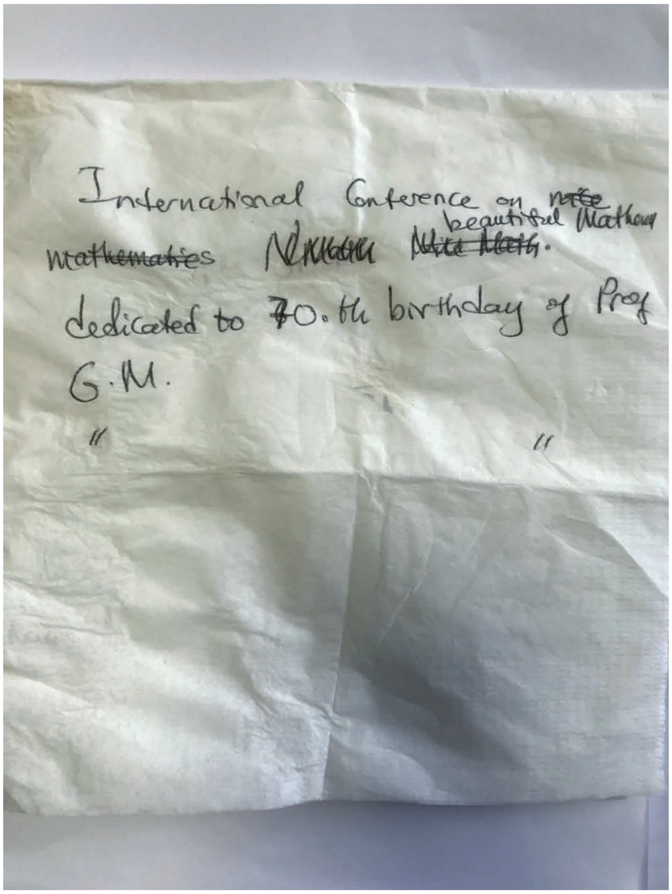






The idea of organizing this conference was appeared in 2017 at Belgrade, Serbia, while speaking with Professor Milovanović.





Our dreams are happening today because we are happy to have the opening of the conference together. Thus, dear distinguished participants, you have given honor to us by attending our conference: MICOPAM 2018.

I would like to thank to the following my colleagues and students who helped me at every stage of the Mediterranean International Conference of Pure & Applied Mathematics and Related Areas:

Plenary Speakers

- Abdelmejid Bayad, (Université d'Evry Val d'Essonne, France)
- Walter Gautschi, (Purdue University, USA)
- Allal Guessab, (University of Pau, France)
- Satish Iyengar, (University of Pittsburgh, USA)
- Burcin Simsek, (Bristol-Myers Squibb Company, USA)
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- Hari M. Srivastava, (University of Victoria, Canada)

Local Organizing Committee: (including especially Co-Chairman Prof. Dr. Mustafa Alkan, Prof. Dr. Veli Kurt, Conference Secretary Asst. Prof. Dr. Irem Kucukoglu, Dr. Ortac Ones, Dr. Neslihan Kilar, Dr. Busra Al, Asst. Prof. Dr. Fusun Yalcin, Asst. Prof. Dr. Ayse Yilmaz, Assoc. Prof. Dr. Ahmet Aykut Aygunes, Dr. Burak Kurt); my precious family: (my wife Saniye, my daughters Burcin and Buket), Professor Milovanović; besides academic staff of Akdeniz University: Rector Prof. Dr. Mustafa Ünal and Vice Rector Prof. Dr. Erol Gürpınar, Dean of Faculty of Science Prof. Dr. Ahmet AKSOY, some staff of Department of Mathematics; Prof. Dr. Ömer Colak, Prof. Dr. Gurhan Yalcin, Prof. Dr. Niyazi Ugur Kockal, and also other friends whose names that I did not mention here.

As for mathematics; Mathematics is the common heritage of everyone; Mathematics is the common language of the world that is always passed from generation to generation by refreshing.

It would be appropriate to say the following:

In addition to the poetic and artistic aspect of mathematics, mathematics has such a spiritual, magical and logical power, all natural science and social science cannot breathe and survive without mathematics.

Mathematics is such a branch of science that other sciences cannot develop without it. Therefore, Mathematics, which is the oldest of Science, has contributed fundamentally to the development of our world civilizations. So, we can enter into the science and technology centers using the power of mathematics and its branches. So, mathematics and its branches create the possibility of bridgework and communication between the Natural Sciences and the Engineering Sciences as well as the Economic and also Social Science.

The aim of the conference is to bring together leading scientists of the pure and applied mathematics and related areas to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other. In Fact, the main purpose of this conference is to bring to the fore the best of research and applications that will help our world humanity and society. Due to the valuable idea of the MICOPAM2018, this conference welcomes speakers having talks whose contents mainly related to the following two subjects: Pure and Computational and Applied Mathematics, Statistics, Mathematical Physics (related to p-adic Analysis, Umbral Algebra and Their Applications), Analysis Algebra Linear and Multi-linear Algebra, Clifford Algebras and Applications, Real and Complex Functions, Orthogonal Polynomials, Special numbers and Functions, Fractional Calculus, q-calculus, Number theory, Combinatorics, Approximation theory, Optimization Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Probability and Statistics and their Applications, Scientific Computation Mathematical Methods in Physics and in Engineering Mathematical Geosciences.

To summarize my speech, this conference has provided a novel opportunity to our distinguished participants to meet each other and share their scientific works and friendships in the above areas.

I am delighted to note that all participants have free and active involvement and meaningful discussion with other participants during the conference at the hotel Sherwood Exclusive Kemer, which contains all shades of green and yellow, around the Taurus Mountains and decorated with turquoise color of the Mediterranean Sea.

It is my great pleasure to thank Professor Gradimir V. Milovanović, because this conference is dedicated to his 70th birthday at Antalya. Happy Birth Day Professor Gradimir V. Milovanović. I hope that his life will be with health and happiness. It is my great pleasure to thank again local organizing committee Consequently, I send my thanks to all distinguished invited speakers, and all participants.

PROF. DR. YILMAZ SIMSEK

Head of the Organizing Committee of MICOPAM 2018

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PREFACE

Why we call the name of the conference as "The Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2018)". Because by the Mediterranean Sea, almost all the countries, the sea and the oceans are connected. For this reason, the first of our distinguished congress is dedicated to the 70th birthday of the respected and humble scientist Prof. Dr. Gradimir V. Milovanović. The subject of this conference includes the many fields of physical mathematics and engineering, especially all branches of mathematics. Therefore, we include Pure and Computational and Applied Mathematics & Statistics and also Mathematical Physics (related to p-adic Analysis, Umbral Algebra and Their Applications).

The aim of the Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2018) is to bring together all the researchers working in various fields of Mathematics, (Mathematical) Physics, Engineering and related areas such as Analysis, Non-linear Analysis, Integral transforms, Number Theory, p-adic Analysis and Applied Algebra, Special Functions, q-analysis and Discrete Mathematics, Probability and Statistics, Mathematical Physics and their applications. Our main aim is also to bring together theoretical, numerical and apply analyst, number theorists, (quantum) physicist working in the above areas and their applications. All of the participants likely lead to significant uncover new connections on these.

A brief description of the contents of Conference Proceedings Book as follows (for details see the Table of Contents).

The First Chapter is about the conference of MICOPAM2018, the second chapter is the full of contributing articles and manuscripts. The Conference Proceedings Book can be used for a variety of areas in Analysis, Non-linear Analysis, Integral transforms, Number Theory, p-adic Analysis and Applied Algebra, Special Functions, q-analysis and Discrete Mathematics, Probability and Statistics, Mathematical Physics and their applications. Therefore, this book is suitable for advanced graduate students and researchers as well as research workers and also practitioners.

Editor

Yilmaz SIMSEK

Associate Editors

Miloljub ALBIJANIĆ, Mustafa ALKAN, Irem KUCUKOGLU

Acknowledgments: Thank to Prof. Dr. Gradimir V. Milovanović, Prof. Dr. Miloljub Albijanić, Prof. Dr. Mustafa Alkan, Asst. Prof. Dr. Irem Kucukoglu, and Dr. Ortaç Öneş who provided most valuable contribution on preparing this book with their LaTeX&PDF processes and Cover Page of the Conference Proceedings Book.

Editor

Yilmaz SIMSEK

ABOUT CONFERENCE

The Mediterranean International Conference of Pure&Applied Mathematics and Related Areas Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary Antalya-Turkey, October 26 – 29, 2018.

The Mediterranean International Conference of Pure&Applied Mathematics and Related Areas (MICOPAM 2018) will be held in Antalya, Turkey, on October 26–29, 2018. The event will be held over four days, with presentations delivered by researchers from the international community, including presentations from keynote speakers and state-of-the-art lectures. The aim of the conference is to bring together leading scientists of the pure and applied mathematics and related areas to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other.

The conference is dedicated to the renowned mathematician Prof. Dr. Gradimir V. Milovanović on the occasion of his 70th anniversary.

Prof. Dr. Gradimir V. Milovanović

Prof. Dr. Gradimir V. Milovanović**,** born in Zorunovac, Serbia, 2 January 1948, is one of the the world leading scientists in the field of numerical analysis, approximation theory and special functions, a longtime professor at the University of Niš, Serbia, and a member of the Serbian Academy of Sciences and Arts.

This conference welcomes speakers whose talk or poster contents are mainly related to the following two subjects:

- Pure and Computational and Applied Mathematics & Statistics
- Mathematical Physics (related to *p*-adic Analysis, Umbral Algebra and Their Applications)

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- Buket Simsek,
- Fusun Yalcin.

SHORT BIOGRAPHY OF PROF. GRADIMIR V. MILOVANOVIĆ

Prof. Gradimir V. Milovanović, full member of the Serbian Academy of Sciences and Arts (SASA), was born on 2nd January, 1948, in Zorunovac, municipality of Knjaževac, to father Vukašin and mother Vukadinka, b. Savić. He finished primary school in his birth place, and high school of Natural-mathematical specialization in Knjaževac. He graduated from the Faculty of Electronics in Niš, in 1971, at the department of Computer Sciences. Postgraduate studies in the area of Applied Mathematics he completed in 1974, and in 1976, defended the PhD thesis and thus acquired the scientific degree of Doctor of Mathematical Sciences.

At the Faculty of Electronics in Niš, he passed all degrees, from an assistant (1971), assistant professor (1976), associate professor (1982), to professor (1986). In 2006 he was elected a Corresponding Member of SASA and a full member in 2012. Since 2014 Prof. Gradimir V. Milovanović is retired (for detailed biographical and bibliographical data see: http://www.mi.sanu.ac.rs/~gvm/).

In his teaching career of several decades, he taught at the Electronic Faculty in Niš, and other faculties in Serbia (Faculty of Electrical Engineering and Faculty of Mathematics in Belgrade, Faculty of Mechanical Engineering and Faculty of Civil Engineering in Niš, Faculty of Sciences and Mathematics in Niš, Kragujevac, etc.). He held courses at graduate, master, and doctorate studies, including Numerical Analysis, Approximation Theory, Special Functions, Operational Research, as well as numerous subjects in the area of Computer Science and Information Technology. He was a visiting professor at Purdue University (USA), Universite de Pau (France), and Universita di Basilicata, Potenza (Italy). He published 23 text-books, among them his *Numerička analiza* (*Numerical Analysis*) in three volumes (Научна књига, Belgrade; first edition 1985 was the first complete text book on this subject in ex Yugoslavia, which has been widely used by numerous generations of students.

In the scientific research he published 7 monographs and about 350 scientific papers (over 150 in journals from the SCI list), with several thousand citations. Most significant monograph works of Milovanović are *Topics in Polynomials: Extremal Problems, Inequalities, Zeros* (coauthors: D. S. Mitrinović and Th. M. Rassias), published at over 800 pages by World Scientific (Singapore, 1994) and known in the world as „Bible of Polynomials“ and the monograph *Interpolation Processes – Basic Theory and Applications* (coauthor: G. Mastroianni) by Springer Verlag, 2008. He supervised 13 Ph.D. Theses and 16 Master Theses, as well as many scientific research projects, including the international projects SCOPES and TEMPUS. As a reviewer of scientific projects, he worked for the Ministry of Science of Serbia, Italy and Montenegro. He participated in the work of commissions for doctorate theses and promotion of professors in many countries (France, Italy, Morocco, Cyprus, Australia, India). He is the founder of the scientific journal *Facta Universitatis: Series Mathematics and Informatics* at the University of Niš and its first Editor-in-Chief. He is Editor-in-Chief of the journals: *Journal of Inequalities and Applications* (Springer), *Publication Mathématique Belgrade*, *Bulletin* (SANU); Associate Editor of the journals: *Optimization Letters* (Springer), *Applied Mathematics and Computation* (Elsevier), as well as a member of the editorial board of a number of journals in Serbia (*AADM*, *FILOMAT*, ...), Romania, Bulgaria, Armenia and India. As an invited lecturer he took part in numerous international conferences worldwide, eg. Bulgaria (Sofia, Borovec), Poland (Warsaw), Hungary (Miskolc, Budapest), USA (Purdue University), Germany (Oberwolfach), Romania (Cluj-Napoca, Timisoara), Italy (Vico Equense, Acquafredda di Maratea, Falerna, Erice, Alba di Canazei), Singapore, Norway (Røros), Denmark (Copenhagen), France

(Marseille), Spain (Granada, Seville, Ubeda), South Africa (Stellenbosch, Port Elizabeth), Morocco (Marrakech, Casablanca), Brazil (Campos do Jordão), South Korea (Gyeongju, Seoul), Turkey (Antalya, Kirsehir, Kusadasi-Aydin), etc.

Milovanović performed various relevant functions: the Head of the Department of Mathematics at the Faculty of Electronics in Niš (1983–2002), Vice Rector of the University of Niš (1989–1991), the Dean of the Faculty of Electronics in Niš (2002–2004), the Rector of the University of Niš (2004–2006), member of the Executive Board of Electric Power Industry of Serbia (2004–2006), the President of the National Council of Serbia for Science and Technology Development (2006–2010), the President of the Scientific Committee for Msathematics, Computer Sciences and Mechanics (2010–2015), etc. He is also a member of several important international organisations, among which are AMS (American Mathematical Society), SIAM (Society for Industrial and Applied Mathematics) and GAMM (Gesellschaft für Angewandte Mathematik und Mechanik). Since 2016 he is the Secretary of the Department of Mathematics, Physics and Geo Sciences in SASA.

MICOPAM 2018

The Mediterranean International Conference of Pure&Applied Mathematics and Related Areas

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Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary

Antalya-Turkey, October 26–29, 2018

http://micopam2018.akdeniz.edu.tr/

Conference Venue: Sherwood Exclusive Kemer-ANTALYA

Proceedings Book of The Mediterranean International Conference of Pure&Applied Mathematics and Related Areas 2018 (MICOPAM 2018)

October 29, 2018

Contents

1 INVITED SPEAKERS

Orthogonal Polynomials on Radial Rays in the Complex Plane

Gradimir V. Milovanović[1,2]

Abstract

We consider some classes of polynomials orthogonal on radial rays in the complex plane with respect to the Hermitian and Non-Hermitian inner products, as well as some applications of such polynomials. Some applications of such polynomials could be done, including an electrostatic interpretation of their zeros.

2010 Mathematics Subject Classifications : 33C45, 33C47, 30C15
Keywords: Orthogonal polynomials, Inner product, Recurrence relation, Numerical construction, Zero distribution.

Introduction

Orthogonal polynomials play a very important role in applications not only in mathematics, but in many other computational and applied sciences, physics, chemistry, engineering, economics, etc. The most important orthogonal polynomials are ones which are orthogonal on the real line with respect to the inner product

$$(p,q) = \int_{\mathbb{R}} f(t)g(t)\mathrm{d}\mu(t) \quad \left(p,q \in L^2(\mathbb{R};\mathrm{d}\mu\right),$$

where $\mathrm{d}\mu$ is a positive measure on $\mathbb{R}$ with finite or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k\,\mathrm{d}\mu(t)$, $k = 0,1,\ldots$, exist and are finite, and $\mu_0 > 0$ (cf. [4], [8]). Because of the property $(tp,q) = (p,tq)$, these orthogonal polynomials $\pi_k(\,\cdot\,) = \pi_k(\mathrm{d}\mu;\,\cdot\,)$ satisfy a three–term recurrence relation

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0,1,2\ldots, \tag{1}$$

with $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$, where the sequences of recursion coefficients α_k and β_k depend on the measure $\mathrm{d}\mu$. Only for certain narrow classes of measures, e.g., for the *classical measures* (Jacobi, generalized Laguerre, Hermite), these coefficients α_k and β_k are known in the explicit form (for a characterization of the classical orthogonal polynomials see [1]). Orthogonal polynomials for which the recursion coefficients are not known we call *strongly non-classical polynomials*.

In eighties of the last century Walter Gautschi developed the so-called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, which includes effective algorithms for numerical generating orthogonal polynomials with respect to an arbitrary measure, strong stability analysis of such algorithms, necessary software for implementing such algorithms and applications (cf. [3], [4], [5]).

This constructive theory opened the door for extensive computational work on orthogonal polynomials and many their applications (construction of many new classes of strongly non-classical polynomials, development of other types of orthogonality, *s* and σ-orthogonality, Sobolev type of orthogonality, multiple orthogonality, orthogonality on some curves in the complex plane (circle, semicircle [6, 7, 9], circular arc), orthogonality on radial rays [10, 11, 12, 13], etc.), applications in diverse areas of applied and numerical analysis (numerical integration, interpolation, integral equations, ...), approximation theory (e.g., moment-preserving spline approximation), integration of fast oscillating functions, summation of slowly convergent series, integration of fast oscillating functions, etc.

Orthogonal Polynomials on Radial Rays

Let $M \in \mathbb{N}$, $a_s > 0$, $s = 1, \ldots, M$, and $0 \le \theta_1 \le \cdots \le \theta_M < 2\pi$. Putting $\varepsilon_s = \mathrm{e}^{\mathrm{i}\theta_s}$, $s = 1, \ldots, M$, we consider M points in the complex plane, $z_s = a_s\varepsilon_s \in \mathbb{C}$, $s = 1, \ldots, M$, with arguments θ_s (see Fig. 1). Some of a_s (or all) may coincide and also can be ∞.

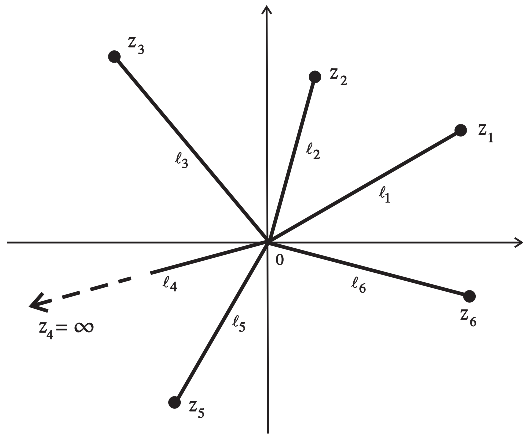

Figure 1: The case of six rays ($M = 6$)

The inner product can be introduced so that it is *hermitian*,

$$(f, g) = \sum_{s=1}^{M} \mathrm{e}^{-\mathrm{i}\theta_s} \int_{\ell_s} f(z)\overline{g(z)}\, |w_s(z)| \mathrm{d}z,$$

where $x \mapsto \omega_s(x) = |w_s(x\varepsilon_s)| = |w_s(z)|$ $(z \in \ell_s;\ s = 1, \ldots, M)$ are weight functions on $(0, a_s)$, i.e., they are nonnegative on $(0, a_s)$ and $\int_0^{a_s} \omega_s(x)\mathrm{d}x > 0$. It can be represented as

$$(f, g) = \sum_{s=1}^{M} \int_0^{a_s} f\big(x\varepsilon_s\big)\overline{g\big(x\varepsilon_s\big)}\, \omega_s(x)\mathrm{d}x,$$

and we can see that $(f, f) > 0$, except when $f(z) = 0$. Polynomials orthogonal with respect to this inner product can be considered. In the symmetric case with even numbers of rays ($M = 2m$) we can obtained analytic results for the recurrence coefficients for all classical weight functions (Jacobi, generalized Laguerre, Hermite).

In the simple symmetric (Legendre) case with four rays ($M = 4$) and

$$(f, g) = \int_0^1 \Big[f(x)\overline{g(x)} + f(\mathrm{i}x)\overline{g(\mathrm{i}x)} + f(-x)\overline{g(-x)} + f(-\mathrm{i}x)\overline{g(-\mathrm{i}x)} \Big] \mathrm{d}x,$$

we can prove the recurrence relation

$$\pi_{N+2}(z) = z^2\pi_N(z) - b_N\pi_{N-2}(z), \quad N \geq 2; \quad \pi_N(z) = z^N, \quad N \leq 3,$$

where the coefficient b_N $(N = 4n + \nu;\ n = [N/4])$ is given by

$$b_{4n+\nu} = \begin{cases} \dfrac{16n^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 0, 1, \\ \dfrac{(4n+2\nu-3)^2}{(8n+2\nu-3)(8n+2\nu+1)} & \text{if } \nu = 2, 3. \end{cases}$$

In the general case, using some kind of the discretized Stieltjes-Gautschi procedure, we can numerically construct the coefficients β_{kj} in the relation

$$\pi_k(z) = z\pi_{k-1}(z) - \sum_{j=1}^{k} \beta_{kj}\pi_{j-1}(z), \quad \beta_{kj} = \frac{(z\pi_{k-1}, \pi_{j-1})}{(\pi_{j-1}, \pi_{j-1})} \quad (1 \leq j \leq k).$$

The following result is related to the zero distribution of $\pi_N(z)$.

Theorem. *All the zeros of the orthogonal polynomial $\pi_N(z)$ lie in the convex hull of the rays $L = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_M$.*

Example. We consider an asymmetric case with five rays $(M = 5)$, defined by points in the complex plane: $z_1 = 6$, $z_2 = 5\mathrm{e}^{9\pi\mathrm{i}/14}$, $z_3 = 2\mathrm{e}^{4\pi\mathrm{i}/5}$, $z_4 = 5\mathrm{e}^{6\pi\mathrm{i}/5}$, $z_5 = 5\mathrm{e}^{7\pi\mathrm{i}/4}$, with weight functions transformed to $(0, 1)$: $\omega_1(x) = 1$ (Legendre weight), $\omega_2(x) = 1/\sqrt{x(1-x)}$ (Chebyshev weight of the first kind), $\omega_3(x) = \sqrt{x(1-x)}$ Chebyshev weight of the second kind), $\omega_4(x) = \sqrt{x/(1-x)}$ (Chebyshev weight of the fourth kind), $\omega_5(x) = \sqrt{(1-x)/x}$ (Chebyshev weight of the third kind), respectively.

Zeros of $\pi_N(z)$ for $N = 20$ and $N = 100$ are presented in Figure 2.

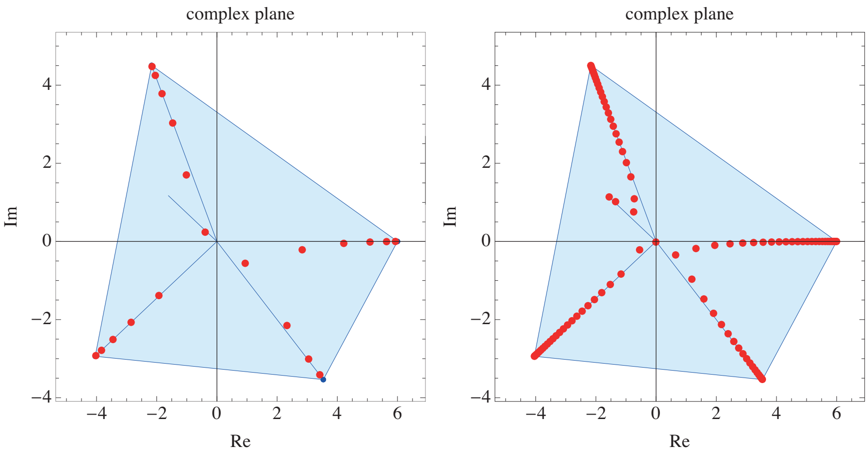

Figure 2: Zeros of $\pi_N(z)$ for $N = 20$ (left) and $N = 100$ (right)

In some symmetric cases, an electrostatic interpretation of the zeros of $\pi_N(z)$ can be done [11].

Orthogonal polynomials on radial rays with respect to a complex-valued moment functional

$$\mathcal{L}(p) = \sum_{s=1}^{M} \int_0^{a_s} p(x\varepsilon_s)\omega_s(x)\mathrm{d}x, \quad p \in \mathcal{P},$$

can be also considered, where $a_s > 0$ are given real numbers, and ε_s and ω_s are as before.

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Diophantine equations and Indices in cubic number fields

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Abstract

Let $F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \in \mathbb{Z}[x, y]$ be an irreducible cubic form and D denotes its discriminant. In this paper, we investigate arithmetic properties of the common indices of algebraic integers in cubic fields. For each integer k such that $v_2(k) \not\equiv 0 \pmod 3$ and $v_2(D) = 3v_2(b^2 - 3ac)$, we prove that the cubic Thue equation $F(x, y) = k$ has no solution $(x, y) \in \mathbb{Z}^2$. As application, we construct parametrized families of twisted elliptic curves

$$E : \; ax^3 + bx^2 + cx + d = ey^2$$

without integer points (x, y).

2010 Mathematics Subject Classifications : 11R04, 11DXX, 11R16, 11R33, 11R09

Keywords: Cubic Thue equations, cubic fields, common index divisors of cubic fields

Introduction

Let $f(x, y) \in \mathbb{Z}[x, y]$ be a homogeneous irreducible polynomial of degree $n \geq 3$ and k be a non-zero integer. In 1909, Thue proved the following fundamental result.

Theorem 1 ([18])**.** *The diophantine equation*

$$f(x, y) = k \tag{1}$$

has only finitely many solutions $(x, y) \in \mathbb{Z}^2$.

However, Thue's proof is not effective. The problem of estimating the number of solutions of (1) has rich history, see for example Siegel [17], Mahler [13], Erdös-Mahler [7], Davenport-Roth [6] and Lewis-Mahler [12].

For each integer k, $w(k)$ denote the number of distinct prime factors of k. In 1933, Mahler [13] proved that if f is irreducible then (1) has at most $C^{1+w(k)}$ primitive solutions where C depending only on f.

If $f(x, y)$ is an irreducible binary cubic form with negative discriminant, Delauney [5] and Nagell [15] showed that the equation $f(x, y) = 1$ has at most five integer solutions (x, y). Now if its discriminant is positive, then Evertse [8] showed that the equation $f(x, y) = 1$ has at most twelve integer solutions (x, y). Recently, Bennett [3] refined Delauney-Nagell-Evertse result as follows: if $f(X, 1)$ has at least two distinct complex roots, then the equation $f(x, y) = 1$ possess at most 10 solutions in integers x and y.

In 1984, Ayad [2] proved that if $f(x,y)$ is a binary form of degree 3 with coefficients in $\mathbb{Z}$, $Aut(f)$ its automorphisms group and $H(f)$ its Hessian, then $Aut(f)$ is trivial except when $H(f)=\lambda g(x,y)$, $\lambda\in\mathbb{Z}^*$ and $g(x,y)$ is equivalent to x^2-xy-y^2. In this last case, $Aut(f)$ is cyclic of order 3 and f is equivalent to one binary form of type :

$$f_{m,n}(x,y)=mx^3-nx^2y-(n+3m)xy^2-my^3,\quad m,n\in\mathbb{Z},$$

so, the number of representations of integer k by $f(x,y)$ is divisible by 3. Note that the case $m=k=1$ is proved by Avanesov [1].
Let n be a rational integer and $\mathbb{K}_n=\mathbb{Q}(\theta)$ be a cyclic cubic number field generated by a root θ of $f_{1,n}(X,1)=X^3-nX^2-(n+3)X-1$ and let $\mathbb{O}_{\mathbb{K}_n}$ be its ring of integers. The polynomial $f_{1,n}(X,1)$ has discriminant $(n^2+3n+9)^2$. If n^2+3n+9 is square-free, then we have the discriminant of $\mathbb{K}_n$, $D(\mathbb{K}_n)=(n^2+3n+9)^2$ and $\mathbb{O}_{\mathbb{K}_n}=\mathbb{Z}[\theta]$ (there exists infinitely many such n, cf. Cusick [4, pp. 63-73]).

In 2011, A. Hoshi [9] studied the case when k is a positive divisor of n^2+3n+9, and gave a correspondence between integer solutions to the parametric family of cubic Thue equations

$$x^3-nx^2y-(n+3)xy^2-y^3=k$$

and isomorphism classes of the simplest cubic fields. For more details on the study of simplest cubic fields see [16].

Recently, Wakabayashi [22], using Baker's method, proved that for any integer $n\geq 1.35\cdot 10^{14}$, the family of parametrized Thue equations

$$x^3-n^2xy^2+y^3=1$$

has only trivial solutions $(x,y)=(0,1),(1,0),(1,n^2),(n,1),(-n,1)$.

A. Togbé [19], using Baker's method and the results obtained by L. C. Washington [23] and O. Lecacheux [11], solved the family of parametrized Thue equations

$$x^3-(n^3-2n^2+3n-3)x^2y-n^2xy^2-y^3=\pm 1,\quad \text{when } n\geq 1.$$

A. Togbé [21, 20] using Baker's method and the results obtained by Y. Kishi [10], solved the two families of parametrized Thue equations

$$\begin{aligned}&x^3-n(n^2+n+3)(n^2+2)x^2y-(n^3+2n^2+3n+3)xy^2-y^3=\pm 1,\\&x^3+(n^8+2n^6-3n^5+3n^4-4n^3+5n^2-3n+3)x^2y\\&\qquad\qquad -(n^3-2)n^2xy^2-y^3=\pm 1,\end{aligned}$$

when $n\geq 0$.

Let a,b,c,d,e be integers. The equation $ax^3+bx^2+cx+d=ey^2$ was studied by Mordell [14, pp.255-261]. He proved the following important result: if the polynomial ax^3+bx^2+cx+d has no squared linear factor in x, then the equation $ax^3+bx^2+cx+d=ey^2$ has only a *finite number of integer solutions.*

Now we state our main result.

Main results

Let N be any integer. We denote by $v_2(N)$ the greatest exponent s such that 2^s divides N. The discriminant of the form

$$F(x,y)=ax^3+bx^2y+cxy^2+dy^3$$

is the invariant

$$D = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2. \tag{2}$$

The binary form F has the quadratic and cubic covariants

$$H(x,y) = A_0x^2 + B_0xy + C_0y^2, \tag{3}$$
$$G(x,y) = A_1x^3 + B_1x^2y + C_1xy^2 + D_1y^3. \tag{4}$$

where

$$\begin{aligned} A_0 &:= b^2 - 3ac, B_0 := bc - 9ad, C_0 := c^2 - 3bd; \\ A_1 &:= 2b^3 + 27a^2d - 9abc, B_1 = 3(b^2c + 9abd - 6ac^2), \\ C_1 &= -3(bc^2 + 9acd - 6b^2d), D_1 = -(2c^3 + 27ad^2 - 9bcd). \end{aligned} \tag{5}$$

the quadratic form H is the Hessian and G is the gradient of F.

Throught this paper, we assume that $D \neq 0$ and $\gcd(a,b,c,d) = 1$. Now we state our main result.

Theorem 2. *Let a,b,c,d and k be integers such that*

$$v_2(D) = 3v_2(A_0), \ \ v_2(k) \equiv 1,2 \pmod 3.$$

Then the cubic Thue Diophantine equations

$$ax^3 + bx^2y + cxy^2 + dy^3 = k$$

has no integer solution (x,y).

Corollary 3. *Let a,b,c,d as in Theorem 2, and e be integers such that $v_2(e) \geq v_2(D)/2$ and $v_2(e) \equiv 1 \pmod 3$. Then the familly of twisted ellitpic curves*

$$E: \ ax^3 + bx^2 + cx + d = ey^2 \tag{6}$$

have no integer points (x,y).

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Inference for first passage times of the Cox-Ingersoll-Ross process

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Abstract

The Cox-Ingersoll-Ross (or Feller) diffusion process has linear drift and a state-dependent diffusion coefficient that vanishes at zero. Earlier studies have shown that it provides a better fit for neural activity than the Ornstein-Uhlenbeck under certain conditions. In this talk we describe inference based on maximum likelihood for this model when the available data are spike trains rather than the neurons subthreshold voltage traces. This work is joint with Bowen Yi.

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On conjectures of Stenger in the theory of orthogonal polynomials

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Abstract

The conjectures in the title deal with the zeros x_j, $j = 1, 2, \ldots, n$, of an orthogonal polynomial of degree $n > 1$ relative to a nonnegative weight function w on an interval $[a, b]$ and with the respective elementary Lagrange interpolation polynomials $\ell_k^{(n)}$ of degree $n-1$ taking on the value 1 at the zero x_k and the value 0 at all the other zeros x_j. They involve matrices of order n whose elements are integrals of $\ell_k^{(n)}$, either over the interval $[a, x_j]$ or the interval $[x_j, b]$, possibly containing w as a weight function. The claim is that all eigenvalues of these matrices lie in the open right half of the complex plane. Ample evidence is provided for the validity of the claim when the integrals are weighted, but not necessarily otherwise. Connections are mentioned with the theory of collocation Runge–Kutta methods in ordinary differential equations.

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Voronoi diagrams, Delaunay triangulations and applications

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Abstract

In this lecture, we present the concepts of a Voronoi diagram (VD) and of a Delaunay triangulation (DT). These two geometrical structures are important tools in many areas like Astronomy, Physics, Chemistry, Biology, Ecology, Economics, Mathematics and Computer Science. Here, we present a set of results showing some of the advantages of their optimality criteria in computing integral approximations, which are based upon a geometric point of view exploiting Delaunay triangulations and Voronoi tessellations. We begin by introducing a new class of cubature formulas for numerical integration (or multidimensional quadrature), that approximate from above (or respectively from below) the exact value of the integrals of every function having a certain type of *convexity*. Under suitable regularity assumptions, we show that all these integral approximations enjoy certain desirable properties. In particular, they can be totally characterized in terms of the approximation error generated by a multidimensional quadratic function. We show that the Delaunay triangulation, the Voronoi tessellation and their generalizations give access to efficient algorithms for computing these cubature formulas. We also briefly discuss some ongoing related research.

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On some new sequence spaces of order α

Ekrem Savaş

Abstract

In this paper we introduce and examine some properties of some new sequence spaces of order α that are defined using modulus function and generalized three parametric real matrix A.

2010 Mathematics Subject Classifications : 40H05, 40C05.
Keywords: Modulus function, Almost convergence, Lacunary sequence, φ-function order α.

Introduction and Background

Let w denote the set of all real and complex sequences $x = (x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $||x|| = \sup_n |x_n|$, respectively. In summability theory, the concept of almost convergence was first introduced by G.G. Lorentz in 1948. Let us observe the outline of it. A linear functional L on l_∞ is said to be a Banach limit [1] if it has the following properties:

1. $L(x) \geq 0$ if $n \geq 0$ (i.e. $x_n \geq 0$ for all n),
2. $L(e) = 1$ where $e = (1, 1, \ldots)$,
3. $L(Dx) = L(x)$, where the shift operator D is defined by $D(x_n) = \{x_{n+1}\}$.

Let B be the set of all Banach limits on l_∞. A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide.

It is easy to verify that if x is a convergent sequence, then $L(x) = lim_n x_n$ for any Banach limits L. In the other words, $L(x)$ takes the same value for any Banach limits L. It is notable that this condition is meaningful not only for convergent sequences, but also for a certain type of bounded sequences. Let $\hat{c}$ denote the space of almost convergent sequences. Lorentz [6] has shown that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n \right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m+1}.$$

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman at al [5] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L|) = 0, \text{ for some } L \right\}.$$

Das and Mishra[4] have introduced the space AC_θ of lacunary almost convergent sequences and the space $|AC_\theta|$ of lacunary strongly almost convergent sequences as follows:

$$AC_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} (x_{k+n} - L) = 0, \text{ for some } L \text{ uniformly in } n \right\}.$$

and

$$|AC_\theta| = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_{k+n} - L| = 0, \text{ for some } L \text{ uniformly in } n \right\}.$$

Note that in the special case where $\theta = 2^r$, we have $AC_\theta = \hat{c}$ and $|AC_\theta| = [\hat{c}]$. which is defined by Maddox [7].

Following Ruckle [10], a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(x)$ for all $x, y \geq 0$,

(iii) f increasing,

(iv) f is continuous from the right at zero.

Maddox [8] introduced and examined some properties of the sequence spaces $w_0(f)$, $w(f)$ and $w_\infty(f)$ defined using a modulus f, which generalized the well-known spaces w_0, w and w_∞ of strongly summable sequences.

Recently E. Savas [11] generalized the concept of strong almost convergence by using a modulus f and examined some properties of the corresponding new sequence spaces.
Let $A = (a_{nk})$ be a nonnegative regular matrix summability method. Connor [2] further extended Maddox's results by giving the following definition:

Definition 1. *Let f be a modulus and A be a nonnegative regular summability method. We let*

$$w(A, f) = \left\{ x : lim_n \sum_{k=1}^{\infty} a_{nk} f(|x_k - L|) = 0 \right\}$$

and

$$w(A, f)0 = \left\{ x : lim_n \sum_{k=1}^{\infty} a_{nk} f(|x_k|) = 0 \right\}.$$

In 1993, Nuray and Savas [9] generalized Connor's definition by using almost convergence:

Definition 2. *Let f be a modulus and A be a nonnegative regular summability method. We let*

$$w(\hat{A}, f)=\left\{x: lim_n \sum_{k=1}^{\infty} a_{nk} f(|x_{k+m}-L|)=0, for\ some\ L,\ uniformly\ in\ m\right\}$$

and

$$w(\hat{A}, f)_0=\left\{x: lim_n \sum_{k=1}^{\infty} a_{nk} f(|x_{k+m}|)=0,\ uniformly\ in\ m\right\}.$$

By a φ-function we understood a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0)=0, \varphi(u)>0$,for $u>0$ and $\varphi(u) \rightarrow \infty$ as $u \rightarrow \infty$, (see, [12], [13]).
On the other hand in [3] a different direction was given to the study of Cesàro-type summability spaces of order α, $0<\alpha \leq 1$ and lacunary statistical convergence of order α.

In the present paper, we introduce and study some properties of the following sequence space of order α that is defined using the φ- function, generalized three parametric real matrix and modulus.

Main Results

Let φ and f be given φ-function and modulus function, respectively and $p=(p_k)$ be a sequence of positive real numbers. Moreover, let $\mathbf{A}=(\mathbf{A}_i)$ be the generalized three parametric real matrix with $A_i=(a_{n,k}(i))$, a lacunary sequence $\theta=(k_r)$ and $0<\alpha \leq 1$ be given. Then we define the following sequence spaces,

$$N_\theta^\alpha(\mathbf{A}, \varphi, f, p)_0=\left\{x=(x_k): \lim_r \frac{1}{h_r^\alpha} \sum_{n \in I_r} f\Big(\left|\sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|)\right|\Big)^{p_k}=0,\ \text{uniformly in}\right.$$

where h_r^α denote the αth power $(h_r)^\alpha$ of h_r, that is $h^\alpha=(h_r^\alpha)=(h_1^\alpha, h_2^\alpha, h_3^\alpha, \ldots.)$. If $x \in N_\theta^\alpha(\mathbf{A}, \varphi, f)_0$, the sequence x is said to be lacunary strong $(\mathbf{A}, \varphi)$- convergent to zero with respect to a modulus f. When $\varphi(x)=x$ for all x, we obtain

$$N_\theta^\alpha(\mathbf{A}, f, p)_0=\left\{x=(x_k): \lim_r \frac{1}{h_r} \sum_{n \in I_r} f(\left|\sum_{k=1}^{\infty} a_{nk}(i) x_k\right|)^{p_k}=0,\ \text{uniformly in } i\right\}.$$

If we take $f(x)=x$, we write

$$N_\theta^\alpha(\mathbf{A}, \varphi, p)_0=\left\{x=(x_k): \lim_r \frac{1}{h_r} \sum_{n \in I_r}\left|\sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|)\right|^{p_k}=0,\ \text{uniformly in } i\right\}.$$

If we take $\mathbf{A}=I$ and $\varphi(x)=x$ respectively, then we have

$$(N_\theta^\alpha)_0=\left\{x=(x_k): \lim_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} f(|x_k|)^{p_k}=0\right\}.$$

In the next theorem we establish inclusion relations between $w^\alpha(A, \varphi, f, p)$ and $N_\theta^\alpha(\mathbf{A}, \varphi,$
We now have

Theorem 3. *Let f be a any modulus function and let φ-function φ, generalized three parametric real matrix $\boldsymbol{A}$, $p=(p_k)$ be a sequence of positive real numbers and the sequence θ be given. If*
$w^{\alpha}(\boldsymbol{A},\varphi,f,p)_0=\Big\{x=(x_k):\lim_m \frac{1}{m}\sum_{n=1}^{m} f\Big(\,|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)|\,\Big)^{p_k}=0,$ *uniforml*
then the following relations are true:
(a) *If* $\liminf_r q_r>1$ *then we have* $w^{\alpha}(A,\varphi,f,p)_0\subseteq N_{\theta}^{\alpha}(\boldsymbol{A},\varphi,f,p)_0$,
(b) *If* $\sup_r q_r<\infty$, *then we have* $N_{\theta}^{\alpha}(\boldsymbol{A},\varphi,f,p)_0\subseteq w^{\alpha}(A,\varphi,f,p)_0$,
(c) $1<\liminf_r q_r\le\limsup_r q_r<\infty$, *then we have* $N_{\theta}^{\alpha}(\boldsymbol{A},\varphi,f,p)_0=w^{\alpha}(A,\varphi,f)_0$.

Proof. (a) Let us suppose that $x\in w^{\alpha}(A,\varphi,f,p)$. There exists $\delta>0$ such that $q_r>1+\delta$ for all $r\ge 1$ and we have $h_r/k_r\ge\delta/(1+\delta)$ for sufficiently large r. Then, for all i,

$$\begin{aligned}\frac{1}{k_r^{\alpha}}\sum_{n=1}^{k_r} f\Big(\Big|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n} &\ge \frac{1}{k_r^{\alpha}}\sum_{n\in I_r} f\Big|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_k}\\ &= \frac{h_r^{\alpha}}{k_r^{\alpha}}\frac{1}{h_r^{\alpha}}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n}\\ &\ge \frac{\delta^{\alpha}}{(1+\delta)^{\alpha}}\frac{1}{h_r^{\alpha}}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n}.\end{aligned}$$

Hence, $x\in N_{\theta}^{\alpha}(\mathbf{A},\varphi,f,p)_0$.

(b) If $\limsup_r q_r<\infty$ then there exist $M>0$ such that $q_r<M$ for all $r\ge 1$. Let $x\in N_{\theta}^{\alpha}(\mathbf{A},\varphi,f,p)_0$ and ε is an arbitrary positive number, then there exists an index j_0 such that for every $j\ge j_0$ and all i,

$$R_j=\frac{1}{h_j^{\alpha}}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n}<\varepsilon.$$

Thus, we can also find $K>0$ such that $R_j\le K$ for all $j=1,2,\ldots$. Now let m be any integer with $k_{r-1}\le m\le k_r$, then we obtain, for all i

$$I=\frac{1}{m^{\alpha}}\sum_{n=1}^{m} f\Big(\Big|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n}\le\frac{1}{k_{r-1}^{\alpha}}\sum_{n=1}^{k_r} f\Big(\Big|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n}=I_1+I_2$$

where
$I_1=\frac{1}{k_{r-1}^{\alpha}}\sum_{j=1}^{j_0}\sum_{n\in I_j} f\Big(\,|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)|\,\Big)^{p_n}$,
$I_2=\frac{1}{k_{r-1}^{\alpha}}\sum_{j=j_{0+1}}^{m}\sum_{n\in I_j} f\Big(\,|\sum_{k=1}^{\infty}a_{nk}(i)\varphi(|x_k|)|\,\Big)^{p_n}$.

It is easy to see that,

$$
\begin{aligned}
I_1 &= \frac{1}{k_{r-1}^{\alpha}} \sum_{j=1}^{j_0} \sum_{n\in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \Big| \Big)^{p_n} \\
&= \frac{1}{k_{r-1}^{\alpha}} \Big(\sum_{n\in I_1} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \Big| \Big)^{p_n} + ... + \sum_{n\in I_{j_0}} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \Big| \Big)^{p_n} \\
&\le \frac{1}{k_{r-1}^{\alpha}} (h_1 R_1 + ... + h_{j_0} R_{j_0}), \\
&\le \frac{1}{k_{r-1}^{\alpha}} j_0 k_{j_0}^{\alpha} sup_{1\le i\le j_0} R_i, \\
&\le \frac{j_0 k_{j_0}^{\alpha}}{k_{r-1}^{\alpha}} K.
\end{aligned}
$$

Moreover, we have for all i

$$
\begin{aligned}
I_2 &= \frac{1}{k_{r-1}^{\alpha}} \sum_{j=j_0+1}^{m} \sum_{n\in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \Big| \Big)^{p_n} \\
&= \frac{1}{k_{r-1}^{\alpha}} \sum_{j=j_o+1}^{m} \Big(\frac{1}{h_j} \sum_{n\in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \Big| \Big)^{p_n} h_j \\
&\le \varepsilon \frac{1}{k_{r-1}^{\alpha}} \sum_{j=j_0+1}^{m} h_j, \\
&\le \varepsilon \frac{k_r^{\alpha}}{k_{r-1}^{\alpha}}, \\
&= \varepsilon q_r^{\alpha} < \varepsilon.M.
\end{aligned}
$$

Thus $I \le \frac{j_o k_{j_o}^{\alpha}}{k_{r-1}^{\alpha}} K + \varepsilon.M$. Finally, $x \in w^{\alpha}(A, \psi, f, p)$.
The proof of (c) follows from (a) and (b). This completes the proof. □

Theorem 4. *Let $0 < \alpha \le \beta \le 1$ and p be a positive real number, then $N_{\theta}^{\alpha}(\boldsymbol{A}, \varphi, f)_0 \subseteq N_{\theta}^{\beta}(\boldsymbol{A}, \varphi, f)_0$.*

Proof. Let $x = (x_k) \in N_{\theta}^{\alpha}(\mathbf{A}, \varphi, f)_0$. Then given α and β such that $< \alpha \le \beta \le 1$ and a positive real number p, we write

$$
\frac{1}{h_r^{\beta}} \sum_{n\in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \Big| \Big)^{p} \le \frac{1}{h_r^{\alpha}} \sum_{n\in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \Big| \Big)^{p}
$$

and we get that $N_{\theta}^{\alpha}(\mathbf{A}, \varphi, f)_0 \subseteq N_{\theta}^{\beta}(\mathbf{A}, \varphi, f)_0$. □

The proof of the following result is a consequence of Theorem 2.2.

Corollary 5. *Let $0 < \alpha \le \beta \le 1$ and p be a positive real number. Then*

i) *If $\alpha = \beta$, then $N_{\theta}^{\alpha}(\boldsymbol{A}, \varphi, f)_0 = N_{\theta}^{\beta}(\boldsymbol{A}, \varphi, f)_0$.*

ii) *$N_{\theta}^{\alpha}(\boldsymbol{A}, \varphi, f)_0 \subseteq N_{\theta}(\boldsymbol{A}, \varphi, f)_0$ for each $\alpha \in (0, 1]$ and $0 < p < \infty$.*

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Matrix biorthogonal polynomials and matrices of measures: Linear Spectral perturbations

Francisco Marcellán[1]

Abstract

In this presentation, linear transformations for matrix biorthogonal polynomials are studied. The orthogonality is understood in a broad sense, and is given in terms of a nondegenerate continuous sesquilinear form, which in turn is determined by a quasidefinite matrix of bivariate generalized functions with a well defined support. The basic tool is the Gauss-Borel factorization of the Gram matrix, and particular attention will be paid to the non-associative character, in general, of the product of semi-infinite matrices. We will focus the attention on the derivation of Christoffel type formulas, which allow to express the perturbed matrix biorthogonal polynomials and its norms in terms of the original ones. In particular, Christoffel and Geronimus transformations, where a right multiplication by a matrix polynomial and the inverse of a matrix polynomial plus adequate mass points, respectively, will be analyzed. The resolvent matrix and connection formulas are given. Then, using spectral techniques and spectral jets, Christoffel-Geronimus formulas for the transformed polynomials and norms are presented.

These linear spectral transformations are considered in the context of the 2D non–Abelian Toda lattice and noncommutative KP hierarchies. The interplay between transformations and integrable flows is discussed. Miwa shifts, τ-ratio matrix functions and Sato formulas are given. Bilinear identities, involving Geronimus-Uvarov transformations, first for the Baker functions, second for the biorthogonal polynomials and its second kind functions as well as for the τ-ratio matrix functions are deduced.

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Generalized Krylov subspace methods for l_p-l_q minimization with application to image restoration

Lothar Reichel[1]

Abstract

This talk presents new efficient approaches for the solution of l_p-l_q minimization problems with $0 < p,\ q \leq 2$, based on the application of successive orthogonal projections onto generalized Krylov subspaces of increasing dimension. The subspaces are generated according to the iteratively reweighted least-squares strategy for the approximation of l_q- and l_q-norms or quasi-norms by using weighted l_2-norms. Computed image restoration examples illustrate the performance of the methods discussed. The talk presents joint work with A. Buccini, G.-X. Huang, A. Lanza, S. Morigi, and F. Sgallari.

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On Extended Stirling polynomials of the second kind and extended Bell polynomials associated with Poisson random variables

Taekyun Kim[1]

Abstract

Recently, several authors have studied the Stirling numbers of the second kind and Bell polynomials. In this paper, we study the extended Stirling polynomials of the second kind and the extended Bell polynomials associated with the Stirling numbers of the second kind. In addition, we note that extended Bell polynomials can be expressed in terms of the moments of the Poisson random variable with parameter $\lambda > 0$.

2010 Mathematics Subject Classifications : 11B73; 11B83

Keywords: Extended Stirling polynomials of the second kind, extended Bell polynomials

Introduction

As is well known, the Stirling numbers of the second kind are defined as

$$x^n = \sum_{n=0}^{\infty} S_2(n,l)(x)_l, \ (n \geq 0). \text{ (see [1, 3, 6, 18]).} \tag{1}$$

The generating function of $S_2(n,l)$ is given by

$$\frac{1}{m!}(e^t-1)^m = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}, \tag{2}$$

The Stirling polynomials of the second kind are defined by the generating function

$$\frac{1}{k!}e^{xt}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k|x)\frac{t^n}{n!}, \tag{3}$$

where $k \geq 0$.

The Bell polynomials are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \tag{4}$$

where $x = 1$, $Bel_n(1) = Bel_n$, $(n \geq 0)$, are called the Bell numbers.

From (2) and (4), we note that

$$e^{x(e^t-1)} = \sum_{m=0}^{\infty} x^m \frac{1}{m!}(e^t-1)^m = \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} S_2(n,m)x^m\Big)\frac{t^n}{n!}. \tag{5}$$

Thus, by (5), we get

$$Bel_n(x) = S_2(n,m)x^m, \ (n \geq 0). \tag{6}$$

The expectation of a Poisson random variable with parameter λ is given by

$$E[X] = \sum_{i=0}^{\infty} iP(i) = \sum_{i=0}^{\infty} ie^{-\lambda}\frac{\lambda^i}{i!} = \lambda. \tag{7}$$

The moments of Poisson random variable X with parameter $\lambda > 0$ is defined by

$$E[X^n] = \sum_{x=0}^{\infty} x^n P(x) = e^{-\lambda}\sum_{x=0}^{\infty} x^n \frac{\lambda^x}{x!}. \tag{8}$$

Main Results

Theorem 1. *For $n \geq 0$, we have*

$$Bel_{n,r}(\lambda) = \sum_{m=0}^{n} \lambda^m S_{2,r}(n,m), \tag{9}$$

and

$$S_{2,r}(n,m|x) = \sum_{k=m}^{n} \binom{n}{k} S_{2,r}(k,m)x^{n-k}. \tag{10}$$

Theorem 2. *For $n, k \geq 0$, we have*

$$S_{2,r}(n,k) = \sum_{l=0}^{k} \binom{n}{l} r^l S_2(n-l,k-l). \tag{11}$$

Theorem 3. *For $n \geq m \geq 0$, we have*

$$S_2(n,m) = \sum_{l=0}^{m} \binom{n}{l} (-1)^l r^l S_{2,r}(n-l,k-l). \tag{12}$$

Theorem 4. *For $n \geq k \geq 0$, we have*

$$S_{2,r}(n,k) = \sum_{l=0}^{k} \binom{n}{l} r^l S_2(n-l,k-l). \tag{13}$$

Theorem 5. *For $n, m, k \geq 0$ with $n \geq m+k$, we have*

$$\binom{m+k}{m} S_{2,r}(n,m+k) = \sum_{l=m}^{n} \binom{n}{l} S_{2,r}(l,m)S_{2,r}(n-l,k). \tag{14}$$

Further Remarks

A random variable X, taking on one of the values $0, 1, 2, \cdots$, is said to be a Poisson random variable with parameter $\lambda > 0$ if $P(i) = P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}, i = 0, 1, 2, \cdots$. Note that $\sum_{i=0}^{\infty} P(i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$. The Bell polynomials $Bel_n(x)$, $(n \geq 0)$, are known to be connected with the Poisson distribution. More precisely, $Bel_n(\lambda)$ can be expressed in terms of the moments of Poisson random variable x with parameter $\lambda > 0$ as

$$Bel_n(\lambda) = E[X^n]. \tag{15}$$

Let X be a Poisson random variable with parameter $\lambda > 0$.

$$E[e^{t(X+r\lambda)}] = \sum_{n=0}^{\infty} Bel_{n,r}(\lambda)\frac{t^n}{n!}. \tag{16}$$

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Formulas and identities on the expected values and moments of special polynomials

Burcin Simsek[1]

Abstract

In this paper, some formulas and identities for expected values and moments of special polynomials including the Bernoulli numbers and polynomials and Hermite type polynomials using characteristic functions of particular distribution functions and taking derivative of the generating functions that are possessed by the Bernoulli numbers and polynomials and Hermite type polynomials.

2010 Mathematics Subject Classifications : 05A15, 11B83, 11B68, 60E05, 60E10, 62E15

Keywords: Bernoulli numbers and polynomials, Hermite polynomials, Generating function, Expected values, Moment generating function, Characteristic function, Distribution functions, Hypergeometric function

Introduction

The theory of characteristic functions and generating functions, moment generating functions, and probability generating functions have been many applications for special numbers and polynomials in many areas, including mathematics, mathematical physics and other related areas (*cf.* [1]-[9]; and the references cited therein). In recent years, there has been many studies illustrating the relations of these functions to special polynomials such as the Bernoulli polynomials, the Euler polynomials, the Hermite polynomials, which is also the probability density function of normal distribution (*cf.* [2], [3], [5], [6], [7], [9]).

Background on the generating functions for Bernoulli polynomials and Hermite type polynomials

The generating functions for the Bernoulli polynomials of order k is defined by

$$F_B(t,x;k) = \left(\frac{t}{e^t-1}\right)^k e^{tx} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{1}$$

One can derive the Bernoulli numbers of order k: $B_n^{(k)} = B_n^{(k)}(0)$ by letting Eq-(1) $x=0$ in Eq-(1). Notice that letting $k=0$ into this equation yields the Bernoulli polynomials: $B_n(x) = B_n^{(1)}(x)$, and one can obtain the Bernoulli numbers easily from the Bernoulli polynomials: $B_n = B_n(0)$ (*cf.* [1], [8]; and the references cited therein).

The probabilistic representation associated with the Bernoulli polynomials is given as follows:

Theorem 1. *([9, p. 749. Theorem 1]). Given a sequence* $\{L_n\}_{n\in\mathbb{N}}$ *of independent random variable, each with the Laplace distribution* $\frac{1}{2}\exp(-|x|)$ $(x\in\mathbb{R})$*, define the random variable* $\mathcal{L}_B$ *by*

$$\mathcal{L}_B=\sum_{k=1}^{\infty}\frac{L_k}{2k\pi}.$$

Then the following probabilistic representation holds true:

$$B_n(x)=\mathbb{E}\left[\left(i\mathcal{L}_B+x-\frac{1}{2}\right)^n\right],$$

where $n\in\mathbb{N}_0=\{0,1,2,\cdots\}$*;* $x\in\mathbb{R}$*, set of real numbers,* $i^2=-1$ *and* $\mathbb{E}$ *denotes the expectation value operator which is defined by*

$$\mathbb{E}\left[g\left(x\right)\right]=\mathbb{E}_X\left[g\left(x\right)\right]=\int_{-\infty}^{\infty}f_X(x)g(x)dx,$$

f_X *is a probability density of the relevant random variable* X *(cf. [7], [9], [3]; and the references cited therein).*

This is further studied in [5] by providing relations between generating functions, characteristic functions, the nth moment of normal distribution $N(\mu,\sigma^2)$ and the Hermite polynomials. There is a closed relation between characteristic function for the normal distribution $N(\mu,\sigma^2)$ and generating function for the Hermite polynomials, which is given by the relation below:

$$f_{\mathcal{H}}(t;\mu,\sigma)=\exp\left(i\mu t-\frac{\sigma^2}{2}t^2\right)=\sum_{n=0}^{\infty}\mathcal{H}_n(\mu,\sigma)\frac{t^n}{n!}\tag{2}$$

(*cf.* [5]), where $\mathcal{H}_n(\mu,\sigma)$ denotes the Hermite type polynomials with variable μ and σ and $\exp\left(i\mu t-\frac{\sigma^2}{2}t^2\right)$ is the characteristics function of normal distribution $N(\mu,\sigma^2)$ (*cf.* [2], [3], [10]; and the references cited therein). By applying some routine calculations in (2), a formula for the polynomials the Hermite type polynomials is given the following corollary.

Corollary 2. *(cf. [5]) Let* $n\in\mathbb{N}_0$*. Then we have*

$$\mathcal{H}_n(\mu,\sigma)={}_2F_0\left[-\frac{n}{2},-\frac{n}{2}+\frac{n}{2};-;-\frac{1}{\left(\frac{i\mu}{\sqrt{2}\sigma}\right)^2}\right](i\mu)^n,$$

where ${}_pF_q\left[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z\right]$ *denotes the hypergeometric function, which is defined by*

$${}_pF_q\left[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z\right]=1+\sum_{n=1}^{\infty}\frac{\prod_{v=1}^{p}\left(a_v\right)_n}{\prod_{m=1}^{q}\left(b_m\right)_n}\frac{z^n}{n!}$$

in which b_m *is not zero or negative integers and* $(\alpha)_n=\alpha\left(\alpha+1\right)\ldots\left(\alpha+n-1\right)$ *and* $(\alpha)_0=1$ *for* $\alpha\neq0$ *(cf. see for detail [4, p. 73, Eq-(2)]).*

The probabilistic representation of the Bernoulli polynomials also given by the following generating function:

$$F_{\mathbb{E}}(t)=\frac{te^{t/2}}{e^t-1}=\sum_{n=0}^{\infty}\mathbb{E}[(\mathcal{L}_B)^n]\frac{(it)^n}{n!}\tag{3}$$

(*cf.* [9, p. 749. Theorem 1]).

Differential equations for generating functions of Berno polynomials and Hermite type polynomials

In this section, give some formulas and identities for the Bernoulli polynomials, the Hermite type polynomials with variable μ and σ, and $\mathbb{E}\left[(\mathcal{L}_B)^n\right]$.

Taking derivative of the generating functions (3) and (2) with respect to t, we have the following equations, respectively:

$$\frac{d}{dt}\left\{F_{\mathbb{E}}(t)\right\} = \left(\frac{1}{t}+\frac{1}{2}\right)F_{\mathbb{E}}(t) - \frac{1}{t}e^{\frac{t}{2}}F_{\mathbb{E}}^2(t), \tag{4}$$

and

$$\frac{\partial^2}{\partial t^2}\left\{f_{\mathcal{H}}(t;\mu,\sigma)\right\} = \left(-\mu-\sigma^2-2i\mu\sigma^2 t-\sigma^4 t^2\right)f_{\mathcal{H}}(t;\mu,\sigma) \tag{5}$$

(*cf.* [6]).

We note that modification Eq-(4) is also given in [6].

Resulted formulas and identities

In this section, we give identity and recurrence relation that involves the Hermite type polynomials with variable μ and σ and the $\mathbb{E}[(\mathcal{L}_B)]$.

Theorem 3. *Let $n \in \mathbb{N}_0$ with $n \geq 2$ Then we have*

$$\mathbb{E}[(\mathcal{L}_B)^n] = \frac{n}{2i\,(n-1)}\mathbb{E}[(\mathcal{L}_B)^{n-1}] - \sum_{j=0}^{n}\binom{n}{j}\frac{i^{j-n}}{2^{n-j}}\mathbb{E}^2[(\mathcal{L}_B)^j],$$

where

$$\mathbb{E}^2[(\mathcal{L}_B)^j] = \sum_{v=0}^{j}\binom{j}{v}\mathbb{E}[(\mathcal{L}_B)^v]\mathbb{E}[(\mathcal{L}_B)^{j-v}].$$

Proof. Combining Eq-(3) with Eq-(4), we get

$$\begin{aligned} & t\sum_{n=1}^{\infty}\mathbb{E}[(\mathcal{L}_B)^n]\,\frac{i^n t^{n-1}}{(n-1)!} - \sum_{n=0}^{\infty}\mathbb{E}[(\mathcal{L}_B)^n]\,\frac{(it)^n}{n!} - \frac{t}{2}\sum_{n=0}^{\infty}\mathbb{E}[(\mathcal{L}_B)^n]\,\frac{(it)^n}{n!} \\ = & -\sum_{n=0}^{\infty}\left(\frac{t}{2}\right)^n\frac{1}{n!}\left(\sum_{n=0}^{\infty}\mathbb{E}[(\mathcal{L}_B)^n]\,\frac{(it)^n}{n!}\sum_{n=0}^{\infty}\mathbb{E}[(\mathcal{L}_B)^n]\,\frac{(it)^n}{n!}\right). \end{aligned}$$

We then use Cauchy product on each component of above equation and applying some routine calculations, we obtain

$$\begin{aligned} & (n-1)\sum_{n=0}^{\infty}\mathbb{E}[(\mathcal{L}_B)^n]\,\frac{(it)^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty}n\mathbb{E}[(\mathcal{L}_B)^{n-1}]i^{n-1}\,\frac{t^n}{n!} \\ = & -\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{2^{n-j}}\mathbb{E}^2[(\mathcal{L}_B)^j]\,i^j\,\frac{t^n}{n!}, \end{aligned}$$

where

$$\mathbb{E}^2[(\mathcal{L}_B)^j] = \sum_{v=0}^{j}\binom{j}{v}\mathbb{E}[(\mathcal{L}_B)^v]\mathbb{E}[(\mathcal{L}_B)^{j-v}].$$

Finally, the comparison of the coefficient of $\frac{t^n}{n!}$ in both sides of the above equation, we reach at the proof of theorem. □

Theorem 4. *Let $n \in \mathbb{N}_0$ with $n \geq 2$. Then we have*

$$\mathcal{H}_{n+2}(\mu,\sigma) = -(\mu+\sigma^2)\mathcal{H}_n(\mu,\sigma) - 2in\mu\sigma^2\mathcal{H}_{n-1}(\mu,\sigma) + n(n-1)\sigma^4\mathcal{H}_{n-2}(\mu,\sigma). \quad (6)$$

Proof. Combining (2) and (5), we obtain

$$\begin{aligned} &\sum_{n=2}^{\infty}\mathcal{H}_n(\mu,\sigma)\frac{t^{n-2}}{(n-2)!} + (\mu+\sigma^2)\sum_{n=0}^{\infty}\mathcal{H}_n(\mu,\sigma)\frac{t^n}{n!} \\ = \quad & -2in\mu\sigma^2\sum_{n=0}^{\infty}\mathcal{H}_n(\mu,\sigma)\frac{t^{n+1}}{n!} + \sigma^4\sum_{n=0}^{\infty}\mathcal{H}_n(\mu,\sigma)\frac{t^{n+2}}{n!}. \end{aligned}$$

After some elementary computations, comparison of the coefficient of $\frac{t^n}{n!}$ in both sides of the above equation provides the proof of theorem. □

Notice that in [6], we give another recurrence relation for the hypergeometric function.

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2 PROCEEDINGS

Plane wave solution for a particle in a time- dependent linear potential

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Abstract

We studied the quantum motion of a particle in the presence of a time-dependent linear potential by using an operator invariant that is quadratic in p and x within the framework of the Lewis-Riesenfeld invariant. The special invariant operator in this work is demonstrated to be Hermitian operator that has a plane-waves as its eigenfunctions.

KEYWORDS: Time-dependent linear potential; Invariant operator; Unitary transformation; plane wave.

Introduction

The study of time-dependent quantum systems has drawn much attention over past decades not only for their fundamental physical perspective but also for their applicability in different areas of physics, such as in quantum transport [1], quantum optics [2], quantum information [3]. Recently, physicists have focused on the exact solution of one dimensional Schrodinger equation with time-dependent linear potential [4]-[9]. Starting first by the work of Guedes [4] who solved the Schrödinger equation in the framework of the Lewis- Riesenfeld approach [10], using a linear Hermitian operator and obtained a particular solution of the plane-wave type. Then, by means of the space-time transformation approach, Feng [5] obtained solutions of the-plane wave type and Airy wave-packet. However, Bauer [6] explained that the solution found by Guedes was simply a special case of the earlier found solution, proposed by Volkov. In this work, we apply the Lewis-Riesenfeld approach to solve the one-dimensional Schrödinger equation with a time-dependent linear potential, using a class of three Hermitian operators which are limiting forms of a general quadratic operator in the form $I(t) = \alpha(t)p^2 + \beta(t)(xp + px) + \gamma(t)x^2 + \eta(t)x + \rho(t)p + \delta(t)$. We show that the eigenstates of this operator depend strongly on the time-function $\alpha(t)$. Setting $\alpha(t)$ identically zero, the corresponding solutions will be of the plane-waves type.

Quadratic invariant and solutions of the Schrödinger equation

The Hamiltonian for the one-dimensional Schrödinger equation with a time dependent linear potential reads

$$H(t) = \frac{p^2}{2m} + f(t)x, \tag{1}$$

where $f(t)$ is a time-dependent function.

The problem is to find the solutions of Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H(t)\,\psi(x,t), \tag{2}$$

by means of the Lewis-Riesenfeld approach [?] and unitary transformations.

According to the theory of Lewis-Riesenfeld [10], a complete set of solutions of the Schrödinger equation (2), with a time-dependent Hamiltonian, is easily found if a nontrivial Hermitian operator $I(t)$ exists and satisfies the Liouville-von Neumann equation,

$$\frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} - \frac{i}{\hbar}\left[I(t), H(t)\right] = 0. \tag{3}$$

Then, $\varphi_\lambda(x,t)$ if are the eigenfunctions $I(t)$ of , corresponding to real time-independent eigenvalues λ,

$$I(t)\,\varphi_\lambda(x,t) = \lambda\varphi_\lambda(x,t),$$

we can find the corresponding solutions of the Schrödinger equation (2) in the form

$$\psi_\lambda(x,t) = e^{i\alpha_{\lambda(t)}}\varphi_\lambda(x,t), \tag{4}$$

where the global phases $\mu_\lambda(t)$ satisfy the following eigenvalues equation

$$\left(\frac{1}{i\hbar}H - \frac{\partial}{\partial t}\right)\varphi_\lambda(x,t) = i\dot{\alpha}_\lambda(t)\,\varphi_\lambda(x,t). \tag{5}$$

In this work, we look for a class of invariant operators which are at most quadratic with respect to position and momentum operators. The general form may be written as

$$I(t) = \alpha(t)\,p^2 + \beta(t)\,(xp+px) + \gamma(t)\,x^2 + \eta(t)\,x + \rho(t)\,p + \delta(t), \tag{6}$$

where the coefficients $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\eta(t)$, $\rho(t)$ and $\delta(t)$ are time-dependent real functions to be determined.

Substituting (1) and (6) into (3) and accomplishing the integrations, we obtain

$$\begin{aligned}
\gamma(t) &= \gamma_0,\\
\beta(t) &= \beta_0 - \frac{\gamma_0}{m}t,\\
\alpha(t) &= \alpha_0 - \frac{2\beta_0}{m}t + \frac{\gamma_0}{m^2}t^2,\\
\eta(t) &= \eta_0 + 2\beta(t)\,f_1(t) + \frac{2\gamma_0}{m}f_2(t),\\
\rho(t) &= \rho_0 - \frac{\eta_0}{m}t + 2\alpha(t)\,f_1(t) + \frac{2\beta(t)}{m}f_2(t),\\
\delta(t) &= \delta_0 + \left(\rho_0 - \frac{\eta_0}{m}t\right)f_1(t) + \frac{\eta_0}{m}f_2(t) + \alpha(t)\,f_1^2(t) + \frac{2\beta(t)}{m}f_2(t)\,f_1(t) + \frac{\gamma_0}{m^2}f_2^2(t
\end{aligned} \quad ($$

Where

$$f_s(t) = \int_0^t dt_s \ldots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)\,dt_1, \tag{8}$$

and the initial coefficients are real parameters that can be fixed judiciously. However, without loss of generality, the parameter δ_0 can be taken as zero.

It is worth noting that the construction of $I(t)$ the instantaneous eigenstates of depends strongly on the function $\alpha(t)$, that is to say, on the choice of the initial parameters α_0, β_0 and γ_0.

Plan wave solution

For $\alpha_0 = \beta_0 = \gamma_0 = 0$, i.e. $\alpha(t) \equiv 0$, without loss of generality, the parameters $\delta_0 = 0$ and $\rho_0 = 0$ and take $\eta_0 = 0$ the invariant (6) becomes linear in the position and momentum operators.

By using (7), it can be written as

$$I(t) = \bar{\rho}(t)\, p + \eta_0 x + \bar{\delta}(t) \tag{9}$$

Where

$$\begin{aligned} \bar{\rho}(t) &= \left(\rho_0 - \frac{\eta_0}{m} t\right) \\ \bar{\delta}(t) &= \left(\rho_0 - \frac{\eta_0}{m} t\right) f_1(t) + \frac{\eta_0}{m} f_2(t) \end{aligned} \tag{10}$$

To obtain its eigenvalues from Eq. (**??**), it is better to introduce the unitary transformation

$$\varphi_\lambda(x,t) = U(t)\, \bar{\varphi}_\lambda(x,t), \tag{11}$$

leading to the new eigenvalues equation

$$\widehat{I}\, \bar{\varphi}_\lambda(x,t) = \lambda \bar{\varphi}_\lambda(x,t), \tag{12}$$

where the transformed invariant, $\widehat{I} = U^{-1} I(t)\, U$, is a time-independent Hermitian operator.

As usual, the unitary operator $U(t)$ is taken as $U(t) = U_1(t)\, U_2(t)$, where $U_i(t)$ are given by

$$\begin{aligned} U_1(t) &= e^{-\frac{i}{\hbar}\left[A(t)x + B(t)x^2\right]}, \\ U_2(t) &= e^{\frac{iC(t)}{\hbar}[xp+px]}, \end{aligned} \tag{13}$$

Where

$$\begin{aligned} A(t) &= f_1(t) - \frac{|\eta_0|\, f_2(t)}{m\bar{\rho}(t)}, \\ B(t) &= -\frac{|\eta_0|}{2\bar{\rho}(t)}, \\ C(t) &= -\ln \bar{\rho}(t). \end{aligned} \tag{14}$$

The transformed invariant $\widehat{I}$ reads

$$\widehat{I} = p. \tag{15}$$

The eigenfunctions of $\widehat{I}$ in x coordinate is well known as

$$\bar{\varphi}_\lambda(x,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i\lambda x}{\hbar}} \tag{16}$$

Inserting (16) into (11) and using again (14) leads to

$$\varphi_\lambda(x,t) = \frac{1}{\sqrt{2\pi\hbar\bar{\rho}(t)}} \exp\left[\frac{i}{\hbar} \frac{\left(\lambda - \bar{\delta}(t)\right) x + \frac{|\eta_0|}{2} x^2}{\bar{\rho}(t)}\right] \tag{17}$$

Now, inserting (17) into (5), accomplishing the temporal and spatial derivation and then identifying the coefficients of the similar operators between the two sides, we get the phase functions as

$$\alpha_\lambda(t) = -\frac{1}{2\hbar m}\int_0^t \left(\frac{\lambda - \bar{\delta}(\tau)}{\bar{\rho}(\tau)}\right)^2 d\tau.$$

Therefore, the solutions of the Schrödinger equation (2) are given by

$$\psi_\lambda(x,t) = \frac{\exp\left(-\frac{1}{2\hbar m}\int_0^t \left(\frac{\lambda-\bar{\delta}(\tau)}{\bar{\rho}(\tau)}\right)^2\right) d\tau}{\sqrt{2\hbar m\bar{\rho}(t)}} \exp\left[\frac{i}{\hbar}\frac{\left(\lambda - \bar{\delta}(t)\,x + \frac{|\eta_0|}{2}x^2\right)}{\bar{\rho}(t)}\right]$$

Conclusion

In summary, we have presented a completely analytical solution of a system with a particle moving in a time-dependent linear potential. The Schrödinger equation for the system with a time-dependent linear potential is investigated on the basis of Lewis-Riesenfeld invariant theory. We confirm that the use of a Hermitian invariant operator leads to the the plane-waves solution.

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A Note on Relations Among Partitions

Busra Al[1] , *Mustafa Alkan*[2]

Abstract

In this paper, we give study the relation among partitions of numbers. We denote the numbers of partitions, odd part, even part, distinct even parts, unequal parts and distinct even parts. Then we investigate some relationships among. We also obtain three new recurrence formulas for the number of partitions of positive integer.

2010 Mathematics Subject Classifications : 05A17, 03E02.

Keywords: Number of Partition, Number of Odd Partition, Number of Odd and Unequal Partition, Number of Even Partition.

Introduction

For centuries, the partition of any positive integer, which is one of the fundamental problems of number theory, has attracted attention of researchers. The history of partition of any positive integer goes back to the discovery of some formulas which were introduced by many famous mathematicians such as S. Ramanujan, Jacobi, Leonard Euler and G.H. Hardy (*cf.* [1]-[9]).

In [3, Theorem 14.4], Euler investigated the generating function of the positive integer, which is also called by the partition function, is defined us

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)} = \sum_{n=0}^{\infty} p(n)x^n. \tag{1}$$

where $\mid x \mid < 1$ and $p(n)$ denotes of the number of partitions of positive integer n. The main problem of recurrence formulas is that $p(n)$ increase faster than the value of n. For this reason, it is more useful to perform operations with small positive integers in order to find the number of any partition. For example, in order to calculate $p(200) = 3.972.999.029.388$, we need to compute all values of $p(n)$ where $1 \leq n \leq 199$ by using the Euler's recurrence formula. It should be note that the recently improved recurrence formulas by J.A.Ewell [5, Theorem 1.2] and Merca [6, Theorem 1] are more usefull than the Euler's recurrence formula.

Some generating functions for the numbers of restricted partitions

In the literature, there are two kind of partitions which are the restricted partitions and the unrestricted partitions. For details about partitions, see the works (cf. [2], [4],[7],[8],[9], [3]).

The generating function for the numbers $Q(n)$, which is the number of all partitions of positive integer n into odd parts, is given as follows:

$$\prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} = \sum_{n=0}^{\infty} Q(n)x^n. \tag{2}$$

The generating function of the number of partitions of an positive integer n into even parts is

$$\prod_{n=1}^{\infty} \frac{1}{1-x^{2n}} = \sum_{n=0}^{\infty} E(n)x^n. \tag{3}$$

Theorem 1 (Euler recurrence formula). *Let $p(0) = 1$ and define $p(n)$ to be 0 if $n < 0$. Then for $n \geq 1$ we have*

$$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(n - w(\frac{3k^2-k}{2})) + p(n - w(\frac{3k^2+k}{2}))\}.$$

(cf. [3]).

In this paper, as usual, we here consider that $p(0) = Q(0) = 1$, $p(n) = Q(n) = 0$ whenever $n \in Q - N$ and $0 < x < 1$. Recall that special case of Jacobi's identity, is given as follows:

$$\prod_{n=1}^{\infty} \left(\left(1-x^{2na}\right)\left(1-x^{2na-a+b}\right)\left(1-x^{2na-a-b}\right)\right) = \sum_{m=-\infty}^{\infty} (-1)^m x^{m(am+b)} \tag{4}$$

(cf. [3]).

The relations among partitions

In this section, we give our main results.

Theorem 2. *For a positive integer n,*

$$p(n) = \sum_{i=0}^{\|\frac{n}{2}\|} E(2i)Q(n-2i).$$

Proof. Beginning with equation (1) proof, we get

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{1}{1-x^n} &= \prod_{n=1}^{\infty} \frac{1}{(1-x^{2n})(1-x^{2n-1})} \\ &= \prod_{m=1}^{\infty} \frac{1}{1-x^{2m}} \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}. \end{aligned}$$

Due to equation (3) and equation (2),

$$\begin{aligned} \sum_{l=0}^{\infty} p(l)x^l &= \sum_{m=0}^{\infty} E(m)x^m \sum_{k=0}^{\infty} Q(k)x^k \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} E(i)Q(n-i)x^n \end{aligned}$$

Thus

$$p(n) = \sum_{i=0}^{\|\frac{n}{2}\|} E(2i)Q(n-2i).$$

$\square$

Theorem 3. *For a positive integer n,*

$$n.Q(n) = o(n) + \sum_{l=1}^{n} 2(-1)^{l+1} Q(n-l^2)$$

where

$$o(n) = \begin{cases} (-1)^m, & \text{if } n = \frac{m}{2}(3m \pm 1) \text{ for some } m \in \mathbb{Z}^+ \\ 0 \quad , & \text{otherwise} \end{cases} .$$

Proof. Let $a = 1$ and $b = 0$ in the equation 4. And observe that the following identities;

$$\prod_{n=0}^{\infty} \left(1 - x^{2n+1}\right)\left(1 - x^{2n+1}\right)\left(1 - x^{2n+2}\right) = 1 + 2\sum_{m=1}^{\infty} (-1)^m x^{m^2} .$$

If both sides are multiplied by $\prod_{n=1}^{\infty} (1 + x^n)$,

$$(1-x)\left(\prod_{n=1}^{\infty} (1 - x^n)\right) = \left(\sum_{n=0}^{\infty} Q(n) x^n\right)\left(1 + 2\sum_{m=1}^{\infty} (-1)^m x^{m^2}\right). \tag{5}$$

The left side of the (5) equation from Euler Pentagonal Number Theorem;

$$\begin{aligned}(1-x)\left(\prod_{n=1}^{\infty} (1 - x^n)\right) &= (1 - 2x - x^3 + \sum_{m=2}^{\infty} (-1)^m x^{\frac{m}{2}(3m\pm1)} \\ &\quad - \sum_{l=2}^{\infty} (-1)^l x^{\frac{l}{2}(3l\pm1)+1}).\end{aligned} \tag{6}$$

To be able to write the series shorter, the common strengths of x in the series should be determined. The following statements can be observed for this:

$$\frac{m}{2}(3m-1) = \frac{t}{2}(3t+1)$$

There is no $m, t \in \mathbb{Z}^+$.

$$\frac{m}{2}(3m-1) = \frac{k}{2}(3k+1) + 1$$

There is no $m, k \in \mathbb{Z}^+$.

$$\frac{t}{2}(3t+1) = \frac{k}{2}(3k+1) + 1$$

There is no $k, t \in \mathbb{Z}^+$.

$$\frac{m}{2}(3m-1) + 1 = \frac{k}{2}(3k+1) + 1$$

There is no $m, k \in \mathbb{Z}^+$.

$$\frac{t}{2}(3t+1) = \frac{l}{2}(3l-1) + 1$$

This equation has a solution for $t = l = 1$ only in positive integers.Therefore, the x forces are different for each series in the (6) equation. On the other hand, if the necessary actions are made on the right side of the equation (5), the following equation is obtained.

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} Q(n)x^n\right)\left(1+2\sum_{m=1}^{\infty}(-1)^m x^{m^2}\right)\\ = \quad & Q(0)+\sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\left[Q(n)+2(-1)^l Q(n-l^2)\right]\right)x^n \end{aligned} \tag{7}$$

Using coefficients (6) and (7), the coefficients x^n can be compared to the equation (5).

$$n=\frac{m}{2}(3m-1) \text{ or } n=\frac{m}{2}(3m+1), \tag{8}$$

If there are $m \in \mathbb{Z}^+$ such that 8, the following equation can be observed by comparing the x^n coefficients in the equation.

$$\sum_{l=1}^{n}\left[Q(n)+2(-1)^l Q(n-l^2)\right]=(-1)^m$$

Otherwise

$$\sum_{l=1}^{n}\left[Q(n)+2(-1)^l Q(n-l^2)\right]=0.$$

□

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Exploring Non-Convex Mixtures

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Abstract

Mixtures with negative weights are often studied in statistics insofar as they can be applied in several practical issues, such as reliability questions. In addition, simple conditions can be imposed to obtain a mixture with negative weights when dealing with shape-extended stable distributions. Therefore, the main purpose of this work is to highlight those mixtures properties as well as to deal with estimation issues. The performance of some usual estimators is also analysed under simulation.

2010 Mathematics Subject Classifications : 60E07, 65C50
Keywords: Pseudo-convex mixtures, shape-extended stable distributions, parametric estimation.

Introduction

Mixtures with negative weights have been studied since last century (cf. [1, 7]), specially in what concerns exponential distribution mixtures. In fact, this distribution has relevant properties in reliability and is easy to deal with.

Parametric estimation is always relevant in this context because it is mandatory for data fitting purposes. Therefore, and with the increasing of computational power, the evaluation of the performance of different estimators under simulation and real data fitting are common questions when studying this type of mixtures (cf. [3, 5, 6]).

In [4, 6] is introduced a shape-extended definition of distributions closed under minimization (maximization). This new family of distributions is later used to define pseudo-convex mixtures generated by shape-extended stable distributions for extremes. The properties of this mixtures will be used in this work for the power-function distribution.

Hence, let us consider $X_1, ..., X_n$ as a sequence of independent and identically distributed (i.i.d.) continuous random variables (r.v.) with distribution function (d.f.) F, and survival function (s.f.) $\overline{F}$. Let $X_{i:n}$ be the associated i-th ascending order statistics. Then $X_{1:n} = \min\{X_1, ..., X_n\}$ and $X_{n:n} = \max\{X_1, ..., X_n\}$. Thus,

$$F_{X_{n:n}}(x) = \mathbb{P}\left(X_{n:n} \le x\right) = \mathbb{P}\left(\bigcap_{i=1}^{n}\{X_i \le x\}\right) = F^{n}(x), \forall x \in \mathbb{R}.$$

Moreover, if

$$F_{X_{n:n}}(x) = F\left(\alpha_n x + \beta_n\right), \forall x \in \mathbb{R} \text{ with } \alpha_n \in \mathbb{R}^+, \beta_n \in \mathbb{R}$$

where α_n and β_n are the scale and location changes, then F is stable for maxima or a max-stable distribution [2]. The previous definition can be extended in order to allow changes to the shape parameter as well (cf. [4, 6]). We say that F is shape-extended stable for maxima (SEmaxS) if

$$F_{X_{n:n}}(x) = F_{\gamma_n}\left(\alpha_n x + \beta_n\right), \forall x \in \mathbb{R}, \tag{1}$$

where $\gamma_n \in \mathbb{R}$ represents the shape parameter change. The SEmaxS family includes the generalized extreme value distribution for maximum, the generalized logistic type I distribution and the power function distribution.

The same procedure can be applied to obtain shape-extended distributions for minimum (SEminS). This family includes generalized extreme value distribution for minima, generalized logistic type II distribution and generalized Pareto distribution. General expressions for moments, densities, hazard rates and other properties were derived in [4] and [6].

From now on we will focus on SEmaxS family, specially in what concerns to the power function. Let X be a SEmaxS distribution with d.f. F. Thus, the r.v. X_M with d.f. F_{X_M} given by

$$F_{X_M}(x) = (1-\omega)\, F(x) + \omega F_{X_{2:2}}(x) = F(x)\left[1 - \omega \overline{F}(x)\right], \tag{2}$$

with $\omega \in [-1, 1]$, is a pseudo-convex mixture (PCM) generated by the SEmaxS distribution F. The above formula can be used with the power function distribution, leading to the results presented in the next section.

Main Results

Let X be a r.v. with power function (PF) distribution with d.f. $F(x) = x^\gamma$, $x \in (0, 1)$ and $\gamma \in \mathbb{R}^+$. As PF is a SEmaxS distribution, then the r.v. X_M (PCM_{PF}) has for d.f.

$$F_{X_M}(x) = x^\gamma(1-\omega) + \omega x^{2\gamma}, \quad \omega \in [-1, 1]. \tag{3}$$

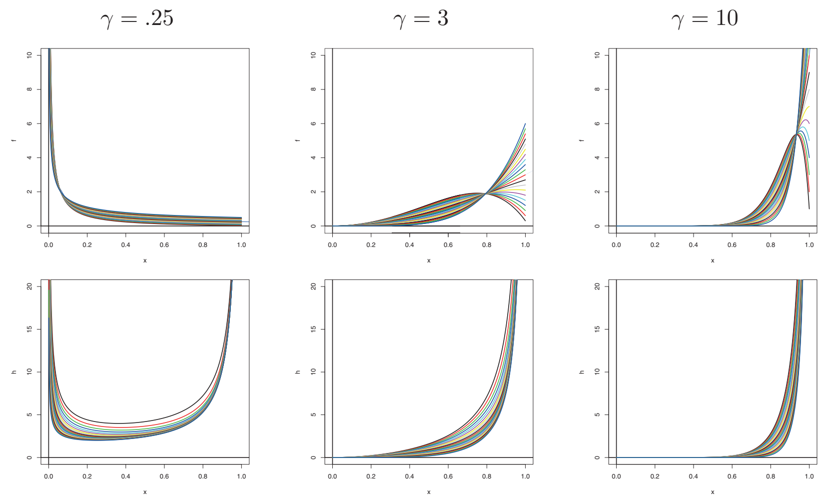

Figure 1 – Density and hazard rate for PCM_{PF} with $\omega = -1 + 0.1k$, $k = 0, 1, \ldots, 20$

Equation (3) was used to obtain the mixture density and, after that, the expressions for the parameters estimators were obtained using the moments method (explicit) and the maximum likelihood method (implicit).

To evaluate the quality of the estimators, a simulation study was performed using 10^3 replicas in the software R. The estimators were derived using the method of moments (MME) and the maximum likelihood (MLE). Their performance was assessed by the bias, the absolute relative bias (ARB) and the mean square deviation (MSD).

Different scenarios were considered, where sample dimension $n \in \{100, 1000\}$, parameters values $\omega \in \{-.75, -.5, -.25, 0, .25, .5, .75\}$ and $\gamma \in \{0.25, 3, 10\}$. The MLE method requires an initial value and, therefore, the method of moments estimates were applied. Besides, in one case, $(\gamma_0, \omega_0) = (\overline{x}/(1-\overline{x}), 0)$ was used in order to assess the sensitivity of the estimator to the initial value.

	MME — Method of moments estimator							**MLE** — Maximum likelihood estimator						
ω	−.75	−.50	−.25	.00	.25	.50	.75	−.75	−.50	−.25	.00	.25	.50	.75
	$\mathrm{PCM_{PF}}$, $\gamma = 3$, $n = 100$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	.0307	.0272	.0112	.0057	.1003	.4067	1.177	.0349	.0193	.0111	.0306	.1575	.4724	1.203
ARB	.0863	.1022	.1297	.1665	.2012	.2362	.4101	.0843	.1006	.1280	.1591	.1899	.2398	.4227
MSD	.1071	.1522	.2484	.3797	.5302	.8637	2.532	.1113	.1527	.2471	.3569	.5064	.8888	2.619
	$\mathrm{PCM_{PF}}$, $\gamma = .25$, $n = 1000$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	–.001	–.002	–.009	–.001	.0213	.0671	.1350	.0000	–.001	.0002	–.001	–.002	.0040	.0449
ARB	.0537	.0848	.1385	.1819	.1814	.2700	.5401	.0257	.0325	.0379	.0519	.0818	.1151	.2032
MSD	.0003	.0008	.0020	.0027	.0031	.0077	.0224	.0001	.0003	.0002	.0003	.0008	.0012	.0046
	$\mathrm{PCM_{PF}}$, $\gamma = 3$, $n = 1000$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	.0026	.0108	.0076	–.004	–.033	.1017	.8567	.0013	.0104	.0081	–.002	–.021	.0576	.5387
ARB	.0265	.0311	.0392	.0533	.0900	.1022	.2856	.0243	.0297	.0389	.0526	.0847	.1085	.2042
MSD	.0100	.0133	.0214	.0425	.1133	.1404	.9575	.0082	.0125	.0213	.0427	.1138	.1609	.6518
	$\mathrm{PCM_{PF}}$, $\gamma = 10$, $n = 1000$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	.0191	.0100	.0238	–.002	–.058	.1758	2.154	.0017	.0077	.0137	–.038	–.101	.1768	1.661
ARB	.0300	.0341	.0404	.0518	.0806	.1095	.2169	.0245	.0296	.0373	.0507	.0834	.1094	.1933
MSD	.1380	.1809	.2552	.4284	1.060	1.660	7.249	.0932	.1425	.2152	.462	1.234	1.799	6.691
	$\mathrm{PCM_{PF}}$, $\gamma = 3$, $n = 1000$ and $(\gamma_0, \omega_0) = (\overline{x}/(1-\overline{x}), 0)$													
Bias	.0044	.0041	.0016	–.001	–.038	.1315	.8539	.0033	.0032	.0025	.0036	–.109	.0369	.4615
ARB	.0266	.0307	.0383	.0509	.0868	.1072	.2847	.0252	.0298	.0381	.0494	.1084	.1188	.1884
MSD	.0102	.0135	.0211	.0367	.1061	.1521	.9584	.0091	.0126	.0210	.0350	.2045	.1885	.6097

Table 1 — γ estimation for $\mathrm{PCM_{PF}}$

	MME — Method of moments estimator							**MLE** — Maximum likelihood estimator						
ω	−.75	−.50	−.25	.00	.25	.50	.75	−.75	−.50	−.25	.00	.25	.50	.75
	$\mathrm{PCM_{PF}}$, $\gamma = 3$, $n = 100$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	–.001	–.005	.0077	.0138	–.031	–.170	–.450	.0691	.0031	.0118	.0012	–.061	–.202	–.460
ARB	.1995	.3793	.9514	–	1.265	.6413	.6110	.0988	.3680	.9322	–	1.169	.6238	.6311
MSD	.0344	.0584	.0952	.1240	.1321	.1556	.3422	.1250	.0602	.0958	.1154	.1223	.1572	.3534
	$\mathrm{PCM_{PF}}$, $\gamma = 0.25$, $n = 1000$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	.0060	.0173	.0696	.0420	–.076	–.301	–.587	.0011	.0015	.0015	.0056	.0172	–.019	–.218
ARB	.0771	.2063	.7688	–	.9535	.6024	.7826	.0571	.1089	.2599	–	.5315	.3283	.3270
MSD	.0056	.0211	.0708	.0856	.0727	.1510	.4030	.0029	.0046	.0066	.0150	.0333	.0386	.1070
	$\mathrm{PCM_{PF}}$, $\gamma = 3$, $n = 1000$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	–.000	–.001	–.003	.0054	.0203	–.041	–.347	.0012	–.001	–.003	.0043	.0140	–.019	–.218
ARB	.0665	.1140	.2798	–	.5757	.2947	.4629	.0558	.1055	.2770	–	.5457	.3105	.3305
MSD	.0039	.0050	.0076	.0141	.0328	.0300	.1533	.0027	.0044	.0075	.0147	.0346	.0355	.1051
	$\mathrm{PCM_{PF}}$, $\gamma = 10$, $n = 1000$ and $(\gamma_0, \omega_0) = (\gamma_{\mathrm{MME}}, \omega_{\mathrm{MME}})$													
Bias	–.007	–.002	–.002	.0014	.0117	–.020	–.263	–.002	–.001	.0003	.0087	.0202	–.018	–.199
ARB	.0847	.1299	.3045	–	.5258	.3207	.3510	.0571	.1048	.2739	–	.5454	.3212	.3073
MSD	.0061	.0069	.0090	.0133	.0283	.0336	.1039	.0028	.0044	.0073	.0148	.0341	.0376	.0957
	$\mathrm{PCM_{PF}}$, $\gamma = 3$, $n = 1000$ and $(\gamma_0, \omega_0) = (\overline{x}/(1-\overline{x}), 0)$													
Bias	–.002	–.001	–.000	–.000	.0237	–.059	–.346	–.001	.0001	–.000	–.002	.0257	–.013	–.187
ARB	.0648	.1138	.2809	–	.5501	.3109	.4619	.0548	.1098	.2790	–	.5318	.3423	.3037
MSD	.0037	.0052	.0078	.0123	.0303	.0333	.1544	.0026	.0046	.0077	.0120	0.0331	.0429	.0990

Table 2 — ω estimation for $\mathrm{PCM_{PF}}$

Conclusion

The performance of γ and ω estimators improves when ω decreases. Good results were achieved when the mixture is not convex $(-1 < \omega < 0)$ albeit the results are not so good when it is convex $(0 < \omega < 1)$. As expected, results were better when the sample dimension increases. Furthermore, the MLE almost always outperforms MME, as is usual for most distributions.

The changes in the γ parameter value do not seem to have great impact on the quality of the estimates, in relative terms. Moreover, different initial values for the MLE gave rise to very similar estimates and therefore do not reveal sensitivity to the initial value.

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Comparing estimation in batched tests using one and two-dimensional arrays via simulation

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Abstract

Pooling individual samples for batch testing is a common procedure for reducing costs. Robotic pooling emergence led to the use of more complex pooling procedures such as two-dimensional arrays. Recently, an algorithm to estimate the prevalence rate was established using this type of arrays without requiring the performance of any individual test. Assuming non-perfect tests, a simulation study is performed to assess estimator's quality and to provide some guidelines for the application of this new procedure.

2010 Mathematics Subject Classifications : 62F10, 62P10
Keywords: Estimation, prevalence rate, pooled samples, array, simulation.

Introduction

Since mid-20th century, the use of pooled samples for screening infected individuals with reduced costs has generated an increasing interest. Dorfman's methodology [1] is possibly the most known procedure and it comprehends two stages. In the first stage a pool of n individuals is homogeneously mixed for batched testing. A negative result determines the classification of all n individuals as non-infected. On the other hand, a positive result means that at least one of the individuals is infected. Hence, further n individuals tests are performed in order to determine who is really infected. This is called a one-dimensional array since it uses non-overlapping pools.

When the goal is to determine how many individuals are infected rather than identify who is infected, individual test is only optional. Thus, procedures with less experimental tests should now be considered. Moreover, more complex schemes of mixing samples can now be easily used with the advent of robotic pooling. The automatization of the process turns the chance of errors due to the mixing process close to zero. Furthermore, costs of mixing samples are usually negligible [5].

Array-based group testing is a two or higher dimensional alternative that uses overlapping pools. In particular, square arrays (a two dimensional array) use a sample of size n^2 placed in a $n \times n$ matrix. We will refer to this kind of array as an array of size n. All individuals within the same row and the same column are gathered for batched testing. Ambiguous results may arise if the experimental test is not gold standard. For instance, in a square array procedure one may have a positive row but all columns may test negative. Clearly, the use of the proportion of infected individuals may be a biased estimate, and sometimes impossible to determine.

An expression for the maximum likelihood (ML) estimator when one-dimensional arrays are used may be found in [9]. However, it only provides meaningful estimates when the inverse of the number of performed tests is inside the interval defined by

$[1 - Se, Sp]$ where Se and Sp stand for the test sensitivity and the test specificity, respectively, for an individual sample.

A first attempt to provide some computational guidelines to derive a ML estimate for the prevalence rate using square arrays is presented in [6] and [7]. Later, [9] argues that the use of simulation to compute the ML estimate may not be advised for square arrays. It is suggested a computational script with Se, Sp and the number of arrays having $i-1$ positive rows and $j-1$ positive columns for $i = 1, 2, \cdots, r+1$ and $j = 1, 2, \cdots, c+1$ stored in a $(r+1) \times (c+1)$ matrix O (observed values) as the inputs. The target function is

$$Dif(p_0|O) = \sum_{i,j} \left(O(i,j) - s \times P_{p_0}(i,j)\right)^2, \qquad (1)$$

where s is the total number of two-dimensional arrays (i.e., $s = \sum_{i,j} O(i,j)$). The matrix P_{p_0} includes estimates of the probability for each entry of matrix O given a prevalence rate p_0. The value that minimizes this function can be used to estimate the prevalence rate.

A short simulation described in [6] points out that two-dimensional arrays may be an option (and sometimes it is the only available option) when the experimental tests have a perfect or at least close to perfect performance.

A simulation study

Our simulation study intends to provide some guidance about the optimal square array size to be used in an estimation problem. We used MatLab R2011 software.

Matrix P_{p_0} was estimated using 200 random replicas of matrices O for each prevalence rate $p_0 = 0.001 \times k$, with $k = 1, \ldots, 400$ (i.e., with p_0 from 0.001 until 0.40 with increments equal to 0.001). The simulated real prevalence ranged from 0.05 to 0.25 with increments of 0.05. The number of square arrays s was set equal to 50. The number of replicas in each simulation was 200.

We restricted the simulation to tests with high specificity ($Sp = 0.98$). Concerning sensitivity, it was considered the values within the set $\{0.6, 0.7, 0.8\}$. A higher sensitivity was not considered as it would lead to a very accurate test, a case in which the use of Dorfman's procedure is advised [9]. The accuracy of the estimates was assessed using the root mean square error (RMSE). Simulation results are displayed in Figure 3 (obtained using IBM SPSS Statistics 25 software).

A high prevalence rate in a high size array generates almost all rows and columns positive. In this way, it was already expected the decrease of the optimal array size with the prevalence rate increase.

The optimal array size is not quite different for each value of the prevalence rate p. For $p = 0.05$ it is equal to 10 or 11 whereas for a $p = 0.25$ it varies from 3 to 5. Hence, as rule of thumb, the optimal array size is approximately equal to $12.3 - 32 \times p_0$. However, this only makes sense when there is an *a priori* knowledge about the value of the true prevalence rate p.

It is surprising to observe that the decrease of the test sensitivity causes a small effect in the RMSE. Hence, the use of the "worse" (cheap) tests is advised in this setting. Figure 3 shows similar accurate estimates for tests that can differ 20% in what concerns to sensitivity!

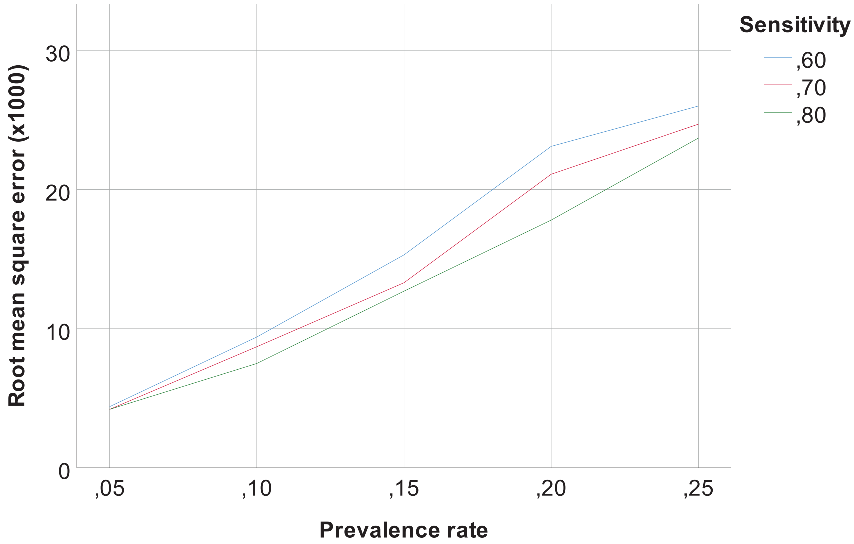

Figure 1: RMSE ($\times$1000)

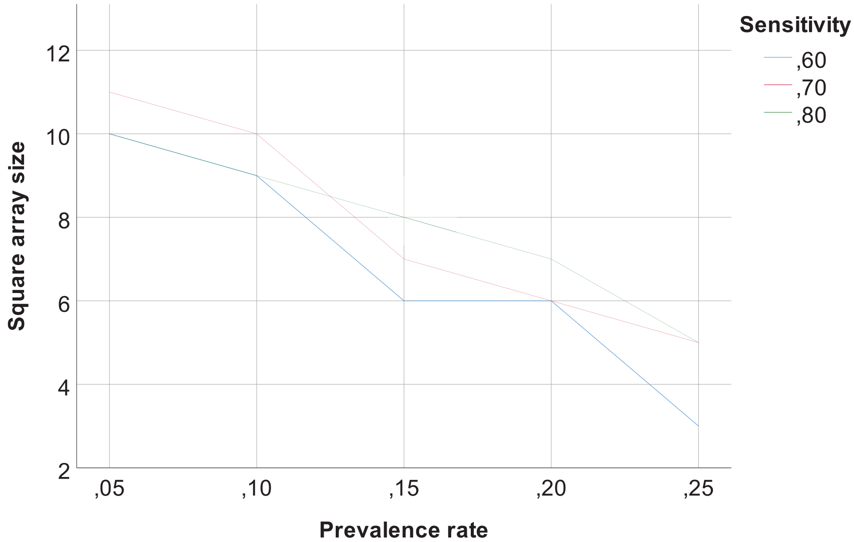

Figure 2: Optimal array size

Figure 3: Root mean square error ($\times$1000) (left) and optimal array size (right) for several prevalence rates

Conclusion

When dealing with a classification problem (screening all the infected individuals) the use of pooled samples is not very common although in some situations being an option to save money [2]. However, when dealing with the estimation problem, the researcher is allowed to use experimental tests with low sensitivity without losing significant accuracy.

The performance of two–dimensional arrays still needs further assessment. In fact, the use of other hierarchical models (with more than one stage) may provide different results for the square array optimal size [4]. However, [8] did not find any improvements when using three-dimensional arrays in the estimation problem.

Square arrays are just a particular case of two-dimensional arrays. Although there are some reasons to believe that they lead to the optimal choice, it remains an open question [3].

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A generalization of incomplete gamma function

Aykut Ahmet Aygunes[a]

Abstract

In this paper, we firstly define the polylogarithms and incomplete gamma function. For our main result, we introduce a generalization of incomplete gamma function. Then, by using the integral representation of polylogarithms, we obtain a relation between polylogarithms and a general case of incomplete gamma function.

2010 MATHEMATICS SUBJECT CLASSIFICATIONS : 11G55, 11M35, 33B15, 33B20.
KEYWORDS: Polylogarithms, incomplete gamma function, a generalization of incomplete gamma function.

Introduction, definitions and preliminaries

Throughout this article, we use the following standard notations:

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Also,

$$\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and the n-th derivative of any function f at z_0 is denoted by $f^{(n)}(z_0)$.

The polylogarithm (or de Jonquiére's function) $Li_s(z)$ (cf. [5]) is defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1)$$

$$(s \in \mathbb{C} \text{ when } |z| < 1; \operatorname{Re}(s) > 1 \text{ when } |z| = 1).$$

where $\Phi(z, s, w)$ is the Hurwitz-Lerch zeta function (cf. [1], [4]) defined by

$$\Phi(z, s, w) = \sum_{n=0}^{\infty} \frac{z^n}{(n+w)^s}$$

$$(s \in \mathbb{C} \text{ when } |z| < 1; \operatorname{Re}(s) > 1 \text{ when } |z| = 1)$$

for $w \in \mathbb{C}\backslash\{0, -1, -2, -3, \cdots\}$.

The integral representation of Hurwitz-Lerch zeta function is as follows (cf. [4]):

$$\Phi(z, s, w) = \frac{(-1)^{s-1}}{\Gamma(s)} \int_0^1 \frac{(\log t)^{s-1} t^{w-1}}{1 - zt} dt \tag{1}$$

or

$$\Phi(z, s, w) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-wt}}{1 - ze^{-t}} dt. \tag{2}$$

Let $N \in \mathbb{N}_0$. For $w = 1$ and $s = N + 1$, by using equation (1), we obtain

$$\begin{aligned} Li_{N+1}(z) &= z\Phi(z, N+1, 1) \\ &= z\frac{(-1)^N}{N!}\int_0^1 \frac{(\log t)^N}{1-zt}dt. \end{aligned} \tag{3}$$

By choosing $z = 1$ in (3), the polylogarithms can be reduced to Riemann zeta function $\zeta(N+1)$ (cf. [3]) given by

$$\zeta(N+1) = \sum_{n=1}^{\infty} \frac{1}{n^{N+1}}$$

where $N > 0$.

Also, we introduce the incomplete gamma function $\Gamma(s, x)$ (cf. [6]):

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1}e^{-t}dt \tag{4}$$

where $\mathrm{Re}(s) > 0$ and $x \in \mathbb{R}$.

By choosing $x = 0$ in (4), we obtain the classical Euler gamma function (cf. [6]) given by

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt.$$

Polylogarithms and incomplete gamma function are useful functions in Analytic Number Theory and Mathematical Physics.

A generalization of incomplete gamma function

In this section we introduce a generalization of incomplete gamma function $\Gamma_{\mu,z}(s, x)$. Then, by using the integral representation of polylogarithms $Li_{N+1}(z)$, we obtain a relation between polylogarithms and a special case of incomplete gamma function.

Recently, some authors have studied on generalizations of incomplete gamma function $\Gamma(\alpha, x; \beta)$ for $\mathrm{Re}(\alpha) > 0$ and $\beta \in \mathbb{C}$.

In [7], Chaudhry and Zubair studied on the generalized incomplete gamma function given by the following integral:

$$\Gamma(\alpha, x; \beta) = \int_x^{\infty} t^{\alpha-1}e^{-t-\beta t^{-1}}dt. \tag{5}$$

In [2], Miller derived several reduction formulas for specializations of a certain generalized incomplete gamma function $\Gamma(\alpha, x; \beta)$ and its associated Kampé De Fériet function.

It is possible to define a generalization of different type from equation (5). Let $\mu \in \mathbb{R}$. Then, we define

$$\Gamma_{\mu,z}(N+1, x) = \int_x^{\infty} \frac{t^N e^{-\mu t}}{1-ze^{-\mu t}}dt.$$

In this section, we claim that $\Gamma_{\mu,z}(s, x)$ is associated with polylogarithms. Firstly, we give the following key theorem for our claim:

Theorem 1. *Let $b > a > 0$ and $N \in \mathbb{N}_0$. Then, we have*

$$\int_{\log a}^{\log b} \frac{t^N e^t}{1 - ze^t} dt = \frac{1}{z} \sum_{k=0}^{N} \binom{N}{k} (-1)^k k! \left\{ (\log b)^{N-k} Li_{k+1}(bz) - (\log a)^{N-k} Li_{k+1}(az) \right\}.$$

Remark 1. *By choosing $a = 1$ in Theorem 1, we have*

$$\int_{0}^{\log b} \frac{t^N e^t}{1 - ze^t} dt = \frac{1}{z} \left\{ -Li_{N+1}(z)(-1)^N N! + \sum_{k=0}^{N} \binom{N}{k} (-1)^k k! (\log b)^{N-k} Li_{k+1}(bz) \right\}. \tag{6}$$

By using the equation (6), we obtain the following corollary:

Corollary 2. *Let $c, \mu \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Then, we have*

$$\Gamma_{\mu,z}(N+1, c) = \int_{c}^{\infty} \frac{v^N e^{-\mu v}}{1 - ze^{-\mu v}} dv = \frac{N!}{\mu^{N+1} z} \sum_{k=0}^{N} \frac{(\mu c)^{N-k}}{(N-k)!} Li_{k+1}(e^{-\mu c} z).$$

Remark 2. *By choosing $\mu = 1$ in Corollary 2, we have*

$$\lim_{z \to 0} \Gamma_{1,z}(N+1, c) = \Gamma(N+1, c).$$

Therefore, we note that $\Gamma_{\mu,z}(N+1, c)$ is a generalization of the incomplete gamma function $\Gamma(N+1, c)$.

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Quadrature Formulas with Multiple Nodes for Fourier Coefficients

Miodrag M. Spalević [1]

Abstract

Gaussian quadrature formulas with multiple nodes and their optimal extensions for computing the Fourier coefficients, in expansions of functions with respect to a given system of orthogonal polynomials, are considered. A numerically stable construction of these quadratures is proposed. Error bounds for these quadrature formulas are derived. We present a survey of recent results on this topic.

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Keywords: Gaussian quadrature formulas with multiple nodes, Fourier coefficients, error bound.

Introduction

Let $\{P_k\}_{k=0}^{\infty}$ be a system of orthonormal polynomials on $[a,b]$ with respect to a weight function ω (integrable, non-negative function on $[a,b]$ that vanishes only at isolated points). The approximation of f by the partial sums $S_n(f)$ of its series expansions $f(x)=\sum_{k=0}^{\infty} a_k(f)P_k(x)$ with respect to a given system of orthonormal polynomials $\{P_k\}_{k=0}^{\infty}$ is a classical way of recovery of f. The numerical computation of the coefficients $a_k(f)$,

$$a_k(f)=\int_a^b \omega(t)P_k(t)f(t)\,dt,$$

requires the use of a quadrature formula. Evidently, an application of the n-point Gaussian quadrature formula with respect to the weight ω will give the exact result for all polynomials of degree at most $2n-k-1$, $k<2n-1$.

Following Bojanov and Petrova [1] and using the same notation, we consider quadrature formulas of the type

$$\int_a^b \omega(t)P_k(t)f(t)\,dt \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji}f^{(i)}(x_j), \quad a<x_1<\cdots<x_n<b, \tag{1}$$

where ν_j are given natural numbers (multiplicities) and $P_k(t)$ is a monic polynomial of degree k.

In [1], for the sake of convenience, Bojanov and Petrova defined the formula (1) to be Gaussian, if it has maximal algebraic degree of precision ADP.

Let

$$\pi_n(\mathbb{R}):=\left\{P(t)\ :\ P(t)=\sum_{k=0}^{n} d_k t^k,\ d_k\in\mathbb{R}\right\}$$

represents the space of all polynomials in one variable of degree at most n. Bojanov and Petrova [1, Section 2] discuss general remarks concerning Gaussian quadrature formulas with multiple nodes, since the study of formulas of type (1) for Fourier coefficients can be reduced to the study of standard multiple node quadratures. We repeat the following theorem established by Ghizzetti and Ossicini [2].

Theorem 1. *For any given set of odd multiplicities $\nu_1, \dots, \nu_n$ ($\nu_j = 2s_j + 1,\ s_j \in \mathbb{N}_0,\ j = 1, \dots, n$), there exists a unique quadrature formula of the form*

$$\int_a^b \omega(t) f(t)\, dt \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \quad a \le x_1 < \cdots < x_n \le b,$$

of ADP $= \nu_1 + \cdots + \nu_n + n - 1$*, which is the well known Chakalov-Popoviciu quadrature formula. The nodes $x_1, \dots, x_n$ of this quadrature are determined uniquely by the orthogonality property*

$$\int_a^b \omega(t) \prod_{k=1}^{n} (t - x_k)^{\nu_k} Q(t)\, dt = 0, \quad \forall Q \in \pi_{n-1}(\mathbb{R}).$$

The corresponding (monic) orthogonal polynomial $\prod_{k=1}^{n}(t - x_k)$ is known in the classical literature as σ-orthogonal polynomial, with $\sigma = \sigma_n = (s_1, \dots, s_n)$, where n indicates the size of the array.

Bojanov and Petrova [1] describe the connection between quadratures with multiple nodes and formulas of type (1). For the system of nodes $\mathbf{x} := (x_1, \dots, x_n)$ with corresponding multiplicities $\bar{\nu} := (\nu_1, \dots, \nu_n)$, they define the polynomials

$$\Lambda(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m), \quad \Lambda_j(t; \mathbf{x}) := \frac{\Lambda(t; \mathbf{x})}{t - x_j}, \quad \Lambda^{\bar{\nu}}(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m)^{\nu_m},$$

set $x_j^{\nu_j} := (x_j, \dots, x_j)$ [x_j repeats ν_j times], $j = 1, \dots, n$, denote by $g[x_1, \dots, x_m]$ the divided difference of g at the points $x_1, \dots, x_m$, and state and prove the following important theorem which reveals the relation between the standard quadratures and the quadratures for Fourier coefficients.

Theorem 2. *For any given sets of multiplicities $\bar{\mu} := (\mu_1, \dots, \mu_k)$ and $\bar{\nu} := (\nu_1, \dots, \nu_n)$, and nodes $y_1 < \cdots < y_k$, $x_1 < \cdots < x_n$, there exists a quadrature formula of the form*

$$\int_a^b \omega(t) \Lambda^{\bar{\mu}}(t; \mathbf{y}) f(t)\, dt \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \tag{2}$$

with ADP $= N$ *if and only if there exists a quadrature formula of the form*

$$\int_a^b \omega(t) f(t)\, dt \approx \sum_{m=1}^{k} \sum_{\lambda=0}^{\mu_m - 1} b_{m\lambda} f^{(\lambda)}(y_m) + \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \tag{3}$$

which has degree of precision $N + \mu_1 + \cdots + \mu_k$. In the case $y_m = x_j$ for some m and j, the corresponding terms in both sums combine in one term of the form

$$\sum_{\lambda=0}^{\mu_m + \nu_j - 1} d_{m\lambda} f^{(\lambda)}(y_m).$$

Main Results

Let us suppose that the coefficients a_{ji} $(j = 1, \dots, n;\ i = 0, \dots, \nu_j - 1)$ in (3) are known. By acting as in the first part of the proof of Theorem 2.1 in [1] we can determine the coefficients c_{ji} $(j = 1, \dots, n;\ i = 0, \dots, \nu_j - 1)$ in (2). Namely, applying (3) to the polynomial $\Lambda^{\bar{\mu}}(\cdot; \mathbf{y})f$, where $f \in \pi_N(\mathbb{R})$, the first sum in (3) vanishes and we can obtain (see [1, Eq. (2.4)])

$$\int_a^b \omega(t)\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\,dt = \sum_{j=1}^{n}\left(\sum_{i=0}^{\nu_j-1} a_{ji}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\right]^{(i)}\Big|_{t=x_j}\right) = \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji}f^{(i)}(x_j),$$

where

$$c_{ji} = \sum_{s=i}^{\nu_j-1} a_{js}\binom{s}{i}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})\right]^{(s-i)}\Big|_{t=x_j} \quad (j = 1, 2, \dots, n;\ i = 0, 1, \dots, \nu_j - 1). \tag{4}$$

In [4], for a Chakalov-Popoviciu quadrature formula of type

$$\int_a^b \omega(t)f(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} a_{\nu i}f^{(i)}(x_\nu), \tag{5}$$

where $a \le x_1 < x_2 < \cdots < x_n \le b$, it was studied its extension to the interpolatory quadrature formula

$$\int_a^b \omega(t)f(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} b_{\nu i}f^{(i)}(x_\nu) + \sum_{\mu=1}^{m}\sum_{j=0}^{2s_\mu^*} c_{\mu j}^* f^{(j)}(x_\mu^*), \tag{6}$$

where x_ν are the same nodes as in (5), and the new nodes x_μ^* and new weights $b_{\nu i}, c_{\mu j}^*$ are chosen to maximize the degree of precision of (6), which is greater than or equal to

$$\sum_{\nu=1}^{n}(2s_\nu + 1) + \sum_{\mu=1}^{m}(2s_\mu^* + 1) + m - 1 = 2\left(\sum_{\nu=1}^{n} s_\nu + \sum_{\mu=1}^{m} s_\mu^*\right) + n + 2m - 1.$$

The interpolatory quadrature formula (6) has in general ADP$= \sum_{\nu=1}^{n}(2s_\nu + 1) + \sum_{\mu=1}^{m}(2s_\mu^* + 1) - 1$ which is higher than the ADP of the quadrature formula (5), i. e. $\sum_{\nu=1}^{n}(2s_\nu + 1) + n - 1$, if

$$2\sum_{\mu=1}^{m} s_\mu^* + m > n.$$

If there exist unique quadrature formulas (5), (6), then Theorem 2 implies that there exist unique quadratures for calculating the integrals

$$\int_a^b \omega(t)f(t)\pi_{n,\sigma}(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu-1} \widehat{a}_{\nu i}f^{(i)}(x_\nu), \tag{7}$$

and

$$\int_a^b \omega(t)f(t)\pi_{n,\sigma}(t)\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu-1} \widehat{b}_{\nu i}f^{(i)}(x_\nu) + \sum_{\mu=1}^{m}\sum_{j=0}^{2s_\mu^*} \widehat{c}_{\mu j}^* f^{(j)}(x_\mu^*), \tag{8}$$

which represent the Fourier coefficients if the given σ-orthogonal polynomial $\pi_{n,\sigma}$ coincides to the corresponding ordinary orthogonal polynomial P_n with respect to the weight function ω, i.e., $\pi_{n,\sigma}(t) \equiv P_n(t)$ on $[a,b]$. Then, the error in (7) can be estimated by the well known method of computing the absolute value of the difference of the quadrature sums in (8) and (7).

Using the above presented method (see (7), (8)) for the case $\omega(t) = 1/\sqrt{1-t^2}$, $t \in [-1,1]$, we have proved in [5] the following statement.

Theorem 3. *Let $n, s \in \mathbb{N}$ and $\omega(t) = 1/\sqrt{1-t^2}$, $t \in [-1,1]$. Then, there exists a unique quadrature formula with multiple nodes for calculating the corresponding Fourier-Chebyshev coefficients $a_n(f) = \int_{-1}^{1} f(t)T_n(t)/\sqrt{1-t^2}\,dt$,*

$$\int_{-1}^{1} \frac{f(t)T_n(t)}{\sqrt{1-t^2}}\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} \widehat{A}_{i,\nu} f^{(i)}(\tau_\nu), \tag{9}$$

with $\mathrm{ADP} = 2sn + n - 1$, *as well as its Kronrod extension*

$$\int_{-1}^{1} \frac{f(t)T_n(t)}{\sqrt{1-t^2}}\,dt \approx \sum_{\nu=1}^{n}\sum_{i=0}^{2s-1} \widehat{B}_{i,\nu} f^{(i)}(\tau_\nu) + \sum_{j=1}^{n+1} \widehat{C}_j f(\hat{\tau}_j),$$

with $\mathrm{ADP} = 2sn + 2n + 1$.

In the special case when $s = 1$ the quadrature formula (9) becomes the well known Micchelli-Rivlin quadrature formula (cf. [3]).

Conclusion

A numerically stable construction of the quadrature formulas with multiple nodes for Fourier coefficients that is proposed in [4], [5] enables us their calculation as well as estimation of its error. A part of those results is presented here.

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On Gaussian rules for some modified Chebyshev weights

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Abstract

In this paper, Gaussian rules for some modified Chebyshev weights introduced by Gautschi and Li in 1993 are considered. Our main concern is providing efficient estimations for the error of quadrature. Those estimations are checked by means of some numerical examples.

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Introduction

In [1], the authors considered a polynomial modification of a given positive measure $d\sigma$ supported on the real axis. Namely, if $n \in \mathbb{N}$ and π_n is the orthogonal polynomial of degree n with respect to $d\sigma$, they deal with the new sequence of polynomials $\{\widehat{\pi}_{m,n}\}$, being orthogonal with regard to the modified measure $d\widehat{\sigma}_n = \pi_n^2 \, d\sigma$. While in general is quite difficult getting explicit expressions for the induced orthogonal polynomials, it is not hard when dealing with the four Chebyshev weights, as pointed out by the authors in [1]. This new family of polynomials, hereafter referred to as "induced" orthogonal polynomials, has a number of applications in constructive approximation of functions, which justifies the interest in studying quadrature rules for approximating integrals with some kind of modified weights. In this note, we focus in estimating the error of Gauss rules for this modified weights in the case of the four Chebyshev weights, and the different bounds we obtain are tested by means of numerical examples.

The problem of estimating the quadrature error for Gauss–type rules has been thoroughly studied in the literature; to only cite a few, see the references [2]–[7].

Main Results

Throughout this note, we deal with integrals of the form

$$I_\sigma(f) = I(f;\sigma,n) = \int f(t)\, d\widehat{\sigma}_n(t)\,,$$

where $d\widehat{\sigma}_n = \pi_n^2 \, d\sigma$, and $d\sigma$ is one of the four Chebyshev weights, namely

$$d\sigma^{[1]}(t) = \frac{dt}{\sqrt{1-t^2}}\,,\ d\sigma^{[2]}(t) = \sqrt{1-t^2}\,dt\,,$$
$$d\sigma^{[3]}(t) = \sqrt{\frac{1-t}{1+t}}\,dt\,,\ d\sigma^{[4]}(t) = \sqrt{\frac{1+t}{1-t}}\,dt\,.$$

by means of Gauss rules

$$I_m(f) = \sum_{j=1}^{m} A_{m,j}\, f(t_{m,j})\,,\; m = 1, 2, \dots\,,$$

which means that the nodes $\{t_{m,j}\}$ are the zeros of the induced orthogonal polynomial $\{\widehat{\pi}_{m,n}\}$. While for the case where $i = 1$ and $n = 1$, whose related weight will be referred hereafter to as $d\sigma^{[I]}$, Gauss rules with an arbitrary number m are considered, otherwise we restrict ourselves to the case where $m = n$ for the sake of simplicity. In addition, since the orthogonal polynomials with respect to the measures $d\sigma^{[3]}$ and $d\sigma^{[4]}$ are easily connected to each other, only the results for $d\sigma^{[3]}$ are shown.

In this sense, our main concern is estimating the error of quadrature. It is well–known that in the usual case where the integrand f is analytic in a neighborhood Ω of a compact interval, say $[-1, 1]$, this error admits the representation

$$R_m(f) = I_\sigma(f) - I_m(f) = \frac{1}{2\pi i} \oint_\Gamma K_m(z)\, f(z)\, dz\,,$$

where the kernel K_m is given by

$$K_m(z) = \frac{\varrho_{m,n}(z)}{\widehat{\pi}_{m,n}}\,,\quad \varrho_{m,n}(z) = \int_{-1}^{1} \frac{\pi_m(t)}{z-t}\, w(t)\, dt\,,$$

Γ being any closed smooth contour contained in Ω and surrounding the real interval $[-1, 1]$. As usual, elliptic contours with foci at ± 1 and sum of the semi–axes equal to $\rho > 1$, are considered. These level contours admit the expression

$$\mathcal{E}_\rho = \{z \in \mathbb{C} : |\phi(z)| = |z + \sqrt{z^2-1}| = \rho\}\,,$$

where the branch of $\sqrt{z^2-1}$ is taken so that $|\phi(z)| > 1$ for $|z| > 1$.

Next, we state our main results. For details about their proofs, as well as other possible error bounds, see [5]. On the sequel, we denote $\rho_f = \sup\{\rho > 1 : f \text{ is analytic on } D_\rho\}$.

Theorem 1. *The following L^∞–type bounds for the error, where $\|f\|_{\mathcal{E}_\rho} = \max_{z\in\mathcal{E}_\rho}$, hold.*

$$r_1^{[I]}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1 \left(\rho^2+1+(-1)^{m/2}\left(\rho^{m+2}+\rho^m\right)\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{\rho^{m+2}\left(\rho-\rho^{-1}\right)\left(\sum_{j=0}^{m/2}(-1)^j\rho^{m-2j}+\sum_{j=0}^{(m-1)/2}(-1)^j\rho^{2j-m}\right)},$$

if m is even, and

$$r_1^{[I]}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1 \left((m+2)\rho^2+m\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{m\rho^{m+2}\left(\rho-\rho^{-1}\right)\left(\sum_{j=0}^{(m-1)/2}(-1)^j\frac{m-2j}{m}\rho^{m-2j}+\sum_{j=0}^{(m-1)/2}(-1)^j\frac{m-2j}{m}\rho^{2j-m}\right)}$$

if m is odd. In the same way,

$$r_1^{(1)}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1\left(3\rho^{2n}+1\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{2^{2n-2}\rho^{3n}\left(\rho-\rho^{-1}\right)\left(\rho^n+\rho^{-n}\right)}. \tag{1}$$

$$r_1^{(2)}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1\left(2\rho^{2n+2}-\rho^{2n}-1\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{2^{2n}\rho^{3n+2}\left(\rho-\rho^{-1}\right)\left(\rho^n+\rho^{-n}\right)},\quad n>1. \tag{2}$$

$$r_1^{(3)}(f) = \inf_{\rho^*<\rho<\rho_f} \frac{\pi a_1\left(2\rho^{2n+1}+\rho^{2n}+1\right)\left(1-\frac{1}{4}a_1^{-2}-\frac{3}{64}a_1^{-4}-\frac{5}{256}a_1^{-6}\right)\|f\|_{\mathcal{E}_\rho}}{2^{2n}\rho^{3n+1}\left(\rho-\rho^{-1}\right)\left(\rho^n+\rho^{-n}\right)}, \tag{3}$$

where $\rho^* > 1$ is a value obtained empirically (see [5] for details), and it was shown to be relatively closed to 1 in all the cases.

Theorem 2. *The following upper bounds for the error of quadrature, based on the Fourier–Chebyshev expansion of the error, hold.*

$$r_2^{(1)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{2^{2n-2}} \frac{1}{\rho^{2n}-1} \|f\|_{\mathcal{E}_\rho}\,,\ n \geq 1. \tag{4}$$

$$r_2^{(2)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{2^{2n}} \left(\frac{1}{\rho^{2n}-1} + \frac{1}{2\rho^{2n+2}}\right) \|f\|_{\mathcal{E}_\rho}. \tag{5}$$

$$r_2^{(3)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{2^{2n}} \left(\frac{1}{\rho^{2n}-1}\right) \|f\|_{\mathcal{E}_\rho}. \tag{6}$$

Theorem 3. *The following L^1–type bounds for the error of quadrature also hold.*

$$r_3^{(1)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{\rho^n \cdot 2^{2n-1}} \sqrt{\frac{7\rho^{-2n}+9\rho^{2n}}{\rho^{4n}-1}}\, \|f\|_{\mathcal{E}_\rho}. \tag{7}$$

$$r_3^{(2)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{\rho^n \cdot 2^{2n+1}} \sqrt{\frac{\rho^{2n-4}+4\rho^{2n}+3\rho^{-2n-4}}{\rho^{4n}-1}}\, \|f\|_{\mathcal{E}_\rho}. \tag{8}$$

$$r_3^{(3)}(f) = \inf_{1<\rho<\rho_f} \frac{\pi}{\rho^n \cdot 2^{2n+1}} \sqrt{\frac{\rho^{2n-2}+4\rho^{2n}+3\rho^{-2n-2}}{\rho^{4n}-1}}\, \|f\|_{\mathcal{E}_\rho}. \tag{9}$$

Numerical experiments and Conclusion

Now, we are concerned with checking the accuracy of the quadratures above, as well as of the bounds given in previous Theorems 1 and 2, when the characteristic example $f_1(z) = e^{\cos(\omega z)}$, $\omega > 0$; it is an entire function and, thus, $\rho_f = +\infty$. In the following Tables, the results obtained by applying our Gauss rules to the Chebyshev weights are displayed, along with the error bounds provided in the above theorems, as well as the actual values of the integrals and the errors. In Tables below the error bounds $r_j^{(i)}$, $i, j = 1, 2, 3$, given in (1)–(9), along with the actual values of the errors and the integrals, are displayed for $\omega = 1$ and some values of n. It is noteworthy that in general the estimates of the error are quite sharp, as well as the accuracy of the respective quadrature rules. More numerical results are displayed in [5].

n, ω	$r_1^{[1]}(f_1)$	$r_2^{[1]}(f_1)$	$r_3^{[1]}(f_1)$	Error$^{[1]}$	$I_\omega^{[1]}(f_1)$
6, 1	4.856(−9)	3.095(−9)	4.643(−9)	5.596(−10)	3.3409...(−3)
10, 1	3.793(−17)	2.444(−17)	3.666(−17)	3.297(−18)	1.3050...(−5)
15, 1	8.548(−28)	5.545(−28)	8.317(−28)	5.915(−29)	1.2744...(−8)
20, 1	8.371(−39)	5.448(−39)	8.172(−39)	4.922(−40)	1.2446...(−11)

n, ω	$r_1^{[2]}(f_1)$	$r_2^{[2]}(f_1)$	$r_3^{[2]}(f_1)$	Error$^{[2]}$	$I_\omega^{[2]}(f_1)$
5, 1	6.844(−8)	6.995(−8)	6.668(−8)	9.110(−9)	3.3409...(−3)
10, 1	6.217(−18)	6.312(−18)	6.111(−18)	5.579(−19)	3.3626...(−6)
15, 1	1.406(−28)	1.423(−28)	1.386(−28)	9.984(−30)	3.1861...(−9)
20, 1	1.379(−39)	1.394(−39)	1.362(−39)	8.296(−41)	3.1115...(−12)

n,ω	$r_1^{[3]}(f_1)$	$r_2^{[3]}(f_1)$	$r_3^{[3]}(f_1)$	Error$^{[3]}$	$I_\omega^{[3]}(f_1)$
5, 1	7.797 − 8)	6.666(−8)	6.707(−8)	1.785(−8)	6.6819...(−3)
10, 1	6.896(−18)	6.110(−18)	6.136(−18)	1.099(−18)	6.5253...(−6)
15, 1	1.542(−28)	1.386(−28)	1.391(−28)	1.972(−29)	6.3723...(−9)
20, 1	1.502(−39)	1.362(−19)	1.366(−19)	1.641(−40)	6.2230...(−12)

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Internality of truncated averaged Gaussian quadratures

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Abstract

When moments or modified moments of the weight function are difficult to compute, generalized averaged Gaussian quadratures can serve as good substitutes. These formulas were introduced by Spalević [3], where it was demonstrated that they may yield a smaller error compared to the Gauss quadrature rules. However, generalized averaged Gaussian quadratures may have external nodes. This would make them unusable when the domain of the integrand is limited to the convex hull of the support of the weight function. In this paper we investigate whether removing some of the last rows and columns of their Jacobi matrices (cf. [2]) will produce quadrature rules with no external nodes. The results that will be presented have been recently published in [1].

KEYWORDS: Truncated averaged Gaussian quadratures, Internality.

Introduction

Let $d\sigma$ be a nonnegative measure with infinitely many points of support. The smallest closed interval that contains the support of $d\sigma$ is denoted by $[a, b]$ with $-\infty \leq a < b \leq \infty$, and we assume that the distribution finction σ has infinitely many points of increase in this interval. If σ is an absolutely continuous function, then $d\sigma(x) = w(x)\,dx$ on $\mathrm{supp}(d\sigma)$, where $w(x) \geq 0$ is a weight function. Let $\mathbb{P}_k$ denote the set of all polynomials of degree at most k and introduce the quadrature formula (abbreviated q.f.)

$$Q_n[f] = \sum_{j=1}^{n} \omega_j f(x_j)$$

with real distinct nodes $x_1 < x_2 < \cdots < x_n$ and real weights ω_j. We say that Q_n is a $(2n - m - 1, n, d\sigma)$ q.f. if the remainder term $R_n[f]$, defined by

$$\int f(x)\,d\sigma(x) = Q_n[f] + R_n[f],$$

satisfies $R_n[f] = 0$ for all $f \in \mathbb{P}_{2n-m-1}$. The rule Q_n then is said to have *algebraic degree of precision* $2n - m - 1$. Here m is an integer such that $0 \leq m \leq n$. If in addition all quadrature weights ω_j are positive, then Q_n is said to be a *positive* $(2n - m - 1, n, d\sigma)$ q.f.. Furthermore, we say that a polynomial $t_n = \prod_{j=1}^{n}(x - x_j)$ generates a $(2n - m - 1, n, d\sigma)$ q.f. if its zeros x_j are real and simple, and the q.f. with nodes $x_1, x_2, \ldots, x_n$ is a $(2n - m - 1, n, d\sigma)$ q.f.. A $(2n - m - 1, n, d\sigma)$ q.f. is *internal* if all its nodes are in the closed interval $[a, b]$. A node not belonging to the interval $[a, b]$ is said to *external*.

It is well known that an ℓ-node Gauss quadrature rule associated with the measure $d\sigma$ can be represented by an $\ell \times \ell$ real symmetric tridiagonal matrix $J_\ell^G(d\sigma)$ determined by the recursion coefficients of the first ℓ orthogonal polynomials associated with the measure $d\sigma$. Spalević [3] proposed that the leading $(\ell-1) \times (\ell-1)$ tridiagonal submatrix of $J_\ell^G(d\sigma)$ be flipped right-left and upside-down, and appended to $J_\ell^G(d\sigma)$ to obtain a new symmetric tridiagonal matrix $J_{2\ell-1,\ell-1}$ of order $2\ell-1$. The latter matrix defines a $(2\ell-1)$-node quadrature formula referred to as a *generalized averaged Gaussian quadrature formula*.

It is the purpose of the present paper to describe extensions of the generalized averaged Gaussian quadrature formulas introduced in [3]. Section 2 discusses the extension of the matrix $J_\ell^G(d\sigma)$ to a real symmetric tridiagonal matrix $J_{k+\ell,k}$ of order $k+\ell$ by appending a fairly arbitrary real symmetric tridiagonal matrix of order k to $J_\ell^G(d\sigma)$. These extensions may yield a smaller quadrature error than the underlying ℓ-node Gaussian quadrature formula. Section 3 is concerned with the possible presence of exterior nodes of generalized averaged Gaussian quadrature formulas. Spalević in [3] showed that the generalized averaged Gaussian quadrature formulas may have one node to the right or to the left of the interval $[a, b]$. It therefore may not be possible to apply these quadrature rules when the integrand is defined on $[a, b]$ only. To remedy this shortcoming, truncated generalized averaged Gaussian quadrature rules were introduced in [2]. These rules are obtained by removing the last few rows and columns of the matrix $J_{2\ell-1,\ell-1}$, which does not affect the degree of precision. We investigate these rules by using results by Peherstorfer [4] on positive quadrature rules. Section 4 presents a detailed analysis of truncated generalized averaged Gaussian quadrature rules obtained by appending only one row and column to the matrix $J_\ell^G(d\sigma)$, and investigates for classical measures $d\sigma$ when these rules are internal. Section 5 presents a few computed examples and Section 6contains concluding remarks.

Main Results

It is shown in [4] that, if

$$t_{k+1}(x) = (x - \tilde{\alpha}_k)t_k(x) - \tilde{\beta}_k t_{k-1}(x)$$

with $\tilde{\alpha}_j$, $\tilde{\beta}_j$ coinciding with α_j, β_j up to $j = n-1-[\frac{m+1}{2}]$, resp. $j = n-1-[\frac{m}{2}]$, then t_n generates a positive quadratic formula (q.f.) with a degree of precision $2n-1-m$.

Then

$$t_n = g_\ell p_{n-\ell} - \tilde{\beta}_{n-\ell} g_{\ell-1} p_{n-\ell-1},$$

where (g_j) is some sequence with

$$g_{j+1}(x) = (x - \tilde{\alpha}_{n-1-j})g_j(x) - \tilde{\beta}_{n-j} g_{j-1}(x).$$

We may choose $g_i = \tilde{p}_i$ as the sequence of orthogonal polynomials with respect to another measure $d\mu$, with the coefficients λ_i and γ_i. The associated $n\times n$ matrix is

$$J_{n,\ell}(d\sigma,d\mu)=\begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & & & & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & & \sqrt{\beta_{n-\ell-1}} & \alpha_{n-\ell-1} & \sqrt{\tilde{\beta}_{n-\ell}} & & & & \\ & & & & \sqrt{\tilde{\beta}_{n-\ell}} & \gamma_{\ell-1} & \sqrt{\lambda_{\ell-1}} & & & \\ & & & & & \sqrt{\lambda_{\ell-1}} & \gamma_{\ell-2} & \ddots & & \\ & & & & & \ddots & \ddots & & & \\ 0 & & & & & & & & \sqrt{\lambda_1} & \gamma_0 \end{bmatrix},$$

Theorem. The q.f. determined by removing the last i rows and columns from the matrix $J_{n,\ell}(d\sigma,d\mu)$ has the same degree of precision.

We now consider appending one row and one column to the matrix associated with the $(\ell+1)$-node Gaussian rule for $d\sigma$.

Suppose that we only replace $\alpha_{\ell+1}$ in $J^G_{\ell+2}(d\sigma)$. This yields a q.f. with the degree of precision $2\ell+2$ generated by the polynomial $t_{\ell+2}$. We know that the zeros of $t_{\ell+2}$ and $p_{\ell+1}$ interlace, so only the smallest and largest zeros of $t_{\ell+2}$ (denoted resp. τ_1 and $\tau_{\ell+2}$) may lie outside $[a,b]$.

Theorem. If $\alpha_{\ell-1}=\alpha_{\ell+1}$, then the obtained q.f. is internal. If $\alpha_{\ell-1}<\alpha_{\ell+1}$, then $a<\tau_{\ell+2}<b$, and if $\alpha_{\ell-1}>\alpha_{\ell+1}$, then $a<\tau_1<b$.

Corollary. If $d\sigma$ is symmetric, then the q.f. is internal because the coefficients α_j vanish.

Next we analyze when the quadrature formula is internal if $d\sigma(x)=w(x)\,dx$, w being one of the classical weight functions.

The previous corollary covers the important special cases over $[-1,1]$ when w is even:

(a) Legendre weight function $w(x)=1$;
(b) Chebyshev weight functions $w(x)=(1-x^2)^{\pm1/2}$.

Consider the generalized Laguerre weight function $w(x)=x^s\,e^{-x}$, $s>-1$, on $[0,\infty)$. Then we have

$$\begin{aligned} \alpha_\ell &= 2\ell+s+1, \quad \beta_\ell=\ell(\ell+s),\\ p_\ell(0) &= (-1)^\ell\,\ell!\binom{\ell+s}{\ell} \quad (s>-1). \end{aligned}$$

Theorem. The corresponding q.f. is internal for $s\geqslant 0$, $\ell\geqslant 2$, and for $s\in(-1,0)$, $\ell\geqslant 3$.
The q.f. is external for $s\in(-1,0)$, $\ell=2$.

Next, let $w^{(\alpha,\beta)}(x)=(1-x)^\alpha(1+x)^\beta$, $\alpha,\beta>-1$ $(\alpha\neq\beta)$ over $[-1,1]$ be the Jacobi weight function. Then

$$\alpha_\ell=\frac{\beta^2-\alpha^2}{(2\ell+\alpha+\beta)(2\ell+\alpha+\beta+2)}, \qquad \beta_\ell=\frac{4\ell(\ell+\alpha)(\ell+\beta)(\ell+\alpha+\beta)}{(2\ell+\alpha+\beta)^2\,((2\ell+\alpha+\beta)^2-1)},$$

$$p_\ell^{(\alpha,\beta)}(1)=2^\ell\binom{\ell+\alpha}{\ell}\Big/\binom{2\ell+\alpha+\beta}{\ell}.$$

Assume w.l.o.g. $\alpha^2 > \beta^2$. Then $\tau_1 \geqslant -1$ is equivalent to

$$\frac{\left[(\alpha+\beta+2\ell-2)(\alpha+\beta+2\ell)+\beta^2-\alpha^2\right] g(\alpha,\beta)}{2(\ell+1)(\ell+1+\alpha)(\alpha+\beta+2\ell-2)(\alpha+\beta+2\ell)} \geqslant 1,$$

where $g(\alpha,\beta) := (\alpha+\beta+2\ell+2)(\alpha+\beta+2\ell+3)$.

Theorem. The q.f. is internal for all $\ell \geq 3$.
If $\ell = 2$, the q.f. is internal whenever

$$\beta + \frac{12}{\alpha+\beta+10} > 0 \quad \text{and} \quad (\beta > \sqrt{13}-4 \text{ or } \alpha > \alpha_0),$$

where $\alpha_0 \approx -0.9419540398$ is the unique zero of

$$\begin{aligned} Q(\alpha) = 993\alpha^6 + 24228\alpha^5 + 113200\alpha^4 - 10400\alpha^3 \\ -1021212\alpha^2 - 1982200\alpha - 1041580 \end{aligned}$$

in the interval $(-1, 0)$.

Consider the weight function

$$w(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha = -3/4, \quad \beta = 3/4,$$

and let $\ell = 4$. Then the generalized averaged rule defined by the matrix $J_{2\ell-1,\ell-1}(d\sigma, d\sigma)$ has one exterior node at about 1.006.

However, truncated rules obtained by removing the last k rows and columns are interior for both $k = 1$ and $k = 2$.

When applying these quadrature rules to the integral

$$\int_{-1}^{1} f(x)dx, \quad f(x) = (5-10x)e^{x-x^2}, \ \ d\sigma(x) = dx$$

(whose actual value is $1 - e^{-10}$), the following results are obtained:

ℓ	$Q_\ell^G[f]$	$Q_{2\ell-1,\ell-1}[f]$	$Q_{\ell+1}^{(1)}[f]$	$Q_{\ell+2}^{(2)}[f]$
4	$4.84 \cdot 10^{-1}$	$-1.16 \cdot 10^{-2}$	$-1.86 \cdot 10^{-1}$	$5.19 \cdot 10^{-2}$
5	$-1.86 \cdot 10^{-1}$	$-6.66 \cdot 10^{-4}$	$4.20 \cdot 10^{-2}$	$-7.29 \cdot 10^{-3}$
6	$4.20 \cdot 10^{-2}$	$5.19 \cdot 10^{-5}$	$-6.41 \cdot 10^{-3}$	$7.09 \cdot 10^{-4}$
7	$-6.41 \cdot 10^{-3}$	$-3.27 \cdot 10^{-6}$	$6.41 \cdot 10^{-4}$	$-2.90 \cdot 10^{-5}$
8	$6.41 \cdot 10^{-4}$	$1.29 \cdot 10^{-7}$	$-2.40 \cdot 10^{-5}$	$-5.48 \cdot 10^{-6}$

Conclusion

An analysis of truncated generalized averaged Gaussian quadrature formulas is presented that sheds light on whether these formulas are interior. Computed examples show that the analysis is sharp in the sense that it cannot be generalized to quadrature rules that are extended more than $Q_{\ell+1}^{(1)}$. Further examples illustrate the performance of generalized averaged Gaussian quadrature formulas and their truncations.

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Error Estimates for Some Product Gauss Rules

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Abstract

Some integrals I^m over m-dimensional regions can be approximated by cubature formulas G_n^m constructed by the product of Gauss quadrature rules G_n. Using corresponding Gauss-Kronrod rules K_{2n+1} or corresponding generalized averaged Gauss rules $\widehat{G}_{2n+1}$ instead of G_n, we construct cubature formulas K_{2n+1}^m and $\widehat{G}_{2n+1}^m$. In order to estimate the error $|I^m - G_n^m|$ we use the differences $|K_{2n+1}^m - G_n^m|$ and $|\widehat{G}_{2n+1}^m - G_n^m|$.

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Introduction

Consider the quadrature formula (q.f.) of the form

$$I(f) = \int_{\mathbb{R}} f(t)d\mu(t) \approx Q_n(f) = \sum_{k=1}^{n} \omega_k f(t_k).$$

The unique optimal interpolatory q.f. with n nodes and (algebraic) degree of exactnes $2n-1$ is Gauss q.f. G_n. The nodes of G_n are the eigenvalues and the weights are proportional to the squares of the first components of the corresponding eigenvectors of tridiagonal symmetric Jacobi matrix with diagonal elements $\alpha_0, ..., \alpha_{n-1}$ and subdiagonal elements $\sqrt{\beta_1}, ..., \sqrt{\beta_{n-1}}$, where αs and βs are coefficient of the three-term recurrence relation, satisfied by the monic orthogonal polynomials.

In order to (economically) estimate the error $|I - G_n|$ we can use the differences $|K_{2n+1} - G_n|$ and $|\widehat{G}_{2n+1} - G_n|$. K_{2n+1} is corresponding Gauss-Kronrod q.f. with degree of exactness $3n+1$, and $\widehat{G}_{2n+1}$ is corresponding generalized averaged Gauss q.f. with degree of exactness $2n+2$, both with $2n+1$ nodes (n nodes of G_n form a subset). K_{2n+1} has higher degree of exactness, but $\widehat{G}_{2n+1}$ exists in some situations when K_{2n+1} does not and its numerical construction is simpler – Spalević in [2] proposed effective numerical procedure for constructing $\widehat{G}_{2n+1}$, where tridiagonal symmetric matrix has diagonal elements $\alpha_0, ..., \alpha_{n-1}; \alpha_n; \alpha_{n-1}, ..., \alpha_0$ and subdiagonal elements $\sqrt{\beta_1}, ..., \sqrt{\beta_{n-1}}; \sqrt{\beta_n}; \sqrt{\beta_{n+1}}; \sqrt{\beta_{n-1}}, ..., \sqrt{\beta_1}$.

Some integrals $I^m = \int_{\Omega^m} f(\boldsymbol{x})\omega(\boldsymbol{x})d\boldsymbol{x}$, $\omega(\boldsymbol{x}) \geq 0$, $\boldsymbol{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $m \geq 2$, over m-dimensional regions Ω^m, can be approximated by cubature formulas (c.f.) G_n^m constructed by the product of q.f. G_n. In order to estimate the error $|I^m - G_n^m|$ we first extend G_n^m to K_{2n+1}^m and $\widehat{G}_{2n+1}^m$, and than use the differences $|K_{2n+1}^m - G_n^m|$ and $|\widehat{G}_{2n+1}^m - G_n^m|$, where K_{2n+1}^m denotes c.f. constructed by the product of corresponding q.f. K_{2n+1}, and $\widehat{G}_{2n+1}^m$ denotes c.f. constructed by the product of corresponding q.f. $\widehat{G}_{2n+1}$.

$I^2 = \int_{-1}^{1}\int_{-1}^{1} \cos(x_1+x_2)dx_1dx_2 = (2\sin 1)^2 \approx 2.832...$					
n	$\lvert I^2 - G_n^2\rvert$	$\lvert I^2 - K_{2n+1}^2\rvert$	$\lvert K_{2n+1}^2 - G_n^2\rvert$	$\lvert I^2 - \widehat{G}_{2n+1}^2\rvert$	$\lvert \widehat{G}_{2n+1}^2 - G_n^2\rvert$
2	2.391e-02	2.979e-07	2.391e-02	2.979e-07	2.391e-02
4	9.455e-07	3.794e-16	9.455e-07	1.086e-13	9.455e-07
6	5.095e-12	8.249e-26	5.095e-12	4.534e-20	5.095e-12
$I^7 = \int_{-1}^{1}\cdots\int_{-1}^{1} \cos(x_1+\cdots+x_7)dx_1\cdots dx_7 = (2\sin 1)^7 \approx 38.237...$					
n	$\lvert I^7 - G_n^7\rvert$	$\lvert I^7 - K_{2n+1}^7\rvert$	$\lvert K_{2n+1}^7 - G_n^7\rvert$	$\lvert I^7 - \widehat{G}_{2n+1}^7\rvert$	$\lvert \widehat{G}_{2n+1}^7 - G_n^7\rvert$
2	1.118	1.408e-05	1.118	1.408e-05	1.118
4	4.468e-05	1.792e-14	4.468e-05	5.131e-12	4.468e-05

Table 1: Selected results for integrals over m-dimensional cube.

Main Results

In all considered cases we first introduce G_n^m constructed by the product of G_n (according to [1]). K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ can be introduced analogously, using corresponding K_{2n+1} and $\widehat{G}_{2n+1}$ instead od G_n. In all examples we first solve I^m analytically, and than show results for $|I^m - G_n^m|$, $|I^m - K_{2n+1}^m|$, $|K_{2n+1}^m - G_n^m|$, $|I^m - \widehat{G}_{2n+1}^m|$, $|\widehat{G}_{2n+1}^m - G_n^m|$, for different values of n. All results are calculated with 40 significant decimal digits.

Cube: $C^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid -1 \le x_l \le 1,\ l = 1, ..., m\}$. Integral of each variable x_l, $l = 1, ..., m$, can be approximated by n-point Gauss q.f. G_n with Legendre weight function $\omega(t) = 1$ on $[-1, 1]$,

$$\int_{-1}^{1} \varphi(t)dt \approx \sum_{k=1}^{n} \omega_k \varphi(t_k),$$

which leads to c.f.

$$I^m \approx G_n^m = \sum_{k_1,...,k_m=1}^{n} \omega_{k_1} \cdots \omega_{k_m} \cdot f(t_{k_1}, ..., t_{k_m}).$$

G_n^m has n^m, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $(2n+1)^m$ nodes. Selected results are shown in table 1.

Simplex: $T^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_l \ge 0,\ l = 1, ..., m,\ x_1 + \cdots + x_m \le 1\}$. Approximating integral of each variable x_l, $l = 1, ..., m$, by n-point Gauss q.f. G_n with Jacobi weight function $\omega(t) = (1-t)^{m-l}$, $l = 1, ..., m$, on $[0, 1]$,

$$\int_{0}^{1} (1-t)^{m-l} \varphi(t)dt \approx \sum_{k=1}^{n} \omega_{k,m-l} \varphi(t_{k,m-l}), \quad l = 1, ..., m,$$

we get c.f.

$$I^m \approx G_n^m = \sum_{k_1,...,k_m=1}^{n} \omega_{k_1,m-1} \cdots \omega_{k_m,0} \cdot f(\Pi(k_1), ..., \Pi(k_1, ..., k_m)),$$

$$\Pi(k_1) = t_{k_1,m-1},$$

$$\Pi(k_1, ..., k_l) = (1 - t_{k_1,m-1}) \cdots (1 - t_{k_{l-1},m-l+1}) t_{k_l,m-l}, \quad l = 2, ..., m.$$

G_n^m has n^m, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $(2n+1)^m$ nodes.

$I^3=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\frac{dx_1dx_2dx_3}{(1+x_1+x_2+x_3)^3}=\frac{8\ln 2-5}{16}\approx 0.034...$					
n	$\lvert I^3-G_n^3\rvert$	$\lvert I^3-K_{2n+1}^3\rvert$	$\lvert K_{2n+1}^3-G_n^3\rvert$	$\lvert I^3-\widehat{G}_{2n+1}^3\rvert$	$\lvert \widehat{G}_{2n+1}^3-G_n^3\rvert$
2	1.237e-04	1.353e-08	1.237e-04	6.196e-08	1.237e-04
4	1.285e-07	2.513e-14	1.285e-07	7.961e-12	1.285e-07
6	1.167e-10	2.024e-18	1.167e-10	2.337e-15	1.167e-10
$I^4=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\int_0^{1-x_1-x_2-x_3}\frac{dx_1dx_2dx_3dx_4}{(1+x_1+x_2+x_3+x_4)^4}=\frac{24\ln 2-16}{144}\approx 0.004...$					
n	$\lvert I^4-G_n^4\rvert$	$\lvert I^4-K_{2n+1}^4\rvert$	$\lvert K_{2n+1}^4-G_n^4\rvert$	$\lvert I^4-\widehat{G}_{2n+1}^4\rvert$	$\lvert \widehat{G}_{2n+1}^4-G_n^4\rvert$
2	1.959e-05	1.131e-09	1.959e-05	1.179e-08	1.960e-05
4	2.111e-08	-	-	1.661e-12	2.111e-08
6	1.937e-11	-	-	5.015e-16	1.937e-11

Table 2: Selected results for integrals over m-dimensional simplex.

$S^3: x_1^2+x_2^2+x_3^2=1,\quad I^3=\int_{S^3}e^{x_1}d\boldsymbol{x}=2\pi(e-1/e)\approx 14.768...$					
n	$\lvert I^3-G_n^3\rvert$	$\lvert I^3-K_{2n+1}^3\rvert$	$\lvert K_{2n+1}^3-G_n^3\rvert$	$\lvert I^3-\widehat{G}_{2n+1}^3\rvert$	$\lvert \widehat{G}_{2n+1}^3-G_n^3\rvert$
2	4.842e-02	5.748e-07	4.842e-02	5.748e-07	4.842e-02
4	1.854e-06	7.429e-16	1.854e-06	2.123e-13	1.854e-06
6	9.855e-12	1.583e-25	9.855e-12	8.746e-20	9.855e-12

Table 3: Selected results for integrals over m-dimensional sphere.

Selected results are shown in table 2. In the cases of I^4, $n=4,6$, q.f. K_{2n+1} doesn't exist and c.f. K_{2n+1}^4 can't be constructed.

Sphere: $S^m=\{\boldsymbol{x}\in\mathbb{R}^m \mid x_1^2+\cdots+x_m^2=r^2\}$. If we introduce spherical coordinates $r,\varphi_1,...,\varphi_{m-1}$, than replace integral of variable φ_{m-1} by $(2n)$-point rectangle formula and approximate integral of each variable φ_{m-l-2}, $l=0,...,m-3$, by n-point Gauss q.f. G_n with Gegenbauer weight function $\omega(t)=(1-t^2)^{l/2}$, $l=0,...,m-3$, on $[-1,1]$,

$$\int_{-1}^1(1-t^2)^{l/2}\varphi(t)dt\approx\sum_{k=1}^n\omega_{k,l}\varphi(t_{k,l}),\quad l=0,...,m-3,$$

we get c.f.

$$I^m\approx G_n^m=r^{m-1}\frac{\pi}{n}\sum_{k=1}^{2n}\sum_{k_1,...,k_{m-2}=1}^{n}\omega_{k_1,m-3}\cdots\omega_{k_{m-2},0}\cdot$$
$$\cdot F(r,\varphi_{1,k_1},...,\varphi_{m-2,k_{m-2}},\frac{\pi}{n}k),$$

$$F(r,\varphi_1,...,\varphi_{m-1})=f(r\cos\varphi_1,...,r\sin\varphi_1\cdots\sin\varphi_{m-1}),$$
$$\varphi_{m-l-2,k}=\arccos t_{k,l},\quad l=0,...,m-3.$$

G_n^m has $2n^{m-1}$, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $2(2n+1)^{m-1}$ nodes. Selected results are shown in table 3.

Ball: $B^m=\{\boldsymbol{x}\in\mathbb{R}^m \mid x_1^2+\cdots+x_m^2\le 1\}$. I^m can be approximated by linear combination of n integrals over m-dimensional spheres of different radii,

$$I^m\approx\sum_{i=1}^nB_i\int_{S_i^m}f(\boldsymbol{x})d\boldsymbol{x},\quad S_i^m=\{\boldsymbol{x}\in\mathbb{R}^m \mid x_1^2+\cdots+x_m^2=r_i^2\}.$$

$B^4: x_1^2 + \cdots + x_4^2 \leq 1, \quad I^4 = \int_{B^4} \sqrt{(x_2^2+x_3^2+x_4^2)^{17}} d\boldsymbol{x} = \frac{524288}{4849845}\pi \approx 0.339...$					
n	$\lvert I^4 - G_n^4 \rvert$	$\lvert I^4 - K_{2n+1}^4 \rvert$	$\lvert K_{2n+1}^4 - G_n^4 \rvert$	$\lvert I^4 - \widehat{G}_{2n+1}^4 \rvert$	$\lvert \widehat{G}_{2n+1}^4 - G_n^4 \rvert$
2	1.084e-01	7.329e-06	1.084e-01	6.606e-05	1.084e-01
4	9.084e-05	9.728e-13	9.084e-05	4.984e-11	9.084e-05
6	4.369e-10	3.459e-16	4.369e-10	1.409e-14	4.369e-10

Table 4: Selected results for integrals over m-dimensional ball.

If m is even, than $r_i^2 = \tau_i$, $2B_i r_i^{m-1} = \lambda_i$, $i = 1, ..., n$, where τ_i and λ_i are nodes and weights of Gauss q.f.

$$\int_0^1 t^{m/2-1}\varphi(t)dt \approx \sum_{i=1}^{n} \lambda_i \varphi(\tau_i).$$

If m is odd, than $r_i = \tau_i$, $B_i r_i^{m-1} = \lambda_i$, $i = 1, ..., n$, where τ_i and λ_i are nodes and weights of Gauss q.f.

$$\int_{-1}^1 t^{m-1}\varphi(t)dt \approx \sum_{\substack{i=-n \\ i\neq 0}}^{n} \lambda_i \varphi(\tau_i).$$

c.f. takes the form

$$I^m \approx G_n^m = \frac{\pi}{2n} \sum_{i=1}^{n} B_i r_i^{m-1} \sum_{k=1}^{4n} \sum_{k_1,\ldots,k_{m-2}=1}^{2n} \omega_{k_1,m-3} \cdots \omega_{k_{m-2},0} \cdot$$
$$\cdot F(r_i, \varphi_{1,k_1}, ..., \varphi_{m-2,k_{m-2}}, \frac{\pi}{2n}k).$$

G_n^m has $(2n)^m$, while corresponding K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have $(4n+2)(4n+1)^{m-1}$ nodes.

Selected results are shown in table 4.

Conclusion

As expected, with the increase of n precision of all three c.f. G_n^m, K_{2n+1}^m, $\widehat{G}_{2n+1}^m$ increases. Also expected, both K_{2n+1}^m and $\widehat{G}_{2n+1}^m$ have better accuracy than G_n^m, while K_{2n+1}^m has better (or the same) accuracy than $\widehat{G}_{2n+1}^m$.

Both differences $|K_{2n+1}^m - G_n^m|$ and $|\widehat{G}_{2n+1}^m - G_n^m|$ give very good estimates of error $|I^m - G_n^m|$. $\widehat{G}_{2n+1}^m$ exists in some situations when K_{2n+1}^m does not, and it's numerical construction is simpler than the construction of K_{2n+1}^m (since the construction of $\widehat{G}_{2n+1}$ is simpler than the construction of K_{2n+1}). So, $\widehat{G}_{2n+1}^m$ might be a better choice than K_{2n+1}^m for estimating error of G_n^m.

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Matrix transformations and generalized almost convergence

Maria Zeltser[1]

Abstract

Maddox generalized the spaces $c_0, c, \ell_1, \ell_\infty$ by adding powers p_k $(k \in \mathbf{N})$ in the definitions of spaces to the terms of elements of sequences (x_k). Using the ideas of Maddox, Nanda ([5]) defined generalizations $f(p)$ and $f_0(p)$ of the spaces of all almost convergent sequences f and of all almost convergent to 0 sequences f_0. In [5] and [6] he characterized some classes of matrix transformations involving theses spaces and Maddox's sequence spaces. We have discovered that almost all results of [5] and [6] are not correct. We give corresponding counterexamples and the correct results. Moreover we give them in greater generality. To obtain these results we reduce these matrix transformations to simpler matrix transformations between Maddox sequence spaces.

Note that variable exponent spaces have recently seen a renaissance and are presently studied extensively.

2010 Mathematics Subject Classifications : 40C05, 40D20
Keywords: Matrix transformation, almost convergence, Maddox sequence spaces.

Introduction

Throughout this note we assume familarity with summability and the standard sequence spaces (see e. g. [1]).

Let $p = (p_k)$ and $q = (q_k)$ be sequences of strictly positive numbers.

The following variable exponent spaces were defined by Maddox ([3]):

$$\begin{aligned} c_0(p) &= \{(x_k) \in \omega |\ |x_k|^{p_k} \to 0\}, \\ c(p) &= \{(x_k) \in \omega |\ |x_k - l|^{p_k} \to 0 \text{ for some } l\}. \end{aligned}$$

When all the terms of (p_k) are constant and equal to $p > 0$ we have $c(p) = c$ and $c_0(p) = c_0$, where c, c_0 are respectively the spaces of convergent and null sequences.

We set

$$t_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \quad (m, n \in \mathbf{N})$$

and note that

$$t_{m,n}(Ax) = \sum_k a(n,k,m) x_k,$$

where

$$a(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} a_{n+i,k} \quad (n,k,m \in \mathbf{N}).$$

Using the ideas of Maddox ([3]), Nanda ([5]) defined generalizations of the spaces of all almost convergent sequences f and of all almost convergent to 0 sequences f_0 as follows

$$f(p) = \{(x_k) \in \omega : \lim_m |t_{mn}(x - le)|^{p_m} = 0 \text{ for some } l \text{ uniformly in } n\}$$

and

$$f_0(p) = \{(x_k) \in \omega : \lim_m |t_{mn}(x)|^{p_m} = 0 \ \text{ uniformly in } n\}.$$

Note that for a bounded positive sequence p the inclusion $f(p) \subset \ell_\infty(p)$ holds.

Also Nanda ([5], [6]) characterized some classes of matrix transformations involving Maddox's and Nakano sequence spaces and the spaces $f(p)$, $f_0(p)$, f, f_0. Unfortunately almost all results are incorrect. In this note we give counterexamples for some incorrect results and state the corresponding correct ones. Moreover we give them in greater generality. For other counterexamples, correct results and proofs see [7].

To prove these results we reduce matrix transformations involving $f(p)$, $f_0(p)$ to simpler matrix transformations between sequence spaces.

We will use the following convention for limits: if the capital letter is used for the index in the limit, then elements of the sequence are finite only for large indices. If a lower letter is used for the index in the limit, then all elements of the sequence are finite. For example $\lim_N \rho_N = 0$ means that $\rho_N < \infty$ for large N and $\rho_N \to 0$, while $\lim_m \rho_m = 0$ means that $\rho_m < \infty$ for all m and $\rho_m \to 0$.

Main Results

In Theorem 2 in [5] Nanda gave a characterization of matrices in $(c_0(p), f_0(q))$:

Theorem 1. Let p be a bounded positive sequence. Then a matrix $A = (a_{nk})$ is in $(c_0(p), f_0(p))$ if and only if
(i) $\exists B \in \mathbf{N}\backslash\{1\} : \sup_m(\sum_k |a(n,k,m)|B^{-1/p_k})^{p_m} < \infty$ $(n \in \mathbf{N})$.
(ii) $\lim_m \sup_n |a(n,k,m)|^{q_m} = 0$ $(k \in \mathbf{N})$.

In the following example we verify that this theorem is not correct:

Example 1. Let $p_n = 1$ $(n \in \mathbf{N})$. Then $c_0(p) = c_0$ and $f_0(p) = f_0$. We consider the matrix $A = (a_{nk})$ with $a_{2^n 2^n} = n$ and $a_{nk} = 0$ for $(n,k) \notin \{(2^i, 2^i)|\ i \in \mathbf{N}\}$ $(n,k \in \mathbf{N})$.

The matrix A satisfies both the conditions in Theorem 1. On the other hand for $(x_n) = ((\log_2 n)^{-1/2}) \in c_0$ we have

$$y_{2^n} = [Ax]_{2^n} = a_{2^n 2^n} x_{2^n} = n \cdot (\log_2 2^n)^{-1/2} = \sqrt{n} \quad (n \in \mathbf{N}).$$

So $(y_n) \notin \ell_\infty \supset f_0$, hence $Ax \notin f_0$. Therefore the statement of Theorem 1 does not hold.

To prove the correct version of Theorem 1 we use the conditions for a matrix to be in $(c_0(p), c_0(q))$ (cf. [4], Theorem 1). We find conditions for a matrix $A = (a_{nk})$ to map $c_0(p)$ to $f_0(q)$ which gives the correct statement of Theorem 1 in the case $q = p$.

Theorem 2. Let p be any positive sequence and let q be a bounded positive sequence. Then a matrix $A = (a_{nk})$ is in $(c_0(p), f_0(q))$ if and only if
(i) $\lim_m \sup_n |a(n,k,m)|^{q_m} = 0$ $(k \in \mathbf{N})$,

(ii) $\lim_N \limsup_m \sup_n (\sum_k |a(n,k,m)|N^{-1/p_k})^{q_m} = 0$.

In Theorem 3 in [5] Nanda gave a characterization of matrices in $(c(p), f)$:

Theorem 3. Let p be a bounded positive sequence. Then a matrix $A = (a_{nk})$ is in $(c(p), f)$ if and only if
(i) $\exists B \in \mathbf{N}\backslash\{1\} : \sup_m (\sum_k |a(n,k,m)|B^{-1/p_k})^{p_m} < \infty$ $(n \in \mathbf{N})$.
(ii) $\exists(\alpha_k) : \lim_m \sup_n |a(n,k,m) - \alpha_k| = 0$ $(k \in \mathbf{N})$,
(iii) $\exists\alpha : \lim_m \sup_n |\sum_k a(n,k,m) - \alpha| = 0$.

The following example demonstrates that Theorem 3 also is not correct:

Example 2. Let $p_n = 1$ $(n \in \mathbf{N})$. Then $c(p) = c$. We consider the matrix $A = (a_{nk})$ with $a_{2^n 2^n} = n$, $a_{2^n,2^n+1} = -n$ and $a_{nk} = 0$ for $(n,k) \notin \{(2^i, 2^i), (2^i, 2^i+1)|\ i \in \mathbf{N}\}$ $(n,k \in \mathbf{N})$.
The matrix A satisfies all the conditions in Theorem 3. On the other hand for $(x_n) \in c_0$ with $x_{2^n} = 1/\sqrt{n}$ and $x_k = 0$ for $k \notin \{2^n|\ n \in \mathbf{N}\}$ we have

$$y_{2^n} = [Ax]_{2^n} = a_{2^n 2^n} x_{2^n} = \frac{n}{\sqrt{n}} = \sqrt{n} \quad (n \in \mathbf{N}).$$

So $(y_n) \notin \ell_\infty \supset f$ hence $Ax \notin f$. Therefore the statement of Theorem 3 does not hold.

We find conditions for a matrix $A = (a_{nk})$ to map $c(p)$ to $f(q)$ which gives the correct statement of Theorem 3 in the case $q = e$. To prove the correct version of Theorem 3 we use the conditions for a matrix A to be in $(c_0(p), c(q))$ (cf. Theorem 5.1, part 9 in [2]).

Theorem 4. Let p and q be bounded positive sequences. Then a matrix $A = (a_{nk})$ is in $(c(p), f(q))$ if and only if
(i) $\exists(\alpha_k) : \lim_m \sup_n |a(n,k,m) - \alpha_k|^{q_m} = 0$ $(k \in \mathbf{N})$,
(ii) $\exists\alpha : \lim_m \sup_n |\sum_k a(n,k,m) - \alpha|^{q_m} = 0$,
(iii) $\exists N \in \mathbf{N} :\ \limsup_m \sup_n \sum_k |a(n,k,m)|N^{-1/p_k} < \infty$,
(iv) $\lim_N \limsup_m \sup_n (\sum_k |a(n,k,m) - \alpha_k|N^{-1/p_k})^{q_m} = 0$.

Conclusion

In [5] Nanda defined generalizations $f(p)$ and $f_0(p)$ of the spaces of all almost convergent sequences f and of all almost convergent to 0 sequences f_0 by adding powers p_k $(k \in \mathbf{N})$ in the definitions of spaces to the terms of elements of sequences (x_k). In [5] and [6] he characterized some classes of matrix transformations involving theses spaces and Maddox's sequence spaces. Unfortunately almost all results are incorrect. In this note we give counterexamples for some incorrect results and state the corresponding correct ones. Moreover we give them in greater generality. For other counterexamples, correct results and proofs see [7].

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New application on (ϕ, δ) monotone sequences

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Abstract

In this paper, we proved a theorem dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series by using (ϕ, δ)-monotone sequences. Also, this theorem includes a new application of the trigonometric Fourier series and a new result.

2010 Mathematics Subject Classifications : 26D15, 40D15, 40F05, 40G99, 42A24.

Keywords: Riesz mean, quasi monotone sequences, (ϕ, δ) monotone sequences, Fourier series, Hölder inequality, Minkowski inequality.

Introduction

A sequence (d_n) is said to be δ-quasi monotone, if $d_n \to 0$, $d_n > 0$ ultimately, and $\Delta d_n \geq -\delta_n$, where $\delta = (\delta_n)$ is a sequence of positive numbers (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). By (u_n) and (t_n) we denote the n-th $(C, 1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1}$$

But since $t_n = n(u_n - u_{n-1})$ (see [7]), condition (1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{2}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{3}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{4}$$

defines the sequence (σ_n) of the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\sigma_n - \sigma_{n-1}|^k < \infty. \tag{5}$$

If we take $p_n = 1$ for all values of n, then $\left|\bar{N}, p_n\right|_k$ summability reduces to $|C,1|_k$ summability. If we write $X_n = \sum_{v=0}^{n} p_v/P_v$, then (X_n) is a positive increasing sequence tending to infinity with n.

In [3], Bor has proved the following theorem dealing with the $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem A. Let $(\lambda_n) \to 0$ as $n \to \infty$ and (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{6}$$

Suppose that there exists a sequence of numbers (μ_n) which is δ-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum \mu_n X_n$ is convergent and $|\Delta\lambda_n| \leq \mu_n$ for all n. If

$$\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k = O\Big(X_m\Big) \quad as \quad m \to \infty, \tag{7}$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Main Results

A sequence (μ_n) is said to be (ϕ, δ)-monotone if and only if $\mu_n \to 0$, $\mu_n \geq 0$ ultimately and $\Delta\mu_n \geq -\delta_{n+1}$, where (δ_n) is a sequence of non-negative numbers, (ϕ_n) is a positive monotone increasing sequence and $\sum \phi_n\delta_n < \infty$ (see [8]). The aim of this paper is to generalize Theorem A by using (ϕ, δ)-monotone sequences. Now, we shall prove the following theorem.

Theorem 2.1 Let $(\lambda_n) \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (μ_n) which is (ϕ, δ)-monotone with $\sum \mu_n\phi_n$ is convergent and $|\Delta\lambda_n| \leq \dfrac{\mu_n}{n}$ for all n. If the conditions (6) and

$$\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k = O\Big(\frac{\phi_m}{m}\Big) \quad as \quad m \to \infty \tag{8}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Lemma 2.1 Under the conditions of Theorem 2.1, we have that

$$|\lambda_n|\phi_n = O\left(1\right) \quad as \quad n \to \infty. \tag{9}$$

Lemma 2.2 If (μ_n) is (ϕ, δ)-monotone with $\sum \mu_n\phi_n$ is convergent, then

$$\sum_{n=1}^{\infty}\phi_n|\Delta\mu_n| < \infty. \tag{10}$$

Proof of Theorem 2.1 Let (T_n) be the sequence of $(\bar{N},p_n)$ means of the series $\sum a_n\lambda_n$. Then by definition, we have

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{r=0}^{v}a_r\lambda_r=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v. \tag{11}$$

Then, for $n\geq 1$,we get

$$T_n-T_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v\lambda_v=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v. \tag{12}$$

By Abel's transformation, we have

$$\begin{aligned}T_n-T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta\Big(\frac{P_{v-1}\lambda_v}{v}\Big)\sum_{r=1}^{v}ra_r+\frac{p_n\lambda_n}{nP_n}\sum_{v=1}^{n}va_v\\ &= \frac{(n+1)p_nt_n\lambda_n}{nP_n}-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{v+1}{v}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\Delta\lambda_vt_v\frac{v+1}{v}\\ &+ \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\lambda_{v+1}t_v\frac{1}{v}=T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}.\end{aligned}$$

To complete the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k<\infty \quad for \quad r=1,2,3,4.$$

First, we get

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=1}^{m}\frac{|\lambda_n|p_n|t_n|^k}{P_n}\\ &= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{r=1}^{n}\frac{p_r}{P_r}|t_r|^k+O(1)|\lambda_m|\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k\\ &= O(1)\sum_{n=1}^{m-1}\mu_n\phi_n+O(1)|\lambda_m|\phi_m=O(1) \quad as \quad m\to\infty,\end{aligned}$$

by the hypotheses of Theorem 2.1 and Lemma 2.1.

Now, applying Hölder's inequality, as in $T_{n,1}$, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\Big\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\Big\}^{k-1}\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{p_v}{P_v}|t_v|^k\\ &= O(1) \quad as \quad m\to\infty,\end{aligned}$$

by the hypotheses of Theorem 2.1 and Lemma 2.1.
Now, using (6), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\Big\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v||t_v|\Big\}^k\\ &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\mu_v^k p_v|t_v|^k\Big\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\Big\}^{k-1}\\ &= O(1)\sum_{v=1}^{m}\mu_v^{k-1}\mu_v p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} = O(1)\sum_{v=1}^{m}\mu_v\frac{p_v}{P_v}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}|\Delta\mu_v|\sum_{r=1}^{v}\frac{p_r}{P_r}|t_r|^k + O(1)\mu_m\sum_{v=1}^{m}\frac{p_v}{P_v}|t_v|^k\\ &= O(1) \quad as \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.
Finally, again using (6), as in $T_{n,1}$, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\Big\{\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}||t_v|\Big\}^k\\ &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v|\lambda_{v+1}|^k|t_v|^k\Big\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\Big\}^{k-1}\\ &= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1) \quad as \quad m\to\infty.\end{aligned}$$

This completes the proof of Theorem 2.1.

An application to trigonometric Fourier series

Let f be a periodic function with period 2π and Lebesque integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}C_n(x) \tag{13}$$

where (a_n) and (b_n) denote the Fourier coefficients.
Write $\Psi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$ and $\Psi_1(t)=\frac{1}{t}\int_0^t\Psi(u)du$.

It is well known that if $\Psi_1(t)\in\mathcal{BV}(0,\pi)$, then $t_n(x)=O(1)$, where $t_n(x)$ is the n-th (C,1) mean of the sequence $(nC_n(x))$ (see [4]). Hence using this fact, we get following result for trigonometric Fourier series.
Theorem 3.1 If $\Psi_1(t)\in\mathcal{BV}(0,\pi)$ and the sequences $(p_n),(\lambda_n)$ and (μ_n) satisfy the conditions of Theorem 2.1, then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N},p_n|_k, k\geq 1$.

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A geometrically convergent modified moving asymptotes method

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Abstract

A new modified moving asymptotes method is presented. In each step of the iterative process, a strictly convex approximating subproblem is generated and explicitly solved, and in doing so we propose a strategy to incorporate a modified second-order information for the moving asymptotes location. This considerably reduces the computational cost of our optimization method and may both stabilize and speed up the convergence of the general process. Under natural assumptions, we prove the geometrical convergence of the associated optimization algorithm. In addition experimental results reveal that the present method is significantly faster compared to the [1] method, Newton's method and the BFGS Method, and it will succeed where these latter diverge simultaneously.

2010 Mathematics Subject Classifications : 65K05, 65K10, 65L10, 90C30, 46N10.

Keywords: Non-convex, non linear optimization Global convergence, Method of moving asymptotes.

Introduction

Consider the unconstrained optimization problem: Find $x_* \in \Omega$ such that

$$f\left(x_*\right) = \min_{x \in \Omega} f\left(x\right), \tag{1}$$

where Ω is an open subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a given non-linear real-valued objective function, typically twice continuously differentiable, which could be non-convex. In order to evaluate the merit of using second order information an extension of the method of moving asymptotes, that accounts for the curvatures, was proposed in [1]. Let us first briefly recall its main idea. Throughout, we assume that f' does not vanish at a given suitable initial point $x^{(0)} \in \Omega$, that is $f'(x^{(0)}) \neq 0$, since if this is not the case we have nothing to solve. Starting from the initial design point $x^{(0)}$ the iterates $x^{(k)}$ are computed successively by solving sub-problems of the form: Find $x^{(k+1)}$ such that

$$f^{(k)}(x^{(k+1)}) = \min_{x \in \Omega} f^{(k)}(x), \tag{2}$$

Throughout this paper we assume that w is a function satisfying the following conditions:

$$w \text{ is a real-valued function, defined and continuous on } \mathbb{R}, \tag{3}$$

$$lim_{|x| \to +\infty} w(x) = 0. \tag{4}$$

Our general modification of moving asymptotes method that we examine herein may be described as follows: Given the iteration point $\tilde{x}^{(k)}$ (at iteration k).

- The objective function f is iteratively approximated at the k-th iteration by the approximating function $\tilde{f}_w^{(k)}$ where:

$$\begin{aligned} \tilde{f}_w^{(k)}(x) &= \tilde{a}^{(k)} + \tilde{b}^{(k)}(x - \tilde{x}^{(k)}) + \\ & \quad \tilde{c}^{(k)} \left(\frac{1}{2} \frac{(\tilde{x}^{(k)} - \tilde{d}^{(k)})^3}{x - \tilde{d}^{(k)}} + \frac{1}{2}(\tilde{x}^{(k)} - \tilde{d}^{(k)})(x - 2\tilde{x}^{(k)} + \tilde{d}^{(k)}) \right). \end{aligned} \tag{5}$$

- The approximating function $\tilde{f}_w^{(k)}$ is first order approximations of the original function f at the current iteration point $\tilde{x}^{(k)}$, i.e.,

$$\tilde{f}_w^{(k)}(\tilde{x}^{(k)}) = f(\tilde{x}^{(k)}), \tag{6}$$

$$(\tilde{f}_w^{(k)})'(\tilde{x}^{(k)}) = f'(\tilde{x}^{(k)}). \tag{7}$$

  In addition to the above conditions (6) and (7), the approximating function should satisfy the more general condition (8) instead of the second order interpolation condition:

$$(\tilde{f}_w^{(k)})''(\tilde{x}^{(k)}) = \left| f''(\tilde{x}^{(k)}) + w(\tilde{x}^{(k)}) f'(\tilde{x}^{(k)}) \right|. \tag{8}$$

Consequently, in the present situation, the approximate parameters $\tilde{a}^{(k)}, \tilde{b}^{(k)}$ and $\tilde{c}^{(k)}$ are here determined for each iteration such that:

$$\tilde{a}^{(k)} = f(\tilde{x}^{(k)}), \tag{9}$$

$$\tilde{b}^{(k)} = f'(\tilde{x}^{(k)}), \tag{10}$$

$$\tilde{c}^{(k)} = \left| f''(\tilde{x}^{(k)}) + w(\tilde{x}^{(k)}) f'(\tilde{x}^{(k)}) \right|. \tag{11}$$

Furthermore, in order to fully determine an explicit expression for the approximating function $\tilde{f}_w^{(k)}$, the parameter $\tilde{d}^{(k)}$ is chosen such that

$$\tilde{d}^{(k)} = \tilde{x}^{(k)} + 2\tilde{\alpha}^{(k)} \frac{f'(\tilde{x}^{(k)})}{\tilde{c}^{(k)}}, \tag{12}$$

where $\{\tilde{\alpha}^{(k)}\}_k$ is a sequence of real numbers with

$$\tilde{\alpha}^{(k)} > 1, (k \in \mathbb{N}). \tag{13}$$

Different rules for how to choose these values (and possible weight functions in (8)) will be discussed later. We note that our method does not use the second order interpolation condition, but instead we have incorporated a first- and second-order information, as given in (11). Moreover, in particular, if you take $w = 0$ and if we suppose that the objective function is convex, then our iterative scheme obviously reduces to the one introduced in [1]. Hence, subsequent iterations of the [1] method are similar, except that in the proposed approximating function $\tilde{f}_w^{(k)}$ the parameters $c^{(k)}$ and $d^{(k)}$ are replaced by those computed in (11) and (12) respectively. It starts at an initial point $\tilde{x}^{(0)}$ and generates successive iterates by

$$f(\tilde{x}^{(k+1)}) \leftarrow \tilde{f}_w^{(k)}(\tilde{x}^{(k+1)}) = \min_{x \in \Omega} \tilde{f}_w^{(k)}(x). \tag{14}$$

For simplicity, we have removed the index w in $\tilde{x}_w^{(k)}$.

We prefer to work with (11) instead of the second order interpolation condition for several reasons. First, as mentioned above, this allows us to apply our method to a large class of objective functions. There is also a significant difference from a numerical point of view: many experimental results reveal that the iterative scheme based on our modification (11) can yield significantly fewer iterations than the [1] method, Newton's method or the BFGS Method itself. In contrast to these three approaches, our method converges even if the starting point is very far from the true solution. In addition, as we will see, the key features of the present method are:

- It does not require us to build a good initial solution close to the exact solution.
- It converges geometrically for a large class of functions w that satisfy condition (4).
- It will succeed where the [1] method, Newton's method and the BFGS Method break down.

Newton's method and the BFGS Method have a well-studied convergence theory that guarantees the convergence to a solution under a standard set of assumptions. For these and other their variants, the interested reader should consult one of the many excellent books on this subject [2, pp. 48–75] and [3, pp. 75–89]. We refer the readers to [1] and the references therein for the method of moving asymptotes.
We have not succeeded in proving that the method can be extended to multiple dimensions, but in practice, we have found it to work in two dimensions.

Main Results

Convergence Analysis

We start this section with a result concerning an explicit expression for the iterative sequence $\left\{\tilde{x}^{(k)}\right\}_k$ generated by the approximating function $\tilde{f}_w^{(k)}$. Here, we continue to denote by $\tilde{c}^{(k)}, \tilde{d}^{(k)}$ and $\tilde{\alpha}^{(k)}$ the coefficients given by (11), (12) and (13) respectively. Note that condition (13), imposed on the parameters $\tilde{\alpha}^{(k)}$, is crucial since it will guarantee strict convexity of the approximating function $\tilde{f}_w^{(k)}$. For brevity, in the following we use the notation:

$$\mathcal{I}_k = \left]-\infty, \tilde{d}^{(k)}\right[\cup \left]\tilde{d}^{(k)}, +\infty\right[, \tag{15}$$

Theorem 1. *With the above notation, let $\Omega \subset \mathbb{R}$ be an open subset of the real line, a given twice continuously differentiable function f in Ω, $\tilde{x}^{(0)} \in \Omega$ and $\tilde{x}^{(k)}$ being respectively the initial and a current point of the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$. Then, for each $k > 0$ the approximating function defined by (5) is a strictly convex function on $\mathcal{I}_k$. In addition, the function $\tilde{f}_w^{(k)}$ has an unique minimum at*

$$\tilde{x}^{(k+1)} \leftarrow \tilde{x}_*^{(k)} = \tilde{d}^{(k)} + (\tilde{x}^{(k)} - \tilde{d}^{(k)})\sqrt{\tilde{s}^{(k)}} \tag{16}$$

where

$$\tilde{s}^{(k)} = \frac{\tilde{\alpha}^{(k)}}{\tilde{\alpha}^{(k)} - 1}. \tag{17}$$

Convergence study

In this Section, we give the main result of this paper, that is sufficient conditions on the data (the point $\tilde{x}^{(0)}$, the function f' in a neighborhood of $\tilde{x}^{(0)}$, the family $f''\left(\tilde{x}^{(k)}\right), k \geq 0$), which guarantee that first derivative of f vanishes in a neighborhood of x^*, first, and secondly, the convergence of the method to this zero.
To establish our convergence results, we need the following assumptions. We assume that there exist positive constants r, M and $\xi < 1$ such that the following assumptions hold:

Assumption 1.

$$B_r := \left\{x \in \mathbb{R} : \left|x - \tilde{x}^{(0)}\right| \leq r\right\} \subset \Omega.$$

Assumption 2.

$$0 < \frac{\tilde{\alpha}^{(k)}}{\tilde{\alpha}^{(k)} - 1} \leq \frac{M}{2}\tilde{c}^k, \quad (k > 0). \tag{18}$$

Assumption 3.

$$\sup_{k\geq 0}\sup_{x\in B}\left|f''(x) - \frac{f'\left(x^{(k-1)}\right)}{x^{(k-1)} - \tilde{x}^{(k)}}\right| \leq \frac{\xi}{M}.$$

Assumption 4.

$$0 < \left|f'\left(\tilde{x}^{(0)}\right)\right| \leq \frac{r}{M}(1-\xi).$$

Throughout this subsection, we assume that Assumptions 1-4 hold. The constants r, M and $\xi < 1$ that appear in the subsequent analysis are always the constants from Assumptions 1-4. Our aim is to show that the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ defined by (19) converges geometrically to a point x^* in the sense that

$$\left|\tilde{x}^{(k)} - x^*\right| \leq \frac{\xi^k}{1-\xi}\left|x^{(1)} - x^{(0)}\right|.$$

Theorem 2. *Assume Assumptions 1-4 hold. Let the assumptions of theorem 1 be valid and let $\tilde{s}^{(k)}$ be defined by (17). Then the sequence $\left\{\tilde{x}^{(k)}\right\}_{k\geq 0}$ given by*

$$\tilde{x}^{(k+1)} = \tilde{d}^{(k)} + (\tilde{x}^{(k)} - \tilde{d}^{(k)})\sqrt{\tilde{s}^{(k)}} \tag{19}$$

is completely contained in the interval B_r, and converges to the unique zero of f' in B_r.

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Omega Invariant and Its Applications in Graph Theory

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Abstract

The last two authors recently defined a new graph invariant denoted by $\Omega(G)$ in terms of a given degree sequence which is also related to the cyclomatic number. It has many important combinatorial applications in graph theory and gives direct information compared to the better known Euler characteristic on the realizability, connectedness, cyclicness, components, chords, loops etc. Many similar classification problems can be solved by means of Ω. In this paper, we study the change of several topological graph indices, the first, second and third Zagreb indices, forgotten index, sigma index and Narumi-Katayama index amongst all possible realizations of a given degree sequence. These results enable us to solve many extremal problems related to graphs.

2010 Mathematics Subject Classifications : 05C07, 05C10, 05C30, 05C35, 57M15.

Keywords: Ω of a graph, degree sequence, graph index, Zagreb index, forgotten index, sigma index, Narumi-Katayama index.

Introduction

Let $G = (V, E)$ be a graph with size m and order n. The degree of a vertex $v \in V(G)$ is denoted by d_v. If the degree of v is one, then it is called a pendant vertex. The biggest vertex degree of G is denoted by Δ, respectively. If u and v are connected to each other by an edge e, this situation will be denoted by $e = uv$. In such a case, the vertices u and v are called adjacent vertices and the edge e is said to be incident with u and v. A graph is connected when there is a path between every pair of vertices. A graph that is not connected is disconnected. A graph having no cycle will be called acyclic and the remaining graphs are called cyclic graphs. An edge connecting a vertex to itself is called a loop, and at least two edges connecting two vertices will be called multiple edges.

Written with multiplicities, a degree sequence is written as

$$DS(G) = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$$

where a_i's are non-negative integers. If we allow the graph to be disconnected and to have isolated points, then we must also add 0^{a_0} at the beginning of $D(S(G))$. Let $D = \{d_1, d_2, d_3, \cdots, d_k\}$ be a set of non-decreasing non-negative integers. A graph G is called a realization of the set D if the degree sequence of G is equal to D.

For a realizable degree sequence, there may be more than one graph having this degree sequence and this is usually the case. For example, the two graphs in Fig. 2 have the same degree sequence:

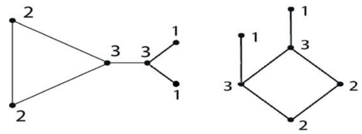

Figure 1 Graphs with the same DS

The most well-known result to determine realizability is the Havel-Hakimi process. For this and some other algorithms, see [1], [2], [3], [6], [7].

The following number have recently been defined by last two authors:

Definition 1. *Let*

$$D = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\} \tag{1}$$

be a set of non-negative integers. $\Omega(D)$ *is defined in terms of the degree sequence* D *as*

$$\begin{aligned}\Omega(D) &= a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 \\ &= \sum_{i=1}^{\Delta}(i-2)a_i.\end{aligned}$$

The number Ω is fixed for all the realizations of a given degree sequence and therefore is a graph invariant. In [4] and [5], several properties of this Ω invariant have been obtained. In [5], a connected realization having a cycle of length $a_2 + a_3 + a_4 + a_5 + \cdots + a_\Delta$, loops, chords and a_1 pendant edges was called a cyclic fundamental realization of D when $\Omega(D) \geq 0$. At the same reference, there are two more definitions for fundamental realizations when $\Omega = -2$ and $\Omega \leq -4$.

Change of graph indices

The following theorem given in [4] will be the main tool for our calculations in this section. It determines the possible realizations of a given degree sequence with zero Ω invariant:

Theorem 2 ([4])**.** *Let* $D = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$ *where* $a_1 > 0$ *and* $a_2,\ a_3,\ \cdots,\ a_\Delta \geq 0$*. If* $\Omega(D) = 0$*, then* D *can be realized as a connected unicyclic graph where the length of this unique cycle could be anything between* 1 *and* $a_2 + a_3 + \cdots + a_\Delta$*.*

The main idea used in the proof was to cut-and-paste process. Start with an $a_2 + a_3 + \cdots + a_\Delta$-gon $C = C_{a_2+a_3+\cdots+a_\Delta}$ so that all vertices on C are placed consecutively from smallest degree to largest degree. Using the fact that $\Omega(D) = 0$, it was shown that we can add one pendant edge to the vertices of degree 3, two pendant edges to the vertices of degree 4, three pendant edges to the vertices of degree 5, and finally $\Delta - 2$ pendant edges to the vertices of degree Δ. This is the fundamental realization defined in [5]. An algorithm was then defined to obtain all unicylic realizations having all positive integers between $a_2 + a_3 + \cdots + a_\Delta$ and 1 as cycle length. Let us denote the realizations obtained at each step of the algorithm by $C_{a_2+a_3+\cdots+a_\Delta-1}$, $C_{a_2+a_3+\cdots+a_\Delta-2}$, $\cdots$, C_1. For example if we start with $C = C_8$, then there are 7 steps giving C_7, C_6, $\cdots$, C_1.

We now calculate the change of six important graph indices amongst all the realizations mentioned in Theorem 2. The first, second and third Zagreb indices of a graph G were defined by

$$M_1(G) = \sum_{v \in V(G)} d_v^2, \tag{2}$$

$$M_2(G) = \sum_{uv \in E(G)} d_u \cdot d_v, \tag{3}$$

$$M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2. \tag{4}$$

The forgotten Zagreb index of G was defined by

$$F(G) = \sum_{v \in V(G)} d_v^3. \tag{5}$$

It is not difficult to see that

$$M_3(G) = F(G) + 2M_2(G). \tag{6}$$

The sigma index was defined as one of the irregularity measures by

$$\sigma(G) = \sum_{uv \in E(G)} (d_u - d_v)^2, \tag{7}$$

and satisfies the relation

$$\sigma(G) = F(G) - 2M_2(G). \tag{8}$$

The last index we deal with in this paper is the Narumi-Katayama index defined by

$$NK(G) = \prod_{v \in V(G)} d_u. \tag{9}$$

Theorem 3. *Let the graph G be one of the realizations of the degree sequence D given in Theorem 2 so that $\Omega(D) = 0$. If $\Delta \leq 9$, then $M_2(G)$ has its smallest and largest values amongst all realizations given in Theorem 2 for C and C_1, respectively. If $\Delta \geq 10$, then $M_2(G)$ has its smallest and largest values amongst all realizations given in Theorem 2 for C_1 and C, respectively.*

Proof. We use the steps of above algorithm to prove the results. We start by a $a_2 + a_3 + \cdots + a_\Delta$-gon C. We first show that the number of pendant edges added to the vertices of C is equal to a_1. Indeed, this number is $a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta$ which is equal, by the definition, to $\Omega + a_1$ which is a_1 in this case. In C, there are a_3 $(1,3)$-vertices, $2a_4$ $(1,4)$-vertices, $3a_5$ $(1,5)$-vertices, $\cdots$, $(\Delta - 2)a_\Delta$ $(1,\Delta)$-vertices, $a_2 - 1$ $(2,2)$-vertices, $a_3 - 1$ $(3,3)$-vertices, $\cdots$, $a_\Delta - 1$ (Δ,Δ)-vertices, one $(2,3)$-vertex, one $(3,4)$-vertex, $\cdots$, one $(\Delta - 1,\Delta)$-vertex, and finally one $(\Delta, 2)$-vertex. So the second Zagreb index of this realization is equal to

$$\begin{aligned} M_2(C) &= \textstyle\sum_{i=2}^{\Delta} \left[(a_i - 1) \cdot i^2 + a_i \cdot (i-2) \cdot (1 \cdot i) + i \cdot (i-1)\right] + 2\Delta - 2 \\ &= \textstyle\sum_{i=2}^{\Delta} 2i \cdot (i-1) \cdot a_i - \dfrac{\Delta^2 - 3\Delta + 2}{2}. \end{aligned}$$

Next, we draw an $a_2 + a_3 + a_4 + \cdots + a_\Delta - 1$-gon by omitting one of the vertices, say v, having the smallest degree on C together with any pendant edges incident with it and the end vertices incident with these edges. To keep the $\Omega(G)$ unchanged, we add a new vertex which we call v again, onto one, say $uv_{a_2+a_3+\cdots+a_\Delta}$, of the $d_{a_2+a_3+\cdots+a_\Delta} - 2$ existing pendant edges. So the second Zagreb index of this new graph $C_{a_2+a_3+\cdots+a_\Delta-1}$ is

$$M_2(C_{a_2+a_3+\cdots+a_\Delta-1}) = \textstyle\sum_{i=2}^{\Delta} \left[(a_i - 1) \cdot i^2 + a_i \cdot (i-2) \cdot (1 \cdot i) + i \cdot (i-1)\right] + 3\Delta - 4.$$

Continuing in the same fashion, we reach to a 1-gon (loop) C_1 which has the second Zagreb index

$$M_2(C_1) = \sum_{i=2}^{\Delta} 2i \cdot (i-1) \cdot a_i - \frac{\Delta^2 + \Delta - 36}{2}.$$

Now define a function f such that

$$\begin{aligned} f(\Delta) &= M_2(C_{a_2+a_3+\cdots+a_\Delta}) - M_2(G_1) \\ &= 2\Delta - 19. \end{aligned}$$

First note that $f'(\Delta) = 2$ and f is always increasing. Also for $\Delta < 19/2$, $M_2(C_{a_2+a_3+\cdots+a_\Delta}) < M_2(G_1)$ and for $\Delta > 19/2$, $M_2(C_{a_2+a_3+\cdots+a_\Delta}) > M_2(G_1)$. This completes the proof. □

Theorem 4. *Let the degree sequence D be realizable and have $\Omega(D) = 0$. Then*

i the first Zagreb index of G is the same for all graphs G given in Theorem 2;

ii the forgotten Zagreb index of G is the same for all graphs G given in Theorem 2.

iii the Narumi-Katayama index of G is the same for all graphs G given in Theorem 2.

Proof. The first two indices are defined as the sum of powers of the vertex degrees. The third one is the product of all vertex degrees. As the degree sequence is the same for all the realizations, each of these three indices takes the same value. □

Corollary 5. *Let the graph G be one of the realizations of the degree sequence D given in Theorem 2 so that $\Omega(D) = 0$. If $\Delta \leq 9$, then $M_3(G)$ has its smallest and largest values amongst all realizations given in Theorem 2 for C and C_1, respectively. If $\Delta \geq 10$, then $M_3(G)$ has its smallest and largest values amongst all realizations given in Theorem 2 for C_1 and C, respectively.*

Proof. It follows from Eqn. 6 and Theorem 3. □

Corollary 6. *Let the graph G be one of the realizations of the degree sequence D given in Theorem 2 so that $\Omega(D) = 0$. If $\Delta \leq 9$, then $\sigma(G)$ has its smallest and largest values amongst all realizations given in Theorem 2 for C and C_1, respectively. If $\Delta \geq 10$, then $\sigma(G)$ has its smallest and largest values amongst all realizations given in Theorem 2 for C_1 and C, respectively.*

Proof. It follows from Eqn. 8 and Theorem 3. □

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Application of Topological Degree Method In Quantitative Behavior of Fractional Differential Equations

Ghaus ur Rahman[1]

Abstract

In this paper, we study existence and uniqueness of positive solution to a classes of fractional order differential equations involving the Caputo derivatives. By using classical fixed point theorem on topological degree methods for condensing map, we obtain sufficient conditions for existence and uniqueness of positive solution. Some conditions devoted to the stability of Hyers-Ullams type are also established. Moreover suitable example will be provided to illustrate the main results.

2010 Mathematics Subject Classifications : 30E25, 26A33, 34A37, 34B05
Keywords: Topological degree method, condensing map, Hyers-Ullams type stability, fractional differential equations

Introduction

The study of fractional differential equations expand from theoretical background of existence and uniqueness of positive solutions to numerical methods for finding solutions. Fractional differential equations have been used in various fields e.g economics, physics and engineering for detail (see[5],[6],[7] and reference therein). It has large applications in image and signal processing phenomenons, nonlinear oscillation of earthquakes. The existence theory has been studied very well by using the tools of classical fixed point theory, for detail (see[2],[3],[4],[8]).

There are two aspects of differential equations, qualitative aspect and quantitative. In qualitative aspect we do not deal with the actual solution of the system of differential equations, rather we discuss the general behavior of the whole family of solution in phase plane. While in the quantitative aspect of system of differential equations we explore the numerical or exact solution of the system. This paper is concerned with the existence, uniqueness and data dependence of solution to the following classes of nonlocal Cauchy problems and boundary value problems of fractional order, using topological degree method.

$$\begin{cases} {}^cD^pu(t) = f(t,u(t)),\ t \in I = [0,1],\ 1 < p \leq 2, \\ u(0) = u_0,\ u(1) = g(u), \end{cases} \tag{1}$$

where $g, h : C(J,\mathbb{R}) \to \mathbb{R}$ are nonlocal functions and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function, ${}^cD^p$ is the Caputo fractional derivatives of order p, $0 < p < 1$. Also A is bounded linear operator from $D(A)$ to $\mathbb{R}$.

Main Results

Existence, uniqueness and data dependence results of the model (1)

In this section, we deal with the existence, uniqueness and data dependence of the problem (1). Initially we assume the some assumptions. Now we study the existence and uniqueness of solution for (1).

Theorem 1. *Fractional differential equation of non-local Cauchy problem*

$$\begin{cases} {}^cD^p u(t) = y(t),\ t \in I = [0,1], \\ u(0) = u_0,\ u(1) = g(u), \end{cases}$$

has a unique solution u, which possess the form $u(t) = \int_0^1 \mathcal{G}(t,s)y(s)ds$, where $\mathcal{G}(t,s)$ is the Green function given by

$$\mathcal{G}(t,s) = \frac{1}{\Gamma(p)} \begin{cases} -t(1-s)^{p-1} + (t-s)^{p-1},\ 0 \le s \le t \le 1, \\ -t(1-s)^{p-1},\ 0 \le t \le s \le 1. \end{cases} \tag{2}$$

Theorem 2. *The operator $F : C(J,R) \to C(J,R)$ is Lipschitz continuous with Lipschitzen $k_g \in [0,1)$. Consequently F is α- Lipschitz with constant k, moreover F obeys the growth condition provided by*

$$||F(u)|| \le |u_0| + C_g||u||^{q_1} + M_1.$$

Theorem 3. *The operator $G : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous, moreover G satisfies the following growth condition*

$$||Gu|| \le \frac{1}{\Gamma(p+1)}[C_f||u||^{q_2} + M_2].$$

Theorem 4. *The operator $G : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is compact, consequently G is α-Lepschitiz with zero constant.*

Theorem 5. *Let (H_1) to (H_3) hold, then (1) has at least one solution $u \in C(J,\mathbb{R})$ and the solution set of (1) is bounded in $C(J,\mathbb{R})$.*

Theorem 6. *Assume that $(H_1) - (H_4)$ hold, then fractional NCP(1) has a unique solution $u \in C(J,\mathbb{R})$ if and only if $k_g + \frac{L_f}{\Gamma(p+1)} < 1$.*

Theorem 7. *Under the assumption $(H_1) - (H_4)$, the BVP (1) is Hyers-Ulam stable and hence generalized Hyers-Ulam stable if*

$$\Upsilon = \left(k_g + \frac{L_f}{\Gamma(\alpha+1)}\right) \neq 1.$$

Conclusion

The present manuscript mainly deals with the existence results for a class of nonlocal Cauchy problem with boundary conditions. The input functions which are designated by $g, h : C(J,\mathbb{R}) \to \mathbb{R}$ are nonlocal functions. Using some fixed point theorem and topological degree theory to exhibit the solutions of the said problems. Moreover, for both problem, sufficient conditions were successfully developed under which they are Hyers-Ullam and generalize Hyers-Ullam stable

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Semilocal convergence of Sakurai-Torii-Sugiura method for simultaneous approximation of polynomial zeros

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Abstract

In this talk, we provide a new semilocal convergence theorem for a fourth-order iterative method for the simultaneous approximation of polynomial zeros due to Sakurai, Torii and Sugiura [1]. This theorem improves and complements the existing result of Petković, Rančić and Milošević [2]. Two numerical examples are given to show some practical applications of our result.

2010 Mathematics Subject Classifications : 65H05
Keywords: Iterative methods, Polynomial zeros, Simultaneous methods, Semilocal convergence, Error estimates

Introduction

Let $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ be a complex polynomial. We consider the zeros $\xi_1, \ldots, \xi_n$ of f as a vector ξ in $\mathbb{C}^n$. More precisely, a vector $\xi \in \mathbb{C}^n$ is said to be a *root vector* of f if $f(z) = a_0 \prod_{i=1}^{n}(z - \xi_i)$ for all $z \in \mathbb{C}$.

In 1991, Sakurai, Torii and Sugiura [1] introduced a fourth-order iterative method for simultaneous finding polynomial zeros. The *Sakurai-Torii-Sugiura method* can be defined in $\mathbb{C}^n$ by the following iteration:

$$x^{(k+1)} = x^{(k)} - \Phi(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where the correction function $\Phi \colon \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$\Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x)) \text{ with } \Phi_i(x) = \begin{cases} \dfrac{2L_i(x)}{L_i(x)^2 - F_i(x)} & \text{if } f(x_i) \neq 0, \\ 0 & \text{if } f(x_i) = 0, \end{cases}$$

with $L_i(x)$ and $F_i(x)$ defined as follows

$$L_i(x) = \frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{1}{x_i - x_j}, \quad F_i(x) = \frac{f''(x_i)}{f(x_i)} - \left(\frac{f'(x_i)}{f(x_i)}\right)^2 + \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}.$$

In 2003, Petković, Rančić and Milošević [2] established a semilocal convergence result for the iteration method (1). They proved that if $f \in \mathbb{C}[z]$ is a polynomial of degree $n \geq 3$ with only simple zeros and $x^{(0)} \in \mathbb{C}^n$ is an initial approximation with distinct components such that

$$\|W_f(x^{(0)})\|_\infty < \frac{\delta(x^{(0)})}{3n+1},$$

then the Sakurai-Torii-Sugiura method (1) is convergent to a root vector of f. Here and throughout, $\|\cdot\|_\infty$ stands for the max-norm on $\mathbb{C}^n$, the function $W_f\colon \mathbb{C}^n \to \mathbb{C}^n$ is the well known Weierstrass' correction defined by

$$W_f(x) = (W_1(x), \ldots, W_n(x)) \quad \text{with} \quad W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)}$$

and the function $\delta\colon \mathbb{C}^n \to \mathbb{R}_+$ is defined by

$$\delta(x) = \min_{i \neq j} |x_i - x_j|. \tag{2}$$

The main purpose of this talk is to present a new semilocal convergence theorem for the Sakurai-Torii-Sugiura iteration (1) which improves and complements the result of Petković et al. [2]. We end the work with two numerical examples that show some practical applications of our result.

Main Result

We define a relation of equivalence $\equiv$ on $\mathbb{C}^n$ by $x \equiv y$ if there exists a permutation $(i_1, \ldots, i_n)$ of the indixes $(1, \ldots, n)$ such that

$$(x_1, \ldots, x_n) = (y_{i_1}, \ldots, y_{i_n}).$$

Now we can define a distance between two vectors $x, y \in \mathbb{C}^n$ as follows:

$$\rho(x, y) = \min_{u \equiv y} \|x - u\|_\infty.$$

Recently, Proinov [3, 4, 5] has developed a general convergence theory for the Picard type iterative methods. Using this theory, in [6] we proved two new semilocal convergence theorems for the Sakurai-Torii-Sugiura method (1). The following result is a consequence of one of these theorems. This result improves and complements the above mentioned result of Petković et al. [2] in several directions.

Theorem 1. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ and $x^{(0)} \in \mathbb{C}^n$ be a vector with pairwise distinct components satisfying*

$$\|W_f(x^{(0)})\|_\infty \leq \frac{8}{(3 + \sqrt{8n - 7})^2}\; \delta(x^{(0)}). \tag{3}$$

Then f has only simple zeros and the iteration (1) is well defined and converges to a root-vector ξ of f with order of convergence four and with error estimate

$$\rho(x^{(k)}, \xi) \leq \alpha(E_f(x^{(k)}))\, \|W_f(x^{(k)})\|_\infty$$

for all $k \geq 0$ such that $E_f(x^{(k)}) < 1/(1 + \sqrt{n-1})^2$, where the functions E_f and α are defined by

$$E_f(x) = \frac{\|W_f(x)\|_\infty}{\delta(x)} \quad \text{and} \quad \alpha(t) = \frac{2}{1 - (n-2)t + \sqrt{(1 - (n-2)t)^2 - 4t}}$$

and the function δ is defined by (2).

Numerical results

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ and $x^{(0)} \in \mathbb{C}^n$ be an initial approximation. In the examples below, we apply the Sakurai-Torii-Sugiura method (1) for computing all the zeros $\xi_1, \ldots, \xi_n$ of f simultaneously.

The main purpose of this section is to show that Theorem 1 can be used for solving the following very important practical problems:

i) numerical proof of the convergence of Sakurai-Torii-Sugiura method;

ii) numerical proof of the guaranteed accuracy of approximations at each iteration.

Let us define the functions E_f and α as in Theorem 1. It follows from Theorem 1 that if there exists an integer $m \geq 0$ such that

$$E_f(x^{(m)}) \leq \mathcal{R}_n = \frac{8}{(3 + \sqrt{8n-7})^2}\,, \tag{4}$$

then the iteration (1) is well-defined and converges to a root vector ξ of f with order of convergence four. Besides, for a given accuracy $\varepsilon > 0$ if there exists an integer $k \geq 0$ such that

$$E_f(x^{(k)}) < \tau_n = \frac{1}{(1 + \sqrt{n-1})^2} \quad \text{and} \quad \varepsilon_k = \alpha(E_f(x^{(k)}))\, \|W_f(x^{(k)})\|_\infty < \varepsilon, \tag{5}$$

then f has only simple zeros and at kth iteration the root vector ξ of f is calculated with guaranteed accuracy ε_k. For each example, we calculate the smallest integer $m \geq 0$ which satisfies the convergence condition (4) and the smallest $k \geq 0$ which satisfies the accuracy criterion (5) with $\varepsilon = 10^{-15}$.

Example 2. *In this example, we consider the polynomial*

$$f(z) = z^5 - 15z^4 + 22z^3 + 438z^2 - 1175z - 1575$$

with zeros ± 5, -1, 7, 9 *and initial guess* $x^{(0)} = (-5.7, -1.8, 4.1, 6.2, -9.8)$ *which are taken from Cholakov and Vasileva [7, Example 5.1]. We can see in Table 1 that the convergence condition* (4) *is satisfied for* $m = 1$ *and the accuracy criterion* (5) *is satisfied for* $k = 3$*. At the third iteration, the zeros of* f *are found with guaranteed accuracy less than* 10^{-21}*. Moreover, at the forth iteration the zeros of* f *are calculated with accuracy* 10^{-91}.

Table 1: Numerical results for Example 2 $(\tau_n = 0.111111)$.

m	$E_f(x^{(m)})$	$\mathcal{R}_n$	k	$E_f(x^{(k)})$	ε_k	ε_{k+1}
1	0.090882	0.104619	3	7.8×10^{-23}	1.5×10^{-22}	1.1×10^{-91}

Example 3. *Consider the polynomial* $f(z) = z^{15} + z^{14} + 1$ *and Aberth's initial approximation* $x^{(0)} \in \mathbb{C}^n$ *defined by*

$$x_\nu^{(0)} = -\frac{a_1}{n} + r_0 \exp(i\,\theta_\nu), \qquad \theta_\nu = \frac{\pi}{n}\left(2\nu - \frac{3}{2}\right), \qquad \nu = 1, \ldots, n,$$

with $a_1 = 1$, $r_0 = 2$ *and* $n = 15$*. As can be seen from Table 2 the convergence condition* (4) *is satisfied for* $m = 5$*, the accuracy criterion* (5) *is satisfied for* $k = 6$ *and at the seventh iteration the zeros of* f *are calculated with an accuracy of at least* 10^{-84}*. In Figure 1, we present the trajectories of approximations generated by 6 iterations.*

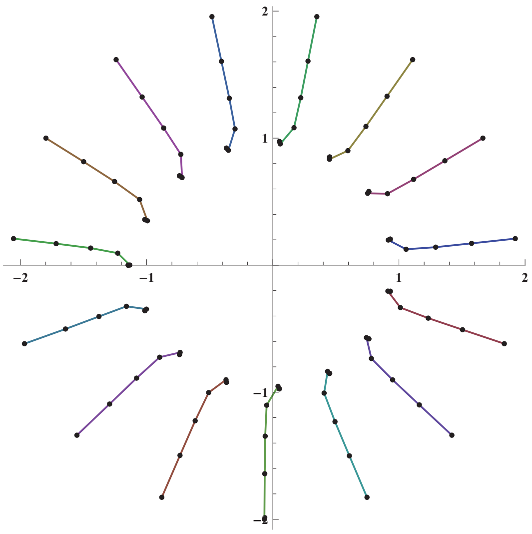

Figure 1: Trajectories of approximations for Example 3.

Table 2: Numerical results for Example 3 ($\tau_n = 0.044477$).

m	$E_f(x^{(m)})$	$\mathcal{R}_n$	k	$E_f(x^{(k)})$	ε_k	ε_{k+1}
5	5.2×10^{-6}	0.043061	6	8.6×10^{-22}	3.7×10^{-22}	1.1×10^{-85}

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On the Upper Second Submodules

Seçil Çeken[1]

Abstract

Let R be a ring with identity and M be a left R-module. The set of all second submodules of M is called the *second spectrum* of M and denoted by $Spec^s(M)$. For each prime ideal p of R we define $Spec^s_p(M) := \{S \in Spec^s(M) : ann_R(S) = p\}$. A second submodule Q of M is called an upper second submodule if there exists a prime ideal p of R such that $Spec^s_p(M) \neq \emptyset$ and $Q = \sum_{S \in Spec^s_p(M)} S$. In this note we give some characterizations of upper second submodules of some module classes.

2010 Mathematics Subject Classifications : 16D10, 16D80
Keywords: Second submodule, Upper Second submodule

Introduction

Throughout this paper all rings will be associative rings with identity elements and all modules will be unital left modules. Unless otherwise stated R will denote a ring. By a proper submodule N of a left R-module M, we mean a submodule N with $N \neq M$. Given a left R-module M, we shall denote the annihilator of M (in R) by $ann_R(M)$.

A non-zero R-module M is called a *prime module* if $ann_R(M) = ann_R(K)$ for every non-zero submodule K of M. A proper submodule N of a module M is called a *prime submodule* of M if M/N is a prime module. If N is a prime submodule of a module M, then $ann_R(M/N) = p$ is a prime ideal of R and in this case N is called a *p-prime submodule* of M. The set of all prime submodules of a module M is called the *prime spectrum* of M and denoted by $Spec(M)$. Also, the set of all p-prime submodules of M is denoted by $Spec_p(M)$ for a prime ideal p of R (see [11]).

In [13], S. Yassemi introduced second submodules of modules over commutative rings as the dual notion of prime submodules. Second modules over arbitrary rings were defined in [2] and used as a tool for the study of attached primes over noncommutative rings. A right R-module M is called a *second module* provided $M \neq (0)$ and $ann_R(M) = ann_R(M/N)$ for every proper submodule N of M. By a *second submodule* of a module, we mean a submodule which is also a second module. If N is a second submodule of a module M, then $ann_R(N) = p$ is a prime ideal of R and in this case N is called a *p-second submodule* of M. Recently, second submodules have attracted attention of many authors and they have been extensively studied in a number of papers (see for example [1], [3], [4], [6], [7], [8], [9]).

In [5], Behboodi and Shojaee introduced the notion of lower prime submodules and investigated a topology on the set of these submodules. A prime submodule Q of a module M is called a *lower prime* submodule if there exists a prime ideal p of R such that $Spec_p(M) \neq \emptyset$ and $Q = \bigcap_{P \in Spec_p(M)} P$. Motivated by this notion, in this paper, we define the concept of upper second submodule and investigate some properties of this module class.

Upper Second Submodules and Upper Dual Zariski Topology

Definition 1. *Let M be a left R-module. A second submodule Q of M is called an upper second submodule if there exists a prime ideal p of R such that $Spec_p^s(M) \neq \emptyset$ and $Q = \sum_{S \in Spec_p^s(M)} S$. The set of all upper second submodules of M is called upper second spectrum of M and denoted by $u.Spec^s(M)$.*

Clearly, if S, $Q \in u.Spec^s(M)$, then $S = Q$ if and only if $ann_R(S) = ann_R(Q)$.

Proposition 2. *Let R be a ring such that R/P is a right or left Goldie ring for every prime ideal P of R. If M is an injective left R-module, then $u.Spec^s(M) = \{(0 :_M p) : p$ is a prime ideal of R and $(0 :_M p) \neq 0\}$.*

Proof. The result follows from [8, Lemma 3.9]. □

We will characterize the upper second submodules of an Artinian module. To do this we need some preliminary results. First, we generalize the notion of p-interior of a submodule which was defined in [3]. Let R be a commutative ring, p be a prime ideal of R and N be a submodule of an R-module M. Following [3], the p-interior of N is defined as the set

$I_p(N) := \cap\{L : L$ is a completely irreducible submodule of M and $rN \subseteq L$ for some $r \in R \backslash p\}$.

Let R be an arbitrary ring, p be a prime ideal of R and M be an R-module. We generalize the p-interior of a submodule N of M as follows.

$I_p^M(N) = \cap\{L : L$ is a completely irreducible submodule of M and $AN \subseteq L$ for some ideal $A \not\subseteq p\}$.

Clearly, $I_p^M(N)$ is a submodule of M and $I_p^M(N) \subseteq N$.

Theorem 3. *Let N be a submodule of a left R-module M such that $ann_R(N) = p$ is a prime ideal of R. If $M/I_p^M(N)$ is a finitely cogenerated R-module, then $I_p^M(N) = \sum_{S \in Spec_p^s(N)} S$, i.e. $I_p^M(N)$ is an upper second submodule of N.*

Proof. First, we show that $ann_R(I_p^M(N)) = p$. Since $I_p^M(N) \subseteq N$, we have $ann_R(N) = p \subseteq ann_R(I_p^M(N))$. Let $s \in ann_R(I_p^M(N))$. Since $M/I_p^M(N)$ is finitely cogenerated, there exist an $n \in \mathbb{Z}^+$ and completely irreducible submodules L_i of M such that $A_iN \subseteq L_i$ for some ideals $A_i \not\subseteq p$ and $I_p^M(N) = \cap_{i=1}^n L_i$. Set $A = A_1...A_n$. Then $AN \subseteq I_p^M(N)$ and so $(sA) = 0$. It follows that $(RsR)A \subseteq p$. Since $A \not\subseteq p$, we have $s \in p$. Therefore, $ann_R(I_p^M(N)) = p$. Now, we show that $I_p^M(N)$ is a second submodule of M. Let B be an ideal of R such that $BI_p^M(N) \subsetneq I_p^M(N)$. Then there exists a completely irreducible submodule L of M. such that $BI_p^M(N) \subseteq L$ and $I_p^M(N) \not\subseteq L$. Hence, for each ideal $C \not\subseteq p$, we have $CN \not\subseteq L$. Now, $AN \subseteq I_p^M(N)$ implies $BAN \subseteq L$. This leads to $BA \subseteq p$. Since $A \not\subseteq p$, we have $B \subseteq p$. Thus, $BI_p^M(N) = 0$. This shows that $I_p^M(N)$ is a p-second submodule of N. Then, clearly, $I_p^M(N) \subseteq \sum_{S \in Spec_p^s(M)} S$. Suppose that there exists a p-second submodule S of N such that $I_p^M(N) \subsetneq S$. Since $I_p^M(N) = \cap_{i=1}^n L_i \subsetneq S$, there exists L_i $(1 \leq i \leq n)$ such that $S \not\subseteq L_i$, but $AN \subseteq L_i$. Since S is p-second and $A \not\subseteq p$, we have $S = AS \subseteq AN \subseteq L_i$, a contradiction. Thus, there does not exist a p-second submodule S of N such that $I_p^M(N) \subsetneq S$. Since $\sum_{S \in Spec_p^s(M)} S$ is a p-second submodule of N, we have $I_p^M(N) = \sum_{S \in Spec_p^s(M)} S$. □

Corollary 4. *Let p be a prime ideal of R and M be an R-module such that $M/I_p^M((0:_M p))$ is a finitely cogenerated R-module. If $I_p^M((0:_M p)) \neq 0$, then $I_p^M((0:_M p))$ is an upper second submodule of M.*

Proof. By Theorem 3, it is enough to show that $ann_R((0:_M p)) = p$. To see this, suppose that $r(0:_M p) = 0$ and $r \notin p$. Hence for each completely irreducible submodule L of M, we have $r(0:_M p) \subseteq L$. Then $(RrR)(0:_M p) \subseteq L$ and $(RrR) \not\subseteq p$. So, $I_p^M((0:_M p)) \subseteq L$. Hence $I_p^M((0:_M p)) = 0$, a contradiction. Therefore $ann_R((0:_M p)) = p$ and by Theorem 3, $I_p^M((0:_M p))$ is an upper p-second submodule of $(0:_M p)$. Let S be a p-second submodule of M. Since $S \subseteq (0:_M p)$, we have $S \subseteq I_p^M((0:_M p))$. Thus $I_p^M((0:_M p))$ is an upper p-second submodule of M. □

Corollary 5. *Let p be a prime ideal of R and M be an R-module such that $M/I_p^M((0:_M p))$ is a finitely cogenerated R-module. Then the following statements are equivalent.*

(1) $ann_R((0:_M p)) = p$.
(2) $I_p^M((0:_M p))$ *is an upper second submodule of M.*
(3) *There exists a second submodule K of M such that $p = ann_R(K)$.*
(4) $I_p^M((0:_M p)) \neq 0$.

Proof. The result follows from Theorem 3 and Corollary 4. □

Corollary 6. *Let M be an Artinian left R-module. Then*
$u.Spec^s(M) = \{I_p^M((0:_M p)) : p$ *is a prime ideal of R and* $Spec_p^s(M) \neq \emptyset\}$.

Proof. The result follows from Corollary 4. □

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A note on combinatorial numbers and polynomials

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Abstract

In this paper, we aim to introduce generating functions for higher-order of the recently introduced family of special numbers and polynomials, which are associated essentially with not only Apostol-type numbers and polynomials, but also combinatorial numbers such as Simsek numbers and polynomials, by Kucukoglu and Simsek [2]. Thereafter, by making use of these functions with their functional equation, we derive some combinatorial sums including the higher-order of λ-Apostol-Daehee polynomials in addition to the higher-order of Simsek numbers and polynomials.

2010 Mathematics Subject Classifications : 05A10, 05A15, 11B83, 26C05, 30D05.

Keywords: Combinatorial sum, Functional Equation, Generating functions, Binomial coefficient, λ-Apostol-Daehee polynomials, Simsek numbers and polynomials, Special numbers and polynomials

Introduction

Combinatorial numbers and polynomials arise in various kind of areas such as mathematics, engineering and mathematical physics. Recently, these kinds of combinatorial numbers and polynomials have been defined by Simsek [5]. In this paper, we focus on these numbers and polynomials. We give some new formulas and relations on these numbers and polynomials.

In [1], higher-order of the Simsek numbers $Y_n(\lambda)$ and the Simsek polynomials $Y_n(x;\lambda)$ were defined by the following generating functions, respectively:

$$\mathcal{F}(t,k;\lambda)=\left(\frac{2}{\lambda(1+\lambda t)-1}\right)^k=\sum_{n=0}^{\infty}Y_n^{(k)}(\lambda)\frac{t^n}{n!}, \tag{1}$$

and

$$\mathcal{F}(t,x,k;\lambda)=\mathcal{F}(t,k;\lambda)(1+\lambda t)^x=\sum_{n=0}^{\infty}Y_n^{(k)}(x;\lambda)\frac{t^n}{n!}. \tag{2}$$

where k is nonnegative integer and λ is a real or complex number.

Note that

$$Y_n(\lambda)=Y_n^{(1)}(\lambda),$$
$$Y_n(x;\lambda)=Y_n^{(1)}(x;\lambda),$$

and

$$Y_n^{(k)}(\lambda)=Y_n^{(k)}(0;\lambda).$$

For detail information about these numbers and polynomials, the reader may consult the recent works [1] and [5], [7].

In [2], generating function for the numbers $I_{n,d}(\lambda,q)$ and the polynomials $I_{n,d}(x;\lambda,q$ which are associated with not only the Apostol-type numbers and polynomials, but also combinatorial numbers and polynomials, were defined as follows, respectively:

$$F_d(t;\lambda,q)=\frac{\log(1+\lambda t)}{(\lambda q)^d(1+\lambda t)^d-1}=\sum_{n=0}^{\infty}I_{n,d}(\lambda,q)\frac{t^n}{n!}, \tag{3}$$

and

$$G_d(t,x;\lambda,q)=(1+\lambda t)^x F_d(t;\lambda,q)=\sum_{n=0}^{\infty}I_{n,d}(x;\lambda,q)\frac{t^n}{n!} \tag{4}$$

where $|\lambda t|<1$.

The generating function for the higher-order of λ-Apostol-Daehee polynomials $\mathfrak{D}_n^{(k)}(x;\lambda)$ was defined by Simsek in [4] as follows:

$$F_{\mathfrak{D}}(t,x;\lambda,k)=\left(\frac{\log\lambda+\log(1+\lambda t)}{\lambda^2 t+\lambda-1}\right)^k(1+\lambda t)^x=\sum_{n=0}^{\infty}\mathfrak{D}_n^{(k)}(x;\lambda)\frac{t^n}{n!}, \tag{5}$$

where k is nonnegative integer and λ is a real or complex number (*cf.* [3], [4], [6]).

Main Results

In this section, we give generating functions for higher-order of the numbers $I_{n,d}(\lambda,q)$ and the polynomials $I_{n,d}(x;\lambda,q)$ which are recently introduced by Kucukoglu and Simsek [2].

Let k be nonnegative integers and λ be real or complex numbers. We define the numbers $I_{n,d}(\lambda,q)$ of order k and the polynomials $I_{n,d}(x;\lambda,q)$ of order k by the following generating functions, respectively:

$$\mathcal{F}_d(t;\lambda,q,k)=\left(\frac{\log(1+\lambda t)}{(\lambda q)^d(1+\lambda t)^d-1}\right)^k=\sum_{n=0}^{\infty}I_{n,d}^{(k)}(\lambda,q)\frac{t^n}{n!}, \tag{6}$$

and

$$\mathcal{G}_d(t,x;\lambda,q,k)=(1+\lambda t)^x\mathcal{F}_d(t;\lambda,q)=\sum_{n=0}^{\infty}I_{n,d}^{(k)}(x;\lambda,q)\frac{t^n}{n!}, \tag{7}$$

where $|\lambda t|<1$.

Note that

$$I_{n,d}(\lambda,q)=I_{n,d}^{(1)}(\lambda,q),$$

$$I_{n,d}(x;\lambda,q)=I_{n,d}^{(1)}(x;\lambda,q),$$

and

$$I_{n,d}^{(k)}(\lambda,q)=I_{n,d}^{(k)}(0;\lambda,q).$$

Now, it is time to obtain some identities and relations involving the numbers $I_{n,d}^{(k)}(\lambda,q)$ and the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$.

Theorem 1 gives us a relation between the numbers $I_{n,d}^{(k)}(\lambda,q)$ and the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$ is given by the following theorem:

Theorem 1. *Let n be a nonnegative integer. Then we have*

$$I_{n,d}^{(k)}(x;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(x)_{n-j}I_{n,d}^{(k)}(\lambda,q).$$

Proof. By using equations (6) and (7) and assuming that $|\lambda t| < 1$, we get

$$\sum_{n=0}^{\infty} I_{n,d}^{(k)}(x;\lambda,q)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(x)_n\lambda^n\frac{t^n}{n!}\sum_{n=0}^{\infty} I_{n,d}^{(k)}(\lambda,q)\frac{t^n}{n!}.$$

By using Cauchy product rule in the above equation and comparing the coefficient $\frac{t^n}{n!}$ on both sides of the final equation, we arrive at the assertion of Theorem 1. □

Theorem 2 gives us a combinatorial sum in order to compute some special values of the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$ with the help of the higher-order of λ-Apostol-Daehee polynomials and the higher-order of Simsek polynomials.

Theorem 2.

$$I_{v,1}^{(k)}(x;\lambda,1) = \sum_{j=0}^{k}\sum_{v=0}^{n}\binom{k}{j}\binom{n}{v}\left(-\frac{\log\lambda}{2}\right)^{j}\mathfrak{D}_{n-v}^{(k-j)}(x;\lambda)\,Y_v^{(j)}(\lambda). \tag{8}$$

Proof. Combining(1), (5) and (7), we obtain the following functional equation:

$$\mathcal{G}_1(t,x;\lambda,1,k) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{\log\lambda}{2}\right)^{j}F_{\mathfrak{D}}(t,x;\lambda,k-j)\,\mathcal{F}(t,j;\lambda).$$

From the above functional equation, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} I_{n,1}^{(k)}(x;\lambda,1)\frac{t^n}{n!} &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{\log\lambda}{2}\right)^{j}\left(\sum_{n=0}^{\infty}\mathfrak{D}_n^{(k-j)}(x;\lambda)\frac{t^n}{n!}\right)\\ &\quad\times\left(\sum_{n=0}^{\infty}Y_n^{(j)}(\lambda)\frac{t^n}{n!}\right).\end{aligned} \tag{9}$$

By using the Cauchy product rule in (9) and comparing the coefficient $\frac{t^n}{n!}$ on both sides of the final equation, we arrive at the assertion of Theorem 2. □

Theorem 3 gives us a combinatorial sum in order to compute the higher-order of λ-Apostol-Daehee polynomials with the aid of the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$ and the higher-order of Simsek polynomials.

Theorem 3.

$$\mathfrak{D}_n^{(k)}(x;\lambda) = \sum_{j=0}^{k}\sum_{v=0}^{n}\binom{k}{j}\binom{n}{v}\left(\frac{\log\lambda}{2}\right)^{k-j}Y_{n-v}^{(k-j)}(\lambda)\,I_{v,1}^{(j)}(x;\lambda,1). \tag{10}$$

Proof. By making use of the binomial theorem in (5) and combining the final equation with (1) and (7), we get the following functional equation:

$$F_{\mathfrak{D}}(t,x;\lambda,k) = \sum_{j=0}^{k}\binom{k}{j}\left(\frac{\log\lambda}{2}\right)^{k-j}\mathcal{F}(t,k-j;\lambda)\,\mathcal{G}_1(t,x;\lambda,1,j). \tag{11}$$

It follows from the above functional equation that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{D}_n^{(k)}(x;\lambda)\frac{t^n}{n!} &= \sum_{j=0}^{k}\binom{k}{j}\left(\frac{\log\lambda}{2}\right)^{k-j}\left(\sum_{n=0}^{\infty}Y_n^{(k-j)}(\lambda)\frac{t^n}{n!}\right)\\ &\quad\times\left(\sum_{n=0}^{\infty}I_{n,1}^{(j)}(x;\lambda,1)\frac{t^n}{n!}\right).\end{aligned} \tag{12}$$

By using Cauchy product rule in (12) and comparing the coefficient $\frac{t^n}{n!}$ on both sides of the final equation, we arrive at the assertion of Theorem 3. □

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Power GCDQ Matrices over Euclidean Domains

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Abstract

In this paper, we use a generalization for the Jordan totient function over Euclidean domains (EDs) to give a full generalization of the power GCD matrices. Their structures, determinants, inverses, and norms defined on arbitrary and factor-closed q-ordered sets are also presented over EDs.

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Preliminaries

Let $T = \{t_1, t_2, ..., t_m\}$ be a well ordered set of m distinct positive integers with $t_1 < t_2 < ... < t_m$ and let (t_i, t_j) be the greatest common divisor (GCD) of t_i and t_j, respectively. The GCD matrix (T) defined on T is an $m \times m$ matrix whose $ij^{th} - entry$ is $t_{ij} = (t_i, t_j)$ and its power GCD matrix defined on T is (T^r) whose ij^{th} entry is $t_{ij} = (t_i, t_j)^r$, where r is a non negative real number. The set $T = \{t_1, t_2, ..., t_m\}$ is said to be factor-closed (FC) set if it contains all the divisors of its elements.

Definition 1. *Let $T = \{t_1, t_2, ..., t_m\}$ be a set of non-zero non-associate elements in an Euclidean domain S with measure q, and let $\{p_1, p_2, ..., p_i, ..\}$ be an ordered listing of all primes in P of S that divide all the elements of T. Assume that $\{p_1, p_2, ..., p_i, ..\}$ has the order inherited from the well ordering of P, then we define an ordering on S via the following scheme: $t_i <_q t_j$ if $q(t_i) < q(t_j)$, and we call it q-ordering.*

We note that the relation $<_q$ is a well-defined linear ordering defined on S. Hence, if the set $T = \{t_1, t_2, ..., t_m\}$ such that $t_1 <_q t_2 <_q ... <_q t_m$, then we say that T is q-ordered. Note that the ring of Gaussian integers $S = \mathbb{Z}[i]$ with $q : S - \{0\} \to N \cup \{0\}$ defined by $q(a + bi) = a^2 + b^2$ is an ED with measure q, where N is the set of positive integers.

Definition 2. *Let S be ED with measure q. A complete set $P = \{p_1, p_2, p_3, ...\}$ is said to be a prime residue system of S if P is a complete, well q-ordered set of non-associate prime elements of S.*

In the following, we study the $GCDQ$ matrices defined over any Euclidean domain S with a prime residue system P and measure q.

Definition 3. *For any non-zero element $x \in S$ with the unique factorization, up to associates, $x = up_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$ define the totally multiplicative function $\phi_s(x) = \prod_{i=1}^{m} q\left(p_i^{\alpha_i - 1}\right)((q(p_i) - 1)$, where $p_i \in P$, $\alpha_i \in N$, and u is a unit in S such that $\phi_s(u) = 1$.*

Theorem 4. *If $x \in S$ and $E(x)$ is a complete set of distinct non-associate divisors d of x in S then* $q(x) = \sum_{d \in E(x)} \phi_s(d)$.

Definition 5. *Let $x = up_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$ be a non-zero element in S. Define the Jordan totient function $J_{k,s}$ on $S - \{0\}$ to be the multiplicative function* $J_{k,s}(x) = \prod_{i=1}^{m} q\left(p_i\right)^{k(\alpha_i - 1)} \left(q(p_i)^k - 1\right)$ *with $J_{k,s}(x) = 1$ if x is unit.*

Theorem 6. *If $x \in S$ and $E(x)$ is a complete set of distinct non-associate divisors d of x in S, then* $q\left(x\right)^k = \sum_{d \in E(x)} J_{k,s}(d)$.

Power GCDQ Matrices over Euclidean Domains

Definition 7. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S, then the r^{th} power GCDQ matrix defined on T is the $m \times m$ matrix $(T^r)_q$ whose ij^{th} entries are defined as $(t_{ij})_r = q\left((t_i, t_j)\right)^r$, where (t_i, t_j) is the greatest common divisor of t_i and t_j in S and r is any non negative real number.*

Example 8. *$T = \{1, 1+i, 2\}$ is q-ordered set in $\mathbb{Z}[i]$ with the measure function $q\left(a + bi\right) = a^2 + b^2$. The 2^{nd} power GCDQ matrix is* $(T^2)_q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 16 \end{bmatrix}$.

Definition 9. *The set $T = \{t_1, t_2, ..., t_m\}$ of non-zero non-associate elements in S is said to be factor-closed (FC) if whenever d divides an element t_i in T, then d is an associate to some element t_j in T.*

Factorizations of Power GCDQ Matrices over Euclidean Domains

In the following, the set $D = \{y_1, y_2, ..., y_n\}$ be the minimal FC set containing T in S, and let $E(t)$ be a complete set of distinct non-associate divisors d of every t in T.

Theorem 10. *Let $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered set of non-zero non-associate elements in S. Then, $(T^r)_q = EG_rE^T$ where G_r is a diagonal matrix and E is a $m \times n$ lower triangular incidence matrix.*

Proof. Consider the $n \times n$ diagonal matrix $G_r = (g_{ii})$ such that $g_{ii} = J_{r,s}(y_i)$ for all $i = 1, 2, ..., n$, and $E = (e_{ij})$ to be the incidence matrix such that $e_{ij} = 1$ if $y_j \in E(t_i)$ and 0 otherwise. Then, $(EG_rE^T)_{ij} = \sum_{k=1}^{n} (e_{ik} J_{r,s}(y_k) e_{jk}) = \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} J_{r,s}(y_k) = \sum_{y_k \in E((t_i, t_j))} J_{r,s}(y_k) = \left(q(t_i, t_j)\right)^r = \left(t_{ij}\right)_r$. □

Theorem 11. *Let $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered set a of non-zero non-associate elements in S. Then,$(T^r)_q = G_r E_r$ where G_r is a $m \times n$ matrix and E_r is an incidence matrix corresponding to the transpose of G_r with $n \geq m$.*

Proof. Consider the matrix $G_r = (g_{ij})_{m\times n}$ such that $g_{ij} = J_{r,s}(y_j)$ if $y_j \in E(t_i)$ and 0 otherwise. Let $E_r = (e_{ij})$ be the incidence matrix such that $e_{ij} = 1$ if $g_{ji} \neq 0$ and 0 otherwise. Then,

$$(G_rE_r)_{ij} = \sum_{k=1}^{n}(g_{ik}e_{kj}) = \sum_{\substack{y_k\in E(t_i)\\ y_k\in E(t_j)}} J_{r,s}(y_k) = \sum_{y_k\in E((t_i,t_j))} J_{r,s}(y_k) = (q(t_i,t_j))^r$$

□

Determinants of Power GCDQ Matrices over Euclidean Domains

Theorem 12. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered FC set of non-zero non-associate elements in S, then* $\det(T^r)_q = \prod_{i=1}^{m} J_{r,s}(t_i)$.

Proof. Since T is FC, then $(T^r)_q = EG_rE^T$ such that E is a lower triangular matrix with diagonal entries $e_{ij} = 1$. Therefore, $\det(T^r)_q = \det(EG_rE^T) = 1\times\det(G_r)\times 1 = \det(G_r) = \prod_{i=1}^{m} J_{r,s}(t_i)$ □

Note that we may prove the above theorem by using the factorizations $(T^r)_q = G_rB_r$ and $(T^r)_q = G_rG_r^T$.

Corollary 13. (***Smith's determinant***) *If $T = \{t_1, t_2, ..., t_m\}$ is FC, then* $\det(T) = \prod_{i=1}^{m} \phi(t_i)$.

Proof. If $S = \mathbb{Z}$ and $r = 1$, then $J_{1,\mathbb{Z}} = \phi$, where ϕ is Euler's totient function, then $\det(T) = \prod_{i=1}^{m} J_{r,s}(t_i) = \prod_{i=1}^{m} J_{1,\mathbb{Z}}(t_i) = \prod_{i=1}^{m} \phi(t_i)$ □

Corollary 14. (***Generalization of Smith's Determinant***) *Let S be ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered FC set of non-zero non-associate elements in S, then* $\det(T)_q = \prod_{i=1}^{m} J_{1,s}(t_i) = \prod_{i=1}^{m} \phi_s(t_i)$.

Theorem 15. *Let S be ED with prime residue system P and measure q. If $\overline{T} = \{y_1, y_2, ..., y_n\}$ is the minimal q-ordered FC set containing $T = \{t_1, t_2, ..., t_m\}$ in S with $m < n$. Let $E_{(k_1,k_2,...,k_m)_r}$ be the submatrix consisting of the $k_1^{th}, k_2^{th}, ..., k_m^{th}$ columns of E for some indices k_i such that $1 < k_1 < k_2 < ... < k_m < n$. Then,*
$$\det(T^r)]_q = \sum_{1\leq k_1<k_2<...<k_m\leq n} \left(\left(\det E_{r_{(k_1,k_2,...,k_m)}}\right)^2 \prod_{i=1}^{m} J_{r,s}(y_{k_i})\right)$$

Proof. Let $G_r = (g_{ij})$ and $E_r = (e_{ij})$ be its corresponding incidence matrix, where $g_{ij} = J_{r,s}(y_j)$ if $y_j \in E(t_i)$ and 0 otherwise. But, G_r is a diagonal matrix whose diagonal entries are $g_{ii} = J_{r,s}(y_1)$ for all $1 \leq i \leq n$, so the ij^{th} entry of G_r may be written as $e_{ij}J_{r,s}(y_j)$ and $(T^r) = EG_rE^T$. Define, for some indices k_i such that $1 < k_1 < k_2 < ... < k_m < n$, the matrices $A_{r_{(k_1,k_2,...,k_m)}}$ and $E_{r_{(k_1,k_2,...,k_m)}}$ to be the submatrices consisting of $k_1^{th}, k_2^{th}, ..., k_m^{th}$ columns of G_r and E respectively, then $G_{r_{(k_1,k_2,...,k_m)}} = E_{r_{(k_1,k_2,...,k_m)}}D_r$, where D_r is the $m \times m$ diagonal submatrix of G_r whose diagonal elements are $d_{ii} = J_{r,s}(y_{k_i})$. Therefore, $\det(G_{r_{(k_1,k_2,...,k_m)}}) =$

$\det(E_{r_{(k_1,k_2,\dots,k_m)}})\left(\prod_{i=1}^{m} d_{ii}\right)$. Now, by applying Cauchy-binet formula we obtain that

$$\begin{aligned}
\det(T^r)_q &= \det G_r E^T \\
&= \sum_{1\le k_1<k_2<\dots<k_m\le n} \left(\left(\det G_{r_{(k_1,k_2,..,k_m)}}\right)\left(\det E_{r_{(k_1,k_2,\dots,k_m)}}\right)^T\right) \\
&= \sum_{1\le k_1<k_2<\dots<k_m\le n} \left(\det(E_{r_{(k_1,k_2,\dots,k_m)}})\left(\prod_{i=1}^{m} J_{r,s}(y_{k_i})\right)\left(\det E_{r_{(k_1,k_2,\dots,k_m)}}\right)^T\right) \\
&= \sum_{1\le k_1<k_2<\dots<k_m\le n} \left(\left(\prod_{i=1}^{m} J_{r,s}(y_{k_i})\right)\left(\det E_{r_{(k_1,k_2,\dots,k_m)}}\right)^2\right)
\end{aligned}$$

□

Corollary 16. *If $(T^r)_q$ is the r^{th} power GCDQ matrix defined on any set T in S, then $\det(T^r)_q \ge J_{r,s}(t_1)J_{r,s}(t_2)...J_{r,s}(t_m)$.*

Theorem 17. (*Generalization of Beslin and Ligh's Result*) *Let S be ED with prime residue system P and measure q. If $(T^r)_q$ is the r^{th} power GCDQ matrix defined on $T = \{t_1, t_2, ..., t_m\}$ in S, then $\det(T^r)_q = \prod_{i=1}^{m} J_{r,s}(t_i)$ if and only if T is FC.*

Proof. If T is FC in S, then, by Theorem ??, $\det(T^r)_q = J_{r,s}(t_1)J_{r,s}(t_2)...J_{r,s}(t_m)$. Conversely, suppose that $\det(T^r)_q = J_{r,s}(t_1)J_{r,s}(t_2)...J_{r,s}(t_m)$ and T is not FC. Let $D = \{t_1, t_2, ..., t_m, t_{m+1}, t_{m+2}, ..., t_{m+s}\}$ be the minimal FC set containing T such that $t_1 <_q t_2 <_q .... <_q t_m$ and $t_{m+1} <_q t_{m+2} <_q ... <_q t_{m+s}$. Since D is not associate to T in S, then there exist at least one element t_{m+1} in D, but not in T such that $t_{m+1} \in E(t)$ for some t in T. Let t_k be the first element in T such that $t_{m+1} \in E(t_k)$ then the submatrix $A_{(1,2,...,k-1,m+1,k+1,...,m)}$ consisting of the 1^{st}, 2^{nd}, ..., $(k-1)^{th}$, $(m+1)^{th}$, $(k+1)^{th}$, ..., m^{th} columns of $A_{m\times(m+s)}$ is a lower triangular matrix of nonzero determinant. Hence, $E_{(1,2,...,k-1,m+1,k+1,....,m)}$ is a {0-1} matrix whose diagonal elements are equal to 1 such that $\det\left(E_{(1,2,...,k-1,k+1,....,m,m+1)}\right) = \pm\det\left(E_{(1,2,...,k-1,m+1,k+1,....,m)}\right) = \pm 1$, because $E_{(1,2,...,k-1,k+1,....,m,m+1)r}$ is obtained from $E_{(1,2,...,k-1,m+1,k+1,....,m)r}$ by performing a certain number of successive column permutations. By Theorem ??, we get

$$\begin{aligned}
\det(T^r)_q &= \sum_{1\le k_1<k_2.....<k_m\le n} \left(\left(\det E_{(k_1,k_2,.....,k_m)_r}\right)^2 \prod_{i=1}^{m} J_{r,s}(t_{k_i})\right) \\
&= \prod_{i=1}^{m} J_{r,s}(t_{k_i}) + \prod_{\substack{i=1\\ i\neq k}}^{m+1} J_{r,s}(t_{k_i}) + ... > \prod_{i=1}^{m} J_{r,s}(t_{k_i})
\end{aligned}$$

and this contradicts the necessary condition that the equality holds. Therefore $D \approx T$ and T is FC. □

Inverses of Power GCDQ Matrices over Euclidean Domains

Definition 18. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S, then the inverse of the power GCDQ matrix defined on T is denoted by $(T^r)_q^{-1}$ such that $(T^r)_q (T^r)_q^{-1} = I_m$.*

Theorem 19. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S. Let $U = (u_{ij})$ such that $u_{ij} = \mu(\frac{t_i}{t_j})$ if $t_j \in E(t_i)$ and 0 otherwise. Let $E = (e_{ij})$ such that $e_{ij} = 1$ if $t_j \in E(t_i)$ and 0 otherwise. Then, $U = E^{-1}$.*

Proof. $(EU)_{ij} = \sum_{k=i}^{m} e_{ik}u_{kj} = \sum_{t_k \in E(t_i)} u_{kj} = \sum_{\substack{t_j \in E(t_k) \\ t_k \in E(t_i)}} \mu(\frac{t_k}{t_j}) = \begin{cases} \sum_{t \in E(\frac{t_i}{t_j})} \mu(t) & \text{if } t_j \in E(t_i) \\ 0 & \text{otherwise} \end{cases}$
$= \begin{cases} 1 & \text{if } \; t_j \approx t_i \\ 0 & \text{otherwise} \end{cases}$. Therefore, $U = E^{-1}$. □

Theorem 20. *If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered FC set of non-zero non-associate elements in S. Then, $(T^r)_q^{-1} = U^T G_r^{-1} U$, where $G_r = diag(J_{r,s}(t_1),\ J_{r,s}(t_2),\ ...,\ J_{r,s}(t_m))$.*

Proof. Since, T is FC, then $(T^r)_q = EG_rE^T$. Therefore, $\left((T^r)_q^{-1}\right)_{ij} = \left(U^T(G_r)^{-1}U\right)_{ij} = \sum_{i=1}^{k} u_{ik} \frac{1}{J_r(t_k)} u_{kj} = \sum_{\substack{t_i \in E(t_k) \\ t_j \in E(t_k)}} \frac{1}{J_r(t_k)} \mu_s\left(\frac{t_i}{t_k}\right) \mu_s\left(\frac{t_j}{t_k}\right)$. □

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A Recurrence Relation for the q-Appell Polynomials

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Abstract

In this work, we investigate some properties of the q-Appell polynomials based upon the q-umbral algebra. We focus on some operators which are useful for obtaining recurrence relations for the q-Appell polynomials. Both methods of umbral calculus and quantum calculus are used in this study.

2010 Mathematics Subject Classifications : 05A40, 05A30, 11B83.
Keywords: q-umbral calculus, q-calculus, q-Appell polynomials.

Introduction

Throughout of this paper, we use the notation

$$[x]_q = \begin{cases} \frac{1-q^x}{1-q}, q \neq 1 \\ x, q = 1, \end{cases}$$

where $0 < q < 1$ when $q \in \mathbb{R}$ and $|q| < 1$ when $q \in \mathbb{C}$.

Derivative operator t is defined by

$$tx^n = \frac{x^n - (qx)^n}{x - qx} = [n]_q \, x^{n-1}.$$

The q-analogue of the exponential series is defined by

$$\varepsilon_q (yt) = \sum_{k=0}^{\infty} \frac{(yt)^k}{[k]_q!}.$$

One must notice that $\varepsilon_q (yt)$ is well defined for all $|yt| < \frac{1}{|1-q|}$ if $|q| < 1$ and for all $yt \in \mathbb{C}$ if $|q| > 1$ or $q = 1$.

For detailed information, see [8], [9], [12], [13].

The main definitions and results about umbral algebra can be found in [13].

Let $\mathcal{P}$ be the algebra of polynomials in the single variable x over the field complex numbers. Let $\mathcal{P}^*$ be the vector space of all linear functionals on $\mathcal{P}$. Let $\langle L \mid p(x) \rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let $\mathfrak{F}$ denotes the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k.$$

This algebra is called *q-umbral algebra*. Each $f \in \mathfrak{F}$ defines a linear functional on $\mathcal{P}$ and for all $k \geqslant 0$, $a_k = \langle f(t) \mid x^k \rangle$.

In the special case,

$$\langle t^k \mid x^n \rangle = [n]_q! \delta_{n,k},$$

where

$$\delta_{n,k} = \begin{cases} 0, n \neq k \\ 1, n = k. \end{cases}$$

Let $f(t)$, $g(t)$ be in $\mathfrak{F}$, we have

$$\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle .$$

The *order* $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $o(f(t)) = 1$ is called a *delta series*. And a series $f(t)$ for which $o(f(t)) = 0$ is called a *invertible series*.

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exist a unique sequence $S_n(x)$ of polynomials satisfying the orthogonality conditions

$$\left\langle g(t)f(t)^k \mid S_n(x)\right\rangle = [n]_q!\delta_{n,k} \tag{1}$$

for all $n, k \geq 0$.

The sequence $S_n(x)$ in (1) is the q-Sheffer polynomials for pair $(g(t), f(t))$, where $g(t)$ must be invertible and $f(t)$ must be delta series. In particular, the q-Sheffer polynomials for pair $(g(t), t)$ is the q-Appell polynomial for $g(t)$.

Every q-Appell polynomials satisfy the identities listed below:

The polynomial $S_n(x)$ is q-Appell for $g(t)$ if and only if

$$\frac{1}{g(t)}\varepsilon_q(yt) = \sum_{k=0}^{\infty} \frac{S_k(y)}{[k]_q!}t^k \tag{2}$$

for all constants $y \in \mathbb{C}$.

The polynomial $S_n(x)$ is q-Appell for $g(t)$ if and only if

$$S_n(x) = g(t)^{-1} x^n \tag{3}$$

The polynomial $S_n(x)$ is q-Appell for $g(t)$ if and only if

$$tS_n(x) = [n]_q S_{n-1}(x) \tag{4}$$

For detailed information, see [13], [12], [4], [5], [6]

q-derivative operator defined by $D_{t,q} : t^n \longrightarrow [n]_q t^{n-1}$,

$$D_{t,q}f(t) = \frac{f(t) - f(qt)}{t - qt},$$

where $q \neq 1$ ([13]).

Main Results

θ operator defined by

$$\theta : x^n \longrightarrow \frac{(n+1)}{[n+1]_q}x^{n+1}.$$

One can observe that

$$\theta t x^n = [n]_q \theta x^{n-1} = n x^n,$$

and so

$$\theta t = xD$$

where D is the ordinary derivative ([13]).

If we investigate the relationship between operators θ and $D_{x,q}$, we get

$$\theta t = \frac{n}{[n]_q} x D_{x,q}. \tag{5}$$

Lemma 1. *Let $S_n(x)$ be a q-Appell polynomial. Then*

$$\theta S_n(x) = \frac{n}{[n]_q} x S_n(x). \tag{6}$$

Proof. See that

$$\theta S_n(x) = \theta \frac{1}{[n+1]_q} t S_{n+1}(x).$$

By using (5) and (4), we get

$$\begin{aligned} \theta S_n(x) &= \frac{1}{[n+1]_q} \frac{n}{[n]_q} x D_{x,q} S_{n+1}(x) \\ &= \frac{1}{[n+1]_q} \frac{n}{[n]_q} x [n+1]_q S_n(x) \\ &= \frac{n}{[n]_q} x S_n(x). \end{aligned}$$

□

Roman ([13]) gave a recurrence formula for generalized Sheffer polynomials. After making some necessary changes, we give the following theorem:

Theorem 2. *Let $S_n(x)$ be a q-Appell polynomials for $g(t)$. Then*

$$(n+1) S_{n+1}(x) = [n+1]_q \left(\theta - \frac{D_{t,q}(g(t))}{g(t)} \right) S_n(x). \tag{7}$$

Conclusion

The family of q-Appell polynomials includes very important polynomials such as q-Bernoulli polynomials, q-Euler polynomials and q-Hermite polynomials. The further investigation on this polynomials is making by the author.

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Cylcic Generalized Group of Units of $\mathbf{Z}[i]/<\beta>$

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Abstract

In this article, we study the structure of the generalized second group of units, $U^2(R)$ of the quotient ring $R = \mathbf{Z}[i]/<\beta>$. In particular, we consider the case where the generalized second group of units, $U^2(R)$ is cyclic.

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Keywords: Group of units, cyclic group and Gaussian primes.

Introduction

The fundamental theorem of finite abelian groups states that any finite abelian group G is isomorphic to a direct product of cyclic groups. That is, $G \cong \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times ... \times \mathbf{Z}_{n_r}$. Hence, the group of units of a finite commutative ring with identity is isomorphic to a direct product of cyclic groups. The decomposition of U_n, the group of units of $\mathbf{Z}_n$, into a product of cyclic groups of prime power order is given in the following theorem.

Theorem 1. *Let $n = p_1^{a_1}.p_2^{a_2}...p_r^{a_r}$ be the decomposition of n into product of distinct prime powers. Then, $U_n \cong U_{p_1^{a_1}} \times U_{p_2^{a_2}} \times ... \times U_{p_r^{a_r}}$. Moreover,*

1. $U_2 \cong \mathbf{Z}_1$.
2. $U_{2^2} \cong \mathbf{Z}_2$.
3. $U_{2^a} \cong \mathbf{Z}_2 \times \mathbf{Z}_{2^{a-2}}$ *, where* $a > 2$.
4. $U_{p^a} \cong \mathbf{Z}_{p-1} \times \mathbf{Z}_{p^{a-1}}$ *, where p is an odd prime.*

Let $G_1, G_2, ..., G_r$ be finite cyclic groups. Then $G_1 \times G_2 \times ... \times G_r$ is cyclic if and only if their orders are pairwise relatively prime, see [4].

Lemma 2. *If $G_1 \times G_2 \times ... \times G_r$ is cyclic, then each G_s is cyclic and* $\gcd(|G_s|, |G_t|) = 1$.

If R is a finite commutative ring with identity, then $U(R)$ denotes its group of units. It is well known that if R decomposes as a direct sum of rings, $R = R_1 \oplus R_2 ... \oplus R_r$, then $U(R) \cong U(R_1) \times U(R_2) \times ... \times U(R_r)$. In [3], El-Kassar and Chehade generalized the concept of the group of units as follows: the multiplicative group $U(R)$ support a ring structure by defining the operations $\oplus$ and $\otimes$ on $U(R)$ that makes $(U(R), \oplus, \otimes)$ a ring isomorphic to $U(R_1) \oplus U(R_2) \oplus ... \oplus U(R_r)$. The ring $U(R_1) \oplus U(R_2) \oplus ... \oplus U(R_r)$ will be denoted by $R^2 \cong U(R)$ and R^1 denote the ring R. They defined $U^2(R)$ to be the group of units of the ring $R^2 \cong U(R)$ so that $U^2(R) = U(R^2) \cong U(U(R))$ and $U^2(R) \cong U^2(R_1) \times U^2(R_2) \times ... \times U^2(R_r)$.

Theorem 3. *Let $\gamma_1, \gamma_2, ..., \gamma_r$ be distinct Gaussian prime integers and let $\beta = \prod_{j=1}^{r} \gamma_j^{n_j}$, then*

$$\mathbf{Z}[i]/ < \beta >\cong \mathbf{Z}[i]/ < \gamma_1^{n_1} > \oplus \mathbf{Z}[i]/ < \gamma_2^{n_2} > \oplus ... \oplus \mathbf{Z}[i]/ < \gamma_r^{n_r} >$$

and

$$U^2\left(\mathbf{Z}[i]/ < \beta >\right) \cong U^2\left(\mathbf{Z}[i]/ < \gamma_1^{n_1} >\right) \times U^2\left(\mathbf{Z}[i]/ < \gamma_2^{n_2} >\right) \times ... \times U^2\left(\mathbf{Z}[i]/ < \gamma_r^{n_r} >\right).$$

We may also write $U^2(\beta) \cong U^2(\gamma_1^{n_1}) \times U^2(\gamma_2^{n_2}) \times ... \times U^2(\gamma_r^{n_r})$, see [1].

For the notations, we henceforth use

- m, n and r always denote positive integers,
- p and p_j always denote prime integers that are congruent to 3 modulo 4,
- γ and γ_j always denote Gaussian prime integers,
- π and π_j always denote Gaussian prime integers of the form $a + bi$ where a and b are non-zero integers,
- $q = q(\pi) = \pi\overline{\pi}$ and $q_j = q_j(\pi_j) = \pi_j\overline{\pi_j}$ are always prime integers that are congruent to 1 modulo 4,
- S_1 and S_2 always denote the respective sets $\{3, 4, 5\}$ and $\{2, 4, d^s, 2d^s\}$, where d is an odd prime integer.

The problem of classifying the group of units of an arbitrary finite commutative ring with identity is an open problem. However, the problem is solved for certain cases. In the case when $R = \mathbf{Z}_n$, it is well-known that U_n is cyclic if and only if $n = 2$, 4, p^α, q^α, $2p^\alpha$ or $2q^\alpha$, see [5]. Also, Cross [2] showed that the group of units of the quotient ring of Gaussian integers, $U(\mathbf{Z}[i]/ <\beta>)$, is cyclic if and only if $\beta = 1+i$, $(1+i)^2$, $(1+i)^3$, p, $(1+i)p$, π^n, $(1+i)\pi^n$. Chehade and Al-Saleh [1] studied the trivial case of the group of units of the ring $\mathbf{Z}[i]/ <\beta>$ and its generalization $U^2(\beta)$. El-Kassar and Chehade [3] showed that $U^2(\mathbf{Z}_n)$ is cyclic if n is a product of at most three prime power factors when $n > 1$. Their results are summarized in the following theorem.

Theorem 4. *$U^2(\mathbf{Z}_n)$ is cyclic if and only if one of the following is true:*

1. *$n = 2^\alpha.3.p$, where $a \leq 3$.*
2. *$n = 3.p$, where $p = 4k + 3$ and $2k + 1 = q^a$.*
3. *$n = 2^4.3$ or $2^a.3^b$, where $a \leq 3$.*
4. *$n = 2^a$, where $a \leq 4$.*
5. *$n = 5$ or $2p^\alpha + 1$, where $2p^\alpha + 1 = q$.*

The structure of the group of units of the ring $\mathbf{Z}[i]/ < \beta >$ given in the below theorem is due to cross [2].

Theorem 5.

1. $U(1 + i) \cong \mathbf{Z}_1$.

2. $U((1+i)^2) \cong \mathbf{Z}_2$.
3. $U((1+i)^n) \cong \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-2}} \times \mathbf{Z}_4$ if $n = 2m$.
4. $U((1+i)^n) \cong \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_4$ if $n = 2m+1$.
5. $U(\pi^n) \cong \mathbf{Z}_{q^n - q^{n-1}}$.
6. $U(p^n) \cong \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^2-1}$.

The goal of this paper is to study cyclic property of the generalized group of units, $U^2(\beta)$, of the ring $\mathbf{Z}[i]/ <\beta>$.

Cyclic Second group of Units

We completely characterize all rings $\mathbf{Z}[i]/ <\beta>$ for which the $U^2(\beta)$ is cyclic. We start with the case where β is a Gaussian prime power integer. Note that $U^2(1+i) \cong \{0\}$ and hence cyclic.

A Prime Power Factor

Lemma 6. *Let $\beta = (1+i)^n$ with $n \geq 2$, then $U^2(\beta)$ is cyclic if and only if $n \in S_1$.*

Proof. If $n = 2m+1$, then $U^2((1+i)^n) \cong U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_{2^{m-1}}) \times \mathbf{Z}_2$ which is cyclic if $U(\mathbf{Z}_{2^{m-1}})$ is cyclic of odd order. Hence, $2^{m-1} \in S_2$. But $|U(\mathbf{Z}_{2^{m-1}})| = \phi(2^{m-1})$ is odd if $m = 2$ and hence $n = 5$. If $m = 1$, then $n = 3$ and $U^2((1+i)^n) \cong \mathbf{Z}_2$ which is cyclic.
Now, if $n = 2m$ then $U^2((1+i)^n) \cong U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_{2^{m-2}}) \times \mathbf{Z}_2$ is cyclic if $U(\mathbf{Z}_{2^{m-1}})$ and $U(\mathbf{Z}_{2^{m-2}})$ are cyclic with relatively prime odd orders. But $U(\mathbf{Z}_{2^{m-1}})$ and $U(\mathbf{Z}_{2^{m-2}})$ are cyclic if $m = 2, 3$ or 4. For $m = 2$, $U^2\left((1+i)^4\right) \cong \mathbf{Z}_2$ is cyclic. The cases $m = 3$ or 4, $\phi(2^{m-1})$ is even and $U^2((1+i)^n)$ is not cyclic. □

Lemma 7. *$U^2(p^n)$ is not cyclic.*

Proof. $U^2(p^n) \cong U(\mathbf{Z}_{p^{n-1}}) \times U(\mathbf{Z}_{p^{n-1}}) \times U(\mathbf{Z}_{p^2-1})$. For $n > 1$, $p^{n-1} > 2$ and $\gcd\left(\phi(p^{n-1}), \phi(p^{n-1})\right) = \phi(p^{n-1})$ is even and hence $U^2(p^n)$ is not cyclic. If $n = 1$, then $U^2(p^n) \cong U(\mathbf{Z}_{p^2-1})$ is cyclic if $p^2 - 1 \in \{2, 4, 2d^s\}$. The cases $p^2 - 1 = 2$ or 4 are dismissed since no integer solution exists. Also, the case when $p^2 - 1 = 4$ is dismissed. Since $p \equiv 3 \pmod 4$, the case $p^2 - 1 = 2d^s$ gives $d^s = 4(2k^2 + 3k + 1)$ which is a contradiction. □

Lemma 8. *$U^2(\pi^n)$ is cyclic if and only if $n = 1$ and $q = 5$.*

Proof. Since $q \equiv 1 \pmod 4$, then $U^2(\pi^n) \cong U(\mathbf{Z}_{q^n - q^{n-1}}) \cong U(\mathbf{Z}_t)$ where $t = 4k(4k+1)^{n-1}$.But $U(\mathbf{Z}_t)$ is cyclic if $t \in S_2$. For $t = 2$, $2k(4k+1)^{n-1} = 1$ and no integer solution exists. For $t = 4$, $k(4k+1)^{n-1} = 1$ which is true if and only if $k = n = 1$ and $q = 5$. The case $t = d^s$ is dismissed since t is even. The case $t = 2d^s$ gives $2k(4k+1)^{n-1} = d^s$ which is also dismissed. Therefore, $U^2(\beta)$ is cyclic if and only if $n = 1$ and $q = 5$. Conversely, Its easy to prove that $U^2(\pi^n)$ is cyclic when $n = 1$ and $q = 5$. □

Summarizing the preceding three lemmas, the following theorem is obtained.

Theorem 9. *If β is a Gaussian prime power integer, then $U^2(\beta)$ is cyclic if and only if one of the following is true:*

1. $\beta = (1+i)^n$ with $n \in S_1$.
2. $\beta = \pi$ with $q = 5$.

Two Prime Power Factors

We study the case when $U^2(\beta)$ is cyclic and β is a product of two Gaussian prime power integers. Since $U^2\left(\gamma_1^{n_1}\gamma_2^{n_2}\right) \cong U^2\left(\gamma_1^{n_1}\right) \times U^2\left(\gamma_2^{n_2}\right)$ and since $U^2(p^n)$ is not cyclic. The next lemma follows directly.

Lemma 10. *If* $\beta \in \{(1+i)^{n_1}p^{n_2}, p_1^{n_1}p_2^{n_2}, p^{n_1}\pi^{n_2}\}$, *then* $U^2(\beta)$ *is not cyclic.*

The remaining cases are $\beta = (1+i)^{n_1}\pi^{n_2}$ or $\beta = \pi_1^{n_1}\pi_2^{n_2}$.

Lemma 11. *If* $\beta = (1+i)^{n_1}\pi^{n_2}$, *then* $U^2(\beta)$ *is not cyclic.*

Proof.
$$U^2(\beta) \cong U^2((1+i)^{n_1}) \times U^2(\pi^{n_2}).$$
is cyclic if $U^2((1+i)^{n_1})$ and $U^2(\pi^{n_2})$ are cyclic with relatively prime orders. By theorem 9, $U^2((1+i)^{n_1})$ is cyclic if $n_1 \in S_1$ and $U^2(\pi^{n_2})$ is cyclic if $n_2 = 1$ and $q = 5$. The possible values of β are $(1+i)^3\pi, (1+i)^4\pi$ or $(1+i)^5\pi$ with $q = 5$. If $\beta = (1+i)^3\pi$, then $m = 1$ and the first factor of the rigth hand side of (**??**) becomes $U^2\left((1+i)^3\right) \cong \mathbf{Z}_2$. Also, $U^2(\pi^{n_2}) \cong U(\mathbf{Z}_{5-5^{1-1}}) \cong \mathbf{Z}_2$. Therefore, $U^2(\beta) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ and hence not cyclic. Similarly, for $\beta = (1+i)^4\pi$ or $(1+i)^5\pi$, $U^2(\beta) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ is not cyclic. □

Lemma 12. *If* $\beta = \pi_1^{n_1}\pi_2^{n_2}$, *then* $U^2(\beta)$ *is not cyclic.*

Proof. Assume $U^2(\beta) \cong U^2(\pi_1^{n_1}) \times U^2(\pi_2^{n_2})$ is cyclic, then theorem 9 gives $n_1 = n_2 = 1$ and $q_1 = q_2 = 5$. So, $U^2(\beta) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ which contradicts the assumption that $U^2(\beta)$ is cyclic. □

Theorem 13. *If* $\beta = \gamma_1^{n_1}\gamma_2^{n_2}$, *then* $U^2(\beta)$ *is not cyclic.*

General Case

Let $\beta = (1+i)^n \left(\prod_{t=1}^{r} p_t^{n_t}\right)\left(\prod_{s=1}^{j} \pi_s^{k_s}\right)$ be the decomposition of β into product of distinct Gaussian prime power integers with n_t and k_s are non-negative integers. We deduce that $U^2(\beta)$ is not cyclic if $n_t > 0$ for some or $j > 1$ with $k_1, k_2 > 0$. The proof of the preceding theorem follows directly from theorems 3, 2, 13 and 9.

Theorem 14. *Let* $\beta = (1+i)^n \left(\prod_{t=1}^{r} p_t^{n_t}\right)\left(\prod_{s=1}^{j} \pi_s^{k_s}\right)$, *then* $U^2(\beta)$ *is cyclic if and only if one of the following is true for every* t, s :

1. $n \in S_1$, $n_t = k_s = 0$.
2. $n = n_t = 0$, $j = k_1 = 1$ wih $q_1 = 5$.

Conclusion

We solved completely the case when the second group of units of the ring $R = \mathbf{Z}[i]/<\beta>$ is cyclic. We showed that $U^2(R)$ is cyclic if and only if β is divisible by only one Gaussian prime integer. Moreover, $U^2(R)$ is cyclic if and only if one of the following is true:

1. $\beta = (1+i)^n$ with $n \in S_1$.
2. $\beta \in \{2+i, 1+2i\}$ or any of their associates.

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Box Coefficients for Discrete Time Systems

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Abstract

Given a discrete-time system if all roots of corresponding characteristic polynomial belong to the open unit disc then the corresponding system is called (Schur) stable. It is well-known that the stability region of two dimensional systems in the parameter space is an open triangle. In this report, we define a multilinear (affine linear with respect to each variable) map from the open square $(-1,1)\times(-1,1)$ onto the stability region. Using this map a new multilinear map from the multidimensional box $(-1,1)\times\cdots\times(-1,1)$ onto the stability region in the multidimensional case and new box coefficients are defined. It should be noted that in order to define the box coefficients the roots of a polynomial must be known. The comparison with the classical reflection coefficients has been made and number of examples are given.

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Keywords: Discrete-time system, Schur stability, Multilinear map, Reflection coefficients

Introduction

Given monic polynomial

$$p(s) = a_1 + a_2 s + \cdots + a_n s^{n-1} + s^n \tag{1}$$

with real coefficients, corresponds n-dimensional vector $p = (a_1, a_2, ..., a_n)^T \in \mathbb{R}^n$. The monic polynomial (1) $p(s)$ is called Schur stable if all roots lie in the open unit disc of the complex plane. The vector p is called Schur stable if the corresponding monic polynomial (1) $p(s)$ is Schur stable. Denote by $\mathcal{D}_n$ the set of all Schur stable n-dimensional vectors.

The set $\mathcal{D}_n$ contains the origin, is open, bounded, nonconvex for $n \geq 3$ and open triangular region with vertices $(-1,0)$, $(1,2)$ and $(1,-2)$ in the plane (a_1, a_2) for $n = 2$ (Figure 2). (see [1]).

In [1], it has been shown that the closure convex hull of $\mathcal{D}_n$ is a polytope with $(n+1)$ known vertices, namely

$$\overline{\text{co}}\mathcal{D}_n = \text{co}\{V^1, V^2, \ldots, V^{n+1}\}, \tag{2}$$

where $\overline{\text{co}}$ stands for the closure of the convex hull, the vectors V^i correspond to the unstable vertex polynomials $(s-1)^i(s+1)^{n-i}$ $(0 \leq i \leq n)$. For example, if $n = 3$, then $V^1 = (1,3,3)^T$, $V^2 = (-1,-1,1)^T$, $V^3 = (1,-1,-1)$, $V^4 = (-1,3,-3)^T$.

In the same paper, it has been shown how to construct $\mathcal{D}_n$ recursively from $\mathcal{D}_1$ and $\mathcal{D}_2$ by using the matrix multiplication.

In [2], a similar result is obtained when the stability region is a region in the complex plane bounded by a finite number of circle arcs.

Another description of Schur stable polynomials is the reflection coefficients which can be obtained by using the backward Levinson's recursion (see [3, 4]). There is one

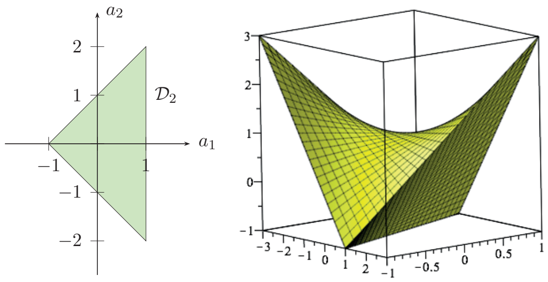

Figure 1: $\mathcal{D}_2 = \{p \in \mathbb{R}^2:$ The polynomial $p(s)$ is Schur stable$\}$ and closure of $\mathcal{D}_3 = \{p \in \mathbb{R}^3:$ The polynomial $p(s)$ is Schur stable$\}$

to one and onto multilinear map $R(k_1, k_2, \ldots, k_n)$ from n-dimensional open cube $(-1,1)^n$ to $\mathcal{D}_n$. For a stable vector $p \in \mathcal{D}_n$, the inverse image $(k_1, k_2, \ldots, k_n) = R^{-1}(p)$ is called its reflection coefficient vector.

Reflection coefficients or Schur-Szegö parameters for polynomials have been widely used in the stability problems of discrete systems [5]. For $k_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ and $n \geq 3$ reflection map $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$(f_1, f_2, \ldots, f_n)^T(k_1, \ldots, k_n) = R_n(k_n) \begin{bmatrix} 0^T \\ R_{n-1}(k_{n-1}) \end{bmatrix} \cdots \begin{bmatrix} 0^T \\ R_1(k_1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $R_j(k_j) = I_{j+1} + k_j E_{j+1}$, I_j is the $j \times j$ identity matrix, $j \times j$ matrix E_j is the following:

$$E_j = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}.$$

The map f is multilinear ([5]), that is affine linear with respect to each component k_i. In the below explicit formulas for f in the cases $n = 3$ is given:

$$\begin{aligned} f_1(k_1, k_2, k_3) &= k_3, \\ f_2(k_1, k_2, k_3) &= k_1 k_2 k_3 + k_1 k_3 + k_2, \\ f_3(k_1, k_2, k_3) &= k_1 k_2 + k_2 k_3 + k_1. \end{aligned}$$

For arbitrary polynomial $f_1 + f_2 s + \cdots + f_n s^{n-1} + s^n$ there exist $k_1, k_2, \ldots, k_n$ such that $f_1 = f_1(k_1, \ldots, k_n), \ldots, f_n = f_n(k_1, \ldots, k_n)$.

The numbers $k_1, k_2, \ldots, k_n$ are called the reflection coefficients of the polynomial $f_1 + f_2 s + \cdots + f_n s^{n-1} + s^n$. The following fact is important:

Proposition 1 ([5]). *Monic polynomial $p(s) = f_1 + f_2 s + \cdots + f_n s^{n-1} + s^n$ is Schur stable if and only if its reflection coefficients satisfy the conditions $|k_i| < 1$ $(i = 1, 2, ..., n)$.*

According the above fact there exists multilinear map f from the open cube $(-1,1)^n$ onto $\mathcal{D}_n$.

By the known extremal property of a multilinear function defined on a box every vertex V^i of $\overline{\text{co}} D_n = \text{co}\{V^1, V^2, \ldots, V^{n+1}\}$ has inverse image which is a vertex of the cube $[-1, 1]^n$.

Box Coefficients

Now we define a multilinear map f from the cube $(-1,1)^n$ to $\mathcal{D}_n$ which is onto. This map is defined recursively starting from $\mathcal{D}_2$ which is an open triangle in the plane (a_1,a_2) with vertices $(-1,0)$, $(1,2)$ and $(1,-2)$. The corresponding multilinear map from $(k_1,k_2)\in(-1,1)\times(-1,1)$ to $\mathcal{D}_2$ is

$$\begin{aligned} a_1 &= f_1(k_1,k_2) = k_2,\\ a_2 &= f_2(k_1,k_2) = k_1k_2 + k_1. \end{aligned}$$

Let $\alpha\in(-1,1)$. The image of the line segment $\{(k_1,k_2):\ -1<k_1<1, k_2=\alpha\}$ under the multilinear map f is a line segment as follows (see Figure 2):

$$\begin{cases} a_1 = k_2 \\ a_2 = k_1k_2 + k_1 \end{cases} \Rightarrow \begin{cases} a_1 = \alpha \\ a_2 = (\alpha+1)k_1 \quad (-1<k_1<1). \end{cases}$$

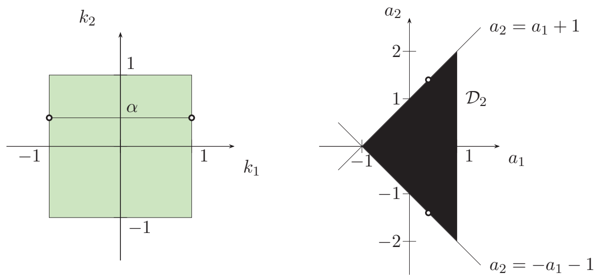

Figure 2: The multilinear map f from the $(-1,1)\times(-1,1)$ to $\mathcal{D}_2$.

For $n=3$, multiply $s^2+(k_1k_2+k_1)s+k_2$ by $s+k_3$ where $k_3\in(-1,1)$ and obtain

$$\begin{aligned} a_1 &= f_1(k_1,k_2,k_3) = k_2k_3,\\ a_2 &= f_2(k_1,k_2,k_3) = k_1k_2k_3 + k_1k_3 + k_2,\\ a_3 &= f_3(k_1,k_2,k_3) = k_1k_2 + k_1 + k_3. \end{aligned}$$

For $n=4$, multiply $s^2+(k_1k_2+k_1)s+k_2$ by $s^2+(k_3k_4+k_3)s+k_4$ where $k_3\in(-1,1)$, $k_4\in(-1,1)$ and obtain

$$\begin{aligned} a_1 &= f_1(k_1,k_2,k_3,k_4) = k_2k_4,\\ a_2 &= f_2(k_1,k_2,k_3,k_4) = k_1k_2k_4 + k_2k_3k_4 + k_1k_4 + k_2k_3,\\ a_3 &= f_3(k_1,k_2,k_3,k_4) = k_1k_2k_3k_4 + k_1k_2k_3 + k_1k_3k_4 + k_1k_3 + k_2 + k_4,\\ a_4 &= f_4(k_1,k_2,k_3,k_4) = k_1k_2 + k_3k_4 + k_1 + k_3 \end{aligned}$$

and so on. By this simple procedure, we define a multilinear map $f(k_1,k_2,\ldots,k_n)$ from $(-1,1)^n$ onto $\mathcal{D}_n$ for any n by the formula $a=f(k)$, where $k=(k_1,k_2,\ldots,k_n)^T$, $f=(f_1,f_2,\ldots,f_n)^T$, $a=(a_1,a_2,\ldots,a_n)^T$.

Note that this formula for box coefficients differs from that used in [5].

Example 2. *Consider the polynomial* $p(s)=\frac{15}{128}-\frac{37}{64}s+\frac{21}{16}s^2-\frac{7}{4}s^3+s^4$ *which roots are* $\frac{1}{4}\pm\frac{1}{2}i$, $\frac{1}{2}$ *and* $\frac{3}{4}$. *For the polynomial* $p(s)$, *the reflection map from [5] gives*

$$\left(-\frac{2914550}{3204073}, \frac{138303512}{223756737}, -\frac{6112}{16159}, \frac{15}{128}\right)^T \approx (-0.909, 0.618, -0.378, 0.117)^T,$$

whereas the multilinear map for $n = 4$ used in this paper gives the box coefficients vector

$$\left(-\frac{8}{21}, \frac{5}{16}, -\frac{10}{11}, \frac{3}{8}\right)^T \approx (-0.3809, 0.3125, -0.9090, 0.375)^T.$$

Example 3. *Consider one application of the box coefficients to stabilising problem. Take the Schur stable polynomial $p(s) = 0.25s^2 + s^3$ with box coefficients vector $(0, 0.25, 0)^T$. By writing -1 and 1 respectively in the first, second and third entry of this vector, we get $U^1 = (-1, 0.25, 0)^T$, $U^2 = (1, 0.25, 0)^T$, $U^3 = (0, -1, 0)^T$, $U^4 = (0, 1, 0)^T$, $U^5 = (0, 0.25, -1)^T$, $U^6 = (0, 0.25, 1)^T$. For $n = 3$, the images of these points under the multilinear map f are the polynomials $p_1(s) = 0.25s - 1.25s^2 + s^3$, $p_2(s) = 0.25s + 1.25s^2 + s^3$, $p_3(s) = -s + s^3$, $p_4(s) = s + s^3$, $p_5(s) = -0.25 + 0.25s - s^2 + s^3$ and $p_6(s) = 0.25 + 0.25s + s^2 + s^3$ respectively. Using the Edge theorem [6], It can be shown that the polytope $\mathcal{P} = \mathrm{co}\{p_1, p_2, \ldots, p_6\}$ is contained by $\overline{\mathrm{co}}\mathcal{D}_3$.*

Let $G(s) = \frac{s+1}{s^2+1.3}$ be a transfer function and $C(s, c) = \frac{c_1 s + c_2}{s}$ be a controller. Then the corresponding closed loop system has characteristic polynomial: $p(s, c) = s^3 + c_1 s^2 + (c_1 + c_2 + 1.3)s + c_2$, where $c = (c_1, c_2)^T \in \mathbb{R}^2$. The polynomial $p(s, c)$ can be written in the matrix form $p(c) = Pc + p^0$, where $P = \left[p^1, p^2\right]$, $p^1 = (0, 1, 1, 0)^T$, $p^2 = (1, 1, 0, 0)^T$, $p^0 = (0, 1.3, 0, 1)^T$.

The intersection of the affine subset $\left\{Pc + p^0 : \ c \in \mathbb{R}^2\right\}$ of $\mathbb{R}^3$ and the polytope $\mathcal{P}$ is

$$\begin{aligned}\mathrm{co}\,\big\{&(-1.15, 0)^T, (-0.92, -0.23)^T, (-0.82, -0.23)^T,\\ &(-0.6, -0.15)^T, (-0.75, 0)^T, (-1.07, 0.02)^T\big\}.\end{aligned}$$

For any inner point c of this convex hull, the polynomial $p(s, c)$ is Schur stable.

Conclusion

Given discrete-time system, by using the roots of the characteristic polynomial box coefficients of this system are defined. These coefficients differ from the classical reflection coefficients. The obtained result can be used in the generation of stable sets of characteristic polynomials.

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Inner differentiability and differential forms on tangentially locally linearly independent sets

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Abstract

The De Rham theorem gives a natural isomorphism between De Rham cohomology [4] and singular cohomology [3] on a paracompact differentiable manifold. This is very important fact as singular cohomology is very topological theory and De Rham cohomology is much more analytical. It comes from differential forms on manifolds and the exterior derivate. Perhaps it is strange to see that analysis and topology can be linked in such a nice way, however Stokes theorem also shows this relationship with the left hand side of the equation coming from topology and the right hand side coming from analysis.

This theorem the well-known mathematician G. de Rham outlined in his theses [2], but not in a form that is common today. In [6] we prove this theorem on a wider family of subsets of Euclidean space, on which we can define inner differentiability.

In our paper we define this family of sets called tangentially locally linearly independent (TLLI) sets, propose inner differentiability on these sets, postulate usual properties of differentiable real functions and moreover show that the integration over sets that are wider than manifolds is possible.

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Keywords: TLLI- tangentially locally linearly independent set, differential form, exact form, closed form

Introduction

The differentiable mappings are usually defined on open sets. On arbitrary set a function is differentiable, if there is a bigger open set that contains the set and the function is differentiable on it. However, this is only an agreement. In this paper we define inner differentiability on a wider family of subsets of Euclidean space called tangentially locally linearly independent-TLLI. In the second Section of the paper we consider this family of TLLI sets and some of their properties. The inner differentiability of real multivariate functions is defined in the third Section. This allows us to postulate in Section 4 the integration over class of sets called cuboidle sets that is wider class of manifolds by defining differential forms on TLLI sets. Section 5 concludes the paper.

Tangentially locally linearly independent and full tangentially locally linearly independent sets

Definition 1. *A set $M \subseteq \mathbb{R}^n$ is called tangentially locally linearly independent (TLLI), if for any arbitrary point $\underline{x}^0 = \left(x_1^0, ..., x_n^0\right) \in M$ is valid:*

if $D_1,...,D_n$ are real functions on the set M and continuous at $\underline{x}^0$ such $\sum_{i=1}^{n}\left(x_i-x_i^0\right)\cdot D_i\left(\underline{x}\right)=0\,,\forall\,\underline{x}\in M,$ then $D_i\left(\underline{x}^0\right)=0\,,\forall\, i\in\{1,...,n\}$.

In [5] is given the proof of the following theorem

Theorem 2. *If $M\subseteq\mathbb{R}^n$ is TLLI set, then all points from the set M are accumulation points of the set M.*

Note that all lines in $\mathbb{R}^2$ are not TLLI sets.
Let $\underline{x}^0\in\mathbb{R}^n$ is an arbitrary point. The line through the point $\underline{x}^0$ and parallel with the x_k- axis, $k\in\{1,...,n\}$, is denoted by:

$$G_k\left(\underline{x}^0\right)=\left\{\left(x_1^0,...,x_{k-1}^0,x_k,x_{k+1}^0,...,x_n^0\right)\;:\;x_k\in R\right\}\,,\;k\in\{1,...,n\}\;.$$

Definition 3. *A set $M\subseteq\mathbb{R}^n$ is full TLLI if any point $\underline{x}^0\in M$ is an accumulation point of all sets $M\cap G_k\left(\underline{x}^0\right)\;,\;k\in\{1,...,n\}$.*

Theorem 4. *Any full TLLI set $M\subseteq\mathbb{R}^n$ is TLLI set.*

Proof. Trivial by the definition of TLLI and full TLLI sets. ■

Note that all open sets and all closed n-dimensional rectangular cuboids in the space $\mathbb{R}^n$ are full TLLI sets.

Differentiability of multivariate real functions

Definition 5. *We say that a multivariate real function $f:M\to\mathbb{R}$, defined on TLLI set $M\subseteq\mathbb{R}^n$ is differentiable at $\underline{x}^0\in M$, if there exist n real-valued functions $D_1,...,D_n$ on the set M and continuous at $\underline{x}^0\in M$ such that:*

$$f\left(\underline{x}\right)=f\left(\underline{x}^0\right)+\sum_{1=1}^{n}\left(x_i-x_i^0\right)\cdot D_i\left(\underline{x}\right)\,,\quad\forall\,\underline{x}\in M \tag{1}$$

Definition 6. *We say that a multivariate real function $f:M\to\mathbb{R}$ is differentiable on the set $M\subseteq\mathbb{R}^n$, if it is differentiable at any point of the set M.*

Theorem 7. *Let $f:M\to\mathbb{R}$ is a real function on the TLLI set $M\subseteq\mathbb{R}^n$ and let f is differentiable at $\underline{x}^0\in M$. Then the values $D_1\left(\underline{x}^0\right),...,D_n\left(\underline{x}^0\right)$ are unique .*

It doesn't mean that the functions $D_1(\underline{x}),...,D_n(\underline{x})$ are unique on the set M.
We say that the values $D_1\left(\underline{x}^0\right),...,D_n\left(\underline{x}^0\right)$ are partial derivatives of the function f at $\underline{x}^0$ and we employ the notation $D_i\left(\underline{x}^0\right)=\frac{\partial f}{\partial x_i}\left(\underline{x}^0\right)=f_{x_i}{}'\left(\underline{x}^0\right)\,,\quad\forall\, i\in\{1,...,n\}$.
In [5] is given the proof of the following theorem

Theorem 8. *Let $f:M\to\mathbb{R}$ is a real function on the TLLI set $M\subseteq\mathbb{R}^n$ and let f is differentiable at $\underline{x}^0\in M$, then f is continuous at $\underline{x}^0\in M$.*

Let $f:M\to\mathbb{R}$ is a real valued function on full TLLI set $M\subseteq\mathbb{R}^n$ and $\underline{x}^0=\left(x_1^0,x_2^0,...,x_n^0\right)$ is a fixed point of the set M.
We define n real univariate functions:
$g_k\left(x_k\right)=f\left(x_1^0,...,x_{k-1}^0,x_k,x_{k+1}^0,...,x_n^0\right)$ **for all $k\in\{1,...,n\}$.**
The domain of these functions g_k for any $k\in\{1,...,n\}$ is the set $A_k=\left\{x_k\in\mathbb{R}:\left(x_1^0,...,x_{k-1}^0,x_k,x_{k+1}^0,...,x_n^0\right)\in\;M\right\}=M\cap G_k\left(\underline{x}^0\right)$.
Since $A_k\,,\,k=1,..,n$ is TLLI set in $\mathbb{R}$, then $x_k\in A_k\,,\;k=1,...,n$ is accumulation point of the set $A_k\,,\;k=1,..,n$.
It is easy to prove that for all real univariate functions g_k, $k\in\{1,...,n\}$ the following theorem is valid:

Theorem 9. *If the function $f : M \to \mathbb{R}$ is differentiable at $\underline{x}^0 \in M$, then all functions g_k, $k = 1, ..., n$ are differentiable at x_k^0, $k = 1, ..., n$ respectively and $g_k'\left(x_k^0\right) = f_{x_k}'\left(\underline{x}^0\right)$.*

Definition 10. *Let $f : M \to \mathbb{R}$ real function on TLLI set $M \subseteq \mathbb{R}^n$. We say that the function f is differentiable with respect to x_k at $\underline{x}^0 \in M$, if the function g_k is differentiable at x_k^0.*

Definition 11. *We say that a function $f : M \to \mathbb{R}$ is continuously differentiable on full TLLI $M \subseteq \mathbb{R}^n$, if it is differentiable on M, and all its partial derivatives are continuous on M.*

If there exists partial derivatives f_{x_k} for some $k = 1, ..., n$, that is differentiable at $\underline{x}^0 \in M$ with respect to some variable x_j , $j = 1, ..., n$ we say that there exists partial derivative of second order of the function f at $\underline{x}^0 \in M$ and it is denoted by $\left(f_{x_k}{}'\right)_{x_j}'\left(\underline{x}^0\right) = f_{x_k x_j}'\left(\underline{x}^0\right) = \frac{\partial^2 f}{\partial x_j \partial x_k}\left(\underline{x}^0\right) = f_{kj}'\left(\underline{x}^0\right)$, where $k = 1, ..., n$ and $j = 1, ..., n$. If there exist partial derivatives of a second order of the function f on the whole set M then it is possible to discuss about their differentiability and partial derivatives of higher order.

Definition 12. *We say that a real multivariate function is r- times differentiable at $\underline{x}^0 \in M$, where $r = 2, 3, ...,$ if there exist an open neighborhood U of that point such that the function f is $r-1$- times differentiable on the set $U \cap M$ and all $r-1$-partial derivatives of f are differentiable at $\underline{x}^0$.*

A function f is r- times differentiable on the set M if it is r- times differentiable at all points of the set M.

The partial derivtives from r-th order of the function f at $\underline{x}^0$ is denoted by $f_{x_{k_1} x_{k_2} ... x_{k_r}}'\left(\underline{x}^0\right) = \frac{\partial^r f}{\partial x_{k_r} ... \partial x_{k_2} \partial x_{k_1}}\left(\underline{x}^0\right)$.

Theorem 13. *Let $f : M \to \mathbb{R}$ is a multivariate real function on a closed rectangulare cuboid $M = \{\underline{x} \in \mathbb{R}^n : \ a_k \le x_k \le b_k, \ a_k, b_k \in \mathbb{R}, \ k = 1, ..., n\}$ and let all partial derivatives of the function f are differentiable with respect to all variables at the point $\underline{x}^0 \in M$. Then, $f_{x_i x_j}\left(\underline{x}^0\right) = f_{x_j x_i}\left(\underline{x}^0\right)$, $i, j = 1, 2, ..., n$.*

Proof. Without losing of generality we can prove the theorem assuming that $\underline{x}^0 = 0$ but first showing that it is enough to validate its statement for any bivariate real function by using the mean value theorem. ■

Differential forms on TLLI sets

Definition 14. *Differential form of k order on the set M (or $k-$form in M) is a mapping ω, $\omega = \sum_{1 \le i_1 < ... < i_k \le n} a_{i_1 ... i_k}\left(\underline{x}\right) dx_{i_1} \wedge ... \wedge dx_{i_k}$, where $a_{i_1 ... i_k} : M \to \mathbb{R}$ are continuous real functions for any $k-$variation $\{i_1, i_2, ..., i_k\}$ of the set of n elements $\{1, 2, ..., n\}$, and we will denote by $\omega = \sum_{\underline{i}} a_{\underline{i}}\left(\underline{x}\right) dx_{\underline{i}}$, where $dx_{\underline{i}} = dx_{i_1} \wedge ... \wedge dx_{i_k}$ and $a_{\underline{i}} = a_{i_1 ... i_k}$ for any variation $\underline{i} = \{i_1, ..., i_k\}$, $1 \le i_1 < ... < i_k \le n$. such that it mapps a real number to any singular $k-$cube $\phi : I^k \to M$(that is continuously differentiable function on cube, i.e., $\phi \in C^1$) obtained by integration:*

$$\omega\left(\phi\right) = \int_{\phi} \omega = \sum_{\underline{i}} \int_{I^k} a_{\underline{i}}\left(\phi\left(\underline{t}\right)\right) \frac{\partial\left(\phi_{i_1}, ... \phi_{i_k}\right)}{\partial\left(t_1, ..., t_k\right)} dt_1 \wedge ... \wedge dt_k,$$

where $\frac{\partial\left(\phi_{i_1}, ... \phi_{i_k}\right)}{\partial\left(t_1, ..., t_k\right)} = \begin{vmatrix} \frac{\partial \phi_{i_1}}{\partial t_1} \cdots\cdots \frac{\partial \phi_{i_1}}{\partial t_k} \\ \vdots \qquad\quad \vdots \\ \frac{\partial \phi_{i_k}}{\partial t_1} \cdots\cdots \frac{\partial \phi_{i_k}}{\partial t_k} \end{vmatrix}$ *is the Jacobian of $\phi = (\phi_1, \phi_2, ..., \phi_n)$.*

If $\Gamma = \sum_\phi n_\phi \phi$ is continuously differentiable $k-$chain on M, then the $k-$form on M, ω maps a real number to the $k-$chain $\Gamma = \sum_\phi n_\phi \phi$

$$\omega(\Gamma) = \int_\Gamma \omega = \sum_\phi \sum_{\underline{i}} n_\phi \int_{I^k} a_{\underline{i}}(\phi(\underline{t})) \frac{\partial(\phi_{i_1}, ...\phi_{i_k})}{\partial(t_1, ..., t_k)} dt_1 \wedge ... \wedge dt_k.$$

Note that if $\phi : I^k \to M$ is degenerated singulare $k-$cube, i.e., there exists singulare $k-1-$cube $\phi^{'} : I^{k-1} \to M$ such that
$\phi(t_1, ..., t_{i-1}, t_i, t_{i+1}, ...t_n) = \phi'(t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$ for some integer i , $1 \le i \le n$, then for any $k-$ form ω on M is valid that $\omega(\phi) = 0$. So, we conclude that a $k-$ form ω on M is a real function from the free abelian group of all nondegenerated continuously differentiable singulare k- cubes, $C_k(M)$.

The set of all $k-$forms for any $k \le n$ on M is denoted by $D^k(M)$, i.e., $D^k(M) = \{\omega \mid \omega : C_k(\mathrm{M}) \to \mathbb{R} \; is \; k - form \; on \; M \}$. If $k > n$ then $D^k(M) = 0$. Moreover, the set of all $k-$forms on M, $D^k(M)$ with respect to sum and product defined in [1] is a vector space.

Next we define an operator $d : D^k(M) \to D^{k+1}(M)$ and state some theorems about its properties that can be easily proved.

Definition 15. *Let $f : M \to \mathbb{R}$ is $0-$form on M, where f is continuously differentiable function. Its differential is a $1-$form on M, $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. Let $\omega = \sum_{\underline{i}} a_{\underline{i}}(\underline{x}) dx_{\underline{i}}$ is an arbitrary $k-$form on M, such that $a_{\underline{i}}$ continuously differentiable real function. Its differential is a $k+1-$form on M, $d\omega = \sum_{\underline{i}} da_{\underline{i}} \wedge dx_{\underline{i}} = \sum_{\underline{i}} \sum_{j=1}^n \frac{\partial a_{\underline{i}}}{\partial x_j} dx_j \wedge dx_{\underline{i}}$.*

Theorem 16. *The mapping $d : D^k(M) \to D^{k+1}(M)$, $k \in \mathrm{Z}$ is linear*

Theorem 17. *Let $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are $0-$forms on M, where f and g are continuously differentiable functions, then $d(fg) = df \cdot g + f \cdot dg$.*

Theorem 18. *Let ω and λ are arbitrary k and $m-$ forms on M, respectively. Then $d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k \omega \wedge d\lambda$.*

Definition 19. *We say that ω is an **exact** differential k- form on M, then there exists $k-1$- form $\lambda \in D^{k-1}(M)$ such that $\omega = d\lambda$. We say that ω is a **closed** differential k- form on M if $d\omega = 0$.*

Definition 20. *We say a set $M \subseteq \mathbb{R}^n$ is **cuboidle**, if for any point $\underline{x} \in M$ there exists rectangular cuboid*

$$K = \left\{ \underline{y} \in \mathbb{R}^n \,\middle|\, a_i \le y_i \le b_i \,,\, a_i, b_i \in \mathbb{R} \,,\, i = \overline{1, n} \right\}$$

such that $K \subseteq M$.

A cuboidle set is TLLI set.

Theorem 21. *Let $\omega = \sum_{\underline{i}} a_{\underline{i}} d_{x_{\underline{i}}}$ is two times differentiable k- form on cuboidle set $M \subseteq \mathbb{R}^n$, i.e., for all indices $\underline{i}$ the functions $a_{\underline{i}} : M \to \mathbb{R}$ are two times differentiable on the set M. Then $dd\omega = 0$ on the set M.*

Proof. The proof of this theorem is obtained by using Theorem 13, Theorem 18 and considering the Definition 20. ■

Theorem 22. *Let $\omega = \sum_{\underline{i}} a_{\underline{i}} d_{x_{\underline{i}}}$ is a differentiable k- form on a cuboidle set $M \subseteq \mathbb{R}^n$. If $\omega = \sum_{\underline{i}} a_{\underline{i}} d_{x_{\underline{i}}}$ is an exact $k-$ form on the set M, then it is closed.*

Proof. Since ω is an exact $k-$ form on the set M, then there exists $k-1$ - form $\lambda \in D^{k-1}(M)$ such that $\omega = d\lambda$. Because ω is a differentiable k- form on a cuboidle set then λ is two times differentiable $k-1$- form on cuboidle set $M \subseteq \mathbb{R}^n$ and by Theorem 17 $dd\lambda = 0$ on the set M.Therefore, $d\omega = dd\lambda = 0$ on the set M, so ω is closed k- form on the set M. ■

The opposite statement of Theorem 22 is not always true, but if we assume additionaly that the cuboidle set $M \subseteq \mathbb{R}^n$ is also star set then it can be shown that any continuously differentiable closed k- form on M is exact.

Conclusion

In our paper we consider a family of sets in n dimensional real space so called TLLI sets that is wider than the family of open sets. Moreover, we define differentiability and differential forms on this family of sets. So we show that it is possible to integrate over singulare cube not only in a manifold as we know by now but in a cuboidle set defined by the TLLI sets. At last we prove and state some theorems which are necessary for the definition of de Rham cohomology [4] in order to complete the proof of the De Rham Theorem in [2] on a wider family than manifolds.

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Some Results On Suborbital Graphs

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Abstract

In this work, we study some special subgraphs of the subgroup Γ^3 of the modular group Γ and give some special number theoretical results.

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Keywords: Modular group, Graph theory, Number theory

Introduction

Let $\Gamma = PSL(2,\mathbb{Z})$ be the modular group acting on the extended rational numbers $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ with the action defined by

$$\mp \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \frac{x}{y} = \frac{ax+by}{cx+dy}$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$. In [1], the subgroup Γ^3 of Γ is defined by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma : ab + cd \equiv 0 \pmod 3 \right\}$$

It follows from definition that the elements of Γ^3 are one of the forms $\left(\begin{smallmatrix} 3a & b \\ c & 3d \end{smallmatrix} \right), \left(\begin{smallmatrix} a & 3b \\ 3c & d \end{smallmatrix} \right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ where a, b, c and $d \not\equiv 0(\text{mod}3)$. Hence, the subgroup Γ^3 acts transitively on the subset $\hat{\mathbb{Q}}$ and the stabilizer of ∞ is the group $\{\mp \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix} \right) : m \in \mathbb{Z}\}$. The diagonal action, given by $g(\alpha, \beta) = (g\alpha, g\beta)$, of the group Γ^3 on $\hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$ defines the suborbitals, which are actually orbits. The orbit $O^3(\alpha, \beta)$ containing (α, β) gives the suborbital graph $G^3(\alpha, \beta)$ defined as follows:

The set of vertices is $\hat{\mathbb{Q}}$, and there is an edge $\gamma \to \delta$ in $G^3(\alpha, \beta)$ if and only if $(\gamma, \delta) \in O^3(\alpha, \beta)$. Since the action is transitive, every suborbital contains a pair $(\infty, \dfrac{u}{n})$ for some $\frac{u}{n} \in \hat{\mathbb{Q}}, (u, n) = 1, n > 0$. As in [2], the congruence subgroup $\Gamma_0^3(n)$ defines the following equivalence relation on $\hat{\mathbb{Q}}$ by $g_1(\infty) \simeq g_2(\infty)$ for $g_1, g_2 \in \Gamma^3$, if $g_1\Gamma_0^3(n) = g_2\Gamma_0^3(n)$.

If $g_1(\infty) = \dfrac{r}{s}$ and $g_2(\infty) = \dfrac{x}{y}$, we have $\frac{r}{s} \simeq \frac{x}{y} \Leftrightarrow ry - sx \equiv 0(\text{mod}n)$

We denote the suborbital graphs by $G^3_{u,n}$ for short, and the subgraphs of $G^3_{u,n}$ whose vertice set is just the equivalence class or block

$$[\infty] = \left\{ \frac{x}{y} \in \hat{\mathbb{Q}} : y \equiv 0 \pmod n \right\}$$

will be denoted by $F^3_{u,n}$.

More knowledge about modular group, subgroups and group actions can be seen in [2-6].

Main Results

Theorem 1. [2] $F^3_{u,n} = F^3_{u',n'}$ *if and only if* $n = n'$ *and* $u \equiv u' (\mathrm{mod} 3n)$

Theorem 2. [2] *There is an edge* $\frac{r}{s} \to \frac{x}{y}$ *in* $F^3_{u,1}$ *if and only if either*

(i) if $r \equiv 0(\mathrm{mod}3)$*, then* $y \equiv \mp us(\mathrm{mod}3)$ *and* $ry - sx = \mp 1$*, or*

(ii) if $s \equiv 0(\mathrm{mod}3)$*, then* $x \equiv \mp ur(\mathrm{mod}3)$ *and* $ry - sx = \mp 1$*, or*

(iii) if $r, s \not\equiv 0(\mathrm{mod}3)$*, then* $x \not\equiv \mp ur(\mathrm{mod}3), y \not\equiv \mp us(\mathrm{mod}3)$ *and* $ry - sx = \mp 1$

The following theorems will be given without proofs.

Theorem 3. *Let* $K = \begin{pmatrix} -23 & 599 \\ -1 & 26 \end{pmatrix}$ *be in* Γ^3*. Then,*

(i) $\forall m \in \mathbb{N}$, $K^m(\frac{1}{0}) \to K^m(\frac{u}{1})$ *in* $F_{u,1}$.

(ii) $\forall m \in \mathbb{N}$, $K^m(\frac{1}{0}) \to K^{m+1}(\frac{1}{0})$ *in* $F_{u,1}$.

(iii) The sequence $\{K^m\}_{m\in\mathbb{N}}$ *is increasing and the path*

$$K(\frac{1}{0}) \to K^2(\frac{1}{0}) \to K^3(\frac{1}{0}) \to \cdots$$

is infinite path.

(iv) The fixed points of K *are* $z_{1,2} = \dfrac{49 \mp \sqrt{5}}{2}$.

Theorem 4. *Let* $K = \begin{pmatrix} -23 & 599 \\ -1 & 26 \end{pmatrix}$ *be in* Γ^3 *and* $a, b \in \mathbb{N}$ *such that* $\frac{23}{1} \leq \frac{a}{b} < \frac{49-\sqrt{5}}{2}$*. Then,*

(i) $\frac{a}{b} < K(\frac{a}{b}) < \frac{49-\sqrt{5}}{2}$,

(ii) $\frac{a}{b} \to K(\frac{a}{b})$ *is an edge in* $F_{23,1}$ *if and only if* $a = \dfrac{49b - \sqrt{5b^2+4}}{2}$ *and there exists some* $t \in \mathbb{N}$ *such that* $5b^2 + 4 = t^2$.

Corollary 5. *Let* $k \in \mathbb{N}$*. Then,*

(i) $\frac{1}{0} \to 23 + \frac{0}{1} \to 23 + \frac{1}{3} \to \cdots \to 23 + \frac{a_k}{b_k} \to 23 + \frac{b_k}{3b_k - a_k} \to \cdots$ *is an infinite path.*

(ii) All above vertices are less than $\dfrac{49-\sqrt{5}}{2}$.

(iii) For the numbers a_k, b_k *in (i), the numbers* $5a_k^2 + 4, 5b_k^2 + 4$ *are perfect square, and* $a_k = \dfrac{3b_k - \sqrt{5b_k^2+4}}{2}$.

Corollary 6. *The integers* $b \in \mathbb{Z}^+ \cup \{0\}$ *in the equality* $5b^2 + 4 = t^2$ *are*

$$0, 1, 3, \cdots, x, y, 3y - x, \cdots$$

Now we take $K = \begin{pmatrix} -23 & 599 \\ -1 & 26 \end{pmatrix}$ and we get that $S = K^{-1} = \begin{pmatrix} 26 & -599 \\ 1 & -23 \end{pmatrix}$.

Theorem 7. *Let* $a, b \in \mathbb{N}$ *such that* $\frac{25}{1} \leq \frac{a}{b} < \frac{49+\sqrt{5}}{2}$. *Then,*

(i) $\frac{a}{b} < S(\frac{a}{b}) < \frac{49+\sqrt{5}}{2}$,

(ii) $\frac{a}{b} \to S(\frac{a}{b})$ *is an edge in* $F_{23,1}$ *if and only if* $a = \frac{49b+\sqrt{5b^2-4}}{2}$ *and there exists* $w \in \mathbb{N}$ *such that* $5b^2 - 4 = w^2$.

Corollary 8. *Let be* $k \in \mathbb{N}$.

(i) $26 - \frac{1}{1} \to 26 - \frac{1}{2} \to 26 - \frac{2}{5} \to \cdots \to 26 - \frac{a_k}{b_k} \to 26 - \frac{b_k}{(3b_k - a_k)} \to \cdots$ *is an infinite path.*

(ii) The vertices are in (i) less than $\frac{49+\sqrt{5}}{2}$.

(iii) For the numbers a_k, b_k *in (i), the numbers* $5a_k^2 - 4, 5b_k^2 - 4$ *are perfect square and* $a_k = \frac{3b_k - \sqrt{5b_k^2 - 4}}{2}$.

Corollary 9. *The numbers* $b_k \in \mathbb{Z}^+$ *making* $5b_k^2+4$ *perfect square are* $0, 1, 3, 8, \cdots, x, y, 3y-x, \cdots$

Corollary 10. *The numbers* $b_k \in \mathbb{Z}^+$ *making* $5b_k^2-4$ *perfect square are* $1, 2, 5, \cdots, x, y, 3y-x, \cdots$

Corollary 11. *Let the sequences* $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ *be* $(0, 1, 3, 8, \cdots, x, y, 3y-x, \cdots)$ *and* $(1, 2, 5, \cdots, x, y, 3y-x, \cdots)$, *respectively. Then, the sequence* $\{c_k\}_{k\in\mathbb{N}}$, *defined by* $(0, 1, 1, 2, 3, 5, 8, \cdots, a_k, b_k, a_{k+1}, b_{k+1}, \cdots)$ *is the Fibonacci sequence.*

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Pascal Trapezoids Emerging from Hypercomplex Polynomial Sequences

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Abstract

The construction of two different representations of special Appell polynomials in $(n+1)$ real variables with values in a Clifford algebra suggested to explore the relation between the respective coefficients. Properties of sequences resulting from such relation and an interesting trapezoidal array of their elements are pointed out.

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Keywords: special sequences, binomial coefficients, Pascal trapezoids, hypercomplex polynomials

Introduction

In this paper we focus on polynomial sequences in $(n+1)$ real variables with values in the real vector space of paravectors in the corresponding Clifford algebra $\mathcal{C}\ell_{0,n}$. We start by introducing some basics of that algebra. The reader can find more details in [4].

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the real Euclidean vector space $\mathbb{R}^n$ endowed with a product according to the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij},\ i,j = 1, \cdots, n,$$

where δ_{ij} is the Kronecker symbol. This non-commutative product generates the associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$. The elements z of $\mathcal{C}\ell_{0,n}$, called *hypercomplex numbers*, are of the form $z = \sum_A z_A e_A$, where $z_A \in \mathbb{R}$ and the basis $\{e_A : A \subseteq \{1, \cdots, n\}\}$ is formed by $e_A = e_{h_1} e_{h_2} \cdots e_{h_r}$, $1 \leq h_1 < \cdots < h_r \leq n$, $e_\emptyset = e_0 = 1$.

The vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by the identification of the real $(n+1)-$tuple $(x_0, x_1, \cdots, x_n)$ with the paravector

$$x = x_0 + \underline{x} = x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}.$$

The conjugate of $x \in \mathcal{A}_n$ is given by $\bar{x} = x_0 - \underline{x}$. The so-called scalar part x_0 and the vector part $\underline{x}$ of x can be written in the form $x_0 = (x+\bar{x})/2$ and $\underline{x} = (x-\bar{x})/2$, respectively. The norm of x is given by $|x| = (x\bar{x})^{\frac{1}{2}} = (x_0^2 + x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. Consequently, the inverse of each non-zero x is $x^{-1} = \bar{x}|x|^{-2}$.

We consider $\mathcal{C}\ell_{0,n}$-valued functions defined as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A}_n \longmapsto \mathcal{C}\ell_{0,n}$$

such that $f(x) = \sum_A f_A(x) e_A,\ f_A(x) \in \mathbb{R}$ and Ω is an open subset of $\mathbb{R}^{n+1},\ n \geq 1$.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is defined by $\bar{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$, with $\partial_0 := \frac{\partial}{\partial x_0}$ and $\partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$. Its conjugate, also called the hypercomplex differential operator, is denoted by $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$.

The analogue of a holomorphic function is now a $\mathcal{C}^1$-function f that is a solution of the differential equation $\bar{\partial} f = 0$ (resp. $f\bar{\partial} = 0$) and is called *left monogenic* (resp. *right monogenic*).

The concept of hypercomplex differentiability as a generalization of complex differentiability reads as follows. A function f defined in Ω is hypercomplex differentiable if and only if it has a uniquely defined areolar derivative f' in each point of Ω (for details, see [10]). A hypercomplex differentiable function f is real differentiable and consequently f' is given by $f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f$. Since a hypercomplex differentiable function f is monogenic, it follows that $f' = \partial_0 f = -\partial_{\underline{x}} f$ (see [9]).

Noting that $\bar{\partial} x = \frac{1-n}{2}$, it is clear that the identity function $f(x) = x$ (and its integer powers) belongs to the class of monogenic functions only if $n = 1$, i.e., in the complex case. Thus, the construction of polynomials which behave with respect to the derivative like simple powers of $x \in \mathcal{A}_n$ is a problem of its own interest. Applying Appell's ideas [3], authors of this paper started a systematic study on Appell sequences in the framework of Hypercomplex Function Theory ([5, 8, 11]).

There are two natural representations of $\mathcal{A}_n$-valued homogeneous polynomials, one by using $(x, \bar{x})$ and the other by using $(x_0, \underline{x})$. Both representations involve coefficients whose relation leads to sequences of nonnegative integers. The main goal of the present paper is to emphasize properties of those sequences and their relation with Pascal's like triangles.

Main Results

We focus on the following two representations of $\mathcal{A}_n$-valued homogeneous monogenic polynomials $\mathcal{P}_k^n(x)$ introduced in [7, 12]:

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n) x^{k-s} \bar{x}^s \tag{1}$$

and

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{2}$$

where the coefficients $T_s^k(n)$ and $c_s(n)$ take the form

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}$$

($(a)_r$ denotes the Pochhammer symbol, i.e., $(a)_0 := 1, (a)_r := \prod_{t=0}^{r-1}(a+t),\ r \geq 1$), and

$$c_s(n) = \begin{cases} \frac{s!!(n-2)!!}{(n+s-1)!!}, & \text{if } s \text{ is odd} \\ c_{s-1}(n), & \text{if } s \text{ is even} \end{cases},$$

respectively.

The sequences $(\mathcal{P}_k^n(x))_{k\geq 0}$ are generalized Appell sequences, with $\mathcal{P}_0^n(x) = 1$, according to the following definition ([7]).

Definition 1. *A sequence of $\mathcal{A}_n$-valued monogenic polynomials $(\mathcal{Q}_k(x))_{k\geq 0}$ is called a generalized Appell sequence, if $\mathcal{Q}_k(x)$ is of exact degree k, for each $k = 0, 1, \ldots,$ and $\partial \mathcal{Q}_k(x) = k\mathcal{Q}_{k-1}(x),\ k = 1, 2, \ldots.$*

The relation between the representations (1) and (2) is intrinsically linked to the one of respective coefficients. In order to point out such relation we start by considering the $(k+1)-$dimensional vectors

$$\mathbf{T}_k(n) = \begin{bmatrix} T_0^k(n) & T_1^k(n) & \ldots & T_{k-1}^k(n) & T_k^k(n) \end{bmatrix}^T$$

and

$$\mathbf{C}_k(n) = \begin{bmatrix} c_0(n) & c_1(n) & \ldots & c_{k-1}(n) & c_k(n) \end{bmatrix}^T.$$

Each component of $\mathbf{T}_k(n)$ can be written as linear combination of the components of $\mathbf{C}_k(n)$ as follows:

$$T_s^k(n) = \tfrac{1}{2^k}\binom{k}{s}\sum_{j=0}^{k}\sigma_{s,j}^k c_j(n), \quad k = 0, 1, \cdots;\ s = 0, \cdots, k \tag{3}$$

where

$$\sigma_{s,j}^k = \sum_{m=0}^{s}(-1)^m\binom{s}{m}\binom{k-s}{j-m}. \tag{4}$$

(cf. [6, Thm. 6]).

Reciprocally, each component of $\mathbf{C}_k(n)$ can also be written in the following way:

$$c_{k-i}(n) = \frac{1}{\binom{k}{i}}\sum_{s=0}^{k}(-1)^s\sigma_{s,i}^k T_s^k(n), \quad k = 0, 1, \cdots;\ i = 0, 1, \ldots, k \tag{5}$$

(cf. [6, Thm. 7]).

The transformation from $\mathbf{C}_k(n)$ to $\mathbf{T}_k(n)$ can be derived in matrix form. Define the diagonal matrix $D_k = \mathrm{diag}[\binom{k}{0}\ \binom{k}{1}\ \cdots\ \binom{k}{k}]$ and the matrix $S_k := \left[s_{ij}^k\right]_{i,j=1}^{k+1}$ such that $s_{ij}^k = \sigma_{i-1,j-1}^k$. Thus, (3) can be written in the form

$$\mathbf{T}_k(n) = N_k\mathbf{C}_k(n),$$

where $N_k = \frac{1}{2^k}D_kS_k$. Analogously, the matrix form of (5) is obtained as

$$\mathbf{C}_k(n) = \tilde{N}_k\mathbf{T}_k(n),$$

where $\tilde{N}_k = D_k^{-1}\tilde{S}_kD$, $D = \mathrm{diag}[1\ -1\ \cdots\ (-1)^k]$ and the entries of $\tilde{S}_k$ are given by $\tilde{s}_{ij}^k = \sigma_{j-1,k-i+1}^k$.

We observe that the connection between $\mathbf{C}_k(n)$ and $\mathbf{T}_k(n)$ relies indeed on the nonnegative integers (4).

The importance of the Pascal's triangle in issues related to Appell polynomials has already been studied. For instance, it appears in the matrix representation of real and hypercomplex Appell polynomials. For details we refer to [1, 2] and references therein. That relation led us to believe that the integers (4) would also be somehow linked to triangles of that type.

Let i, j, k be arbitrary nonnegative integers such that $j \leq k$. For each fixed i, we arrange the numbers $\sigma_{i,j}^k$ in a triangle with rows k and ordered from $j = 0$ to $j = k$ (see Table 1).

Table 1: Triangles associated with $\sigma_{i,j}^k$

$$\begin{array}{c}
\boldsymbol{\sigma_{i,0}^0} \\
\boldsymbol{\sigma_{i,0}^1} \quad \boldsymbol{\sigma_{i,1}^1} \\
\boldsymbol{\sigma_{i,0}^2} \quad \boldsymbol{\sigma_{i,1}^2} \quad \boldsymbol{\sigma_{i,2}^2} \\
\boldsymbol{\sigma_{i,0}^3} \quad \boldsymbol{\sigma_{i,1}^3} \quad \boldsymbol{\sigma_{i,2}^3} \quad \boldsymbol{\sigma_{i,3}^3} \\
\boldsymbol{\sigma_{i,0}^4} \quad \boldsymbol{\sigma_{i,1}^4} \quad \boldsymbol{\sigma_{i,2}^4} \quad \boldsymbol{\sigma_{i,3}^4} \quad \boldsymbol{\sigma_{i,4}^4} \\
\iddots \quad \iddots \qquad\qquad \ddots \quad \ddots \quad \ddots
\end{array}$$

For each fixed value of i, the coeficients $\sigma_{i,j}^k$ satisfy the recurrence relation

$$\sigma_{i,j+1}^{(k+1)} = \sigma_{i,j}^k + \sigma_{i,j+1}^k, \ (0 \leq j \leq k-1, k \geq i) \tag{6}$$

with boundary conditions

$$\sigma_{i,0}^k = 1, \qquad \sigma_{i,k}^k = (-1)^i, \ (k \geq i) \tag{7}$$

and initial values

$$\sigma_{i,j}^{(i)} = \binom{i}{j}(-1)^j, \ j = 1, \ldots, i-1 \tag{8}$$

(cf. [6, Thm. 10]).

Notice that when $i = 0$, formulae (6)-(7)-(8) coincide with the Pascal recurrence $\binom{k+1}{j+1} = \binom{k}{j} + \binom{k}{j+1}$, for all integers $k, j : 0 \leq j \leq k-1$, with initial/boundary values $\binom{k}{0} = \binom{k}{k} = 1$. As is well known, this formula allows the construction of the Pascal triangle. Now, the general formulae (6)-(7)-(8) permit, for each fixed i, the construction of *Pascal trapezoids*, i.e., each triangle (Table 1) encloses a trapezoidal substructure in itself (see Table 2) for the cases $i = 1$ and $i = 2$).

Table 2: Triangles associated with $\sigma_{i,j}^k$, $i = 1, 2$

$$\begin{array}{c}
0 \\
1 \quad -1 \\
1 \quad 0 \quad -1 \\
1 \quad 1 \quad -1 \quad -1 \\
1 \quad 2 \quad 0 \quad -2 \quad -1 \\
1 \quad 3 \quad 2 \quad -2 \quad -3 \quad -1 \\
1 \quad 4 \quad 5 \quad 0 \quad -5 \quad -4 \quad -1 \\
1 \quad 5 \quad 9 \quad 5 \quad -5 \quad -9 \quad -5 \quad -1 \\
\iddots \quad \iddots \quad \iddots \quad \iddots \qquad \ddots \quad \ddots \quad \ddots \quad \ddots \\
\\
\boldsymbol{i{=}1}
\end{array}
\qquad
\begin{array}{c}
2 \\
-2 \quad -2 \\
1 \quad -2 \quad 1 \\
1 \quad -1 \quad -1 \quad 1 \\
1 \quad 0 \quad -2 \quad 0 \quad 1 \\
1 \quad 1 \quad -2 \quad -2 \quad 1 \quad 1 \\
1 \quad 2 \quad -1 \quad -4 \quad -1 \quad 2 \quad 1 \\
1 \quad 3 \quad 1 \quad -5 \quad -5 \quad 1 \quad 3 \quad 1 \\
\iddots \quad \iddots \quad \iddots \quad \iddots \qquad \ddots \quad \ddots \quad \ddots \quad \ddots \\
\\
\boldsymbol{i{=}2}
\end{array}$$

More details on this subject can be seen in [6].

Conclusion

Two particular representations of generalized Appell polynomials in the hypercomplex context lead to special integer sequences satisfying a Pascal recurrence. The construction of Pascal trapezoids resulting from that recurrence has been examined.

Trapezoids presented in this paper, but not obtained as emerging from hypercomplex Appell polynomials, occure also in the expansion of $(1-x)^i(1+x)^{k-i}$. Indeed, $(1-x)^i(1+x)^{k-i} = \sum_{j=0}^{k} \sigma_{i,j}^{k}\, x^j$.

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Some result for binomial convolution sums of restricted divisor functions

Ho Park[1] , *Daeyeoul Kim*[2] , *Ji Suk So*[3]

Abstract

Besge presented the result about the convolution sum of divisor functions. Since then Liouville obtained the generalized version of Besge's formula, which is the binomial convolution sum of divisor functions. In 2004, Hahn obtained the results about the convolution sums of $\sum_{d|n}(-1)^{d-1}d$ and $\sum_{d|n}(-1)^{n/d-1}d$. In this talk, we present the results for the binomial convoltion sums, generalized convolution sums of Hahn, of these divisor functions.

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Introduction

For a positive integer n and a nonnegative integer k, let $\sigma_k(n)$ be the usual divisor function defined by $\sigma_k(n) = \sum_{d|n} d^k$. The well-known identity

$$\sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m) = \frac{1}{12}(5\sigma_3(n) + (1-6n)\sigma_1(n))$$

first appeared in a letter from Besge to Liouville in 1862(see [1]). The generalized version of Besge's identity, that is said to the binomial convolution sums of divisor functions was obtained by Liouville(see [5]).

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m) = \frac{2k+3}{4k+2}\sigma_{2k+1}(n)$$
$$+ \left(\frac{k}{6} - n\right)\sigma_{2k-1}(n) + \frac{1}{2k+1}\sum_{j=2}^{k} \binom{2k+1}{2j} B_{2j}\sigma_{2k+1-2j}(n),$$

where for a nonnegative integer n, B_n is the n-th Bernoulli number.

We introduce another divisor functions. For a positive integer n and a nonnegative integer k,

$$\sigma_k^*(n) = \sum_{\substack{d|n \\ n/d\equiv 1(2)}} d^k \qquad \textit{and} \qquad \sigma_{k,i}(n;2) = \sum_{\substack{d|n \\ d\equiv i(2)}} d^k \quad (i \in \{0,1\}).$$

Recently, Kim and Bayad(see [3]) obtained the convolution sum of $\sigma_k^*(n)$

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(n-m) = \frac{1}{2}\sigma^*_{2k+1}(n) - \frac{n}{2}\sigma^*_{2k-1}(n)$$

and Kim, Bayad and Park(see [4]) computed the convolution sum of $\sigma_{k,1}(n;2)$

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2)$$
$$= 2^{2k-1}\sigma_{2k+1}(n/2) + \frac{2^{2k}}{2k+1} \sum_{\substack{d|n \\ d\equiv 1(2)}} B_{2k+1}\left(\frac{d+1}{2}\right),$$

where for a nonnegative integer n, $B_n(x)$ is the Bernoulli polynomial.

Hahn has defined for $k, r \in \mathbb{N}$

$$\tilde{\sigma}_k(n) = \sum_{d|n} (-1)^{d-1} d^k \qquad \text{and} \qquad \hat{\sigma}_k(n) = \sum_{d|n} (-1)^{n/d-1} d^k.$$

Hahn obtained the convolution sums of the these divisor functions(see [2]).

$$\sum_{m=1}^{n-1} \tilde{\sigma}_1(m)\tilde{\sigma}_1(n-m) = -\frac{1}{4}\tilde{\sigma}_3(n) + \left(\frac{1}{2}n - \frac{1}{4}\right)\tilde{\sigma}_1(n),$$
$$\sum_{m=1}^{n-1} \hat{\sigma}_1(m)\tilde{\sigma}_1(n-m) = -\frac{1}{12}\tilde{\sigma}_3(n) + \left(\frac{1}{4}n - \frac{1}{8}\right)\hat{\sigma}_1(n) - \frac{1}{24}\tilde{\sigma}_1(n).$$

Later, Williams(see [6]) computed

$$\sum_{m=1}^{n-1} \hat{\sigma}_1(m)\hat{\sigma}_1(n-m) = \frac{5}{42}\hat{\sigma}_3(n) - \frac{1}{28}\tilde{\sigma}_3(n) - \frac{1}{12}\hat{\sigma}_1(n).$$

Main Results

In this talk, we present the binomial convolution sums of two version of (restricted) divisor functions. The first binomial convolution sums(Theorem 1) have a combination of $\sigma_k(n)$, $\sigma_k^*(n)$ and $\sigma_{k,1}(n;2)$. The second binomial convolution sums(Theorem 2) are the generalized versions of the results of Hahn and Williams.

Theorem 1. *For each $k, n \in \mathbb{N}$, we have*

(i)
$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2)=\frac{1}{4}\sigma^*_{2k+1}(n)-2^{2k-2}\sigma^*_{2k+1}(n/2)$$
$$-\frac{n}{4}\Big(\sigma_{2k-1,1}(n;2)-2^{2k-1}\sigma^*_{2k-1}(n/2)\Big)+\frac{1}{4}\sum_{\substack{d\mid\frac{n}{2}\\ \frac{n/2}{d}\equiv1(2)}}E_{2k}(2d)+\frac{2}{2k+1}\sum_{\substack{d\mid\frac{n}{2}\\ \frac{n/2}{d}\equiv1(2)}}B_{2k+1}(d)$$
$$+\frac{2^{2k-1}}{2k+1}\Big(\sum_{\substack{d\mid n\\ \frac{n}{d}\equiv1(2)}}B_{2k+1}\Big(\frac{d+1}{2}\Big)-\sum_{\substack{d\mid\frac{n}{2}\\ \frac{n/2}{d}\equiv1(2)}}B_{2k+1}\Big(\frac{2d+1}{2}\Big)\Big),$$

(ii)
$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1}(n-m)=\frac{1}{2}\sigma_{2k+1}(n)+\frac{1}{2}\sigma_{2k+1}(n/2)$$
$$+\frac{3}{4}\sigma_{2k}(n)+\frac{1}{4}\sigma_{2k}(n/2)-\frac{5n}{4}\sigma_{2k-1}(n)-\frac{n}{2}\sigma_{2k-1}(n/2)$$
$$+\frac{1}{2(2k+1)}\Big(3\sum_{d\mid n}B_{2k+1}(d)+\sum_{d\mid n/2}B_{2k+1}(d)\Big),$$

(iii)
$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,1}(n-m;2)=\frac{1}{4}(\sigma_{2k+1}(n)-2^{2k}\sigma(n/2))+\frac{1}{4}\sigma_{2k,1}(n;2)$$
$$-\frac{n}{2}\sigma_{2k-1,1}(n;2)+2^{2k-2}\sigma_{2k-1}(n/2)-\frac{1}{2(2k+1)}\Big(\sum_{d\mid n}B_{2k+1}(d)+2^{2k}\sum_{d\mid\frac{n}{2}}B_{2k+1}(d)$$
$$-2^{2k}\sum_{\substack{d\mid n\\ d\equiv1(2)}}B_{2k+1}\Big(\frac{d+1}{2}\Big)\Big).$$

Theorem 2. *For each $n, k \in \mathbb{N}$, we have*

$$(i) \quad \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m) = \frac{1}{2}\sigma_{2k+1}(n) - \frac{1}{2}\hat{\sigma}_{2k}(n)$$
$$-\frac{1}{2k+1}\sum_{d|n} B_{2k+1}(d) + \frac{2}{2k+1}\sum_{d|\frac{n}{2}} B_{2k+1}(d),$$

$$(ii) \quad \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \tilde{\sigma}_{2k-2s-1}(m)\tilde{\sigma}_{2s+1}(n-m) = -\frac{1}{2}\tilde{\sigma}_{2k+1}(n) - 2^{2k}\sigma_{2k+1}(n/2)$$
$$-\frac{1}{2}\tilde{\sigma}_{2k}(n) + n\tilde{\sigma}_{2k-1}(n) + 2^{2k}\sigma_{2k-1}(n/2) - \frac{1}{2k+1}\Big(\sum_{d|n} B_{2k+1}(d)$$
$$-2^{2k+1}\sum_{d|\frac{n}{2}} B_{2k+1}(d) - 2^{2k+1}\sum_{\substack{d|n\\ d\equiv 1(2)}} B_{2k+1}\Big(\frac{d+1}{2}\Big)\Big),$$

$$(iii) \quad \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \hat{\sigma}_{2k-2s-1}(m)\tilde{\sigma}_{2s+1}(n-m) = -2^{2k-1}\hat{\sigma}_{2k+1}(n/2) - \frac{1}{4}(\hat{\sigma}_{2k}(n) + \tilde{\sigma}_{2k}(n))$$
$$+\frac{n}{2}\hat{\sigma}_{2k-1}(n) + 2^{2k-1}n\sigma^*_{2k-1}(n/2) - 2^{2k-1}\sigma_{2k-1}(n/2) + \sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}} E_{2k}(2d)$$
$$+\frac{8}{2k+1}\sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}} B_{2k+1}(d) + \frac{2^{2k+1}}{2k+1}\left(\sum_{\substack{d|n\\ \frac{n}{d}\equiv 1(2)}} B_{2k+1}\left(\frac{d+1}{2}\right) - \sum_{\substack{d|\frac{n}{2}\\ \frac{n/2}{d}\equiv 1(2)}} B_{2k+1}\left(\frac{2d+1}{2}\right)\right)$$
$$-\frac{1}{2k+1}\sum_{d|n} B_{2k+1}(d) + \frac{1+2^{2k}}{2k+1}\sum_{d|\frac{n}{2}} B_{2k+1}(d) - \frac{2^{2k}}{2k+1}\sum_{\substack{d|n\\ d\equiv 1(2)}} B_{2k+1}\left(\frac{d+1}{2}\right).$$

Conclusion

The value of

$$\sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m) = \frac{1}{12}(5\sigma_3(n) + (1-6n)\sigma_1(n))$$

convolution sum first appeared in a letter from Besge to Liouville in 1862. The evaluation also appears in the work of Glaisher, Lahiri, Lehmer, Ramanujan and Skoruppa. We obtain Besge's formula as a simple application of Liouville's identity. In 2004, Hahn obtained the results about the convolution sums of $\sum_{d|n}(-1)^{d-1}d$ and $\sum_{d|n}(-1)^{n/d-1}d$. In this talk, we present the results for the binomial convoltion sums, generalized convolution sums of Hahn, of these divisor functions.

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Note on Möbius-Bernoulli numbers

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Abstract

Let k be a non-negative integer. We define the Möbius-Bernoulli numbers which is denoted by $M_k(n)$ and double Möbius-Bernoulli numbers $M_k(n, n')$ for some $n, n' \in \mathbb{N}$. In this article, we find fourmula of $M_k(n, n')$ and applications.

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Introduction

The Bernoulli polynomial $B_n(x)$ is usually defined by means of the following generating functions:

$$\frac{ue^{ux}}{e^u - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{u^k}{k!}.$$

Note that $B_k(x)$ are monic polynomials with rational coefficients and Bernoulli numbers $B_k := B_k(0)$. The Bernoulli numbers $B_k^{(n)}$ of order n are defined by

$$\left(\frac{u}{e^u - 1} \right)^n = \sum_{k=0}^{\infty} B_k^{(n)} \frac{u^k}{k!}. \tag{1}$$

For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $k \geq 0$ the number $M_k(n)$ is defined as follows:

$$\sum_{k=0}^{\infty} M_k(n) \frac{t^k}{k!} = \sum_{d|n} \mu(d) \frac{t}{e^{dt} - 1}, \quad |t| < \frac{2\pi}{n}.$$

The Möbius-Bernoulli numbers $M_k(n)$ are analogue Bernoulli numbers with $M_k(1) = B_k$, where $\mu(n)$ is the Möbius function. Let n' be a positive integer, we investigate the double Möbius-Bernoulli numbers $M_k(n, n')$ given by

$$M_k(n, n') = \sum_{j=0}^{k} \binom{k}{j} M_j(n) M_{k-j}(n'). \tag{2}$$

Lemma 1. *Let k be any non-negative integer, and n be a positive integer. Then*

$$M_k(n) = B_k \prod_{p|n} (1 - p^{k-1}). \tag{3}$$

Here, p are prime numbers with $p|n$.

By Lemma 1 and (2), we get

$$M_k(n,n') = \sum_{j=0}^{k} \binom{k}{j} B_j B_{k-j} \prod_{p|n}(1-p^{j-1}) \prod_{q|n'}(1-q^{k-j-1}). \tag{4}$$

Consider the generating function :

$$\sum_{k=0}^{\infty} M_k(n,n')\frac{t^k}{k!} = (\sum_{k=0}^{\infty} M_k(n)\frac{t^k}{k!}\cdot)(\sum_{k=0}^{\infty} M_k(n')\frac{t^k}{k!}) = \sum_{\substack{d|n \\ d'|n'}} \mu(d)\mu(d')\frac{t^2}{(e^{dt}-1)(e^{d't}-1)}. \tag{5}$$

Note that, by definition,

$$\frac{t^2}{(e^{dt}-1)(e^{d't}-1)} = \sum_{k=0}^{\infty} B_k((d,d'))\frac{t^k}{k!}, \tag{6}$$

where $B_k((d,d'))$ are Bernoulli-Barnes numbers (for the general definition, see (8)). By (5) and (6), we have the following result:

Lemma 2. *Let n and n' be positive integers. Then by (4),*

$$M_k(n,n') = \sum_{\substack{d|n, \\ d'|n'}} \mu(d)\mu(d')B_k((d,d')), \tag{7}$$

where $B_k((d,d'))$ are Bernoulli-Barnes numbers defined by (6).

In particular, if n, n' are relative prime then,

$$\mu(dd') = \mu(d)\mu(d').$$

Lemma 3. *Let n, n' be positive integers and n^*, ${n'}^*$ be their square free parts, respectively. Precisely, let $p_1, \cdots, p_r$ and $q_1, \cdots, q_s$ be the distinct prime factors of n and n', respectively. Then, clearly, $n^* = p_1 \cdots p_r$ and ${n'}^* = q_1 \cdots q_s$. Then we have $M_k(n,n') = M_k(n^*, {n'}^*)$ and $M_k(n) = M_k(n^*)$.*

Similarly, we get

$$\begin{aligned} M_k(p_1^\alpha, p_2^\beta) &= B_k^{(2)} - \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}(p_1^{j-1}+p_2^{k-j-1}) + \sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}p_1^{j-1}p_2^{k-j-1} \\ &= B_k^{(2)} - \frac{1}{p_1}\sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}p_1^{j} - \frac{1}{p_2}\sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}p_2^{k-j} + \frac{p_2^{k-1}}{p_1}\sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}(\frac{p_1}{p_2})^j, \end{aligned}$$

where p_1, p_2 are primes and α, β are positive integers. If $p_1 = p_2 = p$, then

$$M_k(p^\alpha, p^\beta) = (1+p^{k-2})B_k^{(2)} - \frac{2}{p}\sum_{j=0}^{k}\binom{k}{j}B_jB_{k-j}p^j.$$

Hence, to compute $M_k(p_1^\alpha, p_2^\beta)$ explicitly, we need to compute

$$\sum_{j=0}^{k}\lambda^j\binom{k}{j}B_jB_{k-j}$$

with $\lambda \in \mathbb{R}$.

Main Results

The Bernoulli-Barnes numbers $B_k(\mathfrak{a})$, defined for a fixed vector $\mathfrak{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$ through

$$\frac{z^n}{(e^{a_1 z} - 1)\cdots(e^{a_n z} - 1)} = \sum_{k=0}^{\infty} B_k(\mathfrak{a}) \frac{z^k}{k!}. \tag{8}$$

Combining (1) and (8), we get the relation between Bernoulli-Barnes numbers and Bernoulli numbers:

$$B_k(\mathfrak{a}) = \sum_{m_1+\cdots+m_n=k} \binom{k}{m_1, \cdots, m_n} a_1^{m_1-1} \cdots a_n^{m_n-1} B_{m_1} \cdots B_{m_n}, \tag{9}$$

where $\binom{k}{m_1,\cdots,m_n} = \frac{k!}{m_n!\cdots m_n!}$.
In particular, for any prime p, we have

$$B_k((p,1)) = \sum_{j=0}^{k} \binom{k}{j} p^{j-1} B_j B_{k-j}.$$

In the following, we want to compute $B_k((p,1))$. By definition,

$$\frac{t^2}{(e^t - 1)(e^{pt} - 1)} = \sum_{k=0}^{\infty} B_k((p,1)) \frac{t^k}{k!}.$$

Lemma 4. *Let p a prime. Then*

$$\frac{1}{(x-1)(x^p-1)} = \frac{1}{p}\frac{1}{(x-1)^2} - \frac{p-1}{2p}\frac{1}{x-1} + \sum_{j=1}^{p-1} \frac{1}{p(\zeta_p^{-j} - 1)} \cdot \frac{1}{\zeta_p^j x - 1},$$

where $\zeta_p = e^{\frac{2\pi i}{p}} \in \mathbb{C}$.

From Lemma 4, we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} B_k((p,1)) \frac{t^k}{k!} \\
&= \frac{t^2}{(e^t - 1)(e^{tp} - 1)} \\
&= \frac{1}{p}\frac{t^2}{(e^t-1)^2} - \frac{p-1}{2p}\frac{t^2}{e^t - 1} + \sum_{j=1}^{p-1} \frac{1}{p(\zeta_p^{-j} - 1)} \cdot \frac{t^2}{\zeta_p^j e^t - 1} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{p} B_k^{(2)} - \frac{p-1}{2p} k B_{k-1} + \sum_{j=1}^{p-1} \frac{1}{p(\zeta_p^{-j} - 1)} k B_{k-1}((\zeta_p^j)) \right) \frac{t^k}{k!},
\end{aligned} \tag{10}$$

where $B_k((\zeta_p^j))$, so called Apostol-Bernoulli numbers, are exactly defined this way. And we assume B_k and $B_k((\zeta_p^j))$ are both zero if $k < 0$.

Extracting the coefficient of both dide of $\frac{t^k}{k!}$ by (10) yields the following theorem.

Theorem 5. *Notations as above, for $k \geq 0$, we have,*

$$B_k((p,1)) = \frac{1}{p} B_k^{(2)} - \frac{p-1}{2p} k B_{k-1} + \sum_{j=1}^{p-1} \frac{k}{p(\zeta_p^{-j} - 1)} B_{k-1}((\zeta_p^j)).$$

For $\omega \in \mathbb{C} - \{0\}$ and x being a variable, the n-th Apostol-Bernoulli polynomial $B_n(x;\omega)$ is defined by the generating function

$$\sum_{n=0}^{\infty} B_n(x;\omega)\frac{t^n}{n!} = \frac{te^{xt}}{\omega e^t - 1}, \quad (|t + \log(\omega)| < 2\pi),$$

where $\omega = |\omega|e^{i\theta}$, $-\pi \leq \theta < \pi$ and $\log(\omega) = \log|\omega| + i\theta$.

Let n be a positive integer and $\omega \neq 1$. It is well known that

$$B_n(x;\omega) = \sum_{k=0}^{n} \binom{n}{k} B_k(0;\omega)x^{n-k}$$

and where, by definition, the Apostol-Bernoulli numbers $B_k((\omega)) = B_k(0;\omega)$

$$B_k((\omega)) = B_k(0;\omega) = \frac{k}{\omega}\sum_{j=0}^{k-1}(-1)^j j!S(k-1,j)(\frac{\omega}{\omega-1})^{j+1}, \quad \text{for } k \geq 0 \text{ (see [1], [3])}, \tag{11}$$

with $S(k,j)$ being Stirling numbers of the second kind.

It is well known that

$$B_k^{(2)} = -kB_{k-1} - (k-1)B_k, \text{ for } k \geq 0. \tag{12}$$

By Theorem 5, (11) and (12), we get

$$\begin{aligned} pB_k((p,1)) &= -(\frac{p-1}{2} + k)B_{k-1} - (k-1)B_k \\ &+ k(k-1)\sum_{j=1}^{p-1}\frac{1}{1-\zeta_p^j}\sum_{l=0}^{k-2}(-1)^l l!S(k-2,l)\left(\frac{\zeta_p^j}{\zeta_p^j - 1}\right)^{l+1}, \quad \text{for } k \geq 0. \end{aligned}$$

Conclusion

In this section, we want to compute $M_k(n,n')$ with n, n' being positive integers. First, assume $(n,n') = 1$. By Lemma 2, we have

$$M_k(n,n') = \sum_{d|n,d'|n'} \mu(dd')B_k((d,d')).$$

To prove Theorem 8, we need a lemma.

Lemma 6. *Let $(d,d') = 1$. Then we have*

$$\begin{aligned} &\frac{1}{(x^d-1)(x^{d'}-1)} \\ &= \frac{1}{dd'}\frac{1}{(x-1)^2} - \frac{d+d'-2}{2dd'}\frac{1}{(x-1)} \\ &+ \sum_{j=1}^{d-1}\frac{1}{d(\zeta_d^{-jd'}-1)}\cdot\frac{1}{(\zeta_d^j x - 1)} + \sum_{j=1}^{d'-1}\frac{1}{d'(\zeta_{d'}^{-jd}-1)}\cdot\frac{1}{(\zeta_{d'}^j x - 1)}, \end{aligned}$$

where $\zeta_d = e^{\frac{2\pi i}{d}}$ and $\zeta_{d'} = e^{\frac{2\pi i}{d'}}$ are roots of unity.

Remark 3. *In Lemma 6, if $d = p$ and $d' = 1$, then we recover Lemma 4.*

Theorem 7. *Let $n \geq 1$ and $x, y \in \mathbb{R}$. Then we have*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} d^k {d'}^{n-k} B_k(x) B_{n-k}(y) \\
=&(1-n)B_n(dx + d'y) + (d(x - \frac{1}{2}) + d'(y - \frac{1}{2}))nB_{n-1}(dx + d'y) \\
&+ nd' \sum_{j=1}^{d-1} \frac{1}{(\zeta_d^{-jd'} - 1)} B_{n-1}(dx + d'y; \zeta_d^j) + nd \sum_{j=1}^{d'-1} \frac{1}{(\zeta_{d'}^{-dj} - 1)} B_{n-1}(dx + d'y; \zeta_{d'}^j).
\end{aligned}$$

Theorem 8. *Let $(n, n') = 1$. Then we have*

$$M_k(n, n') = \sum_{\substack{d|n \\ d'|n'}} \frac{\mu(dd')}{dd'}(1-k)B_k - \sum_{\substack{d|n \\ d'|n'}} \frac{\mu(dd')}{2}(\frac{1}{d} + \frac{1}{d'})kB_{k-1} + A_1 + A_2,$$

where

$$A_1 = k \sum_{\substack{d|n \\ d'|n'}} \frac{\mu(dd')}{d} \sum_{j=1}^{d-1} \frac{1}{\zeta_d^{-jd'} - 1} B_{k-1}(0; \zeta_d^j),$$

$$A_2 = k \sum_{\substack{d|n \\ d'|n'}} \frac{\mu(dd')}{d'} \sum_{j=1}^{d'-1} \frac{1}{\zeta_{d'}^{-jd} - 1} B_{k-1}(0; \zeta_{d'}^j).$$

Proof. By (6) and (7),

$$M_k(n, n') = \sum_{\substack{d|n \\ d'|n'}} \mu(dd')B_k((d, d')) \tag{13}$$

and

$$\sum_{k=0}^{\infty} B_k((d, d'))\frac{t^k}{k!} = \frac{t^2}{(e^{dt} - 1)(e^{d'} - 1)} = \frac{1}{dd'}(\frac{dt}{e^{dt} - 1})(\frac{d't}{e^{d't} - 1}).$$

By formula (9), we have

$$B_n((d, d')) = \sum_{k=0}^{n} \binom{n}{k} d^k {d'}^{n-k} B_k B_{n-k}.$$

In Theorem 7, putting $x = y = 0$, we get

$$\begin{aligned}
&dd'B_k((d, d')) \\
=&(1-k)B_k - (\frac{d + d'}{2})kB_{k-1} + kd' \sum_{j=1}^{d-1} \frac{1}{(\zeta_d^{-jd'} - 1)} B_{k-1}(0; \zeta_d^j) + kd \sum_{j=1}^{d'-1} \frac{1}{(\zeta_{d'}^{-dj} - 1)} B_{k-1}(0; \zeta_{d'}^j).
\end{aligned} \tag{14}$$

By (13) and (14), we get the theorem. ■

Example 9. *We consider a special case of Theorem 8.*

1. *Case $d = 2$, $d' = 1$:*

$$\begin{aligned}
M_k(2, 1) &= \sum_{d|2} \frac{\mu(d)}{d}(1-k)B_k - \sum_{d|2} \frac{\mu(d)}{2}(1 + \frac{1}{d})kB_{k-1} + k\frac{\mu(2)}{2}(\frac{1}{-2})B_{k-1}(0; -1) \\
&= -\frac{1}{4}kB_{k-1} - \frac{1}{2}(k-1)B_k - \frac{k(k-1)}{8}E_{k-2}(0).
\end{aligned}$$

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Remarks on Special Sums Associated with Hardy Sums

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Abstract

A new special finite sum was defined in [11] as $C(h,k;1)$. This special finite sum is related with other well known sums such as Dedekind sums and Hardy sums. In this paper two of useful identities of this sum will be given which are related with Hardy sum $S(h,k)$.

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Keywords: Special Finite Sums, Hardy Sums, The Mean Value Function, The Sawtooth Function.

Introduction

The special finite sums are related with many areas in mathematics, particularly in analytic number theory. They have many applications which are also used in approximation theory and combinatorics. These sums are mostly depend on the greatest integer function $[x]$, and the sawtooth function $((x))$, where

$$((x)) = \begin{cases} x - [x] - 1/2, & if\ x\ is\ not\ an\ integer, \\ 0, & if\ x\ is\ an\ integer. \end{cases}$$

Thus the mathematicians wanted to find out more results about these functions. Hardy sums are related with these functions. There are eleven Hardy sums, which were defined by the famous mathematician Godfrey Harold Hardy. But mostly six of them studied by other mathematicians. There are many useful resulst and applications about them. In this paper some relations will be given about just for the sum $S(h,k)$. The definition of the Hardy sum $S(h,k)$ which will be helpful in the following sections, is given by the next identity: If h and k are integers with $k > 0$, the Hardy sums are defined by

$$S(h,k) \quad = \quad \sum_{j=1}^{k-1} (-1)^{j+1+[\frac{jh}{k}]},$$

In some sources the Hardy sums are also called as Hardy-Berndt sums because of the American matematician Bruce C. Bernd's contributions to the subject. The next reciprocity theorem's proof was given by Hardy [14] and Berndt [6]:

Theorem 1. *Let h and k be coprime positive integers. Then*

$$S(h,k) + S(k,h) = 1, \quad if \quad h+k \quad is \quad odd. \tag{1}$$

Apostol [2] gave the next identity with the help of the equation (1):

Theorem 2. *If both h and k are odd and $(h,k)=1$, then*

$$S(h,k)=S(k,h)=0.$$

The sum $B_1(h,k)$ is defined by Cetin et al. in [11]. It will be helpful for other sections so its definition is as follows:

$$B_1(h,k)=\sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]}\left[\frac{hj}{k}\right], \tag{2}$$

which $(h,k)=1$ and $k>0$. In [9], basic properties of the sum $B_1(h,k)$, and the relations of this sum with other well known sums was studied. The following theorem, which gives the relationship between the sum $B_1(h,k)$ and Hardy sum $S(h,k)$, was also given in [9].

Theorem 3. *If $h+k$ is odd, $k>0$, and $(h,k)=1$, then*

$$B_1(h,k)=\frac{1}{2}(1-h)S(h,k). \tag{3}$$

Main Results

In this section some properties of the sum $C(h,k;1)$, which is related with Hardy sums, are given. In [11], the sum $C(h,k;1)$ was defined as follows:

$$C(h,k;1)=\sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]}j$$

which $(h,k)=1$ and $k>0$.

Now some relations between the sum $C(h,k;1)$ and the Hardy sum $S(h,k)$ will be given for $h+k$ is an odd positive integer.

Theorem 4. *If $h+k$ is odd, $k>0$, and $(h,k)=1$ then the following equality holds:*

$$B_1(h,k)=\frac{h}{k}C(h,k;1)+\frac{1}{2}S(h,k) \tag{4}$$

Proof. With the help of the definition of the sum $B_1(h,k)$ and the definition of the mean value function

$$B_1(h,k)=\sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]}\left(\frac{hj}{k}-\left(\left(\frac{hj}{k}\right)\right)-\frac{1}{2}\right)$$

The last equation is can be also expressed as

$$B_1(h,k)=\frac{h}{k}\sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]}j-\sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]}\left(\left(\frac{hj}{k}\right)\right)+\frac{1}{2}\sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]+1}$$

so the desired result is obtained. ■

The next theorem gives a direct relationship between the sum $C(h,k;1)$ and the Hardy sum $S(h,k)$. In this case usege of the sum $B_1(h,k)$ is not necessary.

Theorem 5. *If $h+k$ is odd, $k>0$, and $(h,k)=1$ then the following equality holds:*

$$C(h,k;1) = -\frac{k}{2}S(h,k)$$

Proof. From the previous theorem, it is known that the equation 4 holds. If the equation 3 is written down into the equation 4, then the desired result is obtained. ■

Conclusion

In this paper, the connection between the sum $C(h,k;1)$ and the Hardy sum $S(h,k)$ is given. This new $C(h,k;1)$ sum has also relations with the Dedekind sums and other Hardy sums. Moreover it has a connection with Fibonacci numbers. So it functions as a bridge bewteen analysis and the number theory. These relations will be very usefull for the future investigations about the new defined special finite sums.

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Parikh Matrices of Binary Picture Arrays

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Abstract

A word is a finite sequence of symbols. Parikh matrix of a word is an upper triangular matrix with 1 in the main diagonal and non-negative integers above the main diagonal which give the counts of certain scattered subwords in the word. On the other hand a picture array which is a rectangular arrangement of symbols is an extension of the notion of word to two dimensions. Parikh matrices associated with a picture array have been introduced and studied. Here we obtain certain properties of Parikh matrices of a binary picture array based on the notions of power and fairness of an array in terms of subwords, extending the corresponding notions studied in the case of words.

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Keywords: Words, Subwords, Parikh matrix, Picture Array

Introduction

"Combinatorics on words" [3] is a comparatively new branch of Discrete Mathematics with applications in many fields. A finite word or simply a word is a finite sequence of symbols in a finite set called an alphabet. The recently introduced notion of Parikh matrix [5] of a word over an ordered alphabet is an extension of the Parikh vector which has played a significant role in the theory of formal languages [6] and is based on subwords (also called scattered subwords) of the word. Parikh matrix is a very interesting and effective tool in the study of certain numerical properties of a word. Intensive work has taken place investigating properties of words based on associated Parikh matrices. On the other hand, a picture array or simply an array, having a rectangular arrangement of symbols in rows and columns, is an extension of a word to two-dimensions [6]. Recently, the notion of Parikh matrix of a word has been extended to row and column Parikh matrices of picture arrays in [7] and their properties have been studied. Here we obtain certain properties of the Parikh matrices of power of an array and also study fairness of an array in terms of subwords, extending the corresponding notions investigated in the case of words.

For notions of formal string language theory and two-dimensional languages, not explained here, the reader is referred to [6]. We recall only some basic notions. A set Σ, called an alphabet, is a finite set of symbols. A word w over Σ is a finite sequence of symbols over Σ. The set of all words over Σ is denoted by Σ^* and λ is the empty word with no symbols. An alphabet $\Sigma = \{a_1, a_2, \cdots, a_k\}$ with an order $a_1 < a_2 < \cdots < a_k$ defined on it, is called an ordered alphabet and we write $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$. A word u is said to be a scattered subword or simply called a subword, of a word $w \in \Sigma^*$ if there exist words $x_1, x_2, \cdots, x_n, y_0, y_1, \cdots, y_n \in \Sigma^*$ (possibly empty) such that $u = x_1x_2\cdots x_n$ and $w = y_0x_1y_1\cdots y_{n-1}x_ny_n$. The length of a word $w \in \Sigma^*$, denoted by $|w|$, is the number of symbols present in w. The number of occurrences of a word u as a subword of w is denoted by $|w|_u$. A picture array (or simply an array) A over Σ of size $m \times n$, $m, n \geq 1$ is a rectangular arrangement of symbols in Σ in m rows and n columns. For example, $\begin{matrix} a & b & a \\ b & a & b \end{matrix}$ is a 2×3 binary array over the binary alphabet $\Sigma = \{a < b\}$. We denote the set of all $m \times n$ arrays over Σ as $\Sigma^{m \times n}$.

Throughout the rest of the paper we consider only a binary ordered alphabet Σ with two symbols and binary arrays over Σ unless specified otherwise. We now recall the definition of Parikh matrix mapping [5] restricting it to a binary alphabet. Let $\mathcal{M}_3$ be the monoid of 3×3 upper triangular matrices with non-negative integer entries and unit diagonal with respect to multiplication of matrices. For a matrix $M \in \mathcal{M}_3$, the $(i, j)th$ entry is denoted by M_{ij}.

Definition 1. *[5] Let $\Sigma_2 = \{a_1 < a_2\}$ be an ordered alphabet. The Parikh matrix mapping, denoted by ψ_3, is the morphism: $\psi_3 : \Sigma_2^* \longrightarrow \mathcal{M}_3$ defined as follows: $\psi_3(\lambda) = I_3$ and for $1 \leqslant k \leqslant 2$, $\psi_3(a_k) = (m_{ij})_{1 \leqslant i,j \leqslant 3}$ where $m_{ii} = 1$ for $1 \leq i \leq 3$, $m_{k(k+1)} = 1$ and all other entries are zero.*

For a word $w = w_1 w_2 \cdots w_n$ with $w_i \in \Sigma_2$, the Parikh matrix of w is obtained by $\psi_3(w) = \psi_3(w_1)\psi_3(w_2) \cdots \psi_3(w_n)$.
If $M_1, M_2 \in \mathcal{M}_3$ are two matrices then the partial sum $M = M_1 \oplus M_2$ is defined [4] as the usual sum of matrices M_1 and M_2 except that the diagonal entries of M by definition have the value 1.
The notion of Parikh matrix of a word has been extended to a picture array by Subramanian et al. [7] by introducing row Parikh matrix and column Parikh matrix of an array, which we recall now again restricting to a binary alphabet.

Definition 2. *Let $\Sigma = \{a_1 < a_2\}$ and the array $A \in \Sigma^{m \times n}$. Let the words in the m rows of A be x_i, $1 \leq i \leq m$ and the vertical words in the n columns of A be y_j, $1 \leq j \leq n$. Let the Parikh matrices of x_i and y_j be respectively $M(x_i)$, $1 \leq i \leq m$ and $M(y_j)$, $1 \leq j \leq n$. Then the row Parikh matrix $M_r(A)$ of A is defined as $M_r(A) = M(x_1) \oplus \cdots \oplus M(x_m)$ and the column Parikh matrix $M_c(A)$ of A is defined as $M_c(A) = M(y_1) \oplus \cdots \oplus M(y_n)$.*

We give an example. Consider the array $A = \begin{matrix} a & b & a \\ b & a & b \end{matrix}$. Denoting the words in the first and second rows as u and v respectively, the row Parikh matrix of A is $M(u) \oplus M(v) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

Main Results

We first obtain a property of the row and column matrices of a binary picture array which is analogous to a corresponding property [4] of the Parikh matrix of a binary word.

Theorem 3. *For $m, n(\geq 1) \in \mathbb{N}$, suppose $M = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_3$. Then*

1. *$r + s = mn$ and $t \leq nr - \sum\limits_{i=1}^{m} r_i^2$, where r_i $(1 \leq i \leq m)$ is the number of a's in the i^{th} row of an $m \times n$ binary array A, if M is the row Parikh matrix of A.*

2. $r + s = mn$ *and* $t \leq mr - \sum_{i=1}^{n} c_i^2$*, where* c_i $(1 \leq i \leq n)$ *is the number of* a*'s in the* i^{th} *column of an* $m \times n$ *binary array* A*, if* M *is the column Parikh matrix of* A*..*

Proof. We prove the first statement and the second statement can be proved in a similar manner. Suppose M be the row Parikh matrix of an $m \times n$ binary array A. Then A has mn symbols, r a's and s b's, we have $r + s = mn$. Let r_i be the number of a's in the i^{th} $(1 \leq i \leq m)$ row of A. Then $\sum_{i=1}^{m} r_i = r$, and the number of b's in the i^{th} row is $(n - r_i)$. Therefore the maximum number of ab's in the i^{th} row is $r_i(n - r_i)$. Thus the maximum number of ab's in the row Parikh matrix of A is $\sum_{i=1}^{m} r_i(n - r_i)$ so that $t \leq nr - \sum_{i=1}^{m} r_i^2$. ∎

Corollary 4. *Let* M *be as in Theorem 3. If* M *is the row (respy. column) Parikh matrix of an* $m \times n$ *array, then* $r + s = mn$ *and* $t \leq nr - \frac{r^2}{m}$ *(respy.* $t \leq mr - \frac{r^2}{n}$*).*

This result follows from Theorem 4 using the Cauchy Schwarz inequality ($\sum_{i=1}^{m} r_i^2 \geq \frac{(\sum_{i=1}^{m} r_i)^2}{m}$

Parikh matrix of a word raised to an arbitrary power has been studied in [1]. Here we consider power of an array.

Definition 5. *Let* A *be an* $m \times n$ *array. Then* $p \times q$ *power of* A*, denoted by* $A^{p\times q}$*, is the* $pm \times qn$ *picture array such that* $A_{ij}^{p\times q} = A_{(i \text{ mod } p)(j \text{ mod } q)}$*.*

Theorem 6. *Let* $M = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ *be the row Parikh matrix of a binary* $m \times n$ *array* A *over* $\{a < b\}$*. Then the row Parikh matrix of the power* $A^{p\times q}$ *is given by* $\begin{pmatrix} 1 & pqr & \alpha \\ 0 & 1 & pqs \\ 0 & 0 & 1 \end{pmatrix}$*, where* r_i *and* s_i *denote respectively the number of a's and b's in the* i^{th} *row of* A *and* $\alpha = qpt + \frac{qp(p-1)}{2} \sum_{i=1}^{m} r_i \cdot s_i$

Proof. We have $A^{p\times q} = (A^{p\times 1})^{1\times q}$.

Now $A^{p\times 1}$ is the vertical concatenation of A p times. Let r_i, s_i and t_i denote the number of a's, b's and ab's in the i^{th} row x_i, $(1 \leq i \leq m)$ of A. Then The i^{th} row of $A^{p\times 1}$ is x_i^p.

Using the formula in ([1], Theorem 3.1), the Parikh matrix of x_i^p is given by $\psi_3(x_i^p) = \begin{pmatrix} 1 & pr_i & pt_i + \frac{p(p-1)}{2} r_i \cdot s_i \\ 0 & 1 & ps_i \\ 0 & 0 & 1 \end{pmatrix}$ Therefore the row Parikh matrix of $A^{p\times 1}$ is $= \begin{pmatrix} 1 & pr & pt + \frac{p(p-1)}{2} \sum_{i=1}^{m} r_i \cdot s_i \\ 0 & 1 & ps \\ 0 & 0 & 1 \end{pmatrix}$. Using the fact that $A^{p\times q}$ is the horizontal concatenation of $A^{p\times 1}$ q times, the required result can be proved. ∎

Fair words and their properties have been studied in [2]. We extend the the notion of fair words to two dimensional arrays as follows.

Definition 7. *An $m \times n$ binary array $A \in \Sigma$ is said to be fair if the total number of subword ab equals the total number of subword ba in the rows (respy. columns) in A.*

A weak ratio property for an array is introduced in [7]. We recall this notion restricting it to binary arrays.

Definition 8. *Let A and B be two $m \times n$ binary arrays over $\Sigma = \{a < b\}$. Denoting the number of occurrences of a symbol x in an array X by $|X|_x$, the arrays A and B are said to satisfy a weak ratio property if $\frac{|A|_a}{|B|_a} = \frac{|A|_b}{|B|_b} = k$ where k, is non-zero constant.*

Theorem 9. *For any two fair $m \times n$ binary arraysA and B over $\Sigma = \{a < b\}$ having weak ratio property, the product arrays [6] AB and BA are also fair.*

We omit a formal proof due to space restrictions but only note that the weak ratio property ensures fairness of the product of the corresponding words in the product arrays. Other properties Parikh matrices of arrays such as restricted shuffle of arrays, geometric operations on arrays, will be considered in an extended version of this work.

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Vietoris' number sequence and its generalizations through hypercomplex function theory

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Abstract

The so-called Vietoris' number sequence is a sequence of rational numbers that appeared for the first time in a celebrated theorem by Vietoris (1958) about the positivity of certain trigonometric sums with important applications in harmonic analysis (Askey/Steinig, 1974) and in the theory of stable holomorphic functions (Ruscheweyh/ Salinas, 2004). In the context of hypercomplex function theory those numbers appear as coefficients of special homogeneous polynomials in $\mathbb{R}^3$ whose generalization to an arbitrary dimension n lead to a n-parameter generalized Vietoris' number sequence that characterizes hypercomplex Appell polynomials in $\mathbb{R}^n$.

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Introduction

The Vietoris' number sequence $\mathcal{S}$ is the following sequence of rational numbers

$$1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{5}{16},\ \tfrac{35}{128},\ \tfrac{35}{128},\ \tfrac{63}{256},\ \tfrac{63}{256},\ \tfrac{231}{1024},\ \tfrac{231}{1024},\ldots. \tag{1}$$

which by means of the *generalized central binomial coefficient* $\binom{k}{\lfloor \frac{k}{2}\rfloor}$ can be written in compact form (cf. [3]) as $\mathcal{S}=(c_k)_{k\geq 0}$, where

$$c_k = \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2}\rfloor} = \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2}\rfloor}}{(1)_{\lfloor \frac{k+1}{2}\rfloor}}. \tag{2}$$

Here, as usual, $\lfloor\cdot\rfloor$ denotes the floor function and $(\cdot)_k$ is the raising factorial in the classical form of the Pochhammer symbol.

Seemingly this sequence appeared, for the first time, in the context of positive trigonometric sums in a celebrated paper of L. Vietoris [11]. Askey's version [2, p. 5] of Vietoris' theorem is the following:

Theorem 1 (L. Vietoris)**.**

$$\sum_{k=1}^{n} a_k \sin k\theta > 0,\quad 0<\theta<\pi,\quad \textit{and}\quad \sum_{k=0}^{n} a_k \cos k\theta > 0,\quad 0\leq\theta<\pi,$$

where

$$a_{2k}=a_{2k+1}=\frac{(\frac{1}{2})_k}{k!},\qquad k=0,1,\ldots. \tag{3}$$

We call attention to the fact that because of (3), the coefficients in the *sine* sum are exactly the elements of $\mathcal{S}$ in (2) or, explicitly, in (1). Compared with the traditional way of defining the coefficient sequence by (3), the use of the properties of the generalized central binomial coefficient allows a unique representation (2) with consecutively running index k.

In the context of hypercomplex function theory, the sequence $\mathcal{S}$ characterizes a special homogeneous polynomial sequence that can be considered in higher dimensions as the counterpart of the sequence of holomorphic powers.

In the sequel we will use the following basic concepts and notations. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ endowed with a non-commutative product according to the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij},\ i, j = 1, 2, \ldots, n,$$

where δ_{ij} is the Kronecker symbol. This generates the associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$, whose elements are of the form $a = \sum_A a_A e_A,\ a_A \in \mathbb{R}$, with $A \subseteq \{1, \cdots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$ and $e_\emptyset = e_0 = 1$. In general, the vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by identifying the element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with the element (paravector)

$$x = x_0 + \sum_{k=1}^{n} e_k x_k = x_0 + \underline{x} \in \mathcal{A}_n := \operatorname{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}.$$

Its conjugate is $\bar{x} = x_0 - \underline{x}$ and the norm of x is given by $|x| = (x\bar{x})^{1/2} = (\bar{x}x)^{1/2} = \left(\sum_{k=0}^{n} x_k^2\right)^{1/2}$.

We consider $\mathcal{C}\ell_{0,n}$-valued functions defined as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A}_n \longmapsto \mathcal{C}\ell_{0,n}$$

such that $f(x) = \sum_A f_A(x) e_A,\ f_A(x) \in \mathbb{R}$ and Ω is an open subset of $\mathbb{R}^{n+1}$, $n \geq 1$.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is $\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$, with $\partial_0 := \frac{\partial}{\partial x_0}$, and $\partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$. A C^1-function f is called *left (right) monogenic*, or simply *monogenic* in $\mathbb{R}^{n+1}$ if it is a solution of the differential equation $\overline{\partial} f = 0$ $(f\overline{\partial} = 0)$.

Notice that the operator $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ is the conjugate generalized Cauchy-Riemann operator and acts as derivative of a monogenic function (cf.[9]). Therefore the hypercomplex derivative of a monogenic function f can be calculated as $\partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}}) f = \partial_0 f$, i.e. in the same way as the complex derivative of a holomorphic function.

Main Results

In the center of our attention is the sequence of paravector-valued monogenic polynomials $(\mathcal{P}_k^n)_{n\in\mathbb{N}}$ such that

$$\partial \mathcal{P}_k^n(x) = k\, \mathcal{P}_{k-1}^n(x), \qquad x \in \mathcal{A}_n, \qquad k = 1, 2, \ldots. \tag{4}$$

Choosing as initial value $\mathcal{P}_0^n = 1$, the recurrence (4) together with the requirement of monogeinity,

$$\overline{\partial} \mathcal{P}_k^n(x) = 0, \qquad x \in \mathcal{A}_n,$$

lead to the explicit representation

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \qquad x \in \mathcal{A}_n, \tag{5}$$

where

$$c_s(n) := \frac{\left(\frac{1}{2}\right)_{\lfloor\frac{k+1}{2}\rfloor}}{\left(\frac{n}{2}\right)_{\lfloor\frac{k+1}{2}\rfloor}},\ s = 0, \ldots, k,\ k = 1, 2, \ldots. \tag{6}$$

See [4, 7, 8] for details.

The equality (4) for monogenic homogeneous polynomials generalizes to higher dimensions the classical concept of Appell polynomials (cf.[1]). We remark that the initial value of a Clifford algebra-valued Appell polynomial sequence can be a real, a Clifford number or a vector-valued monogenic polynomial of a fixed degree (a *monogenic constant*) (see [6, 10]).

Taking into account that for $n = 2$, (6) gives exactly the rational numbers (2) that constitute the Vietoris' number sequence $\mathcal{S}$, the coefficients sequence $(\mathcal{S}(n))_{n\in\mathbb{N}}$ characterizing the Appell polynomials (5) and whose general term is given by (6) is a n-parameter generalization of $\mathcal{S}$.

Moreover, the representation (6) in terms of quotients of numbers represented by the Pochhammer symbol suggests the use of the well known Gauss' hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k\,(b)_k}{(c)_k}\frac{z^k}{k!},\quad |z| < 1,\quad a, b \in \mathbb{C},\quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

to derive a generating function of the generalized Vietoris' sequence $(\mathcal{S}(n))_{n\in\mathbb{N}}$. In fact, in [5] the following result was obtained.

Theorem 2. *Let $G(.,n)$ be the following real-valued function depending on a parameter $n \in \mathbb{N}$:*

$$G(t;n) = \begin{cases} \frac{1}{t}\left[(1+t)\,{}_2F_1(\frac{1}{2}, 1; \frac{n}{2}; t^2) - 1\right], & \text{if } t \in]-1, 0[\cup]0, 1[\\ 1, & \text{if } t = 0. \end{cases}$$

Then, for any fixed $n \in \mathbb{N}$, $G(.,n)$ is a one-parameter generating function of the sequence $\mathcal{S}(n)$.

It is clear that we can obtain a closed formula for the generating function of the sequence $(\mathcal{S}(n))_{n\in\mathbb{N}}$ as long as a closed formula for the corresponding hypergeometric series is known. As examples we list some cases where closed formulae can be easily obtained:

1. $n = 1$

   In this case, $c_k(1) = 1$ $(k \geq 0)$ and the corresponding generating function is given by

   $$G(t;1) = \tfrac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2}, 1; \tfrac{1}{2}; t^2) - 1\right] = \tfrac{1}{1-t},$$

   because ${}_2F_1(\frac{1}{2}, 1; \frac{1}{2}; t^2)$ reduces to the geometric function.

2. $n = 2$

   Recalling (2), we have $c_{2k}(2) = c_{2k-1}(2) = \frac{\left(\frac{1}{2}\right)_k}{k!}$ and

   $$G(t;2) = \tfrac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2}, 1; 1; t^2) - 1\right] = \tfrac{\sqrt{1+t}-\sqrt{1-t}}{t\sqrt{1-t}}.$$

3. $n = 3$

   The generalized Vietoris' numbers are $c_{2k}(3) = c_{2k-1}(3) = \frac{1}{2k+1}$ and the corresponding generating function is given by

   $$G(t;3) = \tfrac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2}, 1; \tfrac{3}{2}; t^2) - 1\right] = \tfrac{1}{t}\left(\tfrac{t+1}{t}\ln\sqrt{\tfrac{1+t}{1-t}} - 1\right).$$

4. $n = 4$

   In this case, $c_{2k}(4) = c_{2k-1}(4) = \frac{\left(\frac{1}{2}\right)_k}{(k+1)!}$ and

$$G(t;4) = \frac{1}{t}\left[(1+t)\,{}_2F_1(\tfrac{1}{2},1;2;t^2) - 1\right] = \frac{2t+1-\sqrt{1-t^2}}{t(1+\sqrt{1-t^2})}.$$

Conclusion

By providing a link between the Vietoris' number sequence and hypercomplex Appell polynomials, we were able to define one-parameter generalized Vietoris' number sequences and obtain their generating functions.

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On the construction of fuzzy topology induced by a fuzzy metric

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Abstract

In this study we establish a way to construct a stratified fuzzy topology from a fuzzy metric by means of some particular level topologies. Then we compare their induced fuzzy topology with the one induced by the Lowen functor w.

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Introduction

One of the basic branches of fuzzy theory is fuzzy topology which has grow into an area of active research in recent years owing to the wide range of applications. After Chang [1] introduced the fuzzy topological space, the concepts in the general topology began to move into fuzzy topological space. In Chang's topology, a fuzzy topology itself was a crisp subset of the family of all fuzzy subsets of X. Many researchers have worked to generalize the theory of general topology to the fuzzy setting with crisp methods. In 1976, Lowen [5] introduced a more natural definition of fuzzy topology in order to obtain generalized version of some topological concepts such as continuity and compactness. Lowen's definition is as follows:

Definition 1. *[5] An stratified fuzzy topological space is an ordered pair $(X, \mathcal{T})$ such that X be a set and $\mathcal{T} \subset I^X$ satisfy the following condition*

LT1 *$\mathcal{T}$ contains all constant fuzzy sets in X*

LT2 *If $\lambda_1, \lambda_2 \in \mathcal{T}$, then $\lambda_1 \wedge \lambda_2 \in \mathcal{T}$*

LT3 *If $\lambda_i \in \mathcal{T}$ for all $i \in I$ then $\bigvee_{i\in I} \lambda_i \in \mathcal{T}$.*

After the introduction of the concept of a fuzzy metric by Kramosil and Michalek [4], topological structure of fuzzy metric space attracted the attention of many researchers [3, 7, 10]. Definition of fuzzy metric space as follows:

Definition 2. *[4] A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty)set, $*$ is a continuous t-norm and $M : X \times X \times [0, \infty) \to [0, 1]$ is a map satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:*

(F1) $M(x, y, 0) = 0$

(F2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$

(F3) $M(x, y, t) = M(y, x, t)$

(F4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ *for all* $t, s > 0$

(F5) $M(x,y,\cdot):[0,\infty)\to[0,1]$ *is left continuous.*

In most paper the topology induced by a fuzzy metric was crisp topology underlying set. However, recently few researchers [3, 7, 8, 13] have addressed the problem of construction of fuzzy-type topological structures induced fuzzy metric. In general, most study consist of to determine a fuzzifying topology.

Main Result

Construction of fuzzy topology

In this section we intend to construct a stratified fuzzy topology with the help of level topologies. For this we consider the family $B_\alpha=\{B(x,r,t):x\in X, r>\alpha, t>0\}$. In the following lemma we show that B_α is a base of a topology T_α.

Lemma 3. *Let* $(X,M,\wedge)$ *be a fuzzy metric space and* $\alpha\in[0,1)$. *Then the family* $B_\alpha=\{B(x,r,t):x\in X,\ r>\alpha,\ t>0\}$ *is a base for topology* T_α.

Proof. Let $z\in B(x,r_1,t)\cap B(y,r_2,s)$ where $r_1,r_2>\alpha$. Then we have $M(x,z,t)>1-r_1$ and $M(y,z,s)>1-r_2$. There exists $t_0<t$ and $s_0<s$ such that $M(x,z,t_0)>1-r_1$ and $M(y,z,s_0)>1-r_2$.
Let $r=\min\{r_1,r_2\}$ and $p=\min\{t-t_0,s-s_0\}$. We consider $B(z,1-r,p)$. We claim that $B(z,1-r,p)\subset B(x,r_1,t)\cap B(y,r_2,s)$. Let $u\in B(z,r,p)\subset B(z,r,t-t_0)$. Then $M(u,z,t-t_0)>1-r$.
Therefore $M(x,u,t)\geq M(x,z,t_0)\wedge M(z,u,t-t_0)>(1-r_1)\wedge(1-r)=1-r_1$. Then $u\in B(x,r_1,t)$ and we have $B(z,r,p)\subset B(x,r_1,t)$.
On the other hand let $u\in B(z,r,p)\subset B(z,r,s-s_0)$. Then $M(u,z,s-s_0)>1-r$. Therefore $M(y,u,s)\geq M(y,z,s_0)\wedge M(z,u,s-s_0)>(1-r_2)\wedge(1-r)=1-r_2$. Then $u\in B(y,r_2,s)$. We have $B(z,r,p)\subset B(y,r_2,s)$. Hence $B(z,r,p)\subset B(x,r_1,t)\cap B(y,r_2,s)$. ■

The topology T_α is chacterized in the following theorem.

Theorem 4. *Let* $(X,M,\wedge)$ *be a fuzzy metric spaces,* $G\subset X$. *Then* $G\in T_\alpha$ *if and only if for each* $x\in G$ *there exists* $t>0$ *and* $r\in(0,1)$ *such that* $B(x,r,t)\subset G$.
Equivalently;

$$G\in T_\alpha \ if and only if \ \ G=\bigcup_{\substack{r>\alpha\\ B(x,r,t)\subset G}} B(x,r,t). \tag{1}$$

and this topology is
$T_\alpha=\{G\subset X\ :\ \forall x\in G\ \exists t>0 \text{ and } r\in(0,1) \text{ such that } B(x,r,t)\subset G\}$.

From the definition of T_α, we have $T_0=T^M$.
Since $B_\alpha\subset B_\beta$ whenever $\beta<\alpha$, we have the following corollary.

Corollary 5. $\{T_\alpha:\alpha\in[0,1)\}$ *is a decreasing family of topologies.*

Theorem 6. *Let* $(X,M,\wedge)$ *be a fuzzy metric spaces and* $\{T_\alpha:\alpha\in[0,1)\}$ *be the family of topologies induced by this metric. Then*

$$\mathcal{T}^M:=\{\lambda:\lambda^\alpha\in T_\alpha \text{ for all } \alpha\in[0,1)\}$$

is the finest stratified fuzzy topology satisfying $\imath_\alpha\left(\mathcal{T}^M\right)=T_\alpha$.

As a consequence we have $\imath\left(\mathcal{T}^M\right) = T^M$.

Definition 7. *[10] Let $(X, M, \wedge)$ be a fuzzy metric space, $x \in X$, $r \in (0,1)$, $t > 0$ and $\beta \in (0,1)$. Then the fuzzy set $\beta B(x,r,t)$ is called β open ball with the center x and radius r and defined by*

$$\beta B(x,r,t)(y) = \begin{cases} \beta, & y \in B(x,r,t) \\ 0, & \text{other} \end{cases}.$$

In the following proposition we show that the collection of β open balls is a base for $w\left(T^M\right)$.

Proposition 8. *Let $(X, M, \wedge)$ be a fuzzy metric space. Then the family $\mathcal{B}_1 = \{\beta B(x,r,t) : x \in X,\ r \in (0,1), t > 0,\ \beta \in (0,1)\}$ is a base for $w\left(T^M\right)$.*

Proof. Obviously, $\mathcal{B}_1 \subset w\left(T^M\right)$. Let $\lambda \in w\left(T^M\right)$ and $\lambda(x) > 0$. Since λ is lower semicontinuous for all $\varepsilon \in (0,1)$ satisfying $\lambda(x) - \varepsilon > 0$ there exists $r \in (0,1)$ and $t > 0$ such that $\lambda(y) \geq \lambda(x) - \varepsilon$ for all $y \in B(x,r,t)$. Choose $\beta = \lambda(x) - \varepsilon$, we get $\beta B(x,r,t) \leq \lambda$. ■

On the other hand if M is co-principle, it can be easily shown that $T_\alpha = T_\beta$ for all $\alpha \neq \beta$. By the Proposition **??** in [11] and Theorem 6, we have the following corollary:

Corollary 9. *The fuzzy topological space $(X, \mathcal{T}^M)$ is topologically generated if M is co-principle, i.e $\mathcal{T}^M = w\left(\imath\left(\mathcal{T}^M\right)\right)$.*

In the next example we show that $\mathcal{T}^M \neq w\left(\imath\left(\mathcal{T}^M\right)\right)$ in general.

Example 10. *Let $f : [0,1) \to \left(\frac{1}{2}, 1\right]$ be a nondecreasing left continuous surjective function. Consider the fuzzy metric space $(X, M, \wedge)$ in [9] M is given by*

$$M(x,y,t) = \begin{cases} 0, & t = 0 \\ f\left(\frac{t}{|x-y|}\right) & x \neq y, t \geq 0 \\ 1 & x = y,\ t \geq 0 \end{cases}.$$

Let $\alpha = \frac{1}{4}$ and $r = \frac{1}{3} > \alpha$. Then $B\left(x, \frac{1}{3}, t\right) = \{x\}$ and $\frac{2}{3}B\left(x, \frac{1}{3}, t\right) \in w\left(\imath\left(\mathcal{T}^M\right)\right)$. On the other hand for $\alpha = \frac{1}{2} < \beta = \frac{2}{3}$ we have $\left[\frac{2}{3}B\left(x, \frac{1}{3}, t\right)\right]^{\frac{1}{2}} = \{x\} \notin T_{\frac{1}{2}}$, $T_{\frac{1}{2}}$ is trivial topology since $B(x,r,t) = X$ for $r > \frac{1}{2}$. It follows that $\frac{2}{3}B\left(x, \frac{1}{3}, t\right) \notin \mathcal{T}^M$.

On the other hand, by considering a relation between β and r, we can construct a base for the fuzzy topology $\mathcal{T}^M$ in the following proposition:

Proposition 11. *Let $(X, M, \wedge)$ be a fuzzy metric space. Then the family $\mathcal{B}_2 = \{\beta B(x,r,t) : x \in X,\ r \in (0,1),\ t > 0,\ \beta \in (0,r)\}$ is a base for $\mathcal{T}^M$.*

Proof. First we show that $\mathcal{B}_2 \subset \mathcal{T}^M$. If $\alpha < \beta$ then $[\beta B(x,r,t)]^\alpha = B(x,r,t)$. Since $r > \beta$ we have $r > \alpha$. It follows that $B(x,r,t) \in \mathcal{B}_2 \subset T_\alpha$. Hence $\beta B(x,r,t) \in \mathcal{T}^M$.
Let $\lambda \in \mathcal{T}^M$ and $\lambda(x) > 0$. Then $[\lambda]^\alpha \in T_\alpha$ and $x \in [\lambda]^\alpha$ for all $\alpha \in [0,1)$ satisfy $\lambda(x) > \alpha > 0$. By the definition of T_α there exists $r > \alpha$ such that $B(x,r,t) \subset [\lambda]^\alpha$. That is $\lambda(y) > \alpha$ for each $y \in B(x,r,t)$. It follows that $\alpha B(x,r,t) \leq \lambda$. ■

Conclusion

In this study we focused on investigate of the relationship between fuzzy metric spaces and Lowen-type fuzzy topology. We present a construction to determine a fuzzy topology by the help of an ordered family of induced topologies. We have to restrict to consider only fuzzy metric in order to construct the fuzzy topology, since there are some difficulties as pointed out in [7]. We compare the induced fuzzy topology with the fuzzy topology induced by the Lowen functor w and show that they are different for non-coprinciple fuzzy metric spaces.

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Multi Hypergroups

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Abstract

In this paper, we first define the notion of multi hypergroups over a canonical hypergroup and investigate some of their properties.

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Keywords: multigroup, hypergroup, multi hypergroup

Introduction

Algebraic hyperstructures are a generalization of classical algebraic structure. As a generalization of algebraic structures, hyper structure was introduced by Marty [5] in 1934. Since then this theory has enjoyed a rapid development. Also theory of multisets is an important generalization of classical set theory. Many studies have investigated the theory of multisets. So, it may be interesting to study multi hypergroups.
In this presentation, we first define the notion of multi hypergroups over a canonical hypergroup and investigate some of their properties. In this section, we first give some fundamental definitions and results from literature. For more details, we refer to the references quoted in [2, 3].

Let H be a nonempty set and let $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A hyperoperation on H is a map $* : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, *)$ is called a hypergroupoid. If A and B are nonempty subsets of H, then we denote
$A * B = \cup\{a * b \mid a \in A, b \in B\}$,
$A * x = A * \{x\}$,
$x * A = \{x\} * A$.
A hypergroupoid $(H, \circ)$ is called semihypergroup if for all a, b, c of H we have $a \circ (b \circ c) = (a \circ b) \circ c$.
A hypergroupoid $(H, \circ)$ is called quasihypergroup if for all a of H we have $a \circ H = H \circ a = H$.
A hypergroup is a hypergroupoid which is both a semihypergroup and a quasihypergroup.

A canonical hypergroup is a nonempty set H endowed with an additive hyperoperation $\circ$, satisfying the following axioms:

i) for every $x, y, z \in H,\ x \circ (y \circ z) = (x \circ y) \circ z$,

ii) for every $x, y \in H,\ x \circ y = y \circ x$,

iii) there exists $0 \in H$ such that $0 \circ x = \{x\}$ for all $x \in H$,

iv) for every $x \in H$ there exists a unique element $x' \in H$ such that $0 \in x \circ x'$,
(We shall write x^{-1} for x' and we call it the opposite of x.)

v) $z \in x \circ y$ implies $y \in x^{-1} \circ z$ and $x \in z \circ y^{-1}$.

For the sake of simplicity of notations we write $0 \circ x = x$ instead of $0 \circ x = \{x\}$. It can be easily proved that 0 is unique. For $A \subseteq H$, the set $\{a^{-1} \mid a \in A\}$ is denoted by A^{-1}.

A nonempty subset K of a hypergroup $(H, \circ)$ is called a subhypergroup of H if $K \circ K \subseteq K$ and K is a hypergroup under the hyperoperation $\circ$. In other words, it is a hypergroup according to the hyperoperation on H. K provides the following conditions:

1) $a \circ b \subseteq K$ for all $a, b \in K$.
2) $a \circ K = K \circ a = K$, for all $a \in K$.

A subhypergroup of a hypergroup $(H, \circ)$ is called a canonical subhypergroup of H if it is a canonical hypergroup with respect to the hyperoperation $\circ$ of H.

Definition 1.1 [1] A multiset M drawn from the set X is represented by a Count function C_M defined as $C_M : X \longrightarrow N$ where N represents the set of non negative integers.

Definition 1.2 [6] Let $\{M_i | i \in I\}$ be a nonempty family of msets drawn from the set X. Then

Their intersection, denoted by $\bigcap\limits_{i \in I} M_i$ where

$C_{\bigcap\limits_{i \in I} M_i}(x) = \bigwedge\limits_{i \in I} C_{M_i}(x), \forall x \in X$

Their union, denoted by $\bigcup\limits_{i \in I} M_i$ where

$C_{\bigcup\limits_{i \in I} M_i}(x) = \bigvee\limits_{i \in I} C_{M_i}(x), \forall x \in X.$

Definition 1.3 [6] Let X be a group. A multiset G over X is said to be a multigroup over X if the Count function G or C_G satisfies the following two conditions.

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y), \forall x, y \in X$

(ii) $C_G(x^{-1}) \geq C_G(x), \forall x \in X$.

The set of all multigroups over X is denoted by $MG(X)$.

Main Results

Definition 2.1 Let $(H, \circ)$ be a canonical hypergroup. A multigroup G over H is said to be a multi hypergroup if the Count function G or C_G satisfies the following two conditions.

(i) $\bigwedge\limits_{z \in x \circ y} C_G(z) \geq C_G(x) \wedge C_G(y), \forall x, y \in X$

(ii) $C_G(x^{-1}) \geq C_G(x), \forall x \in X$

The set of all multi hypergroups over H is denoted by $MHG(H)$.

Example 2.2 Let $A(n) = \{e_0, e_1, ..., e_k(n)\}$, where $k(n) = \begin{cases} \frac{n}{2}, & \text{if } 2|n, \\ \frac{n-1}{2} & \text{if } 2 \not| n. \end{cases}$ For all e_s, e_t of $A(n)$, define $e_s \circ e_t = \{e_p, e_v\}$, where $p = min\{s+t, n-(s+t)\}$, $v = |s-t|$. So $(A(n), \circ)$ is a canonical hypergroup.

Let $G = \{e_0, e_0, e_0, e_1, e_1, e_2, e_2\}$ be a multiset over $A(5)$ canonical hypergroup. In this hypergroup,

$e_0 \circ e_0 = \{e_0\},\ e_0 \circ e_1 = \{e_1\},\ e_0 \circ e_2 = \{e_2\}$

$e_1 \circ e_1 = \{e_0, e_2\},\ e_1 \circ e_2 = \{e_1, e_2\},\ e_2 \circ e_2 = \{e_0, e_1\}$

e_0 is identity element of $A(5)$ and $e_0^{-1} = e_0$, $e_1^{-1} = e_1$, $e_2^{-1} = e_2$. So,

$3 = C_G(e_0) \wedge C_G(e_0) \leq \bigwedge_{z \in e_0 \circ e_0} C_G(z) = 3$

$2 = C_G(e_0) \wedge C_G(e_1) \leq \bigwedge_{z \in e_0 \circ e_1} C_G(z) = 2$

$2 = C_G(e_0) \wedge C_G(e_2) \leq \bigwedge_{z \in e_0 \circ e_2} C_G(z) = 2$

$2 = C_G(e_1) \wedge C_G(e_1) \leq \bigwedge_{z \in e_1 \circ e_1} C_G(z) = 2$

$2 = C_G(e_1) \wedge C_G(e_2) \leq \bigwedge_{z \in e_1 \circ e_2} C_G(z) = 2$

$2 = C_G(e_2) \wedge C_G(e_2) \leq \bigwedge_{z \in e_2 \circ e_2} C_G(z) = 2$

Thus G is a multi hypergroup over H.

Theorem 2.3 Let $(H, \circ)$ be a canonical hypergroup and G be a multi hypergroup over H. Then $C_G(e) \geq C_G(x)$, $\forall x, y \in X$.

Proof. Since $C_G(x^{-1}) \geq C_G(x)$ and $C_G(x) \geq C_G(x^{-1})$, then $C_G(x) = C_G(x^{-1})$. Thus

$C_G(x) = C_G(x) \wedge C_G(x^{-1}) \leq \bigwedge_{z \in x \circ x^{-1}} C_G(z) \leq C_G(e)$.

Definition 2.4 Let $(H, \circ)$ be a canonical hypergroup and $G \in MHG(H)$. Then $G_n = \{x \in H | C_G(x) \geq n, n \in N\}$.

Theorem 2.5 If G is a multi hypergroup over H, then G_n is a subhypergroup of H, for all $n \in N$.

Proof. Let $x, y \in G_n$, so $C_G(x) \geq n$ and $C_G(y) \geq n$.

$n \leq C_G(x) \wedge C_G(y) \leq \bigwedge_{z \in x \circ y} C_G(z)$

Since $C_G(z) \geq n$ for all $z \in x \circ y$, then $x \circ y \subseteq G_n \ldots (*)$

Now, for all $x \in G_n$ $x \circ G_n = G_n$ should be shown. It is easily seen $x \circ G_n \subseteq G_n$.

Let $a \in G_n$. Since $x \in G_n$, then $C_G(x) = C_G(x^{-1}) \geq n$. So $x^{-1} \in G_n$ and $x^{-1} \circ a \subseteq G_n$ form $(*)$.Then

$x \circ (x^{-1} \circ a) \subseteq x \circ G_n$

$(x \circ x^{-1}) \circ a \subseteq x \circ G_n$

$a \in x \circ G_n$

$G_n \subseteq x \circ G_n$. Thus $G_n = x \circ G_n$. G_n is a subhypergroup of H.

Theorem 2.6 Let $(H, \circ)$ be a canonical hypergroup. If G_n is a canonical subhypergroup of H, for all $n \in N$, then G is a multi hypergroup over H, define $C_G(x) = \sum_{n \in N} \chi_{G_n}(x)$.

Proof. Let $x, y \in H$ and $x \in G_p$ and $y \in G_q$ so that $x \notin G_{p+n}$ $y \notin G_{q+n}$, for all $n \in N$. Let $min\{p, q\} = p$. So $y \in G_p$. Since G_p is a subhypergroup, then $x \circ y \subseteq G_p$. We obtain $C_G(z) \geq p$. Thus

$\bigwedge_{z \in x \circ y} C_G(z) \geq p = C_G(x) \wedge C_G(y)$

Since G_n is a canonical subhypergroup, $C_G(x^{-1}) \geq C_G(x)$. As a result, G is a multi hypergroup over H.

Theorem 2.7 Let $(H, \circ)$ be a canonical hypergroup. If G_1 and G_2 are multi hypergroups over H, then $G_1 \cap G_2$ is a multi hypergroup over H.

Proof. For all $x, y \in H$,

$$\begin{aligned}
\bigwedge_{z\in x\circ y} C_{G_1\cap G_2}(z) &= \bigwedge_{z\in x\circ y} (C_{G_1}(z)\wedge C_{G_2}(z)) \\
&= (\bigwedge_{z\in x\circ y} C_{G_1}(z))\wedge(\bigwedge_{z\in x\circ y} C_{G_2}(z)) \\
&\geq (C_{G_1}(x)\wedge C_{G_1}(y))\wedge(C_{G_2}(x)\wedge C_{G_2}(y)) \\
&= (C_{G_1}(x)\wedge C_{G_2}(y))\wedge(C_{G_1}(x)\wedge C_{G_2}(y)) \\
&= C_{G_1\cap G_2}(x)\wedge C_{G_1\cap G_2}(y)
\end{aligned}$$

$$\begin{aligned}
C_{G_1\cap G_2}(x^-1) &= C_{G_1}(x^-1)\wedge C_{G_2}(x^-1) \\
&\geq C_{G_1}(x)\wedge C_{G_2}(x) \\
&= C_{G_1\cap G_2}(x)
\end{aligned}$$

Thus $G_1 \cap G_2$ is a multi hypergroup over H.

If $\{G_i|i \in I\}$ be a family of multi hypergroups over a hypergroup H, then their intersection $\bigcap_{i\in I} G_i$ is a multigroup over H but their union is not a multi hypergroup in general.

Example 2.8 $G_1 = \{e_0, e_0, e_0, e_1\}$ and $G_2 = \{e_0, e_0, e_0, e_2\}$ are two multi hypergroup over $A(6)$ canonical hypergroup.

Really, $A(6) = \{e_0, e_1, e_2, e_3\}$ and

$e_0 \circ e_0 = \{e_0\},\ e_0 \circ e_1 = \{e_1\},\ e_0 \circ e_2 = \{e_2\},\ e_0 \circ e_3 = \{e_3\}$

$e_1 \circ e_1 = \{e_0, e_2\},\ e_1 \circ e_2 = \{e_1, e_3\},\ e_1 \circ e_3 = \{e_1\}$

$e_2 \circ e_2 = \{e_0, e_2\},\ e_2 \circ e_3 = \{e_1\},\ e_3 \circ e_3 = \{e_0\}$

Here, $G_1 \cup G_2 = \{e_0, e_0, e_0, e_1, e_2\}$

Since $\bigwedge_{z\in e_1\circ e_2} C_G(z) = 0 \ngeq 1 = C_G(e_1) \wedge C_G(e_2)$. Then $G_1 \cup G_2$ is not a multi hypergroup over $A(6)$.

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New Results on Edge and Vertex Deletion in Graphs

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Abstract

Recently the authors defined a new graph characteristic similar to the well-known Euler characteristic to determine several topological and combinatorial properties of a given graph. This new characteristic is defined only by means of a given degree sequence, This number gives direct information on the realizability, number of realizations, connectedness, being acyclic or cyclic, number of components, chords, loops, pendant edges, faces, bridges etc. In this paper, the effect of the deletion of vertices and edges in graphs on this new characteristic is studied.

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Introduction

Throughout this paper, we assume that $G = (V, E)$ be a graph with $\mid V(G) \mid= n$ vertices and $\mid E(G) \mid= m$ edges. For a vertex v, the degree of v is denoted by by d_v. In particular, a vertex with degree one will be called a pendant vertex. As usual, the biggest vertex degree in a graph will be denoted by Δ. An edge e connecting two neighbouring vertices u and v will be denoted by $e = uv$ and the vertices u and v are called adjacent vertices while the edge e is said to be incident with u and v. If there is a path between every pair of vertices in a graph G, then G is called connected.

A degree sequence D is $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots , \Delta^{(a_\Delta)}\}$, where a_i's are non-negative integers. When the graph has vertices of degree 0, the corresponding graph will certainly be disconnected and the degree sequence will also have zeroes.

Let $D = \{d_1, d_2, d_3, \cdots , \Delta\}$ be a set of non-decreasing non-negative integers. We say that a graph G is a realization of the set D if the degree sequence of G is equal to D. It is well known that there is at least one graph having a given degree sequence. Some realizations of a given degree sequence could be connected and some could be disconnected. A graph having no cycle will be called acyclic. For example, all trees are acyclic. The remaining graphs are called cyclic graphs. An edge connecting a vertex to itself is called a loop, and at least two edges connecting two vertices will be called multiple edges. The number a_1 of leaves of any tree T is given by

$$a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta - a_1 = -2. \qquad (1)$$

The authors tried to determine the conditions which give some information about the topological and combinatorical properties of the given graph and came up with similar sums. Trying to unify those sums resulted in noticing a number which gives more information than expected, see [1]:

Definition 1. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a set which also is the degree sequence of a graph G. The $\Omega(G)$ of the graph G is defined only in terms of the degree sequence as*

$$\Omega(G) = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = \sum_{i=1}^{\Delta}(i-2)a_i. \tag{2}$$

Ω is even by the definition. It is shown, in [1] and [2], that all graphs with $\Omega(G) \leq -4$ are disconnected, and if $\Omega(G) \geq -2$, then the graph could be connected or disconnected. As most of the degree sequences can be realizable in different ways, the number of ways the given degree sequence can be realizable as a connected/disconnected graph is determined.

Also it is shown that if the realization is a connected graph and $\Omega(G) = -2$, then certainly the graph should be acyclic. Similarly, it is shown that if the realization is a connected graph G and $\Omega(G) \geq 0$, then certainly the graph should be cyclic. Also, when $\Omega(G) \leq -4$, the components of the disconnected graph could not all be cyclic and if all the components of G are cyclic, then $\Omega(G) \geq 0$.

In [2], the same authors obtained solutions of some extremal graph theory problems by studying the maximum number of componenets and the maximum number of loops in three types of realizations of a given degree sequence. In [1], the authors gave a new result using Ω to determine the realizability of a degree sequence:

Corollary 2. *Let D be a set of non-negative integers. If $\Omega(D)$ is odd, then D is not realizable.*

In [1], the authors determined the number r for connected graphs:

Theorem 3. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. If D is realizable as a connected planar graph G, then the number r of faces (closed regions) is given by $r = \frac{\Omega(G)}{2} + 1$.*

As Ω is additive over the set of the components of G, they generalized this result to all graphs with c components as $r = \frac{\Omega(G)}{2} + c$. This implies the following useful result, see [1]:

Corollary 4. *For each graph G, we have $c \geq -\frac{\Omega(G)}{2}$. Equivalently, $c \geq n - m$.*

Effect of Vertex and Edge Deletion on Ω

In this section, we study edge-deleted, vertex-deleted, path-contracted and cycle-contracted graphs and calculate the change in Ω for these operations. Given a graph G. Let $u, u_1, u_2, \cdots, u_k$ be some of the vertices of G and let $e, e_1, e_2, \cdots, e_t$ be some of the edges of G. The graph obtained by deleting the vertices $u_1, u_2, \cdots, u_k$ together with all the incident edges to these vertices is denoted by $G - \{u_1, u_2, \cdots, u_k\}$. Similarly, the graph obtained by deleting the edges $e_1, e_2, \cdots, e_t$ will be denoted by $G - \{e_1, e_2, \cdots, e_t\}$. We shall call these operations vertex deletion and edge deletion. When only one vertex v or one edge e is deleted from G, the resulting graph will shortly be denoted by $G - v$ and $G - e$, respectively. Recall that when we delete a pendant edge e from a graph G, the pendant vertex in G at the end of this pendant edge becomes an isolated vertex in $G - e$. Therefore there are many occasions where we face with isolated vertices in a given graph. If these vertices had no effect, then we could easily omit them. When studying with Ω, the contribution of each isolated vertex of G to $\Omega(G)$ is -2:

Lemma 5. *Let $G = \{v\}$ be a graph consisting of one vertex v and no edges. Then*

$$\Omega(G) = \Omega(\{v\}) = -2.$$

Proof. As the degree sequence of $G = \{v\}$ is $\{0^{(1)}\}$, the result follows by the definition of Ω. ■

Recall that we had already shown the equality $\Omega(G) = -2$ for all connected acyclic graphs. In some sense, a graph consisting of a single vertex can be counted as acyclic. So this result is expected.

When the graph has isolated vertices, we need to adjust our definition of Ω as follows. Let G be a graph with degree sequence $DS(G) = \{0^{(a_0)}, 1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. Then the Ω of G can be reformulated by

$$\Omega(G) = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 - 2a_0 = \sum_{i=0}^{\Delta}(i-2)a_i. \tag{3}$$

Let G be a graph and let $u_1, u_2, \cdots, u_n \in V(G)$. We denote $\Omega(G) - \Omega(G - u_1)$ by $\Delta(G, u_1)$ and $\Omega(G) - \Omega(G - \{u_1, u_2, \cdots, u_n\})$ by $\Delta(G, \{u_1, u_2, \cdots, u_n\})$. That is

$$\Delta(G, u_1) = \Omega(G) - \Omega(G - u_1)$$

and

$$\Delta(G, \{u_1, u_2, \cdots, u_n\}) = \Omega(G) - \Omega(G - \{u_1, u_2, \cdots, u_n\}).$$

Then we have

Theorem 6. *Let the graph G have no loops. Deleting a vertex $v \in G$ of degree d_v reduces $\Omega(G)$ by $2d_v - 2$. That is*

$$\Delta(G, v) = 2d_v - 2.$$

Proof. Let the neighbours of v be $v_1, v_2. \cdots, v_k$ where $k = d_v$, see Fig. 1. Let $d_1, d_2, \cdots, d_k$ be the degrees of $v_1, v_2, \cdots, v_k$ in G, respectively. The contribution of v and its neighbours in G to $\Omega(G)$ is

$$d_1 - 2 + d_2 - 2 + \cdots + d_k - 2 + d_v - 2 = d_1 + d_2 + \cdots + d_k + d_v - 2k - 2$$

and the contribution of the neighbours of v in $G - v$ is

$$d_1 - 3 + d_2 - 3 + \cdots + d_k - 3 = d_1 + d_2 + \cdots + d_k - 3k.$$

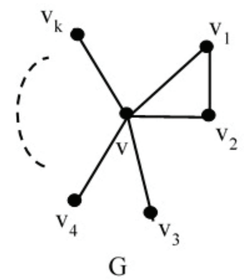

Figure 1 The neighbours of v

So the decrease in $\Omega(G)$ is

$$\begin{aligned}\Omega(G)-\Omega(G-v) = &\ d_v-2k-2+3k\\ = &\ d_v-2d_v-2+3d_v\\ = &\ 2d_v-2.\end{aligned}$$

■

Theorem 7. *Let v be a vertex in a graph G which is incident to $l \geq 1$ loops. Let $d_v = k$ be the degree of v. Then deleting v from G reduces $\Omega(G)$ by $\Delta(G,v) = 2(k-l-1)$.*

Proof. Let G be a graph with loops at a vertex $v \in V(G)$, see Fig. 2.

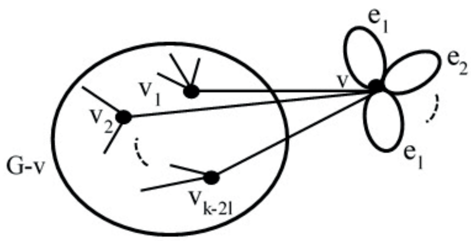

Figure 2 A graph G with l loops at the same vertex

The contribution of v and its neighbours $v_1, v_2, \cdots, v_{k-2l}$ to $\Omega(G)$ is

$$d_{v_1}-2+d_{v_2}-2+\cdots+d_{v_{k-2l}}-2+d_v-2.$$

Now delete v from G, see Fig. 3:

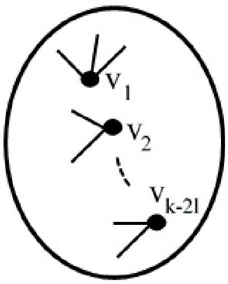

Figure 3 A graph $G-v$

The contribution of the neighbours of v to $\Omega(G-v)$ is

$$d_{v_1}-3+d_{v_2}-3+\cdots+d_{v_{k-2l}}-3$$

as the degree of each neighbour is reduced by 1 in $G-v$. Therefore the decrease in $\Omega(G)$ will be

$$\begin{aligned}\Delta(G,v) = &\ [d_{v_1}+d_{v_2}+\cdots+d_{v_{k-2l}}+k-2(k-2l+1)]\\ &\ -[d_{v_1}+d_{v_2}+\cdots+d_{v_{k-2l}}-3(k-2l)]\\ = &\ 2(k-l-1).\end{aligned}$$

■

Lemma 8. *If $u_1, u_2, \cdots, u_n$ are the vertices of G no two of them are adjacent, then*

$$\Delta(G, \{u_1, u_2, \cdots, u_n\}) = \sum_{i=1}^{n} \Delta(G, u_i).$$

That is, $\Delta(G, \{u\})$ is additive on every set of non-adjacent vertices.

Theorem 9. *Let G be a graph. Deleting an edge reduces $\Omega(G)$ by 2.*

Proof. Let $e = uv$ be an edge and d_u and d_v be the degrees of u and v, respectively. When the edge e is deleted, the numbers of d_u's and d_v's in $DS(G)$ decrease by 1 and the numbers of $d_u - 1$'s and $d_v - 1$'s increase by 1. Therefore the Ω decreases by 2. ∎

As a special case, we have

Theorem 10. *Let G be a graph. Deleting a loop reduces $\Omega(G)$ by 2.*

Proof. Let L be a loop in G with its unique vertex is of degree d. Deleting L reduces the number of d's in $DS(G)$ by 1 and increases $d - 2$'s in $DS(G)$ by 1. As a result, $\Omega(G)$ is reduced by 2. ∎

Theorem 11. *Contracting a path which does not belong to a cycle preserves $\Omega(G)$.*

Proof. Contracting a path which does not belong to a cycle does not change the number $r(G)$ of regions bounded by the edges of the graph G. So $\Omega(G)$ does not change as $\Omega(G) = 2(r(G) - c(G))$ where $c(G)$ is the number of components of G. ∎

The following special case follows directly:

Corollary 12. *Contracting bridges and pendant edges preserves $\Omega(G)$.*

Proof. Clear from Theorem 11 as both a pendant edge and a bridge are paths of length one which does not belong to any cycle. ∎

Note that although $\Omega(G)$ is preserved when the bridges and pendant edges are contracted, the $DS(G)$ does not have to stay unchanged. Because of Theorem 11 and Corollary 12, instead of calculating Ω of any graph G, we can first contract all bridges, paths which do not belong to any cycle, and pendant edges to obtain a smaller graph G' and we could calculate $\Omega(G')$ as $\Omega(G) = \Omega(G')$.

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A Survey on Z Transforms and q-Analysis

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Abstract

The Z-transforms have many applications in physics, engineering and mathematics. In mathematics, these transforms extensively apply to partial differential equations, numerical analysis, difference equations and approximation theory. Also, summation of infinite series easily obtain by using Z- transforms. Recently, some mathematicians derived Z-transforms of notations and functions in q- analysis. In this study, we give some results for q- type functions by aid of Z-transforms.

2010 Mathematics Subject Classifications : $81Q10, 35A22, 39A13$
Keywords: Z-transforms, q-Analysis, Difference equations.

Introduction

Mathematical transforms, Laplace, Fourier and Z-Transforms, appear in many fields of mathematics, especially in ordinary and partial differential equations, in mathematical statistics, in mathematical physics and also in numerical analysis. These transforms are also used in applied sciences such as engineering, computering and physics. For instance, the Fourier transforms are known to be suitable for measuring the resistance and strength of the earthquakes in geophysical engineering. Laplace transforms are used in electrical circuits, current and load calculation, network currents and loads in electrical networks, in signaling problems, harmonic pendulum problems in environments.

Z- transform method may be traced back to A. De Moivre around the year 1730 when he introduced the concept of "generating functions" in probability theory. Closely related to generating functions is the Z-transform, which may be considered as the discrete analogue of the Laplace transform. The Z-transform is widely used in the analysis and design of digital control and signal processing. The Z-transforms have many applications in physics, engineering and mathematics. In mathematics, these transforms extensively apply to partial differential equations, numerical analysis, difference equations and approximation theory. Also, summation of infinite series easily obtain by using Z- transforms. Applications of Z- transforms are important in sampling system theory. It is also common in warning modulation systems and recycling systems where digital computers are one of the common elements.

Z- transform of a sequence $x[n]$ is given by the sum

$$X(z)=\sum_{n=0}^{\infty} x[n]\, z^{-n} \tag{1}$$

for all z such that(1) converges. Here, z is a complex variable and the set of values of z for which the sum (1) converges is called the region of convergence (ROC) of the z-transform. This transform have important properties as follows:

1. Linearity property

$$Z\left[c_1x_n+c_2y_n\right]=c_1X\left(z\right)+c_2Y\left(z\right).$$

   where c_1 and c_2 are constant.

2. Delay shift property

$$Z\left[x\left[n-N\right]u\left[n-N\right]\right]=X\left(z\right)z^{-N}.$$

3. Advance shift property

$$Z\left[x\left[n+N\right]\right]=z^{N}\left(X\left(z\right)-x\left[0\right]-x\left[1\right]z^{-1}-x\left[2\right]z^{-2}-...-x\left[N-1\right]z^{-N+1}.\right)$$

4. Multiplication by n property

$$Z\left[nx_n\right]=-z\frac{d}{dz}X\left(z\right).$$

q- Analysis is an mathematical structure that can be studied and used in many areas such as approximation theory, analytic number theory and numerical analysis. Recently, the Z-transforms are used in q-matrices and q- type mathematical tools. For example, Z-transform of some q-sequences obtained as follows:

$$\begin{bmatrix} q\text{-Sequences} & \left(Z\left|q\right.\right)\text{-Transform} \\ \left[n\right]_q & \frac{z}{(z-1)_q^2} \\ \begin{bmatrix} n \\ k \end{bmatrix}_q & \frac{z}{(z-1)_q^{k+1}} \\ e_q^x & \frac{z}{z-e_q^x} \\ E_q^x & \frac{z}{z-E_q^x} \\ \sin_q\left(x\right) & \frac{z\sin_q(x)}{z^2-2z\cos_q(x)+1} \\ \cos_q\left(x\right) & \frac{z\cos_q(x)}{z^2-2z\cos_q(x)+1} \end{bmatrix}$$

Main Results

In this section, we give some new results with related to q- type functions and series under the Z-transforms as follows:

Corollary 1 Let be $f_1(n)=\frac{1}{[n]_q!}$ and $f_2\left(x\right)=\frac{q^{\frac{n.(n-1)}{2}}}{[n]_q!}$. The Z- transform of $f(n)$ is defined

$$\begin{aligned} Z\left(\frac{1}{\left[n\right]_q!}\right) &= e_q^{-z} \\ Z\left(\frac{q^{\frac{n.(n-1)}{2}}}{\left[n\right]_q!}\right) &= E_q^{-z} \end{aligned}$$

where e_q^{-z} and E_q^{-z}, respectively, are called first and second type q- exponential functions (cf. [2]).

Corollary 2 The Z- transform of $\cosh_q x$ and $COSH_q x$ are defined

$$Z(\cosh_q x) = \frac{z(z-\cosh_q x)}{z^2 - 2z\cosh_q x + 1},$$
$$Z(COSH_q) = \frac{z(z-COSH_q x)}{z^2 - 2zCOSH_q x + 1}$$

where $\cosh_q x = \frac{e_q^x + e_q^{-x}}{2}$and $COSH_q x = \frac{E_q^x + E_q^{-x}}{2}$.

Corollary 3 The Z- transform of $\sinh_q x$ and $SINH_q x$ are defined

$$\sinh_q x = \frac{z\sinh_q x}{z^2 - 2z\cosh_q x + 1}$$
$$SINH_q x = \frac{zS\dot{I}NH_q x}{z^2 - 2z\cosh_q x + 1}$$

where $\sinh_q x = \frac{e_q^x - e_q^{-x}}{2}$and $SINH_q x = \frac{E_q^x - E_q^{-x}}{2}$.

Conclusion

The Z- transforms are used for important applications in electric-electronical engineering, applied mathematics and statistics. Hence we investigate the applications of our new transforms for q- type functions and series. Our new results may apply to many branches of mathematics and the other sciences such as analytic number theory, approximation theory and sampling theory.

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Prediction of modulus of elasticity by using artificial bee colony optimization

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Abstract

In the study, approximate functions have been developed to estimate modulus of elasticity for concrete concerning compressive strength, unit weight, water-cement ratio, consistency, cement amount, fine aggregate coarse aggregate ratio and air content parameters. The developed functions have non-linear form, discrete variables, and the minimum truncation error. In order to determine the functions, curve fitting applications were converted to the optimization problem. The objective function of the optimization problem is the maximization of determination coefficient. Artificial bee colony (ABC) optimization algorithm was utilized to solve the optimization problem. Six different function types were derived using combinations of the proposed parameters. Obtained results were compared to actual data in order to test the performance of the ABC algorithm. Function coefficients were also compared to discuss the efficiency of the proposed parameters in the functions.

Keywords: Concrete, modulus of elasticity, optimization, artificial bee colony

Introduction

The equation models are obtained using a nonlinear regression model. In order to solve the nonlinear regression model, one of the well-known metaheuristic optimization technique called Artificial Bee Colony (ABC) optimization method is used. The ABC method is previously used to solve regression models such as symbolic regression [1], support vector regression system [2, 3], stepwise regression correlation [4].

In the literature, there are few studies available which contain a numerical model for prediction of behaviour of concrete. N. Ahmadi-Nedushan predicted elastic modulus of normal and high strength concrete using adaptive-network-based fuzzy inference system (ANFIS) and optimal nonlinear regression models [5]. Yan and Shi predicted elastic modulus of normal and high strength concrete by support vector machine [6]. Aydin et al. predicted concrete elastic modulus using an adaptive neuro-fuzzy inference system [7]. Topu and Sardemir predicted elastic modulus of waste AAC aggregate concrete using artificial neural network [8]. In these studies, the numerical models depend on few parameters, and the ABC method is not utilized. According to a used method and comprehensive equation model, the current study is evaluated as a novel study.

Mathematical model

In the study, the equation models have been investigated to determine the modulus of elasticity of concrete with respect compressive strength, unit weight, water-cement ratio, consistency, cement amount, fine aggregate course aggregate ratio and air content. The regression models were evaluated as optimization problems to obtain the best equation models, which is described as follows:

Find the most appropriate equation constants $\vec{x} = [x_1, x_2, \ldots, x_8]$ in order to maximize coefficient of determination(R2);

$$R^2(\vec{x}) = \frac{\sum_{i=1}^{n}(E_{(est)i} - E_i)^2}{\sum_{i=1}^{n}(E_{(est)i} - E_{ave})^2} \tag{1}$$

where E is the modulus of elasticity of concrete obtained from experimental tests, E_{ave} is the average value of the modulus of elasticity of concrete obtained from experimental tests, E_{est} is estimated modulus of elasticity the from equation model and n is the specimen number in the experimental tests. In the study, six different equation models are optimized which are described as follows:

$$E(\sigma_B, \gamma) = x_1 \cdot \sigma_B^{x_2} \cdot \gamma^{x_3} \tag{2}$$

$$E(\sigma_B, \gamma, w/c) = x_1 \cdot \sigma_B^{x_2} \cdot \gamma^{x_3} \cdot (w/c)^{x_4} \tag{3}$$

$$E(\sigma_B, \gamma, w/c, S) = x_1 \cdot \sigma_B^{x_2} \cdot \gamma^{x_3} \cdot (w/c)^{x_4} \cdot S^{x_5} \tag{4}$$

$$E(\sigma_B, \gamma, w/c, S, C) = x_1 \cdot \sigma_B^{x_2} \cdot \gamma^{x_3} \cdot (w/c)^{x_4} \cdot S^{x_5} \cdot C^{x_6} \tag{5}$$

$$E(\sigma_B, \gamma, w/c, S, C, A) = x_1 \cdot \sigma_B^{x_2} \cdot \gamma^{x_3} \cdot (w/c)^{x_4} \cdot S^{x_5} \cdot C^{x_6} \cdot A^{x_7} \tag{6}$$

$$E(\sigma_B, \gamma, w/c, S, C, A, AC) = x_1 \cdot \sigma_B^{x_2} \cdot \gamma^{x_3} \cdot (w/c)^{x_4} \cdot S^{x_5} \cdot C^{x_6} \cdot A^{x_7} \cdot AC^{x_8} \tag{7}$$

In the equations, $\sigma_B, \gamma, w/c, S, C, A$ and AC respectively represent compressive strength unit weight, water cement ratio, consistency, cement amount, fine aggregate course aggregate ratio and air content.

In order to find the best equation constants $\vec{x}$, the ABC method is utilized. The ABC method is developed by Karaboga and Basturk [9, 10, 11] adopting the natural behaviour of honey worker bees. In the theory, the bees are classified into three groups which are called as employed bees, onlooker bees and scout bees. Employed bees search neighbourhood of the nest to find nectar sources and share information about the nectar sources with the onlooker bees. The onlooker bees select the best nectar sources and collect nectars from the nectar sources. If the nectar is consumed in the nectar source, the scout bees searches new nectar source instead of the consumed nectar source.

In the optimization problem, the candidate equation model is represented by the nectar source. The location of the nectar source represents the values of the equation constants $\vec{x}$, the quality of the nectar sources represents the R^2 value of the equation. Consuming of the nectar source means that the equation cannot improve its R^2 value until the defined number of iteration (limit of nectar source). According to the definitions the optimization algorithm can be defined as follows;

Step 1: Optimization and problem parameters called number of nectar source (NS), nectar source limit (NLS), number of function constants (NC) and the maximum function evaluations (MFE) are defined.

Step 2: Initial equation models are generated randomly using the following equation.

$$(x_j)_i = round(lb_j + (ub_j - lb_j \cdot rnd); i = 1, 2, \ldots, NS; j = 1, 2, \ldots, NC \tag{8}$$

where; lb_j and ub_j respectively are the lower and the upper boundaries of the equation constants, rnd is a random number generated from interval the [0-1] and round is a function which rounds the result with respect to predefined decimal. Then R^2 values of the equation models are calculated and saved into the algorithm database.

Step 3: The equation models in the algorithm database are modified concerning the following formula;

$$(x_j)_i^{new} = round((x_j)_i + ((x_j)_k - (x_j)_i) \cdot (rnd - 0.5)); \tag{9}$$

Where, k is an index of the equation model which is randomly chosen in the algorithm database. The R^2 values of the new equations are calculated and compared to their previous versions. If the R^2 value of the modified equation is lower than the previous equation, the previous equation remains, and trial number of the equation (nectar source) is increased by one. If the R^2 value of the modified equation is higher than a previous equation, the modified equation substitutes with the previous equation and its trial number become zero. This process is called Greedy Selection.

Step 4: The selection probabilities (Pr) of the equation models are as follows:

$$Pr_i = \frac{R_i^2}{\sum j = 1NSR_j^2} \tag{10}$$

Then the equations models are chosen according to these probabilities. Chosen equations are modified, and greedy selection is applied in the same way described in step 3.

Step 5: trial number of all equations are controlled in this step. If the trial number of the equation exceeds NLS, the equation removed from the algorithm database and the algorithm adds a new equation which is generated randomly. After this process, the algorithm goes back to the step 2. The algorithm repeats by the time that MFS is reached.

Text example

Twenty seven different concrete mixtures are provided from a previous research [12] for finding the equation models. the lower boundaries, the upper boundaries, and increments of the equation constants are shown in Table 1. The initial parameters of the ABC method are given in Table 2.

The equation models were optimized using the developed algorithm. The optimum equation constants and R^2 values of the models are illustrated in Table 3.

Conclusion

The algorithm established is effective because R^2 values are above 0.95. The scale factor (x_1) value of the equations varies between 3500-4500. When the distribution of this value is examined, it is not possible to associate with the number of equation coefficient. The coefficients of some parameters were always positive. For this reason, it can be said that these parameters are directly proportional to the modulus of elasticity. Although the water-cement ratio coefficient is positive, the modulus of elasticity is inversely proportional to this parameter since water-cement ratio values are

Table 1: Limitations of the function constants

Coefficient	Upper Boundary	Lower Boundary	Increment
x1	5000	1000	10
x2	0.5	0	0.001
x3	0.5	0	0.001
x4	0.5	0	0.001
x5	0.5	-0.5	0.001
x6	0.5	0	0.001
x7	0.5	0	0.001
x8	0	-0.5	0.001

Table 2: Initial parameters of the ABC method

Parameter	Value
NS	100
NLS	30
NC	3–8
MFE	50000

Table 3: the optimum equation constants and R^2 values of the models.

Coefficient	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5	Eq.6
x_1	3980	3980	4000	3760	3760	3760
x_2	0.363	0.42	0.442	0.441	0.441	0.441
x_3	0.085	0.067	0.069	0.1	0.1	0.1
x_4	N.A	0.076	0.094	0.078	0.078	0.078
x_5	N.A	N.A	-0.03	-0.124	-0.124	-0.124
x_6	N.A	N.A	N.A	0.013	0.013	0.013
x_7	N.A	N.A	N.A	N.A	0	0
x_8	N.A	N.A	N.A	N.A	N.A	0
R^2	0.95678	0.961034	0.969338	0.979121	0.979121	0.979121

N.A: Not available

in the range of 0-1. When the number of parameters used in the equation increased, R^2 values were also higher. As a result, the use of more parameters will allow more accurate prediction of the modulus of elasticity of the concrete.

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Boundedness of the B-Maximal Commutators on B-Morrey Spaces

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Abstract

In this paper the boundedness of the B-maximal commutators generated by the generalized translation operator is proved on the B-Morrey space

2010 Mathematics Subject Classifications : 42B20, 42B35

Keywords: Commutators, Maximal Commutators, Generalized shift operator, Maximal function, B-Morrey Space, BMO Space

Introduction

The Laplace-Bessel differential operator

$$\Delta_\nu = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + (\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n}\frac{\partial}{\partial x_n}), \qquad \nu > 0$$

is known as an important operator in analysis and its applications. The relevant harmonic analysis, known as Fourier-Bessel harmonic analysis associated with the Bessel differential operator

$$B_\nu = \frac{d^2}{dt^2} + \frac{2\nu}{t}\frac{d}{dt}, \quad \nu > 0$$

amounts to pioneering works by Delsarte, Levitan and was developed in subsequent publications by many mathematicians such as Kipriyanov, Klyuchantsev, Löfström, Peetre, Trimeche, Stein, Gadjiev, Aliev, Guliyev, Hasanov, Bayrakci and others (see[3, **?**, 6, 7, 8, 12, 15, 16, 17, 18]).

Let $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $B(x, r)$ denote the open ball centered at x of radius r, $|B(x, r)|$ be the Lebesque measure of the ball $B(x, r)$ and $\mathbb{R}^n_+ = \{x : x = (x_1, ...x_n) \in \mathbb{R}^n, x_n \geq 0\}$, $B_+(x, r) = \{y \in \mathbb{R}^n_+ : |x - y| < r\}$. For a measurable set $E \subset \mathbb{R}^n_+$ let $|E|_\nu = \int\limits_E x_n^{2\nu} dx, \ \nu > 0$.

Denote by T^y $(y \in \mathbb{R}^n_+)$, generalized translation operator acting according to the law:

$$T^y f(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int_0^\pi f(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}) \ \sin^{2\nu-1}\alpha \ d\alpha,$$

where $x = (x', x_n)$, $y = (y', y_n)$ and $x', y' \in \mathbb{R}^{n-1}$. We remark that T^y is closely connected with Bessel differential operator B_ν , see [3, 6, 10] for details.

The weighted space $L_{p,\nu} \equiv L_{p,\nu}(\mathbb{R}^n_+)$, $1 \le p < \infty$ consists of equivalence classes of measurable functions on $\mathbb{R}^n_+$ such that $\int\limits_{\mathbb{R}^n_+} |f(x)|^p x_n^{2\nu} dx < \infty$ and the $L_{p,\nu}-$norm of $f \in L_{p,\nu}$ is defined by

$$\|f\|_{L_{p,\nu}} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p x_n^{2\nu} dx \right)^{1/p}.$$

In the case $p = \infty$, we denote by L_∞ the space of all essentially bounded functions with norm $\|f\|_{L_\infty} = ess \sup_{x \in \mathbb{R}^n_+} |f(x)|$.

In the theory of partial differential equations, Morrey spaces play important role. The classical Morrey spaces $L_{p,\lambda} \equiv L_{p,\lambda}(\mathbb{R}^n)$, $1 \le p < \infty$, $0 \le \lambda \le n$ introduced by Morrey [13] in 1938 in relation to the study of partial differential equations are defined to be the subset of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, $1 \le p < \infty$ for which

$$\|f\|_{L_{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\ r>0} \left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(x)|^p x_n^{2\nu} dx \right)^{1/p}$$

is finite. These spaces are an expansion of L_p in the sense that $L_{p,0} \equiv L_p$ and $L_{p,n} \equiv L_\infty$. If $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey and Morrey type spaces. For instance, In [4], Chiarenza and Frasca showed the boundedness of the Hardy-Littlewood maximal operator, the Riesz potential and the Calderon-Zygmund singular integral operator and D.R. Adams [1] proved Hardy-Littlewood-Sobolev theorem on Riesz potentials in Morrey spaces. For the properties and applications of the Morrey spaces, we refer the readers to [13, 14, 15].

The B-Morrey space $L_{p,\lambda,\nu} \equiv L_{p,\lambda,\nu}(\mathbb{R}^n_+)$ associated with the Laplace-Bessel differential operator, introduced in [7] are defined as the set of locally integrable functions with the finite norm

$$\|f\|_{L_{p,\lambda,\nu}} = \sup_{x \in \mathbb{R}^n_+,\ r>0} \left(\frac{1}{r^\lambda} \int\limits_{B_+(0,r)} T^y |f(x)|^p y_n^{2\nu} dy \right)^{1/p},$$

where $1 \le p < \infty$, $0 \le \lambda \le n + 2\nu$. Note that $L_{p,o,\nu} \equiv L_{p,\nu}$, $L_{p,n+2\nu,\nu} \equiv L_{\infty,\nu}$ and if $\lambda < 0$ or $\lambda > n + 2\nu$, then $L_{p,\lambda,\nu} = \Theta$. The boundedness of the B-maximal operator and the Hardy-Littlewood-Sobolev theorem for the B-Riesz potentials on these spaces is proved by Guliyev and Hasanov [7, 8].

Moreover, we denote by the weak B-Morrey space $WL_{p,\lambda,\nu} = WL_{p,\lambda,\nu}(\mathbb{R}^n_+)$, $1 \le p < \infty$, $0 \le \lambda \le n + 2\nu$, defined by Guliyev [7] as the locally integrable functions with finite norm

$$\|f\|_{L_{p,\lambda,\nu}} = \sup_{t>0} t \sup_{x \in \mathbb{R}^n_+,\ r>0} \left(\frac{1}{r^\lambda} \int\limits_{\{y \in B_+(0,r):\ T^y|f(x)|>t\}} y_n^{2\nu} dy \right)^{1/p}$$

and by the B-maximal operator M_ν is defined by

$$(M_\nu f)(x) = \sup_{r>o} \frac{1}{|B(0,r)|_\nu} \int\limits_{B_+(0,r)} T^y |f(x)| y_n^{2\nu} dy.$$

The boundedness of the B-maximal operator M_ν on the B-Morrey spaces is proved by V.Guliyev and J. Hasanov [7, 8].

The space of functions of Bounded Mean Oscillation BMO, plays an important role in harmonic analysis, introduced by John and Nirenberg in 1961, in the study of partial differential equations. These space has been to be the "good space" to study instead of L_∞. In fact, many of the operators, which are ill-behaved on L_∞, are bounded on BMO.

Denote by $B\text{-}BMO = B\text{-}BMO(\mathbb{R}^n_+)$ space is defined the following

$$\|f\|_{B-BMO} = \sup_{x\in\mathbb{R}^n_+,\ r>0} \frac{1}{|B(0,r)|_\nu} \int\limits_{B_+(0,r)} |T^y f(x) - f_{B_+(0,r)}|\ y_n^{2\nu} dy$$

where $f_{B_+(0,r)} = \frac{1}{|B(0,r)|_\nu} \int\limits_{B_+(0,r)} T^y f(x) y_n^{2\nu} dy$.

Let $b \in L^{loc}(\mathbb{R}^n)$. Suppose that T is a linear or sublinear operator on some measurable function space, then the commutator formed by T for the measurable function f is defined by

$$[b,T]\, f(x) = b(x)Tf(x) - T(bf)(x).$$

The L_p boundedness for the commutator $[b,T]$ is obtained by Coifman and Meyer [5] when T is standart Calderon-Zygmund singular integral operator and $b \in BMO$. In recent years, Alvarez, Babgy, Kurtz and Perez [2] developed the idea of Coifman and Meyer and established a general boundedness criteria for the commutators of linear operator.

In [9] Hasanov, A. Mashiyev, Bayrakci proved that maximal commutators, commutators of singular integral operators and B-Riesz potentials associated with the Laplace-Bessel differential operator $\Delta_B = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum\limits_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ are bounded on B-Morrey Spaces.

Given a measurable function b the B-maximal commutator generated by the generalized translation operator is defined by

$$M_{b,\nu} f(x) = \sup_{r>o} \frac{1}{|B(0,r)|_\nu} \int\limits_{B_+(0,r)} |T^y\left(b(x) - b(y)\right) f(x)|\ y_n^{2\nu} dy,$$

for all $x \in \mathbb{R}^n_+$.

In this paper the boundedness of the B-maximal commutator generated by the generalized translation operator is proved on the B-Morrey space $L_{p,\lambda,\nu}$ for all $1 < p < \infty$ and $0 \le \lambda < n + 2\nu$, $b \in B\text{-}BMO$.

Main Results

Theorem 1. *a) Let $1 < p < \infty$, $0 \le \lambda < n + 2\nu$. Then the B-maximal commutator $M_{b,\nu}$ is bounded on $L_{p,\lambda,\nu}$ if and only if $b \in B\text{-}BMO$.*

b) Let $b \in B\text{-}BMO$. Then the B-maximal commutator $M_{b,\nu}$ is bounded from $L_{1,\lambda,\nu}$ to $WL_{1,\lambda,\nu}$.

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Facility Location Determined by an Iterative Technique

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Abstract

For many decades, mathematical models and methodologies have been essential in order to provide the understanding of countless engineering processes and technologies. Using mathematical models in many academic research, it was demonstrated that numerous infrastructural problems can be solved, such as locating several facilities related to a specific industry. In this study, a facility location was determined by considering some parameters. A mathematical problem named Weber problem supported the decision of the process of allocating the factory. Additionally, limitations and weights were taken into account in order to solve the mathematical problem. Moreover, in accordance with the findings, a discussion between this study and the previous studies has been made. As a result, by optimizing the case, the losses such as time wasting and distance traveled unnecessarily are minimized.

KEYWORDS: Algorithm, Iterative technique, Optimization, Weber problem

Introduction

Researchers [1] studied on locating undesirable facilities in terms of waste or potential pollution effects. They utilized maps such as land use, soil, river, road network maps, etc. in order to overlay them in a geographical information system. By deciding the musts and the needs (e.g. a facility that is 3km away from cultivable lands, within a distance of few kilometers from an asphalt road, etc.) for locating an industrial waste facility, alternative areas were revealed using the tools of a geographical information system.

Another study [2], in order to site an alumina-cement plant, suggested a multicriteria estimation method that was to compare a couple of alternative sites analyzing the important factors of accessibility to raw materials in terms of distance, water and power supply, and the land concerns. The model proposed a location of a plant by the integration of typical inputs of a facility such as transport, water, power, fuel consumption, and land.

A study [3] proposed an approach of type-2 fuzzy sets and compared their method with few other fuzzy approaches in order to solve a single-facility location problem. Additionally, several criterion of fixed and variable costs such as the costs of land, transportation, raw material, energy, environment and insurance were considered in their study.

A recent study [4] utilized an analytical hierarchy process and Geographic Information System (GIS) techniques to find out a decision making process for site selection of a common industrial area. While the candidate areas were weighted by the criteria related to the industry, the characteristics of the areas were evaluated according to their sustainability concerns.

Recently, another study [5] proposed a methodology of an optimization model in the area of multi-objective decision making problems having conflicting objectives. The study sought for the set of Pareto solutions rather than optimal solutions to determine a facility location.

Methodology

Weber Problem (WP) is an operational research that constructed by Alfred Weber in 1909 [6]. In this operation research problem, the optimization of location is the main purpose. This problem mainly solved by minimizing the weighted Euclidean distances in order to find an optimal location serving the demand locations [7], [8], [9], [10]. A brief description of WP can be seen as follows.

$$\min \sum_{j=1}^{n} w_j |\bar{x} - \bar{x}_j| \tag{1}$$

subject to, $\bar{x} = (x, y)$ and $\bar{x}_j = (x_j, y_j)$. Additionally, the location points of $\bar{x}_j$ are the points of the surface of E^2. Thus the description (1) discovers the most suitable location of $\bar{x}$. The following part introduces an effective solution of WP.

The Application of Weber Problem to Weiszfeld Algorithm

WP provides a useful mathematical problem to apply in many regional location assessments. Most importantly, WP can be solved by the Weiszfeld Method or Weiszfeld Algorithm (WM), which was built by Enrich Weiszfeld [11]. In the market, there are computer programs or software which are used for running and achieving solutions from WM. Because the computer program of LINGO can achieve the solution of WP by applying WM [12], it is used as a tool for analyzing the location and thus optimizing in this study. The reason of optimizing the problem in this study by using the computer software of LINGO is it is able to analyze and detect the optimum location of a solution point with regards to the weighted Euclidean distances of demand points in a predetermined surface of a topography [13]. The program goes on the iterations until all the demand points in the analysis are satisfied in terms of their weights [14]. As soon as the location of solution does not change according to the iterations running repeatedly, the iterations are stopped and the final decision of the optimization is reported by the program.

The location allocation problem, which is a mathematical problem, discussed in this study previously and the way of assessing the solution can be possessed by WM. Since the location of a facility and the weighted importance (i.e. including the capacity of the facility, production rate, accessibility to the resources etc.) are the inputs, the methodology discussed in this section can be transformed to apply in the problem of this study. Related formulation can be found as follows:

$$\min \sum_{i=1}^{n} a_i D_i \tag{2}$$

such that $a_i D_i$ is the weighted Euclidean distance to the location of facility (X_f, Y_f) where n is the number of resources around the facility. Also the position of the location should be $(X_f, Y_f) \geq 0$.

The Case Study

The data are derived synthetically and include totally 17 resources that the facility needs to receive materials and goods. In the resources there are two types of goods having specific weights. The weights show the importance or the amount needed from that particular good in the resource. While the good-1 has the weight of 0.35, the good-2 has the weight of 0.65. Therefore, the total weight of the goods from each source is equal to 1.00. Moreover, the location information such as positional data of those 17 resources are taken into account in order to determine the location of the facility. Location information provides a very important parameter of cost which can be described as distance in this study.

Main Results

For assessing the results of the optimization, the data were processed in LINGO 16.0 x64 [12]. The data as input and the objective function (2) was entered into the software. Further, the solver was run by a computer with the properties of Intel®Core^{TM} i7-2640M CPU @ 2.80GHz. When the program was run with the help of the inputs, the coordinate of (X_f, Y_f) which is the optimum solution was determined as (0.6687733, 0.6278632). For the Mercator projection, the spatial results of 0.6687733 and 0.6278632 corresponds to $38°19'04.4''$ N, $33°49'10.4''$ E respectively. Additionally, though this is the local optimal solution with totally 0.00 infeasibilities reported by the solver, this local minimum is actually the global minimum. The reason is that our problem is convex. Consequently, the results are verified.

Conclusion

Several infrastructural problems including the locations of many buildings, structures or even facilities can be solved by using mathematical models. A facility location was determined by a commonly used iterative methodology of WM by considering some parameters of the weights and the distances. For this case, a mathematical problem of WP supported the model used in this study. Therefore, according to the mathematical evaluation in this paper, the location of the facility is proposed with the help of the spatial results. This article can contribute to not only the regional planning, also the state planning which is vital for the economic development of a country. As an outcome, important losses for many industries such as time wasting and extra traveled distances or unnecessary mileage use can be minimized by applying the technique in this study.

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Assigning Convenient Paths by an Approach of Dynamic Programming

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Abstract

In order to increase the welfare of the public and provide an understanding of lots of engineering processes, decision makers have been executing several approaches of mathematical methods. In this study, transportation area is focused for finding one of the most essential issues in trip assignment: determining the shortest paths in transportation. Implementing a mathematical approach called dynamic programming in this study, a shortest path transportation problem has been solved for a case. A mathematical problem named dynamic programming illustration supported the decision of the process of the shortest paths assignment. Furthermore, in accordance with the findings, a discussion between this study and the previous studies completed has been made. As a result, by the decision of finding the shortest paths in the case and illustrating them, not only the travel distances are minimized but also the travel times between the origins and destinations.

KEYWORDS: Algorithm, Dynamic programming, Shortest path, Transportation

Introduction

Mathematical methodologies and techniques have been used for many important scientific areas for many years. For the particular engineering area, transportation engineering, the mathematical methods and related algorithms are commonly applied to real cases. For instance, on a transportation network of a region may need a solution to solve its travel assignment issues. At this point, network flow problems which have been used to identify the amount of flows on transportation networks take place. Since the problem needs a specific mathematical structure and solution algorithms to solve, many approaches have been studied on the transportation problem [1]. As the formulation of the transportation problem has been firstly discussed [2, 3], the algorithm of the solution was close to the general-used simplex method [1, 4, 5]. The problem was applied and still being applied to many engineering optimization processes [5, 6]. For example, while a research proved that the transportation problem can solve a case of shipment ways and times [7], another research demonstrated that directing the services on a residential or commercial network can be done by the transportation problem [8, 9].

A wide range of the studies of transportation problem takes place in determining the shortest path. In this case, the shortest route or the least cost path problems are solved in transportation related issues by again implementing the mathematical algorithms [1, 10, 11]. For instance, one of the most popular practice is to assign the least-cost path by assuming all of the nodes in a network are for the same travel purpose [12, 13]. Those studies used linear-integer mathematical procedures.

Methodology

In transportation related issues, many parameters including the cost (i.e. distance) between an origin and a destination location may be exposed to changes very often. In such non-static cases, the dynamic programming which is one of the best way to find the shortest routes on a transportation network can be confidently applied [1, 14]. Since the dynamic programming is an approach to discover an optimal solution in large scale algorithmic problems by dividing the problem into smaller pieces, it is an efficient way to solve relatively large scale transportation problems. In the following, to demonstrate the methodology and the ability of the dynamic programming with its application steps in order to find the optimal solution, a case study is introduced.

The Case Study

In this study, the data are derived synthetically on a transportation network and include totally 15 nodes with 28 arcs connecting particular origin and destination nodes. Although the calculations of dynamic programming may take long time, computer programs or software in the market may be used for quick solutions. Because the computer program of LINGO can achieve the solution of the dynamic programming to reveal the shortest path, it is used as a tool for analyzing the shortest route and thus optimizing in this study [15]. The script illustrating the dynamic programming already publicized is applied in order to find the shortest transport paths in the network of the case. The reason of running the dynamic programming to assess the shortest paths by using the particular software in this study is that it is able to analyze the network and detect the optimum transportation paths.

In the case, the crucial interest is to identify the shortest path from the first origin node (i.e. *node*-1) to the final destination node (i.e. *node*-15). However, since there may be trips joining to the network from the internal nodes (e.g. *node*-2, *node*-3, and the others), the shortest paths heading to the final destination from those nodes are also in the concern. Therefore, the dynamic programming works in this manner by the following steps mainly: *Stage*-1 seeks an answer of which node (*node*-12, *node*-13, or *node*-14) should be used to arrive at the final destination node (i.e. *node*-15) along the shortest path, because *node*-12, *node*-13, or *node*-14 are the only ones should be visited just before the final destination for our case study. Then, using the results of *Stage*-1, *Stage*-2 seeks an answer of which node should be used to arrive at the nodes of 12, 13, or 14 to reach to *node*-15 along the shortest route. Further, the stages do seek the answers until the first departing point of the network which is *node*-1. The last stage in this manner should be $Stage\text{-}final$ seeking an answer of where should one go from *node*-1 so that the route reaches the final destination node can be the shortest route.

Main Results

For achieving the results of the dynamic programming, the data in the case study were processed in LINGO 17.0 x64. The script as well as the data were entered into the software [15]. Further, the solver was run by a processor with the properties of Intel®Core^{TM} i7-2640M CPU @ 2.80GHz.

As the program was run by evaluating the inputs, the least cost path was determined as follows in Table 1. The path starts from *node*-1 as the origin node and terminates at the destination node of 15. In other words, one should be visiting the nodes of 1, 2, 4, 9, 14, and 15 respectively in order to access *node*-15 from *node*-1 with the least cost path.

Table 1: The least cost path used from the origin *node*-1 to the main destination *node*-15.

node-1	to	*node*-2
node-2	to	*node*-4
node-4	to	*node*-9
node-9	to	*node*-14
node-14	to	*node*-15

Additionally, while totally 0.000 infeasibilities are reported by the solver, the shortest routes with their costs (i.e. distances) from each node to the final destination node of 15 are also successfully provided, as can be seen in Table 2. For example, the shortest route distance in the network from *node*-9 as an origin to the destination *node*-15 is 71 units totally. Likewise, the shortest path distance from *node*-15 to *node*-15 should be zero as computed well by the solver.

Table 2: The costs of the shortest paths from the origins to the main destination *node*-15.

from	***distance***	***from***	***distance***	***from***	***distance***
node-1	185	*node*-6	124	*node*-11	127
node-2	141	*node*-7	195	*node*-12	85
node-3	131	*node*-8	166	*node*-13	41
node-4	109	*node*-9	71	*node*-14	39
node-5	160	*node*-10	93	*node*-15	0

Conclusion

By applying mathematical techniques and models, several problems related to transportation engineering come up with solutions including the assignment of the shortest paths according to the structure of a network topology. The shortest transportation routes were determined by a commonly used methodology of dynamic programming considering the parameters of a network in a case given. Surely, the main origin can create trips for a final destination, while the other nodes in the network can also create trips to the same destination. Since dynamic programming can solve such large issues, this article demonstrates the application of it for the solutions of transportation. The contribution of this article can be on not only making valuable decisions on selecting the transportation ways for non-static phases of regional traffic, but also increasing the comfort of many commuters by proposing the most convenient routes. Thus, the time loss which is vital for many travelers can be minimized, and also the economic development of a country is certainly contributed.

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Lorentz-Schatten Characteristic of Compact Inverses of First Order Normal Differential Operators

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Abstract

In this work, the Lorentz-Schatten characteristic of compact inverses of normal extensions of a minimal operator generated by linear differential-operator expression for first order in the Hilbert space of vector-functions at finite interval in terms of unbounded normal operator coefficients is investigated.

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Keywords: Differential and normal operators, s-numbers of compact operator, Lorentz-Schatten operator classes

Introduction

It is known that operator theory plays an exceptionally important role in modern mathematics and physics, quantum mechanics, deformation theory and etc. And also spectral analysis of operators is one of the most important area of modern mathematical physic. In addition the investigation of normal extensions of densely defined closed formal normal operators in any Hilbert space is among the fundamental mathematical problems arising in any physical model. It should be noted that the detail analysis of selfadjoint extensions of any linear closed densely defined having equal deficiency indexes in Hilbert space of vector-functions has been given in [1].

Let us remember that a linear densely defined closed operator T in any Hilbert space H is called formally normal if $D(T) \subset D(T^*)$ and $\|Tx\|_H = \|T^*x\|_H$ for all $x \in D(T)$. If a formally normal operator has no formally normal non-trivial extension, then it is called maximally formally normal operator. If a formally normal operator $T : D(T) \subset H \to H$ satisfies the condition $D(T) = D(T^*)$, then it is called normal operator (see [2]). The general theory of normal extensions of linear unbounded densely (and non-densely) defined formally normal operators has been given in [2]. Some application of this theory to the theory of differential operators in Hilbert space of vector-functions can be found in [3], [4], [5] (see references in it).

The general theory of singular numbers and operator ideals was given by A. Pietsch in [6], [7] and the case of linear compact operators was investigated by I. C. Gohberg and M. G. Krein in [8]. However, the first result in this area can be found in the works of E. Schmidt [9] and J. von Neumann, R. Schatten [10] who used these concepts in the theory of non-selfadjoint integral equations.

Later on, the main aim of the mini-workshop hold in Oberwolfach (Germany) was to present and discuss some modern applications of the functional-analytic concepts of $s-$numbers and operator ideals in areas like numerical analysis, theory of function spaces, signal processing, approximation theory, probability of Banach spaces and statistical learning theory (see [11]).

Let $\mathcal{H}$ be a Hilbert space, $S_\infty(\mathcal{H})$ be a class of linear compact operators in $\mathcal{H}$ and $s_n(T)$ be the $n-th$ singular numbers of the operator $T \in S_\infty(\mathcal{H})$. in [6]. The Lorentz-Schatten operator ideals are defined as

$$S_{p,q}(\mathcal{H}) = \left\{ T \in S_\infty(\mathcal{H}) : \sum_{n=1}^{\infty} n^{\frac{q}{p}-1} s_n^q(T) < \infty \right\},\ 0 < p \leq \infty,\ 0 < q < \infty$$

and

$$S_{p,\infty}(\mathcal{H}) = \left\{ T \in S_\infty(\mathcal{H}) : \sup_{n \geq 1} n^{\frac{1}{p}} s_n(T) < \infty \right\},\ 0 < p \leq \infty$$

in [6], [7], [12], [13].

In this work, the problem of belonging to Lorentz-Schatten classes of the inverses (consequently of resolvent operators) of the normal extensions of the minimal operator generated by differential-operator expression for first order in the Hilbert space of vector-functions at finite interval in terms of unbounded normal operator coefficients is studied.

On the Singular Numbers of Inverses of Normal Extensions of the Minimal Operator

Let H be a separable Hilbert space and $L^2 = L^2(H,(a,b))$ be a Hilbert space of $H-$valued vector-functions at finite interval.

In the space $L^2(H,(a,b))$ consider the following linear differential-operator expression for first order in form

$$l(u) = u'(t) + Au(t), \tag{1}$$

where:
(1) $A : D(A) \subset H \to H$ is a linear unbounded normal operator,
(2) real part A_R of the operator A satisfies the condition $A_R \geq E$, where E denotes the identity operator in H.

By standard method the minimal $L_0(L_0^+)$ and maximal $L(L^+)$ operators corresponding to differential expression $l \left(l^+ = -\dfrac{d}{dt} + A^* \right)$ in $L^2(H,(a,b))$ can be easily defined (see [4]). In this case the minimal operator L_0 is formally normal, but it is not maximal in $L^2(H,(a,b))$.

Now let $U(t,s),\ t,\ s \in [a,b]$ be a family of evolution operators corresponding to the homogeneous equation

$$\begin{aligned} &U_t'(t,s)f + iA_I U(t,s)f = 0,\ t,\ s \in [a,b], \\ &U(s,s)f = f,\ f \in D(A) \end{aligned}$$

where, A_I indicates imaginary part of A.

Now give a few auxiliary three propositions from [4] .

Theorem 2.1. *Let $A_R^{1/2}[D(L) \cap D(L^+)] \subset W_2^1(H,(a,b))$. Each normal extension L_n of the minimal operator L_0 in L^2 is generated by the differential-operator expression (1) with the boundary condition*

$$(u)(b) = U(b,a)W(u)(a), \tag{2}$$

where W and $A_R^{1/2} W A_R^{-1/2}$ are unitary operators in H. The unitary operator W is determined uniquely by the extension L_n, i.e. $L_n = L_W$.

On the contrary, the restriction of the maximal operator L to the linear manifold of vector-functions $u(t) \in D(L) \cap D(L^+)$ that satisfy condition (2) for some unitary operator W, where $A_R^{1/2} W A_R^{-1/2}$ also unitary operator in H, is a normal extension of the minimal operator in the space L^2.

Theorem 2.2. *If $A_R^{-1} \in S_\infty(H)$ and the operator L_W is any normal extension of minimal operator L_0, then $L_W^{-1} \in S_\infty(L^2)$. In addition, in this case for any $\lambda \in \rho(L_W)$, $R_\lambda(L_W) \in S_\infty(L^2)$.*

Theorem 2.3. *If $A_R^{-1} \in S_\infty(H)$ and $\lambda_n(A_R) \sim cn^\alpha$, $0 < c$, $\alpha < \infty$ and L_W is any normal extension of minimal operator L_0, then*

$$s_n\left(L_W^{-1}\right) \sim dn^{-\beta},\ 0 < d < \infty,\ \beta = \frac{\alpha}{1+\alpha}.$$

Lorentz-Schatten Characteristic of Inverses of Normal Extensions of the Minimal Operator

Now give the main results of this work in following theorems.

Theorem 3.1. *Let $A_R^{-1} \in S_\infty(H)$, $\lambda_n(A_R) \sim cn^\alpha$, $0 < c$, $\alpha < \infty$ and L_W be any normal extension of the minimal operator L_0. In order to $L_W^{-1} \in S_{p,q}(L^2)$, $0 < q < \infty$ the necessary and sufficient condition is $p > 1 + \dfrac{1}{\alpha}$.*

Proof. In this case from mentioned above Theorem 2.3. it is known that

$$s_n\left(L_W^{-1}\right) \sim dn^{-\beta},\ 0 < d < \infty,\ \beta = \frac{\alpha}{1+\alpha}.$$

Consequently, for the convergence of the series $\sum\limits_{n=1}^{\infty} n^{\frac{q}{p}-1} s_n^q\left(L_W^{-1}\right)$, $0 < p,\ q < \infty$, i.e., $\sum\limits_{n=1}^{\infty} n^{\frac{q}{p}-\frac{\alpha q}{1+\alpha}-1}$ the necessary and sufficient condition is $1 + \dfrac{\alpha q}{1+\alpha} - \dfrac{q}{p} > 1$. From this implies that $\dfrac{1}{p} < \dfrac{\alpha}{1+\alpha}$. From last inequality it is obtained that $p > 1 + \dfrac{1}{\alpha}$.

Theorem 3.2. *If $A_R^{-1} \in S_\infty(H)$, $\lambda_n(A_R) \sim cn^\alpha$, $0 < c$, $\alpha < \infty$ and L_W be any normal extension of the minimal operator L_0. In order to $L_W^{-1} \in S_{p,\infty}(L^2)$, $0 < p \leq \infty$ the necessary and sufficient condition is $p > 1 + \dfrac{1}{\alpha}$.*

Proof. In this case from Theorem 2.3 it is known that $s_n\left(L_W^{-1}\right) \sim dn^{-\beta}$, $0 < d < \infty$, $\beta = \dfrac{\alpha}{1+\alpha}$.

Then for the validity of following condition $\sup\limits_{n\geq 1} n^{\frac{1}{p}} s_n\left(L_W^{-1}\right) < \infty$, that is,

$$\sup_{n\geq 1} n^{\frac{1}{p}} n^{\frac{-\alpha}{1+\alpha}} = \sup_{n\geq 1} n^{\frac{1}{p}-\frac{\alpha}{1+\alpha}} < \infty,$$

the necessary and sufficient is $\dfrac{1}{p} < \dfrac{\alpha}{1+\alpha}$. From this it is obtained that $p > 1 + \dfrac{1}{\alpha}$.

Corollary 3.2. *Under the conditions of above two theorems* $L_W^{-1} \notin S_{p,q}(L^2)$, $0 < p \leq 1,\ 0 < q \leq \infty$.

Corollary 3.2. *Under the conditions of Theorem 3.1,* $L_W^{-1} \in S_p(L^2)$ *if and only if* $1 + \dfrac{1}{\alpha} < p < \infty$, *where* $S_p(\cdot)$ *denotes a Schatten-von Neumann operators ideal.*

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A note on Hermite Base Euler Type Polynomials

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Abstract

In this paper, by using generating functions with their mutliplication series analysis, we derive some identities and relations including the Milne-Thomson polynomials and numbers, Hermite numbers and polynomials.

2010 Mathematics Subject Classifications : 11B65, 11B68, 11B83, 33C45
Keywords: Euler numbers and polynomials, Hermite polynomials, Generating functions.

Introduction

In this paper we investigate and study on some well-known families of numbers and polynomials including Euler numbers and polynomials, and Hermite numbers and polynomials and also Euler type numbers and polynomials.

The Hermite type polynomials with variable x and y are defined by means of the following generating functions [21]

$$F_H(t;x,y) = e^{xt+y^j t} = \sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!}. \tag{1}$$

Observe that the polynomials $H_n^{(j)}(x,y)$ are also so called that the Hermite-Kampe de Feriet on Gould-Hopper polynomials [2].

From equation (1), we easily see that

$$H_n^{(j)}(x,y) = \sum_{k=0}^{\infty} \frac{n! x^{n-jk} y^k}{k!(n-jk)!}$$

(*cf.* [2], [13], [19]). The Apostol-Euler polynomials are defined by means of the following generating functions

$$F_E(t,x) = \frac{2e^{tx}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_n(x,\lambda)\frac{t^n}{n!}. \tag{2}$$

By equation (2), we have the Apostol-Euler numbers E_n see as follows:

$$E_n(\lambda) = E_n(0,\lambda)$$

(*cf.* [1]-[19]).

In [18], Simsek defined the following polynomials $W_n(x;\lambda)$:

$$F_W(t,x) = \frac{e^{tx}}{(\lambda e^t + \lambda^{-1}e^{-t} + 2)^k} = \sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!}. \tag{3}$$

Setting $x = 0$ in (3), we get the number $W_n^{(k)}(\lambda)$. In [18], Simsek defined higher order $W_n(\lambda)$ numbers by following generating function:

$$F_W(t) = \left(\frac{1}{\lambda e^t + \lambda^{-1}e^{-t} + 2}\right)^k = \sum_{n\geq 0} W_n^{(k)}(\lambda)\frac{t^n}{n!}. \tag{4}$$

In [18], Simsek proved that

$$W_n(\lambda) = \frac{\lambda}{4}\sum_{j=0}^{n}\binom{n}{j}\sum_{l=0}^{j}\binom{j}{l}E_l(\lambda)E_{j-l}(\lambda).$$

The following recurrence relation was given by Simsek ([18]):

$$\sum_{m=0}^{n}\binom{n}{m}W_{n-m}^{(k)}(\lambda)\,y_2(m,k;\lambda) = 0$$

where

$$y_2(m,k;\lambda) = \frac{1}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}2^{k-j}\sum_{l=0}^{j}\binom{j}{l}(2l-j)^n\lambda^{2l-j}$$

(*cf.* [15]).

Hermite base $W_n(x;\lambda)$ polynomials

In this section, we study on Hermite base $W_n(x;\lambda)$ polynomials which defined by

$$F_{HW}(t,x;\lambda;k,j) = \frac{e^{xt+yt^j}}{(\lambda e^t + \lambda^{-1}e^{-t} + 2)^k} = \sum_{n=0}^{\infty} {}_HW_n^{(j,k)}(x,y;\lambda)\frac{t^n}{n!}. \tag{5}$$

By combing equation (5) with equations (1) and (4), we get

$$\sum_{n=0}^{\infty}H_n^{(j)}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}W_n^{(j,k)}(\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}{}_HW_n^{(j,k)}(x,y;\lambda)\frac{t^n}{n!}.$$

By using cauchy product rule in the above equation, we get

$$\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}H_l^{(j)}(x,y)\,W_{n-l}^{(k)}(\lambda)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}{}_HW_n^{(j,k)}(x,y;\lambda)\frac{t^n}{n!}. \tag{6}$$

We now equating coefficients $\frac{t^n}{n!}$ in the both sides of equaiton (6), we get computation formula for the polynomials ${}_HW_n^{(j,k)}(x,y)$ by the following theorem:

Theorem 1. *Let $j, k \in N_0$. Then we have*

$${}_HW_n^{(j,k)}(x,y;\lambda) = \sum_{l=0}^{n}\binom{n}{l}H_l^{(j)}(x,y)\,W_{n-l}^{(k)}(\lambda).$$

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Statistical Classification of Turuncova Marbles with Physical-Mechanical Properties, Finike, Antalya

Burcu Aydin[1] , *Fusun Yalcin*[2] , *Ozge Ozer*[1] , *M. Gurhan Yalcin*[1]

Abstract

In the study area that is one of the most important tourism cities of Turkey and is valuable in terms of economic natural resources, marble holds an important place. Limra marble quarries located in Turuncova (Finike, Antalya) region. The aim of study is to find some physico-mechanical properties of Limra marbles and to explain these data statistically. Analysis results found out that the data obtained in laboratory conditions were in a definite value and the data of the marbles were in physico-mechanical relations. According to the results of statistical analysis, an R2 value is found as 1. This value, which indicates that there is sufficient and appropriate number of data for statistics, also means that data are very high. Anova analysis is regression analysis and error rate of this data is found as 0. In the data used in the analyses, error rate was not found.

2010 Mathematics Subject Classifications : 62H20, 62H30, 62P30, 91C20
Keywords: SPSS, Statistics, Limra Marble, Turuncova, Finike, Antalya.

Introduction

Physical and chemical properties of marbles with natural building block are always subject of curiosity for researchers. Marbles are classified in line with intended purpose and determination of physico-mechanical properties. Analyses like capillary mass water absorption, specific mass, total porosity, compactness, uniaxial compressive strength and mass extinction made in laboratory were made by using TSI standards and were classified to intended purposes. Mechanic properties like triaxial stress compressive strength, deformation stress strength, and friction angle are analyzed for determining marble quality and these data are explained with statistical process. In statistical analysis, reliability of selected experiments was searched (Jiyang and others, 2016) There are also studies on marble powder, not just marbles. Physico-mechanical properties in marble powder was determined and variance analysis (ANOVA) were made to these values, also it was made correlations with different examples (Kelestemur and others, 2014; Benzannache and others., 2017) Statistic results were correlated with experimental findings and these studies achieved successful results. There are some researches in different areas related to marbles of Antalya region (Yalcin and oth., 2015, 2016; Yalcin and Akturk 2017) These can be listed as follows: Current status and future projection of Western Mediterranean marble sector ; Contributions of cities of the western Mediterranean (Antalya, Isparta and Burdur) to export Turkeys natural stones and marble"; Antalya, Burdur and Isparta examples on the importance of block marbles and processed on the prices When literature was examined, an academic study of marble of Turuncova Finike was not found. The aim is to determine the physico-chemical properties of marble samples taken from different levels and locations from a marble quarry in the study area, to evaluate these data with statistical techniques and to classify marble belonging location by using similarities and differences among examples.

Examples, Physico-Mechanical Properties and Statistical Analyses

In this study, samples were taken from Limra marble-quarry in Turuncova region (Finike, Antalya) (Figure 1) In the civil engineering laboratory of Akdeniz University, according to TSI, experiments of ‘’total porosity, compactness, water absorption amount, mass loss and compressive strength” were done.

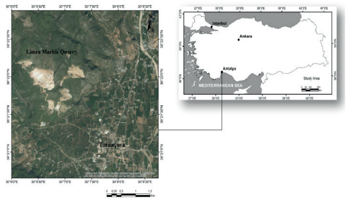


Figure 1: Site Location Map Of Limra Marble-Quarry In Turuncova Region (Finike, Antalya)

Physico-mechanical properties of eight (8) marble samples from different locations of Limra marble quarries are given in the table (table 1). Classification of examples from land: TS 1910 (1977). Natural building stones used as covering, Turkish Standards, TS 2513 (1977). Natural building stones used as covering, TS EN 13755 (2003). Natural stones, testing methods were made in the atmosphere pressure as to determination of water absorption.

Table 1. Physico-mechanical properties of Turuncova region marbles (Finike, Antalya)

Numbers Code	Total porosity (P) (%)	Composity K) (%)	Water observation SE(%)	Mass lossed due to cycling exposure KK (%)	Compressive string PA(MPa)
A1	4,63	95,37	1,73	0	174,69
A2	4,56	95,44	1,68	0	59,41
A3	4,45	95,55	1,32	0	81,06
A4	4,17	95,83	1,56	0	127,67
B1	4,61	95,39	1,2	0	159,65
B2	3,31	96,69	1,38	0	133,23
B3	3,18	96,82	1,47	0	125,07
B4	4,22	95,78	1,48	0	134,78

Values of physico-mechanical properties were examined by using SPSS program. The data of the samples were analyzed in five (5) different ways.

‘’Descriptive statistics’ was chosen as first analysis and ‘numerical value range, minimum-maximum values, cumulative values, arithmetic average, standard deviation and variance’ of each example was calculated. (Table 2)

Table 2. Descriptive Statistics

	Range	Minimum	Maximum	Sum	Mean		Std. Deviation	Variance
P	1,45	3,18	4,63	33,13	4,1413	,20491	,57957	,336
K	1,45	95,37	96,82	766,87	95,8588	,20491	,57957	,336
SE	,53	1,20	1,73	11,82	1,4775	,06310	,17847	,032
KK	,00	,00	,00	,00	,0000	,00000	,00000	,000
PA	115,28	59,41	174,69	995,56	124,4450	13,41011	37,92952	1438,649

The aim of second analysis made by SPSS program was to determine relation of the correlation coefficient on variables (Buyukozturk, 2003). This 'Correlation' analysis takes place in Table 3.

Table 3. Correlation analysis

	p	k	se	kk	Pa
p	1				
k	-1,000**	1			
se	,166	-,166	1		
kk	.,158b	-150	0,7	1	
Pa	-,043	,043	-,084	0,9	1

Third analysis is correlation coefficient analysis. The statistical analysis for the determination of the R2 in the table of 'Model Summary' shows how much the data reflects the truth (Table 4). The result of the analysis of the value of 1.00 shows that the value of the analysis has positive relation while the result of the analysis of the value of -1.00 shows that the value of the analysis has negative relation. The result of the analysis of the value of 0 shows that there is no relation between relations (Tutus M., Kılıc M. A., 2008). Values between 0,70-0,30 of correlation coefficient is in middle level, values between 0,30-0,00 have low level relation. (Tutus M., Kılıc M. A., 2008).

Table 4. 'Model Summary' values of Testing of Limra Marbles

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	1,000a	1,000	1,000	,00000

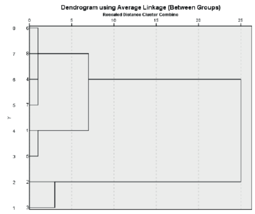
Fourth analysis is Anova analysis. (Table 5.) 'Anova' analysis is a regression analysis. Values of regression analysis' results mean that one of the analysis data is independent and the other one is dependent (Tutus M., Kılıc M. A., 2008)

Table 5. 'Anova' values of Testing of Limra Marbles

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	2,351	3	,784	.	0
	Residual	,000	4	,000		
	Total	2,351	7			

Fifth analysis made by SPSS-26 program is 'Classify' analysis (Chart 1) This statistical analysis made by Hierarchial cluster technique found similar characteristics and clustering of 8 different samples.

Chart 1. 'Classify' Diagram Of Testing Of Limra Marbles



Results and Discussion

According to TSI standards, 8 (eight) different examples from Limra marbles Turuncova region (Finike, Antalya) were examined in line with intended purpose in the university laboratory. Physico-mechanical properties were examined with helping of SPSS program. Error rate was found as %0 and accuracy rate R2 value was found as 1 in the 'Model Summary'. High correlation coefficient was acquired from data. In the dendrogram chart, samples are classified into three groups of close smiliraties in their physico-mechanical properties. Group 1 samples (A4, B2, B3, B4) have the lowest total porosity and highest composity properties. They show average water absorption and compressive string values compared to group 2 and 3. Group 2 samples (A1, B1) have the lowest composity, highest total porosity and compressive string properties. Group 3 samples (A2, A3) have the lowest compressive string property. It is understood that statistical methods are successful and can be used in this field.

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Euler-Catalan's Number Triangle and its Application

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Abstract

In this paper we study labeled binary trees of size n with m ascents on the left branch. For this combinatorial object we present the relation of the generated number triangle to Catalan's and Euler's triangles. On the basis of properties of Catalan's and Euler's triangles, we obtain an explicit formula that counts the total number of such trees and an exponential generating function.

2010 Mathematics Subject Classifications: 11Y55, 05A15, 05C05
Keywords: Number triangle, Labeled binary tree, Catalan's triangle, Euler's triangle, Generating function

Introduction

Combinatorial objects like permutations, combinations, partitions, graphs, trees, paths, etc. play an important role in mathematics and computer science and also have many applications in practice. Knuth [1] gives an overview of the formation and development of the direction related to designing combinatorial algorithms.

Sometimes combinatorial objects can be described by using number triangles, for example, for their enumerating. A number triangle is a doubly indexed sequence in which the length of each row corresponds to the index of the row. There are many well-known number triangles such as Pascal's triangle, Catalan's triangle, Euler's triangles, Stirling's triangles, etc., whose elements have a whole set of combinatorial interpretations [2]. To define a number triangle, it is necessary to specify the rules for generating elements of this triangle. For example, it can be some expression in the form of an explicit formula, a recurrence relation or a generating function.

In this paper we study labeled binary trees with ascents on the left branch. For this combinatorial object we obtain an explicit formula that counts the total number of such trees and an exponential generating function. Also we present the relation of the obtained number triangle to Catalan's and Euler's triangles.

Main Results

Let us consider the following combinatorial object: a labeled binary tree of size n with m ascents on the left branch. Figure 1 shows all possible variants of such trees for $n = 3$ and $m = 1$.

Theorem 1. *The number of labeled binary trees of size n with m ascents on the left branch is*

$$EC_{n,m} = \begin{cases} 1, & \text{for } n = m = 0; \\ \sum\limits_{k=m+1}^{n} C_{n,k} E_{k,m} P_{n,n-k}, & \text{otherwise}, \end{cases}$$

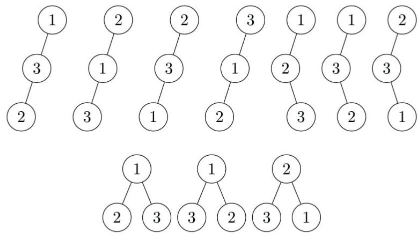

Figure 1: All labeled binary trees of size 3 with 1 ascent on the left branch.

where $C_{n,m}$ is the transposed Catalan's triangle, $E_{n,m}$ is Euler's triangle, $P_{n,m}$ is the number of k-permutations of n.

Proof. The number of binary trees of size n with m nodes on the left branch is defined by the elements of the transposed Catalan's triangle (the sequence $A033184$ in OEIS [3]) that are denoted as $C_{n,m}$. According to [4], an explicit formula for $1 \leq m \leq n$ is

$$C_{n,m} = \frac{m}{n}\binom{2n-m-1}{n-1}.$$

Then we consider labeled version of these binary trees (for each node there is an associated unique value from 1 to n). We need to count the number of ways to label the given binary tree of size n with k nodes on the left branch such that it has exactly m ascents on the left branch. For this, it is necessary that the labels of k nodes on the left branch form a permutation of k elements with m ascents and the remaining labels of $n-k$ nodes form all possible permutations.

The number of permutations of n elements with m ascents is defined by the elements of Euler's triangle (the sequence $A173018$ in OEIS [3]) that are denoted as $E_{n,m}$. According to [2], an explicit formula for $0 \leq m \leq n$ is

$$E_{n,m} = \sum_{k=0}^{m} (-1)^k (m-k+1)^n \binom{n+1}{k}. \tag{1}$$

The number of permutations of m elements given from a set of n elements (the sequence $A008279$ in OEIS [3]) is denoted as $P_{n,m}$. According to [2], an explicit formula for $0 \leq m \leq n$ is

$$P_{n,m} = \frac{n!}{(n-m)!}. \tag{2}$$

Hence, combining (1) and (2), we get the number of ways to label the given binary tree of size n with k nodes on the left branch such that it has exactly m ascents on the left branch

$$E_{k,m} P_{n,n-k}. \tag{3}$$

If we consider all variants of binary trees of size n with k nodes on the left branch for k from $m+1$ to n and get the number of ways to label them using (3), we obtain the total number of labeled binary trees of size n with m ascents on the left branch

$$\sum_{k=m+1}^{n} C_{n,k} E_{k,m} P_{n,n-k}.$$

∎

Table 1: Several first values of Euler-Catalan's triangle.

$n \setminus m$	0	1	2	3	4	5	6	7
0	1							
1	1	0						
2	3	1	0					
3	19	10	1	0				
4	193	119	23	1	0			
5	2721	1806	466	46	1	0		
6	49171	34017	10262	1502	87	1	0	
7	1084483	770274	255795	47020	4425	162	1	0

The sequence of $EC_{n,m}$ forms a number triangle. Due to the connection with Catalan's and Euler's triangles, let us call it Euler-Catalan's triangle.

Theorem 2. *The number of labeled binary trees of size n with m ascents on the left branch is defined by the following exponential generating function:*

$$EC(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} \frac{EC_{n,m}}{n!} x^n y^m = \frac{y-1}{y-e^{C(x)(y-1)}}, \tag{4}$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2}$ is the generating function of the Catalan numbers.

Proof. Let us consider $EC(x,y)$ as the composition of generating functions $EC(x,y) = E(C(x),y)$, where

$$E(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} \frac{E_{n,m}}{n!} x^n y^m = \frac{y-1}{y-e^{x(y-1)}}$$

is the exponential generating function of Euler's triangle [2].

According to [5], if we have the composition $A(x) = R(F(x)) = \sum\limits_{n\geq 0} a_n x^n$ of generating functions $R(x) = \sum\limits_{n\geq 0} r_n x^n$ and $F(x) = \sum\limits_{n>0} f_n x^n$, then

$$a_n = \begin{cases} r_0, & \text{for } n = 0; \\ \sum\limits_{k=1}^{n} F^{\Delta}(n,k) r_k, & \text{otherwise,} \end{cases} \tag{5}$$

where $F^{\Delta}(n,k)$ is the composita of the generating function $F(x)$.

The composita of $C(x) = \frac{1-\sqrt{1-4x}}{2}$ is [6]

$$C^{\Delta}(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1} = C_{n,k}.$$

Using (5) for the composition $A(x) = EC(x,y) = E(C(x),y)$, we get

$$a_n = \begin{cases} 1, & \text{for } n=0; \\ \sum\limits_{k=1}^{n} C^{\Delta}(n,k) \left(\sum\limits_{m\geq 0} \frac{E_{k,m}}{k!} y^m \right), & \text{otherwise}, \end{cases} =$$

$$= \begin{cases} 1, & \text{for } n=0; \\ \sum\limits_{m\geq 0} y^m \frac{1}{n!} \sum\limits_{k=m+1}^{n} C_{n,k} E_{k,m} P_{n,n-k}, & \text{otherwise}, \end{cases} =$$

$$= \begin{cases} 1, & \text{for } n=0; \\ \sum\limits_{m\geq 0} y^m \frac{EC_{n,m}}{n!}, & \text{otherwise}. \end{cases}$$

Hence, we obtain the desired result

$$EC(x,y) = E(C(x),y) = \frac{y-1}{y-e^{C(x)(y-1)}} = \sum_{n\geq 0} \sum_{m\geq 0} \frac{EC_{n,m}}{n!} x^n y^m.$$

■

Conclusion

Using properties of Catalan's and Euler's number triangles, we have obtained the explicit formula that counts the total number of labeled binary trees of size n with m ascents on the left branch. Also for this combinatorial object we have got the exponential generating function.

Acknowledgements

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Integral representations of generating functions for combinatorial numbers and polynomials

Yilmaz Simsek[1]

Abstract

In this paper, we give integral represantations of generating functions for special numbers and polynomials including Cauchy numbers and polynomials and combinatorial numbers and polynomials. Furthermore, with the help of these generating functions and their functional equations, we derive a new identity for these numbers and polynomials and series representation of logarithm function.

2010 Mathematics Subject Classifications : 05A10, 05A15, 11B83, 26C05, 30D05, 40C10

Keywords: Stirling numbers, Cauchy numbers, Bernoulli numbers of the second kind, Combinatorial numbers and polynomials, Generating function, Functional equation, Integral representation, Series representation.

Introduction

It is well known that special numbers and polynomials containing combinatorial numbers and polynomials, Cauchy numbers and polynomials, and also Stirling numbers are used very effectively in almost all fields of science, mainly mathematics. Therefore, special numbers and polynomials are studied in this paper. By integrating the generating functions for special numbers and polynomials including combinatorial numbers and polynomials, and Cauchy numbers and polynomials, both a series expansion of the logarithmic function and an identity containing these numbers and polynomials are obtained.

In order to give proof of the results mentioned in the above, generating functions for combinatorial numbers and polynomials, Cauchy numbers and polynomials, and also some formulas and relations are given below.

We [6] defined the following generating function for the polynomials $Y_{n,2}(x;\lambda)$:

$$H(t,x;\lambda)=\frac{2(1+\lambda t)^{x}}{\lambda^{2}t+2(\lambda-1)}=\sum_{n=0}^{\infty}Y_{n,2}(x;\lambda)\frac{t^{n}}{n!}. \tag{1}$$

From the above equation, we have

$$Y_{n,2}(x;\lambda)=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}j!\frac{\lambda^{n+j}}{2^{j}(\lambda-1)^{j+1}}(x)_{n-j} \tag{2}$$

where $(x)_n$ denotes the falling factorial defined by $(x)_0=1$ and

$$(x)_n=x(x-1)\dots(x-n+1)$$

(*cf.* [6]).

Substituting $x = 0$ into (1), we have the combinatorial numbers $Y_{n,2}(\lambda)$:

$$\begin{aligned} Y_{n,2}(\lambda) &= Y_{n,2}(0;\lambda) \\ &= (-1)^n n! \frac{\lambda^{2n}}{2^n (\lambda-1)^{n+1}} \end{aligned} \tag{3}$$

(*cf.* [6]).

Recently, these numbers and polynomials and other related special numbers and polynomials have been studied by many authors, see for detail (*cf.* [2], [5]-[8]).

Stirling numbers of the first kind, $S_1(n,k)$ are given by

$$S(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k) \frac{t^n}{n!},$$

and also

$$(x)_n = \sum_{j=0}^{n} x^j S_1(n,j), \tag{4}$$

(*cf.* [1], [4]).

Cauchy numbers, or Bernoulli numbers of the second kind, $C_n = b_n(0)$ are defined by the following generating function:

$$F_c(t) = \frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!} \tag{5}$$

or

$$C_n = \int_0^1 (x)_n \, dx = \sum_{j=0}^{n} \frac{S_1(n,j)}{j+1}, \tag{6}$$

(*cf.* [1, p. 294], [3, p. 1908], [4, p. 114]). There are various different relations between Stirling numbers of the first kind and Cauchy numbers.

Cauchy polynomials are defined by

$$C_n(x) = \int_0^x (u)_n du$$

(*cf.* [1], [3], [4]).

Integral representations for generating functions and formulas

In this section, we give integral representations of Eq-(1). We give a series expansion of the logarithmic function and an identity including combinatorial numbers and polynomials and Cauchy numbers and polynomials.

Integrate equation (1) with respect to u from 0 to t, we get

$$\int_0^t H(u,0;\lambda)\, du = \sum_{n=0}^{\infty} \frac{Y_{n,2}(\lambda)}{n!} \int_0^t u^n du.$$

Thus

$$\frac{2}{\lambda^2}\ln\left(\frac{\lambda^2}{2(\lambda-1)}t+1\right)=\sum_{n=0}^{\infty}\frac{Y_{n,2}(\lambda)}{(n+1)!}t^{n+1}.$$

Combining the above equation with (3), after some elementary computation, we obtain series expansion of the logarithmic function $\ln\left(\frac{\lambda^2}{2(\lambda-1)}t+1\right)$ as follows:

Theorem 1.

$$\ln\left(\frac{\lambda^2}{2(\lambda-1)}t+1\right)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\left(\frac{\lambda^2}{2(\lambda-1)}\right)^{n+1}t^{n+1}. \tag{7}$$

We note that, in theory of calculus, there are many different proofs of well known formula for series expansion of the logarithmic function in Eq-(7).

Integrate equation (1) with respect to u from 0 to x, we get

$$\int_0^x H(t,u;\lambda)\,du=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^x Y_{n,2}(u;\lambda)\,du.$$

Thus

$$\frac{H(t,x;\lambda)-H(t,0;\lambda)}{\ln(\lambda t+1)}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^x Y_{n,2}(u;\lambda)\,du.$$

From the above equation, we get the following functional equation:

$$F_c(\lambda t)\left(H(t,x;\lambda)-H(t,0;\lambda)\right)=\lambda\sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}\int_0^x Y_{n,2}(u;\lambda)\,du.$$

Combining the above equation with (2) and (5), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}Y_{n,2}(x;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\lambda^n C_n\frac{t^n}{n!}-\sum_{n=0}^{\infty}Y_{n,2}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\lambda^n C_n\frac{t^n}{n!}\\ =\;&\lambda\sum_{n=0}^{\infty}n\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}j!\frac{\lambda^{n+j-1}}{2^j(\lambda-1)^{j+1}}C_{n-j-1}(x)\frac{t^n}{n!},\end{aligned}$$

By using the Cauchy product rule in the left hand side of the above equation, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\lambda^j C_j\left(Y_{n-j,2}(x;\lambda)-Y_{n-j,2}(\lambda)\right)\frac{t^n}{n!}\\ =\;&\lambda\sum_{n=0}^{\infty}n\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}j!\frac{\lambda^{n+j-1}}{2^j(\lambda-1)^{j+1}}C_{n-j-1}(x)\frac{t^n}{n!},\end{aligned}$$

Comparing the coefficient $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.

$$\begin{aligned}&\sum_{j=0}^{n}\binom{n}{j}\lambda^j C_j\left(Y_{n-j,2}(x;\lambda)-Y_{n-j,2}(\lambda)\right)\\ =\;&n\lambda\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}j!\frac{\lambda^{n+j-1}}{2^j(\lambda-1)^{j+1}}C_{n-j-1}(x).\end{aligned}$$

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Special numbers arised from trigonometric and hyperbolic functions

Neslihan Kilar[1] , *Yilmaz Simsek*[2]

Abstract

The aim of this paper is to give not only recent development on well-known numbers and polynomials including the Bernoulli numbers, the Euler numbers, the tangent numbers, the cotangent numbers and the others, but also identities and relations associated with these numbers. Furthermore, by using trigonometric function identity, combinatorial identity is deriven.

2010 Mathematics Subject Classifications : 05A15, 11B68, 11B83, 26C05.
Keywords: Bernoulli numbers and polynomials, Euler numbers and polynomials, Tangent numbers, Cotangent numbers.

Introduction

By using generating functions including tangent and cotangent functions, we give some new relations and formulas including not only tangent numbers and cotangent numbers and polynomials, but also other well-known families such as the Bernoulli numbers and polynomials, the Euler numbers and polynomials and others.

The Bernoulli polynomials and the Euler polynomials are defined by means of the following generating functions, respectively:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1}$$

where $|t|<2\pi$ and

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n\left(x\right)\frac{t^n}{n!} \tag{2}$$

where $|t|<\pi$ (*cf.* [1]-[13]; and the references therein).

It is clear that $B_n(0)=B_n$ and $E_n(0)=E_n$ which denote the Bernoulli numbers and the Euler numbers, respectively (*cf.* [1]-[13]; and the references therein).

The tangent numbers T_n are defined by

$$\tan\left(t\right)=t+\frac{t^3}{3}+\frac{2t^5}{15}+\frac{17t^7}{315}+...=\sum_{n=1}^{\infty}T_{2n-1}\frac{t^{2n-1}}{(2n-1)!} \tag{3}$$

where $|t|<\frac{\pi}{2}$ (*cf.* [1], [3], [4], [14]; and the references therein) and also

$$\tan\left(t\right)=\sum_{n=0}^{\infty}\left(-1\right)^{n+1}\frac{2^{2n+1}E_{2n+1}}{(2n+1)!}t^{2n+1}$$

where $|t|<\pi$ (*cf.* [1], [9], [10]).

The cotangent numbers C_n are defined by

$$\frac{t}{2}\cot\left(\frac{t}{2}\right)=\sum_{n=0}^{\infty}C_{2n}t^{2n}=\sum_{n=0}^{\infty}(-1)^{n}B_{2n}\frac{t^{2n}}{(2n)!} \tag{4}$$

(*cf.* [6]; and the references therein).

Identities and formulas for the special numbers

In this section we give some identities and formulas for the special numbers including tangent and cotangent numbers.

Theorem 1. *Let n be a positive integer. Then we have*

$$T_{2n-1}=(2n-1)!\left(1-2^{2n}\right)2^{2n}C_{2n}.$$

Proof. Combining following well-known trigonometric identity

$$t\tan(t)=t\cot(t)-2t\cot(2t)$$

(*cf.* [6]) with (3) and (4), we get

$$\sum_{n=0}^{\infty}(2n)\,T_{2n-1}\frac{t^{2n}}{(2n)!}=\sum_{n=0}^{\infty}C_{2n}\left(1-2^{2n}\right)2^{2n}t^{2n}.$$

Comparing the coefficients of t^{2n} on both sides of the above equation, we arrive at the desired result. ■

Theorem 2. *The following identity holds true:*

$$\sum_{k=0}^{n}\binom{2n}{2k}+\sum_{k=0}^{n-1}(-1)^{k}\binom{2n}{2k+1}2^{2n-2k-2}T_{2k+1}=2^{2n}.$$

Proof. Combining following well-known trigonometric identity

$$2\cos^{2}(t)-2\cos(2t)=\tan(t)\sin(2t) \tag{5}$$

(*cf.* [6]) with Eq-(3), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\binom{2n}{2k}-2^{2n}\right)\frac{t^{2n}}{(2n)!}\\ =\;&\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{1+k}\binom{2n}{2k+1}2^{2n-2k-2}T_{2k+1}\frac{t^{2n}}{(2n)!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^{2n}}{(2n)!}$ on both sides of the above equation, we arrive at the desired result. ■

Theorem 3. *The following identity holds true:*

$$\sum_{k=0}^{n}\binom{2n+2}{2k+1}=\sum_{k=0}^{n}(-1)^{k}\binom{2n+2}{2k+1}2^{2n-2k}T_{2k+1}.$$

Proof. Combining following well-known trigonometric identity

$$2\sin^2(t) = \tan(t)\sin(2t) \tag{6}$$

(*cf.* [6]) with Eq-(3), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n+2}{2k+1}\frac{t^{2n+2}}{(2n+2)!}\\ = \quad &\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{2n+2}{2k+1}2^{2n-2k}T_{2k+1}\frac{t^{2n+2}}{(2n+2)!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^{2n+2}}{(2n+2)!}$ on both sides of the above equation, we arrive at the desired result. ■

Observation

We now give a well-known combinatorial sums with the help of trigonometric identity. Using (5) and (6), we have

$$2\cos^2(t) - 2\cos(2t) = 2\sin^2(t). \tag{7}$$

By using Taylor series of the $\sin(t)$ and $\cos(t)$, we get

$$\sum_{n=1}^{\infty}(-1)^n\left(\sum_{k=0}^{n}\binom{2n}{2k} - 2^{2n}\right)\frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-1}\binom{2n}{2k+1}\frac{t^{2n}}{(2n)!}.$$

After some elementary calculations, comparing the coefficients of $\frac{t^{2n}}{(2n)!}$ on both sides of the above equation, the following combinatorial sum is obtained:

$$\sum_{k=0}^{n-1}\binom{2n}{2k+1} + \sum_{k=0}^{n}\binom{2n}{2k} = 2^{2n}.$$

Combining the well-known Pascal's rule, which is a very important combinatorial identity:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

in the above equation, we get

$$\sum_{k=0}^{n-1}\binom{2n+1}{2k+1} = 2^{2n} - 1.$$

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This paper is presented in "**The Mediterranean International Conference of Pure&Applied Mathematics and related areas**" dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary, Antalya-Turkey, October 26-29, 2018.

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Box Plots Analiysis of Elements in the Lara Beach Sand

Fusun Yalcin[1]

Abstract

The study was carried out on Lara sand sediments located on the coast of Antalya to determine the chemical distribution of elements and assess possible heavy metal contamination level and its accumulation index and origin in the beach sand. 21 elements were evaluated using statistical technique to determine the distribution of the metals. Multivariate statistical was applied to understand the the homogeneity of the samples and determine samples in which elements show anomalous concentrations with respect to the other samples. Samples were distinguished in 5 groups of close similarities based on the chemical distribution.

2010 Mathematics Subject Classifications : 62H25, 62P30, 86A32
Keywords: Multivariable Statistic, SPSS, Lara, Beach Sand, Antalya

Introduction

The evaluation of heavy metal concentrations in beach sand and soil sediments has pulled the interest of many geo and environmental scientist over time who seek to establish the origin, determine the potential risk they may pose to the ecosystem and develop solutions on how to ameliorate this risk. This is simple due to the harmful nature of these elements. The existence is natural but their concentration is is either influenced by both natural and anthropogenic activities. Sediments and soil act as sink, scavenger and traps to these metals which are non-biodegradable (*cf.* [1], [4], [5]).

Along the coastline of Lara, the rapid population increase in the city has also lead to an expanding urbanization and a fast development in the touristic industry in the region. Within the last decade, hotel of at least the five star category constructed in this region have exceeded (*cf.* [2]). These generally have increase anthropogenic pressure on the natural Lara coastline (*cf.* [6]).

In this study, the sand samples were collected and their heavy metals content were determined. The distribution of these elements was evaluated using the multivariate statistical approach, whereby samples with anomalous concentrations elements to their normal distribution were evaluated.

Material and Method

To carry out the beach sand sediment geochemical and contamination survey of the Lara beach, 47 beach sand samples were collected from the beach. The samples were prepared using standard sample preparation procedure (*cf.* [6]) and analyzed with XRF technique at the ACME Laboratory Limited. Geochemical data obtained from the XRF technique per sample consists of variables (elements) expressed in concentrations.

The Data was evaluated using both the quantitative and qualitative approaches. The quantitative evaluation included simple statistical case summary and interquartile distribution of variable in the analyzed cases. The Principal Component Analysis (PCA) and Factor Analysis (FA) were applied to study the process controlling the concentrations of the variables alongside cluster analysis and correlational relationship of the coexisting variables.

Results and Discussion

21 elements were identified in the samples using XRF technique. The most abundant element is Ca. However, in sample L11 Si most abundant and it is slightly higher in L (6 & 12) as presented on Table 2. In order of abundance, Ca is followed by Si and Fe respective with Al and Mg having an almost similar concentration in the samples, as illustrated in Figure 2. At a two decimal point evaluation the average concentrations of Cu, Rb, Y and Pb is zero. Ca, Fe, Al, Mg, Ti, Mn, Sr, Ni, Zn, Rb, K and Ba have a higher median to mean value and a corresponding skewness less than zero, except Ca having skewness slightly greater than zero. On the other hand, Si, Na, Cr, Zr, Cu, Y, Pb, S and P have a lower median to mean values with skewness greater than zero at a three decimal point evaluation.

Skewness < 0 is referred to as left skewed distribution. This indicates an element's concentration in most of the samples are concentrated on the right of its mean value in the samples, with extreme values to the left; while Skewness > 0 Is referred to as right skewed distribution. It implies an element's concentration in most the samples is distributed on the left of its mean concentration with extremely values to the right (*cf.* [3]). However, the distribution of Ca reverse skewed value (> 0 instead of < 0) is accounted for by the extremely anomalous high content of Ca in sample. Checkout and compare World Avg.

Figure 1: Simple statistical evaluation of chemical data by the SPSS 23

	Range	Minimum	Maximum	Sum	Mean		Std. Deviation	Variance
P	1,45	3,18	4,63	33,13	4,1413	,20491	,57957	,336
K	1,45	95,37	96,82	766,87	95,8588	,20491	,57957	,336
SE	,53	1,20	1,73	11,82	1,4775	,06310	,17847	,032
KK	,00	,00	,00	,00	,0000	,00000	,00000	,000
PA	115,28	59,41	174,69	995,56	124,4450	13,41011	37,92952	1438,649

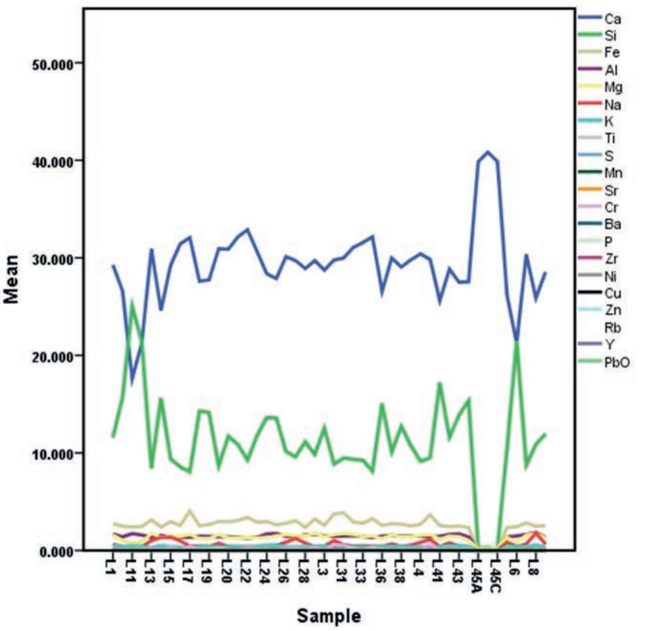

Figure 2: Concentration Distribution of elements in the samples

Interquartile Distribution

Interquartile distribution of the variables within samples, reveals samples L45 (A – C) indicates very high anomalous concentrations for Ca and low anomalous concentration for all other elements. On the opposite, Si shows high anomalous concentrations in samples L (6, 11 & 12), while Ca has low anomalous concentration. In samples L (17, 30, 31 & 40), Fe, Ti and Cr have high anomalous concentrations. Anomalous concentration for Pb is observed for samples L (8, 24, 36 and 43). Sr, Ba and Zr in all samples are well distributed across the interquartile range (Figure 2).

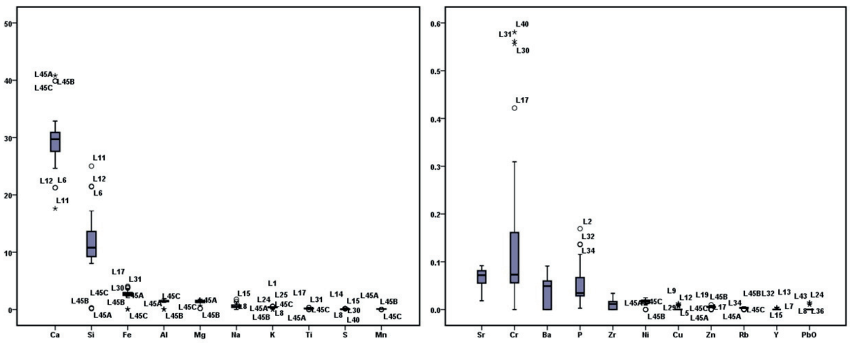

Figure 3: Box Plot Analiysis of samples within their interquartile range.

Conclusion

The study was carried on the Lara beach located on west coast of Antalya to assess the geochemical content of the samples and determine the level of contamination by heavy metals. 21 elements were identified in the 47 samples analyzed. The elements are unequally distributed with Ca, Si, Cr, Pb, Cu and Fe showing anomalous values in some samples. Calcium is the most abundant element in all the samples except in samples L (11, 6 and 12) where SI is most abundant. Heavy metals present in the samples Cr, Mn, Ni, Cu, Zn and Pb. Most of the heavy metals such as Cu and Pb exist in insignificant quantities in most samples. Heavy metal content of the beach is of no risk to the beaches.

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Effects of Wavelet Families and Filter Coefficients on EEG Frequency Spectrum

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Abstract

This paper presents the effects of different wavelet families for numerous filter coefficients on electroencephalography (EEG) signals. Subjects consist of five healthy volunteers were participated in this study, where EEG signals of the subjects recorded with 64 different channels. Subjects were asked to stand-still and relaxed with eyes closed position. The collected data from the subjects, was decomposed with 9th level of wavelet packet transform for different wavelet families such as daubechies, coiflets, symlets and Fejer-Korovkin wavelets. The root mean square energies of the signals were calculated for each subject and wavelets to select suitable wavelet coefficient to examine the EEG signals. The near-harmonics effects of alpha waves at higher bands were decreased, when the higher wavelet coefficient of daubechies, symlets and Fejer-Korovkin wavelets were applied.

2010 Mathematics Subject Classifications : 92C55, 65T60.
Keywords: Wavelet, Near-harmonics, Spectrum, EEG

Introduction

Electroencephalography (EEG) is an electrophysiological monitoring method that record electrical activity of the human brain. EEG signals mostly analyzed through the frequency spectrum which has characterized peaks at various frequency. Generally, spectrum subdivided into frequency band intervals such as delta (0-4Hz), theta (4-8Hz), alpha (8-13Hz), beta (13-30Hz) and gamma (30-50Hz) bands. Among these bands, alpha band is the most salient band during the eyes closed recordings. Because, there is a characterized activity around occipital lobe. This feature was first detected in healthy adult EEG at 1933 [1]. However, the same feature can be observed in higher frequency bands like beta and even gamma which can be seen in Figure 1. In resting eyes closed EEG recordings, there are various peaks that appear like harmonics. An early study investigated and predicted the near-harmonic relationships that might be caused from alpha waves [2]. In order to diminish or at least decrease these near-harmonic effects different wavelets and filter coefficient were used to the extract frequency spectrum.

Method

EEG data were recorded at Akdeniz University, Faculty of Medicine, Neurology Department, EEG Laboratory, Antalya, Turkey. EEG signals were recorded by Nihon Kohden device with a EEG cap that has 64 channel including Ag/AgCl electrodes.

Unipolar signals were recorded with 200 Hz sampling frequency. Five healthy subject that have no history of neurological or psychiatric disorders, participated in this study. Also, an informed consent was obtained from the participants included in the study. All procedures performed in accordance with the ethical standards of the institutional and/or national research committee. Subjects were asked to sit in comfortable chair with eyes closed and stand still as little as possible.

The collected data of the subjects were decomposed into frequency bands for different wavelets via generalized formula below [3],

$$W_{m,j,n}(t) = 2^{-m/2} W_j(2^{-2}t - n), \tag{1}$$

where $j \in N$ represents the node index in each m level. The root mean square value of the decomposition components can be calculated as

$$w_{rms,m,j} = \sqrt{\frac{1}{N} \sum |w_{m,j}(r)|^2} \tag{2}$$

Moreover, the total wavelet energy for each node can be defined as

$$E = \sum_{j=0}^{2^M-1} |w_r ms, m, j|^2. \tag{3}$$

Daubechies(db), Symlets(sym), Coiflets(coif) and Fejer-Korovkin(fk) wavelets were used with five different filter coefficient in ascending order, in order to examine the effect of coefficients.

Main Results

The main effect of the ascended filter coefficients is the decrements at the peak of beta bands which can be seen easily from Figure 2.

Conclusion

The near-harmonics of the alpha waves were reduced and the wavelet packet energies were increased at the higher filter coefficients. Nevertheless, db, sym and fk wavelet gives the best result among mother wavelet types. Consequently, artifacts caused by near-harmonics which might led to false conclusions with the activity in theta band, can be avoided. Therefore, above-mentioned type of wavelets should be used in EEG spectrum analysis with higher filter coefficients.

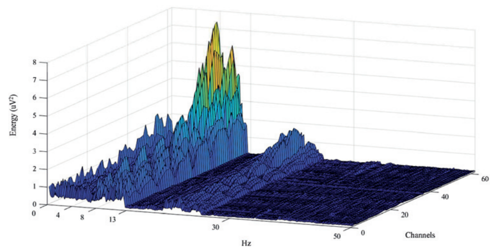

Figure 1: Mean wavelet packet energy data with 'db4'

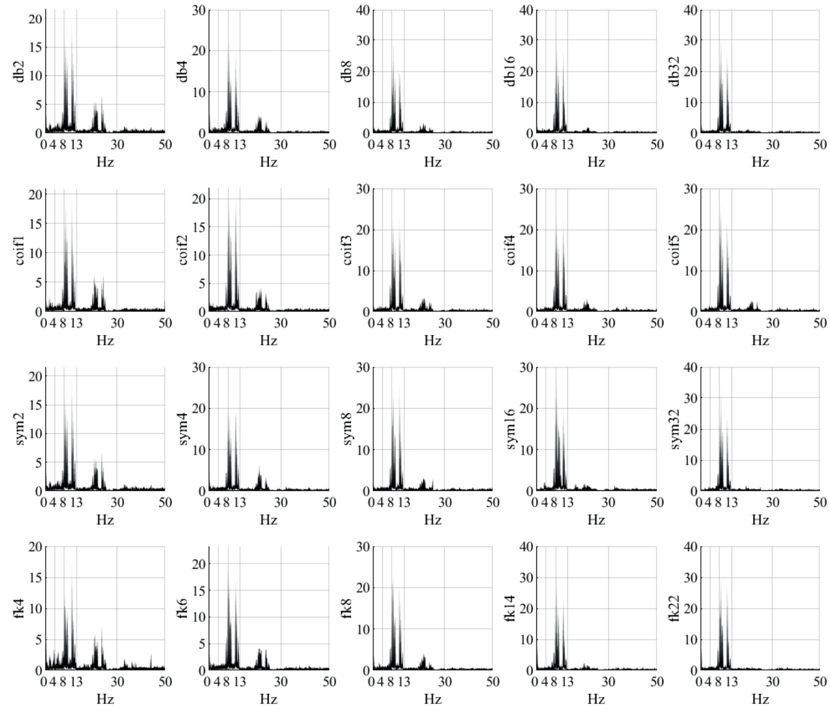

Figure 2: Wavelet packet energy of the first subject with different filter coefficient.

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Observations On Statistical Tests Used In Neuroscience

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Abstract

It is difficult to understand and interpret the results of neuroscience studies without the help of statistical methods. Therefore, some statistical methods used in neuroscience are examined in this paper. Some basic properties of these methotds are given. Consequently, brain activity measurement devices are briefly given. Finally, we survey some remarks and observations on Neurostatistics with their applications.

2010 Mathematics Subject Classifications : 62F03, 62G10.
Keywords: Statistical tests, Neuroscience, Brain activity.

Introduction

The statistical solutions in neuroscience are used in biology, in medicine, in engineering and in many other related areas. Therefore, recently, several investigators have been known to work on neurostatistics (*cf.* [1]-[8]; and the references cited therein). In order to clearly understand neuroscience, there are various statistical tests have been used.

In this paper, we will briefly examine some of the statistical methods used in the data.

By using statistical models including general linear model (GLM), Student's t tests and analyses of variance (ANOVAs), Smith [8] gave experimental design in neuroscience. obtained from devices that measure brain activity.

Some devices used for brain activity measurement

Brain function with its relations to cognition and behavior are known by the use of various complementary methods (*cf.* [5]; and the references cited therein). Here, we mention some well-known devices used for brain activity measurement.

Functional Magnetic Resonance Imaging: fMRI is a device that allows measurement and observation of brain activity during an ongoing procedure (*cf.* [2]; and the references cited therein).

Electroencephalogram: EEG is a test which records the electrical signals of the brain. EEG signal depends on both the amplitude and spatial synchronization of brain neural activity (*cf.* [7]; and the references cited therein).

Transcranial Magnetic Stimulation: TMS is indirect and non-invasive method used to induce excitability changes in the motor cortex via a wire coil generating a magnetic field that passes through the scalp. Recently, TMS is become a efective and important method to investigate brain functioning in humans (*cf.* [6]; and the references cited therein).

Electromyography EMG signal represents the linear transformation of motor neuron discharge times by the compound action potentials of the innervated muscle fibers and is often represented as a source of knowledge about to neural activation of muscle (*cf.* [4]; and the references cited therein).

Some statistical tests used in neuroscience

In this section, we give brief introduction about statistical tests used in neuroscience.

Hypothesis testing

Hypothesis testing are used the test the difference between two groups. There two groups hypothesis testing which are parametric hypothesis testing and non-parametric hypothesis testing. We breiefly give some information about these hypothesis testing as follows.

Parametric hypothesis tests

Parametric hypothesis tests (distributions of data should be normal distribution)

Student t Test

Paiered t Test

ANOVA (Analysis Of Variance)

(*cf.* [1], [3]; and the references cited therein).

Non-parametric hypothesis tests

Non-parametric hypothesis tests (distributions of data should be non-normal distribution)

Mann Whitney U Test

Wilcoxon Signed-Rank Test

Kruskal-Wallis Test

(*cf.* [1], [3]; and the references cited therein).

Relational Statistical Methods

These tests are examined the study of relationship between two or more variables.

Correlation

Regression Analysis

(*cf.* [1], [3]; and the references cited therein).

Conclusion

In this paper, some statistical tests used in the field of neuronscience are mentioned. Due to the very small number of researches in neuroscience assocated with brain activity measurement with the help of statistical tests, other approaches or techniques may be included in the methods mentioned above.By using approaches based on distribution functions, new statistical tests that can be used in neuroscience may be developed. As a result, with the help of these new statistical tests, better results may be obtained in the analysis and interpretation of the data to be obtained in neuroscience. This study may have the potential to form a preliminary information for those working in these areas.

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On Relations between Subgroups of a Group and Submodules of a Module over Group Rings

Ortaç Öneş[1] , *Mustafa Alkan*[2] , *Mehmet Uc*[3]

Abstract

This study deals with relationships between submodules of a module over a group ring and subgroups of a group. We find some connections among RG-submodules of M with regard to normal subgroups and elements of G.

2010 Mathematics Subject Classifications : 16S34, 20C05
Keywords: Group ring, Augmentation Ideal, Group module

Introduction

Let R be a commutative ring with unity and G a finite group. Let us recall the group ring, denoted by RG.

RG is defined as the set of all formal linear combinations of the form

$$r = \sum_{g \in G} r_g g$$

where $r_g \in R$ and $r_g = 0$ for almost everywhere.

For two elements, $r = \sum\limits_{g \in G} r_g g$ and $s = \sum\limits_{g \in G} s_g g \in RG$, we have that $r = s$ if and only if $r_g = s_g$ for all $g \in G$.

The sum of two elements in RG is componentwise as

$$r + s = \sum_{g \in G} r_g g + \sum_{g \in G} s_g g = \sum_{g \in G} (r_g + s_g) g$$

Also, for two elements $r = \sum\limits_{g \in G} r_g g$ and $s = \sum_{g \in G} s_g g$ in RG, their products are defined as follows:

$$rs = \sum_{g,h \in G} (r_g s_h)(gh).$$

The homomorphism $\beta : RG \to R$ given by $\beta\left(\sum\limits_{g \in G} r_g g\right) = \sum\limits_{g \in G} r_g$ is called the augmentation mapping of RG and its kernel, denoted by $\triangle_R(G)$, is called the augmentation ideal of RG. One can observe that

$$\triangle_R(G) = \left\{ \sum_{g \in G} r_g (g - 1) : g \in G, g \neq 1, r_g \in R \right\}.$$

Besides,

$$\triangle_R(G,H)=\left\{\sum_{h\in H} r_h(h-1): r_h\in RG\right\}$$

is the left ideal of RG. One can observe that the ideal $\triangle_R(G,G)$ coincides with the ideal $\triangle_R(G)$.

For a left ideal I of RG, $\bigtriangledown(I)$ is a subgroup of G as follows:

$$\bigtriangledown(I)=\{g\in G: g-1\in I\}=G\cap(1+I)$$

For a subgroup H of G, it is clear that $\bigtriangledown(\triangle_R(G,H))=H$.

Let τ be a group homomorphism from G to $End(M)$. For all $g\in G$ and $m\in M$, the multiplication mg is defined as

$$mg=\tau(g)(m)$$

M is an RG-module with this multiplication. The group homomorphism τ in the multiplication is called a representation of G for M over R.

If $\tau(g)=1_{End(M)}$ for all $g\in G$, the structure of RG-module is the same with the structure of R–module.

In [1], it was studied on modules over group rings and Alkan proved the following property:

Let M be an RG-module and let H be a normal submodule of a group G. Then

$$\triangle_M(H)=\left\{\sum_{h\in H}\alpha_h\,(h-1)\mid \alpha_h\in M\right\}$$

is an RG-submodule of M and

$$\triangle_M(H)=M.\triangle_R(G,H)$$

In [12], the connections between normal subgroups of a group G and ideals of a group ring RG were studied and a new result on a normal subgroup of G corresponding to an ideal of RG was obtained. With this study, we generalize the results obtained in [12] to module and obtain more results between RG-submodules of a module M over a group ring RG and subgroups of a group G.

This study deals with the connections between normal subgroups of G and RG-submodules of M. We find some relationships among RG-submodules of M with regard to normal subgroups and elements of G. We focus on normal subgroups of G and prove that

$$\triangle_R(G,\bigtriangledown(N))=\sum_{i=1}^{n}(\triangle_R(G,<x_i>))$$

if

$$\bigtriangledown(N)=<x_1,x_2,...,x_n>$$

where $x_i\in G$ and N is an RG-submodule of M.

Main results

Lemma 1. *Let H_1 and H_2 be normal subgroups of a group G. Then $\triangle_M(G,<H_1\cup H_2>)=\triangle_M(G,H_1)+\triangle_M(G,H_2)$, where $<H_1\cup H_2>$ is the set generated by H_1 and H_2.*

Lemma 2. *Let H_1 and H_2 be normal subgroups of a group G. Then $\triangle_M(G, H_1 \cap H_2) \subseteq \triangle_M(G, H_1) \cap \triangle_M(G, H_2)$.*

By the standard argument, one can easily prove Lemma 1 and Lemma 2.

Theorem 3. *Let $\triangle_N(H) = \left\{ \sum_{h \in H} n_h(h-1) : a_h \in N \right\} \subseteq \triangle_M(G)$ be an RG-submodule of M, where N is a RG-submodule of M. Then we have $\triangle_{\Sigma_{i=1}^k N_i}(H) = \sum_{i=1}^{k} \triangle_{N_i}(H)$.*

Lemma 4. *Let N be an RG-submodule of an RG-module M. Then $\bigcup_{i=1}^{n} \triangledown(N_i) \subseteq \triangledown\left(\sum_{i=1}^{n} N_i\right)$, where n is a positive integer.*

Proof. Using distributive law, we have the following:

$$\begin{aligned}
\bigcup_{i=1}^{n} \triangledown(N_i) &= \bigcup_{i=1}^{n} (G \cap (1+N_i)) = (G \cap (1+N_1)) \cup ... \cup (G \cap (1+N_n)) \\
&= (G \cap ((1+N_1) \cup ... \cup (1+N_n)) \\
&\subseteq (G \cap (1+N_1+...+N_n)) = \triangledown\left(\sum_{i=1}^{n} N_i\right)
\end{aligned}$$

■

Theorem 5. *Let $x_i \in G$ and N be an RG-submodule of an RG-module M. If $\triangledown(N) = < x_1, x_2, ..., x_k >$, then we have*

$$\triangle_M(G, \triangledown(N)) = \sum_{i=1}^{k} (\triangle_M(G, < x_i >))$$

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On One-sided Prime Submodules

Ortaç Öneş[1]

Abstract

The purpose of this study is to introduce some properties of one-sided prime submodule of modules. We examine the relationships among left O-prime ideal, one-sided prime submodule and the set $\rho_m(P)$, where P is a submodule of a left R-module M and $m \in M$.

2010 Mathematics Subject Classifications : 16N40, 16N60, 16N80
Keywords: Strongly Nilpotent Element, Prime Submodule, One-Sided Prime submodule

Introduction

The concept of prime ideal forms an important part to characterize ring and has been studied for long time by many authors ([1],[6],[10]). In a commutative ring with unity, the set of nilpotent elements forms an ideal equaling to the intersection of all prime ideals. This notion has been generalized in [6] to modules. Let N be a proper submodule of an R-module M. The radical of N in M is defined to be the intersection of all prime submodules of M containing N and it is denoted by $rad_M(N)$. The envelope submodule $RE_M(N)$ of N in M is a submodule of M generated by the set

$$E_M(N) = \{rm : r \in R \ and \ m \in M \ such \ that \ r^n m \in N \ for \ some \ n \in \mathbb{N}\}.$$

If $rad_M(N)$ is equal to $RE_M(N)$, then it is said that N satisfies the radical formula in M. Althought some useful characterizations for modules by using this concept were proved, unfortunately, there are not enough useful results about the radical formula and radical submodule in noncommutative case.

Let N be a submodule of an R-module M.

i) A set $\eta(a) = \{a, a_1, ...\}$ is said to be an sequence of an element a of R if for all $i \in \mathbb{N}$, $a_{i+1} \in a_i R a_i$ and $a_0 = a$.

ii) Let $a \in R$, $m \in M$. Then an element am of M is said to be a strongly nilpotent on N if for all subsets $K = \{a_i \in R : a_0 = a \text{ and } a_{i+1} \in a_i R a_i,\ i \in \mathbb{N}\}$ of R, $0 \in N \cap Km$. We use the notation $W_M(N)$ to denote the submodule generated by the strongly nilpotent elements on N. Now it is clear that $W_M(N) = RE_M(N)$ when R is a commutative ring.

In [12], to examine the radical formula in noncommutative case, a generalization of prime ideal was defined.

Let P be a left ideal of R. Following [12], P is said to be a one-sided prime ideal (left O-prime ideal) if for any left ideals I, J such that $PJ \subseteq P$ and $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$ holds. It is clear that every maximal left ideal is one-sided prime ideal. In this sense, the class of one-sided prime ideals is different from other known classes. As a main result of [12], it can be stated that any left O-radical ideal in a noncommutative ring R satisfying the ascending chain condition on left O-radical ideals is the intersection of a finite number of left O-prime ideals. In particular any left ideal in R is the intersection of a finite number of left O-prime ideals.

In this study, we generalize the definition of left O-prime ideal in a noncommutative ring to the module and it is called one-sided prime submodule. Every left prime submodule of a left R-module M is one-sided prime but the converse of it is not true. It is clear that the set of one-sided prime submodule is different from the set of prime submodule.

Besides, the set $\{r \in R : rm \in P\}$, where P is a submodule of a left R-module M and $m \in M$ is denoted by $\rho_m(P)$. To give a direct connection between left O-prime ideal and one-sided prime submodule, we focus on the set $\rho_m(P)$ for $m \in M$. Clearly, it is a left ideal of R and $\rho_m(P) = R$ if and only if $m \in P$.

Let N be a submodule of an R-module M. In analogy to the definition of radical submodule in a module of a submodule, we define the O-radical of N, which is the intersection one-sided prime submodules of M containing N, denoted by $O\text{-}rad_M(N)$. In particular, N is said to be an O-radical submodule of M if $O\text{-}rad_M(N) = N = W_M(N)$.

In this study, we give some properties of one-sided prime submodule of modules, which is the module version of left O-prime ideal of a noncommutative ring. We also define the set $\rho_m(P)$, where P is a submodule of a left R-module M and $m \in M$ and examine the relationships among left O-prime ideal, one-sided prime submodule and this set.

Main results

Throughout the paper, all rings will be associative rings with identity and all modules will be unital left modules.

We give some properties of one-sided prime submodules in a left R-module M as follows:

Lemma 1. *Let P a prime ideal of a prime ring R and let $M = R \oplus R$ be an R-module. Then $N = 0 \oplus P$ is a one-sided prime submodule of M but N is not a prime submodule of M.*

Lemma 1 also states that the concept of one-sided prime submodule is different from the concept of prime submodule.

The following lemma gives some properties of the set $\rho_m(P)$, where P is a submodule of a left R-module M and $m \in M$.

Lemma 2. *Let M and M^* be R-modules, $\varphi : M \to M^*$ an R-epimorphism and P is a submodule of M. Then we have the following statements:*
i) $\rho_m(P) \subseteq \rho_{f(m)}(f(P))$ for some $m \in M$.
ii) $\rho_{f(m)}(f(P)) \subseteq \rho_m(P)$ when $Kerf \subseteq P$.

Proposition 3. *Let M and M^* be R-modules, $\varphi : M \to M^*$ an R-epimorphism and $Ker\varphi \subseteq P$. Then P is a one-sided prime submodule of M if and only if $\varphi(P)$ is a one-sided prime submodule of M^*.*

With Proposition 3, the following corollary can be stated.

Corollary 4. *Let M be an R-module. Then P is a one-sided prime submodule of M if and only if P/N is a one-sided prime submodule of an R-module M/N for all $N \subseteq P \subseteq M$.*

The following theorem gives us a connection between O-radical submodules and submodules of M.

Theorem 5. *Let M be a finitely generated R-module and let N, L be submodules of M. Then $O\text{-}rad_M(N) + O\text{-}rad_M(L) = M$ if and only if $N + L = M$.*

Proof. Suppose that $O\text{-}rad_M(N) + O\text{-}rad_M(L) = M$ and $N + L \neq M$. Thus, there exists a maximal submodule T of M such that $N + L \subseteq T$. Since T is a one-sided prime submodule of M, we have $O\text{-}rad_M(N) \subseteq T$ and $O\text{-}rad_M(L) \subseteq T$. Then

$$O\text{-}rad_M(N) + O\text{-}rad_M(L) \subseteq T.$$

This is a contradiction. Then $N + L = M$.

Since $N \subseteq O\text{-}rad_M(N)$, $L \subseteq O\text{-}rad_M(L)$ and $N + L = M$, it follows that

$$O\text{-}rad_M(N) + O\text{-}rad_M(L) = M.$$

■

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