ELSEVIER

International Journal of Plasticity 20 (2004) 981–1038

INTERNATIONAL JOURNAL OF Plasticity

www.elsevier.com/locate/ijplas

Thermodynamic framework for coupling of non-local viscoplasticity and non-local anisotropic viscodamage for dynamic localization problems using gradient theory

George Z. Voyiadjis[a,*], Rashid K. Abu Al-Rub[a], Anthony N. Palazotto[b]

[a] *Department of Civil and Environmental Engineering, Louisiana State University, Baton Rouge, LA 70803, USA*

[b] *Department of Aeronautics and Astronautics, Air Force Institute of Technology, WPAFB, OH 45433-7765, USA*

Recieved in final revised form 10 March 2003

Abstract

This study develops a general consistent and systematic framework for the analysis of heterogeneous media that assesses a strong coupling between rate-dependent plasticity and anisotropic rate-dependent damage for dynamic problems within the framework of thermodynamic laws and gradient theories. The proposed formulation includes thermo-elasto-viscoplasticity (rate-dependent plasticity) with anisotropic thermo-viscodamage (rate-dependent damage); a dynamic yield criterion of a von Mises type and a dynamic damage growth criterion; the associated flow rules; thermal softening; non-linear strain hardening; strain-rate hardening; strain hardening gradients; and strain-rate hardening gradients. Since the material macroscopic thermomechanical response under dynamic loading is governed by different physical mechanisms on the meso- and macroscale levels, the proposed three-dimensional kinematical model is introduced with manifold structure accounting for discontinuous fields of dislocation interactions (plastic flow) and crack and void interactions (damage growth). The gradient theory of rate-independent plasticity and rate-independent damage that incorporates macroscale interstate variables and their higher-order gradients is generalized here for rate-dependent plasticity and rate-dependent damage to properly describe the change in the internal structure and in order to investigate the size effect of statistical inhomogeneity of

* Corresponding author. Tel.: +1-255-578-8668; fax: +1-225-578-9176.
E-mail address: voyiadjis@eng.lsu.edu (G.Z. Voyiadjis).

0749-6419/$ - see front matter
doi:10.1016/j.ijplas.2003.10.002

the evolution-related rate- and temperature dependent materials. The idea of bridging length-scales is made more general and complete by introducing spatial higher-order gradients in the temporal evolution equations of the internal state variables that describe hardening in coupled viscoplasticity and viscodamage models, which are considered here physically and mathematically related to their local counterparts. Furthermore, the constitutive equations for the damaged material are written according to the principle of strain energy equivalence between the virgin material and the damaged material; that is, the damaged material is modeled using the constitutive laws of the effective undamaged material in which the nominal stresses and strains are replaced by their effective ones. In addition, computational issues concerned with the current gradient-dependent formulation of initial-boundary value problems are introduced in a finite element context. A weak (virtual work) formulation of the non-local dynamic viscoplastic and viscodamage conditions is derived, which can serve as a basis for the numerical solution of initial boundary value problems in the sense of the finite element method. Explicit expressions for the generalized tangent stiffness matrix and the generalized nodal forces are given. The model presented in this paper can be considered as a feasible thermodynamic approach that enables one to derive various coupled gradient viscoplasticity and viscodamage theories by introducing simplifying assumptions.

Keywords: Non-local theory; Anisotropic viscodamage; Viscoplasticity; Gradient theory; Heterogeneous media

1. Introduction

Experimental observations indicated that in general the processes of cold-working, forming, machining of mechanical parts, etc. can cause an initial evolution of defects in the virgin material state in the form of localized zones, such as the nucleation of certain amount of cracks, voids, dislocations, and shear bands. Those localized defects of plasticity and damage induced in the material structure along with the subsequent defects that occur during deformation process leads to a heterogeneous (non-uniform) material behavior. Further loading of materials of this type will cause failure mechanisms to occur at localized zones of plasticity and damage. In those localized zones, many defects may undergo irreversible growth; coalescence of pre-existing cracks and voids may occur; propagation of dislocations may proceed; and new defects may nucleate and their ultimate coalescence results in failure. Moreover, intensive interaction mechanisms of the evolved defects may take place at those localized zones; such as dislocation-dislocation interaction, microdamage-microdamage interaction, crack dominated-dislocation interaction, dislocation dominated-crack interaction, dislocation/crack-grain boundary interaction, etc.

As the plasticity and damage defects localize over narrow regions of the continuum, the characteristic length-scale governing the variations of those defects and their average interactions over multiple length-scales falls far below the scale of the local state variables of classical plasticity and damage theories used to describe the response of the continuum. This leads to the loss of the statistical homogeneity in the representative volume element (RVE) and causes strong scale effects; in such a way that all the macroscopic response functions of interest (e.g. the Helmholtz free

energy, Ψ; the dissipation potential, Π; the Cauchy stress tensor, $\boldsymbol{\sigma}$; the small strain tensor, $\boldsymbol{\varepsilon}$; the stiffness tensor; $\mathbf{E}$; etc.) are sensitive to the distribution, size, and orientation of the micro-, meso- and macrostructural defects within the RVE. The plasticity and damage evolution processes are, therefore, statistically inhomogeneous at the macroscale level (at the RVE scale). This suggests that the macroscopic inelastic deformations and failure are governed by mechanisms at different scale levels (non-locality) which gives rise to the gradient effects. Thus, the gradient effect is important when the characteristic dimension of the plastic and/or damage deformation zone is of the same order as the material intrinsic length-scale, which is in the order of microns for commonly used materials (Gao et al., 1999a,b). For example dislocation interactions are observed on a mesolevel with length-scale $\sim 0.1 - 10$ μm affecting strongly the material behavior on the macrolevel with length-scale $\leqslant 100$ μm. Therefore, different methodologies rather than the local theories are necessary to adequately capture the decrease in the length-scale from the macroscale to the mesoscale level. The use of the non-local theories is increasing steadily in order to appropriately overcome this problem, which takes into account the influence of the nth nearest neighbor of the material points or the long-range microstructural interaction.

The motivation for introducing the non-local or gradient theory stems from the well-known fact that the use of classical rate-independent plasticity theory or local theory to solve both static and dynamic problems do not possess an intrinsic length-scale. This leads to numerical stability problems, such as mesh size and mesh alignment sensitivities (e.g. Bammann et al., 1999; Glema et al., 2000; Li et al., 2002), particularly, in problems exhibiting strain localization phenomena. However, several regularization approaches have been proposed in the constitutive modeling to accommodate this problem. They include: viscoplastic models (e.g. Perzyna, 1963, 1966, 1971, 1986, 1988; Needleman, 1988; Wang et al., 1996; Dornowski and Perzyna, 2000; Glema et al., 2000) thermal dissipation models (e.g. LeMonds and Needleman, 1986a,b); non-local models (e.g. Aifantis, 1984; Pijaudier-Cabot and Bazant, 1987; Bazant and Pijaudier-Cabot, 1988; Bammann et al., 1999; Voyiadjis and Deliktas, 2000a; Voyiadjis et al., 2001; Voyiadjis and Dorgan, 2001); and strain-gradient models (e.g. Aifantis, 1992; Zbib and Aifantis, 1992; de Borst and Mühlhaus, 1992; de Borst et al., 1993; Fleck and Hutchinson, 1997, 2001, Hwang et al., 2003; Wang et al., 2003).

In the literature, many non-local plasticity and damage models were proposed to introduce intrinsic length-scale measures in the constitutive equations, which can be grouped into two classes: integral models and gradient models.

Kroner (1967) and Eringen and Edelen (1972) incorporated non-local terms through integral equations of elasticity. Pijaudier-Cabot and Bazant (1987) extended this concept to continuum damage mechanics. Bazant and Ozbolt (1990) also addressed non-local anisotropic damage formulation, which are based on non-local tensorial variables. However, integration in the non-local integral models requires a global averaging procedure with resulting equations that cannot be easily linearized (de Borst and Pamin, 1996). This makes the non-local integral models computationally inefficient.

However, the integral approach generally involves an infinitely extended zone of non-local action which may be approximated by truncated Taylor series expansion, giving rise to the so-called gradient theories. Gradient approaches typically retain terms in the constitutive equations of higher-order gradients with coefficients that represent length-scale measures of the deformation microstructure associated with the non-local continuum. Aifantis (1984) was one of the first to study the gradient regularization in solid mechanics. The gradient methods suggested by Lasry and Belytschko (1988) and Mühlhaus and Aifantis (1991) provide an alternative approach to the non-local integral equations. The gradient terms in plasticity models are introduced through the yield function (e.g. Mühlhaus and Aifantis, 1991; de Borst and Pamin, 1996; Fleck and Hutchinson, 2001; Chen and Wang, 2002). The gradient damage theory has been developed for isotropic damage (e.g. Pijaudier-Cabot and Bazant, 1987; Peerlings et al., 1996) and for anisotropic damage (e.g. Voyiadjis and Deliktas, 2000a; Kuhl et al., 2000; Voyiadjis and Dorgan, 2001). This is not a review paper and therefore the above list of authors represents a sample rather than a comprehensive list. However, a fairly complete review of this type of modeling has been given by Aifantis (1995) and Bammann et al. (1999).

Gao et al. (1999a,b) proposed by the analysis of indentation experiments that the intrinsic material length parameter of strain gradient plasticity decreases as the plastic flow stress increases. However, for most metals, the flow stress increases with the strain rate and decreases with temperature increase. This causes the intrinsic material length-scale to decrease with increasing strain-rates, but to increase with temperature decrease. For example, the plastic zone ahead of the crack tip will decrease with increasing yield stress, which for small scale yielding is of the order of microns. Therefore, the consideration of strain-rate effect and temperature variation on gradient plasticity and damage, particularly in dynamic problems, becomes more necessary. Existing theories of gradient plasticity and damage, however, have failed to explain such behavior. Moreover, although it has been shown that the viscoplasticity theory regularizes the solution by introducing implicitly length-scale through the viscous parameter, the numerical results still show a mesh dependency (Wang et al., 1996; Wang and Sluys, 2000; Glema et al., 2000). This serves as our first motivation for this study. Very limited work has been carried out to investigate the influence of strain-rate effect and temperature variation on the gradient-enhanced plasticity and/or damage. In fact very few viscoplastic and/or viscodamage gradient-enhanced models have been proposed until now. Fremond and Nedjar (1996) proposed a combined gradient- and rate-dependent damage model for quasi-brittle materials and performed also two-dimensional analysis. Wang et al. (1998) proposed a gradient viscoplasticity model used to analyze stationary and propagative instabilities. Aifantis et al. (1999) and Oka et al. (2000) proposed a gradient-dependent viscoplastic constitutive model for water-saturated clay, where gradients of the volumetric viscoplastic strain were introduced into the constitutive equations. Di Prisco et al. (2002) modified a pre-existing elasto-viscoplastic constitutive model for granular soils according to gradient and non-local approaches. Gurtin (2002) generalized a rate-independent constitutive theory that includes dependences on a tensorial measure of geometrically necessary dislocations to single-crystal viscoplasticity

using the gradient theory. Moreover, Gurtin (2003) developed a throritical concept for small-deformation viscoplasticity that allows for dependences on plastic strain-gradients. His theory accounts for microstuctural behavior on two length-scale levels through microstresses whose working accompanies flow. In the same framework of Gurtin (2003) theory, Saczuk et al. (2003) proposed a theoretical concept for the analysis of large-deformation viscoinelasticity evolution in heterogeneous media depending on macro- and micro-deformation gradients in plasticity and damage. Taylor et al. (2002) have used the gradient-dependent theory of plasticity to study the effect of void distribution on the ductility and flow stress of viscoplastic materials.

Often, ductile materials undergo a strong plastic deformation, which has a major influence on the damage evolution and vice versa. Thus, our second motivation for this study evolves from the fact that no consistent model realizing a strong coupling between plasticity/viscoplasticity and damage/viscodamage has been published yet. However, there are many models with weak coupling. The models that adopt two separate uncoupled damage and plastic loading surfaces with two independent associated flow rules present a week coupling between plasticity and damage. Those models are being extensively used by many authors (e.g. Chow and Wang, 1987; Simo and Ju, 1989; Lemaitre and Chaboche, 1990; Hansen and Schreyer, 1994; Zhu and Cescetto, 1995; Murakami et al., 1998; etc.). However, relatively strong coupling between plasticity and damage can be achieved by using one single smooth generalized yield surface and an associated flow rule for the plasticity and damage evolutions (e.g. Gurson, 1977; Tvergaard, 1982; Tvergaard and Needleman, 1984; Rousselier, 1987; Ehlers, 1995; Hesebeck, 2001; Mahnken, 2002). Those models obviously cannot describe all loadings correctly since a hydrostatic stress will certainly cause damage before any plastic deformation can be noticed. In addition most of those models are restricted to low damage levels or dilute distribution of defects and therefore they fail to account for the interaction of the defects adequately. Another approach to achieve this strong coupling is by using separate plasticity and damage surfaces with separate non-associated flow rules in such a way that both damage and plasticity flow rules are dependent on both the plastic and damage potentials. Voyiadjis and Deliktas (2000b) introduced a formulation for such an approach for a local based theory and Voyiadjis et al. (2001) extended it to a gradient based formulation. The later approach is adopted in this work, where the strong coupling between plasticity and damage is implemented by using two damage mechanisms. One mechanism is coupled with plasticity, while the other one occurs independent of plastic deformation. The dissipation function of the latter occurs in both the elastic and plastic domains. Abu Al-Rub and Voyiadjis (2003) introduced a formulism for such an approach for a local based theory.

To mention some of the important contributions to phenomenological damage modeling, we start with the effective stress concept of Kachanov (1958), who was the first to introduce for the isotropic case a one-dimensional variable, which may be interpreted as the effective surface density of microdamages per unit volume (Voyiadjis and Venson, 1995). Following Kachanov's pioneered work researchers in different fields applied continuum damage mechanics to their areas in fields like

brittle materials (Krajcinovic and Foneska, 1981; Krajcinovic, 1983, 1996) and ductile materials (Lemaitre and Chaboche, 1990; Lemaitre, 1992; Kachanov, 1986; Murakami et al., 1998). In the 1990′s coupling of continuum damage mechanics to plasticity have appeared (e.g. Voyiadjis and Kattan, 1992a,b, 1999; Lubarda and Krajcinovic, 1995; Voyiadjis and Abu-Lebdeh, 1993; Zhu and Cescetto, 1995; Voyiadjis and Deliktas, 2000b; Armero and Oller, 2000; Menzel et al., 2002; Nesnas and Saanouni, 2002).

The first objective of the present paper is to develop a consistent and systematic gradient-enhanced model in order to study the influence of strain-rate and temperature variation on the material intrinsic length-scales. We introduce a dipolar (i.e. strain-rate-gradient) material model that gives implicit and explicit length-scale measures in the governing equations through the use of coupled viscoplasticity–viscodamage theory and gradient theory, respectively. The second objective of this paper is to derive a general thermodynamic framework for the modeling of heterogeneous media that assesses a strong coupling between viscoplasticity and viscodamage evolution for dynamic problems with consideration of the discontinuities on the macroscale level.

This can be effectively achieved through a thermodynamic framework for the development of a continuum thermo-elasto-viscoplastic and thermo-viscodamage based model. The constitutive equations are derived from the first and second laws of thermodynamics, the expression of Helmholtz free energy, the Clausius–Duhem inequality, the maximum dissipation principle, generalized normality, and the thermomechanical heat balance equation. The non-locality is introduced here through the damage variable and the viscoplasticity and viscodamage hardening variables. The first-order gradients in the gradient-dependent theory are disregarded and the second-order gradients are mainly considered in this work. The local viscoplasticity and viscodamage hardening variables and their corresponding second-order gradients are considered mathematically dependent on each other, but each gives different physical interpretations that guide one to different evolution equations allowing one to computationally introduce independently the macroscale and mesoscale levels influence.

The outline of this paper is as follows: in Section 2, the effective configuration concept in damage theory as proposed by Kachanov (1958) is recalled. In Section 3, we outline a general thermodynamic framework for the non-local elasto-viscoplastic and viscodamage material behavior with thermal effects using the gradient-dependent theory. In Sections 4 and 5 the rate-type constitutive stress-strain relation and the thermomechanical heat balance equation are derived, respectively. Finally, in Section 6 computational aspects of the current gradient-dependent formulation are introduced in a finite element context.

2. Effective configuration

Continuum damage models based on the effective stress space were introduced by Kachanov (1958) and later by Rabotnov (1968) who were the first to introduce for

the isotropic case a one-dimensional variable, which may be interpreted as the effective surface density of microdamage per unit volume (Voyiadjis and Venson, 1995). To illustrate the Kachanov's concept, consider a uniform bar subjected to a uniaxial tensile stress, σ, as shown in Fig. 1. The cross-sectional area of the bar in the stressed damaged configuration is A and it is assumed that both voids and cracks appear as damage in the bar. The uniaxial tensile force, T, acting on the bar is easily expressed using the formula $T = \sigma A$. In order to use the principles of continuum damage mechanics, we consider a fictitious undamaged configuration (effective configuration) of the bar as shown in Fig. 1(b). In this configuration all types of damage, including both voids and cracks, are removed from the bar. The effective stressed cross-sectional area of the bar in this configuration is denoted by $\bar{A}$ and the effective uniaxial stress is $\boldsymbol{\sigma}$. The bars in both the damaged and the effective undamaged configurations are subjected to the same tensile force T. Therefore, considering the effective undamaged configuration, we obtain $T = \bar{\sigma}\bar{A}$. Equating the two expressions of T obtained from both configurations, one obtains the following expression for the effective uniaxial stress $\bar{\sigma}$ (Kachanov, 1958; Rabotnov, 1968):

$$\bar{\sigma} = \frac{\sigma}{1-\phi} \text{ where } \phi = \frac{A-\bar{A}}{A} \tag{1}$$

However, Eq. (1) is a mathematical definition of the effective stress although it may be interpreted as the average stress acting on an effective area of the material (Kachanov, 1986; Lemaitre and Chaboche, 1990; Lemaitre, 1992). In order to give it a general physical meaning, it is necessary to use the corresponding damage-free material (i.e. virgin material) in the mesoscale to represent the 'effective' concept of Eq. (1) for a macroscopically damaged material. Thus, a proper correlating hypothesis between two material scale levels can be obtained through using the non-local damage variable $\hat{\phi}$. It is then important to emphasize that Kachanov's definition given by Eq. (1) can be generalized to a non-local one, for the case of isotropic

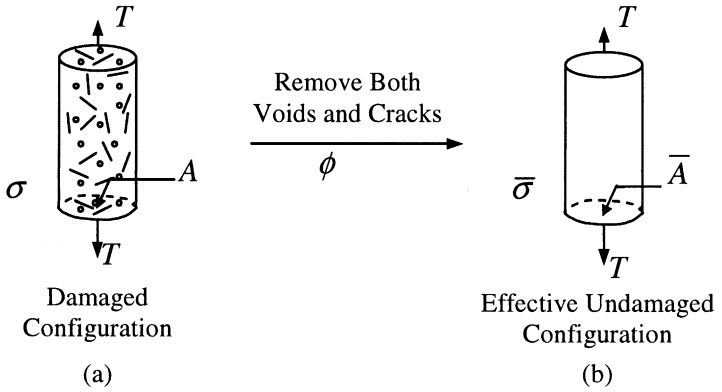

Fig. 1. A cylindrical bar subjected to uniaxial tension: both voids and cracks are removed simultaneously (Voyiadjis and Kattan, 1999; Kattan and Voyiadjis, 2001).

damage and under a general state of stress, as follows:

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{1-\hat{\phi}} \tag{2}$$

where the possible approaches in interpreting the non-local quantity $\hat{\phi}$ will be discussed thoroughly later in this paper.

Many researchers tend to adopt the traditional simple isotropic scalar damage variable, "$(1-\phi)$", which is generalized here as $(1-\hat{\phi})$ where $\hat{\phi}$ is a non-local quantity, to model the material micro-damage mechanism, in which all components of the material stiffness are degraded by the same scalar damage parameter, ϕ (or $\hat{\phi}$ in this work). However, to ensure a more general formulation of the principles of damage mechanics, the case of anisotropic damage will be assumed in this work. In this case different levels of damage are related to the principal directions, and thus a simple scalar damage parameter is no longer sufficient to quantify damage in all directions. Instead, the anisotropic phenomenon of the microdamage (i.e. microcracks and microvoids) distribution in the material is interpreted using a non-local symmetric second-order damage tensor, $\hat{\boldsymbol{\phi}}$.

The linear elastic constitutive equations for the damaged material are expressed by Eq. (1). That is, the damaged material is modeled using the constitutive laws of the effective undamaged material in which the Cauchy stress tensor, $\boldsymbol{\sigma}$, can then be replaced by the effective stress tensor, $\boldsymbol{\sigma}$ (Murakami and Ohno, 1981) as follows:

$$\bar{\sigma}_{ij} = \hat{M}_{ikjl}\sigma_{kl} \tag{3}$$

where $\hat{\mathbf{M}}$ is the fourth-order damage-effect tensor. $\hat{\mathbf{M}}$ which is then expressed in terms of $\hat{\boldsymbol{\phi}}$ characterizes the notion of non-uniform distribution and interaction of microdamage (microcracks and microvoids) over multiple length-scales at which first and second nearest neighbor effects of non-local character are significant, similar to the homogenization theory.

Many different expressions for $\hat{\mathbf{M}}$ have been proposed in the literature in order to symmetrize the effective stress tensor, $\boldsymbol{\sigma}$. A comprehensive review of the most widely used expressions are presented by Voyiadjis and Park (1997). The following expression for $\hat{\mathbf{M}}$, which is proposed by Cordebois and Sidoroff (1979) for the local concept of damage, is used here due to its attractiveness in the mathematical formulation, such that:

$$\hat{M}_{ikjl} = 2\Big[\Big(\delta_{ik} - \hat{\phi}_{ik})\delta_{jl} + \delta_{ik}\Big(\delta_{jl} - \hat{\phi}_{jl})\Big]^{-1} \tag{4}$$

where $\hat{\boldsymbol{\phi}}$ is the non-local damage variable whose evolution will be defined later and δ_{ij} is the Kronecker delta. Note that the fourth-order tensor $\hat{\mathbf{M}}$ exhibits the major symmetries only (i.e. $\hat{M}_{ikjl} = \hat{M}_{jlik}$).

The elastic-damage stiffness, $\mathbf{E}$, is given by Voyiadjis and Park (1999) as follows:

$$E_{ijkl} = \hat{M}^{-1}_{imjn}\bar{E}_{mnpq}\hat{M}^{-1}_{pkql} \tag{5}$$

where

$$\widehat{M}^{-1}_{ikjl} = \frac{1}{2}\left[\left(\delta_{ik} - \widehat{\phi}_{ik}\right)\delta_{jl} + \delta_{ik}\left(\delta_{jl} - \widehat{\phi}_{jl}\right)\right] \tag{6}$$

and $\bar{\mathbf{E}}$ is the fourth-order elastic moduli tensor given by:

$$\bar{E}_{ijkl} = K^e\delta_{ij}\delta_{kl} + 2G^e\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \tag{7}$$

where K^e is the bulk-modulus and G^e is the shear-modulus. It is noteworthy that the elastic-damage stiffness, $\bar{\mathbf{E}}$, exhibits the major and minor symmetries similar to the elastic stiffness, $\bar{\mathbf{E}}$.

In order to derive the transformation relations between the damaged and the hypothetical undamaged (effective configuration) states of the material, the elastic energy equivalence hypothesis (Sidoroff, 1981) is utilized here. This hypothesis assumes that each of the elastic energy density in terms of effective and nominal stress and corresponding strain quantities must be equal. Thus, the elastic strain energy is equated to the effective elastic strain energy such that:

$$\bar{\varepsilon}^e_{ij} = \widehat{\mathrm{M}}^{-1}_{ikjl}\varepsilon^e_{kl} \tag{8}$$

where $\bar{\varepsilon}^e$ is the effective elastic strain.

For small strain problems an additive decomposition of the rate of the total strain tensor, $\dot{\boldsymbol{\varepsilon}}$, can be assumed with $\dot{\boldsymbol{\varepsilon}}^e$ being the elastic component and $\dot{\boldsymbol{\varepsilon}}^{vi}$ being the corresponding viscoinelastic component such that:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^{vi}_{ij} \tag{9}$$

The elastic strain, $\boldsymbol{\varepsilon}^e$, is the reversible part of the total strain which is attributed to the elastic distortions, cracks closure, and voids contraction upon unloading (but not healing), while the viscoinelastic strain, $\boldsymbol{\varepsilon}^{vi}$, designates the irreversible part of the total strain which is attributed to viscoplastic distortions and viscodamage strains characterized by the lack of cracks closure and voids contraction which cause permanent deformations. The lack of cracks closure and voids contraction can be due to the constraints set up by the interacting (micro)-cracks, (micro)-voids, dislocation movements, and external/ internal interfaces.

3. Non-local constitutive modeling for dynamic loading

3.1. Internal state variables

In this work, thermal, elastic, viscoplastic (rate-dependent plasticity), and viscodamage (rate-dependent damage or creep damage) material behavior is considered. This means that the stress path, strain rate, temperature material dependence, and the nonlinear material response are all considered in this work. Thus the dependent constitutive variables are functions of the elastic strain tensor, $\boldsymbol{\varepsilon}^{\mathrm{e}}$, the absolute

temperature, T, the temperature gradient vector, $\nabla_i T$, and n_{int}- of phenomenological internal state variables, $\aleph_k$ $(k = 1, ..., n_{\text{int}};\ n_{\text{int}} \geqslant 1)$. Hence, within the thermodynamic framework and considering the assumption of infinitesimal displacements/strain relationships, the Helmholtz free energy density function can be written as (Coleman and Gurtin, 1967; Lubliner, 1990; Lemaitre and Chaboche, 1990; Doghri, 2000):

$$\boldsymbol{\Psi} = \tilde{\boldsymbol{\Psi}}\left(\varepsilon_{ij}^{e},\ T,\ \nabla_i T;\ \aleph_k\right) \tag{10}$$

Since the main objective is to develop the rate type constitutive equations for a thermoviscoplastic and thermoviscodamage material, the effects of strain rate, viscoplastic strain hardening/ softening, viscodamage strain hardening/ softening, micro-damage mechanisms, and thermomechanical coupling have to be considered. In order to describe such mechanisms, a finite set of internal state variables, $\aleph_k$, representing either a scalar or a tensorial variable are assumed, such that:

$$\aleph_k = \tilde{\aleph}_k\left(\Xi_n,\ \nabla^2 \Xi_n\right) \tag{11}$$

where Ξ_n is a set of viscoplasticity and viscodamage hardening internal state variables, and $\nabla^2 \Xi_n$ is the corresponding second-order gradient (Laplacian) of Ξ_n. The state variables in this gradient-enhanced approach are no longer independent; therefore, special care must be taken to properly account for state variable coupling between Ξ_n and $\nabla^2 \Xi_n$ (Lacey et al., 2000). Moreover, setting Ξ_n and $\nabla^2 \Xi_n$ as dependent internal state variables allows one to computationally introduce the effects of the material defects in the mesoscale on the macroscale response. Also, introducing those higher-order variables in the Helmholtz free energy allows the two different physical phenomena in the meso- and macroscales to be identified separately with different evolution equations. This approach is considered in this work.

We make use here of the postulate of the isotropic influence (de Borst et al., 1993; Askes et al., 2000) of the averaging of the evolution equations of the assumed internal state variables, Ξ_n, over a representative volume element (RVE), which will be discussed thoroughly in the subsequent sections. Thus, the first-order gradients are disregarded and the second-order gradients (Laplacian) are mainly considered in this work. The set of the macro internal state variables, Ξ_n, is postulated as follows:

$$\Xi_n = \tilde{\Xi}_n\left(p, \alpha_{ij}, r, \Gamma_{ij}, \phi_{ij}\right) \tag{12}$$

where p denotes the accumulative equivalent viscoplastic strain and $\boldsymbol{\alpha}$ denotes the flux of the residual stress (backstress). p Is associated with the isotropic hardening and $\boldsymbol{\alpha}$ with the kinematic hardening in the viscoplastic flow process. Similarly, r denotes the accumulative viscodamage and $\boldsymbol{\Gamma}$ denotes the flux of the residual stress (kinematic hardening) in the viscodamage growth process. These viscoplasticity and viscodamage hardening variables are introduced in the Helmholtz free energy density in order to provide sufficient details of the deformation defects (cracks, voids, mobile and immobile dislocation densities) and their interactions, and to properly

(i.e. physically) characterize the material microstructural behavior. Those variables will provide an adequate characterization of these defects in terms of size, orientation, distribution, spacing, interaction among defects, and so forth. Moreover, in order to be able to achieve this, the macroscale discontinuities influence needs to be addressed and implemented properly in the modeling of the material behavior.

For the strain-softening regime of the material behavior, the non-homogenous states of deformation can appear as localized regions. A suitable description of the evolution of such zones can be obtained with the use of a non-local or gradient theory, of which examples are given by Aifantis (1984); Mühlhaus and Aifantis (1991); Bazant et al. (1984); and Voyiadjis et al. (2001). The gradient theory introduces in the material constitutive equations higher-order deformation gradients with coefficients that represent length-scale measures that characterize microstructural links with the non-local continuum. An attempt is made here to account for the non-uniform macroscale viscoplastic and viscodamage distribution on the overall macroscale response by assuming the thermoelastic Helmholtz free energy density $\boldsymbol{\Psi}$ to depend not only on the macroscopic response associated with the internal variables Ξ_n, but also on its macroscopic spatial higher-order gradients $\nabla^2\Xi_n$. Both Ξ_n and $\nabla^2\Xi_n$ are considered dependent on each other. However, each gives different physical interpretations that guide one to different evolution equations for Ξ_n and $\nabla^2\Xi_n$. This postulate is motivated through the fact that certain internal state variables such as the statistically-stored dislocation and geometrically-necessary dislocation densities do not necessarily have the same evolution equations, but they are dependent on each other. They affect each other such that this interaction is introduced in their evolution equations. The set of internal state variables $\nabla^2\Xi_n$ is postulated as follows:

$$\nabla^2\Xi_n = \nabla^2\tilde{\Xi}_n\left(\nabla^2 p, \nabla^2\alpha_{ij}, \nabla^2 r, \nabla^2\Gamma_{ij}, \nabla^2\phi_{ij}\right) \tag{13}$$

where $\nabla^2(\cdot)$ denotes the second-order gradient or Laplacian of $(\cdot)$. The assumed dependence of the Helmholtz free energy on the distinct variables $\nabla^2\Xi_n$ is also motivated by the necessity to include length-scale measures into the equations of state that link the mesoscale interactions to the macroscale viscoplasticity and viscodamage, which can not be captured by Ξ_n variables alone.

Material deformation in metals enhances dislocation formation, dislocation motion, and dislocation storage. Dislocation storage causes material hardening. Stored dislocations generated by trapping each other in a random way are referred to as statistically-stored dislocations (SSD), while the stored dislocations required for compatible deformation within the polycrystal are called geometrically-necessary dislocations (GND), which their presence causes additional storage of defects and increases the deformation resistance by acting as obstacles to the SSD (Gao et al., 1999a,b). The viscoplasticity internal variables $\nabla^2 p$ and $\nabla^2\boldsymbol{\alpha}$ are related to the total dislocation density and the microdamages (i.e. microcracks and microvoids) evolution, which introduces the viscoplasticity long-range microstructural interaction between SSD and GND, dislocations and microdamages, and dislocations and external/ internal interfaces. An example of $\nabla^2 p$ is the forest hardening mechanism arising from the dislocations themselves or the presence of GNDs that are necessary

for maintaining the deformation compatibility (Ashby, 1970). The presence of microcracks/microvoids as lattice defects, which impedes the movement of dislocations, is an example of $\nabla^2\boldsymbol{\alpha}$ (Acharya and Bassani, 2000; Bassani, 2001). Both $\nabla^2 p$ and $\nabla^2\boldsymbol{\alpha}$ play as an explicit link between the viscoplasticity hardening at the mesoscale and the behavior of the homogenous equivalent material at the macroscale. The use of the gradient theory is made to achieve this bridging.

On the other hand, the viscodamage hardening presented by the internal state variables $\nabla^2 r$ and $\nabla^2\boldsymbol{\Gamma}$ accounts for the incompatibility of the microcrack and microvoid nucleation, growth, and coalescence at high strain rates resulting from microcrack and microvoid growth arrested by other microdefects. The damage variable $\nabla^2\boldsymbol{\phi}$ reflects the long-range microstructural deterioration due to nucleation, growth, and coalescence of voids, cavities, and microcracks. It may also account for internal embedded crack-tip stress variations introduced by crack pile-ups and, moreover, for the lack of a proper statistical distribution of microcracks and microvoids due to viscodamage localization.

The determination of the evolution of the assumed internal state variables is the main challenge of the modern constitutive modeling. This can be effectively achieved, so far, through the thermodynamic principles for the development of a continuum thermo-elasto-viscoplastic and thermo- viscodamage based model. That is, use is made of the balancing laws, the conservation of mass, linear and angular momenta, and the first and second laws of thermodynamics. Those fundamental laws of continuum mechanics can be written as follows (Coleman and Gurtin, 1967; Lubliner, 1990; Lemaitre and Chaboche, 1990; Doghri, 2000):

(i) Conservation of mass

$$\dot{\rho} + \rho v_{i,i} = 0 \tag{14}$$

(ii) Balance of linear momentum

$$\sigma_{ij,j} + \rho b_i = \rho \dot{v}_i \tag{15}$$

(iii) Balance of moment of momentum

$$\sigma_{ij} = \sigma_{ji} \tag{16}$$

(iv) Conservation of energy (first law of the thermodynamics)

$$\rho \dot{e} = \sigma_{ij}\dot{\varepsilon}_{ij} + \rho r_{ext} - q_{i,i} \tag{17}$$

(v) and the Clausius-Duhem inequality

$$\sigma_{ij}\dot{\varepsilon}_{ij} - \rho\left(\dot{\Psi} + \eta\dot{T}\right) - \frac{1}{T} q_i \nabla_i T \geqslant 0 \tag{18}$$

where ρ, υ, b, e, r_{ext}, η, and q are the mass density, the velocity vector, the body force vector, the internal energy density, the density of external heat, the specific entropy, and the heat flux vector, respectively. Meanwhile, e, Ψ, T, and η are related by:

$$\Psi = e - T\eta \tag{19}$$

For the purpose of describing the viscoinelastic behavior of the materials involved in the dynamic loading, an additive decomposition of the specific free energy function, Ψ, into thermoelastic, thermoviscoplastic, and thermoviscodamage parts is assumed here, such that:

$$\begin{aligned}\tilde{\Psi}\left(\varepsilon^e_{ij}, T, \nabla_i T; \aleph_k\right) = {} & \tilde{\Psi}^{te}\left(\varepsilon^e, T, \nabla_i T, \phi_{ij}, \nabla^2\phi_{ij}\right) + \tilde{\Psi}^{tvp} \\ & \times \left(T, \nabla_i T, p, \nabla^2 p, \alpha_{ij}, \nabla^2\alpha_{ij}\right) + \tilde{\Psi}^{tvd} \\ & \times \left(T, \nabla_i T, r, \nabla^2 r, \Gamma_{ij}, \nabla^2\Gamma_{ij}, \phi_{ij}, \nabla^2\phi_{ij}\right)\end{aligned} \tag{20}$$

where Ψ^{te} is the thermoelastic stored energy, while $\Psi^{\mathrm{tvp}}$and Ψ^{tvd} are the energies stored due to material hardening in viscoplasticity and viscodamage deformation mechanisms, respectively. The decomposition of Ψ into Ψ^{te} and Ψ^{tvp} is presently considered well established (e.g. Lubliner, 1990; Lemaitre and Chaboche, 1990; Voyiadjis and Kattan, 1999; Doghri, 2000). The addion of Ψ^{tvd}, as proposed by some authors (e.g. Voyiadjis and Park, 1999; Voyiadjis and Kattan, 1999) for simpler models than the one presented here, is an assumption based on the fact that damage has a distinct morphology that is different from the other deformation mechanisms. Furthermore, it should be noted that Eq. (20) is a partially decoupled form of the specific free energy Ψ. There is no state coupling between viscoplasticity and elasticity, but the state coupling of viscodamage with the thermoelastic and thermoviscoplastic stored energies strongly appears in the above decomposition. In particular, the damage variables $\boldsymbol{\phi}$ and $\nabla^2\phi$ appear in all portions of Ψ and the other internal state variables are expressed in the current, deformed, and damaged configurations. $\boldsymbol{\phi}$ and $\nabla^2\phi$ appear implicitly in Ψ^{tvp} through the evolution of its set of internal state variables as will be shown in Section 3.6.

Although the boundary conditions have not been explicitly stated here, they will be discussed in the thermomechanical finite element formulation of Section 6. Further, with the present definition of Ψ the state laws and the internal viscoinelastic dissipation are given below.

3.2. General thermodynamic formulation

According to the definition given above for Ψ, the time derivative of Eq. (10) with respect to its internal state variables is given by:

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\varepsilon^e_{ij}}\dot{\varepsilon}^e_{ij} + \frac{\partial\Psi}{\partial T}\dot{T} + \frac{\partial\Psi}{\partial\nabla_i T}\nabla_i\dot{T} + \frac{\partial\Psi}{\partial\aleph_k}\dot{\aleph}_k \tag{21}$$

where from Eqs. (11–13) one can consider the following:

$$\frac{\partial \Psi}{\partial \aleph_k}\dot{\aleph}_k = \frac{\partial \Psi}{\partial \Xi_n}\dot{\Xi}_n + \frac{\partial \Psi}{\partial \nabla^2 \Xi_n}\nabla^2 \dot{\Xi}_n \tag{22}$$

with

$$\frac{\partial \Psi}{\partial \Xi_n}\dot{\Xi}_n = \frac{\partial \Psi}{\partial p}\dot{p} + \frac{\partial \Psi}{\partial \alpha_{ij}}\dot{\alpha}_{ij} + \frac{\partial \Psi}{\partial r}\dot{r} + \frac{\partial \Psi}{\partial \Gamma_{ij}}\dot{\Gamma}_{ij} + \frac{\partial \Psi}{\partial \phi_{ij}}\dot{\phi}_{ij} \tag{23}$$

and

$$\begin{aligned}\frac{\partial \Psi}{\partial \nabla^2 \Xi_n}\nabla^2 \dot{\Xi}_n = {} & \frac{\partial \Psi}{\partial \nabla^2 p}\nabla^2 \dot{p} + \frac{\partial \Psi}{\partial \nabla^2 \alpha_{ij}}\nabla^2 \dot{\alpha}_{ij} + \frac{\partial \Psi}{\partial \nabla^2 r}\nabla^2 \dot{r} + \frac{\partial \Psi}{\partial \nabla^2 \Gamma_{ij}}\nabla^2 \dot{\Gamma}_{ij} \\ & + \frac{\partial \Psi}{\partial \nabla^2 \phi_{ij}}\nabla^2 \dot{\phi}_{ij}\end{aligned} \tag{24}$$

Substituting the rate of the Helmholtz free energy density, Eq. (21), into the Clausius–Duhem inequality, Eq. (18) along with Eq. (9), one obtains the following thermodynamic constraint:

$$\begin{aligned}&\left(\sigma_{ij} - \rho\frac{\partial \Psi}{\partial \varepsilon^e_{ij}}\right)\dot{\varepsilon}^e_{ij} + \sigma_{ij}\dot{\varepsilon}^{vi}_{ij} - \rho\left(\frac{\partial \Psi}{\partial T} + \eta\right)\dot{T} - \rho\frac{\partial \psi}{\partial \nabla_i T}\nabla_i \dot{T} - \rho\frac{\partial \Psi}{\partial \aleph_k}\dot{\aleph}_k \\ &- \frac{q_i}{T}\nabla T \geqslant 0\end{aligned} \tag{25}$$

Assuming that the axiom of entropy production holds, then the above inequality equation results in the following thermodynamic state laws:

$$\sigma_{ij} = \rho\frac{\partial \Psi}{\partial \varepsilon^e_{ij}}; \quad \eta = -\frac{\partial \Psi}{\partial T}; \quad \frac{q_i}{\dot{T}} = \rho\frac{\partial \psi}{\partial \nabla_i T}; \quad \sum_k = \rho\frac{\partial \Psi}{\partial \aleph_k}(k = 1, ..., 10) \tag{26}$$

The above equations describe the relations between the state variables (observable and internal) and their associated thermodynamic conjugate forces. These thermodynamic forces conjugate to their state variables are listed in Table 1, where $\Sigma_k = \{\mathbf{Y}, \mathbf{Y}^g, R, R^g, \mathbf{X}, \mathbf{X}^g, K, K^g, \mathbf{H}, \mathbf{H}^g\}$ $(k = 1, ..., 10)$ are the conjugate forces corresponding to the viscoplastic and viscodamage internal state variables $\aleph_k = \{\boldsymbol{\phi}, \nabla^2\boldsymbol{\phi}, p, \nabla^2 p, \boldsymbol{\alpha}, \nabla^2\boldsymbol{\alpha}, r, \nabla^2 r, \boldsymbol{\Gamma}, \nabla^2\boldsymbol{\Gamma}\}$ $(k = 1, ..., 10)$, respectively. The stress $\boldsymbol{\sigma}$ is a measure of the elastic changes in the internal structure, while Y and Y^{g} are measures of the elastic-damage changes in the internal structure resulting from crack closure and voids contraction during the unloading process. The conjugate forces R, R^g, X and X^{g} are measures of viscoplastic changes in the internal structure, while K, K^g, H and H^{g} are measures of the viscodamage changes in the internal structure.

The state laws of the assumed internal state variables, Eq. $(26)_4$, are obtained from Table 1 and are expressed in Table 2 in terms of their associated internal state variables. The superscript 'g' in Tables 1 and 2 indicates the thermodynamic conjugate

Table 1
Thermodynamic state variables and their corresponding conjugate forces

State variables		Associated conjugates
Observable	Internal	
$\boldsymbol{\varepsilon}$		$\boldsymbol{\sigma}$
T		η
	$\boldsymbol{\varepsilon}^{e}$	$\boldsymbol{\sigma}$
	$\boldsymbol{\varepsilon}^{vi}$	$-\boldsymbol{\sigma}$
	∇T	$\mathbf{q}$
	$\aleph_1=\boldsymbol{\phi}$, $\aleph_2=\nabla^2\boldsymbol{\phi}$	$\Sigma_1=-\mathrm{Y}$, $\Sigma_2=-\mathrm{Y}^{g}$
	$\aleph_3=p$, $\aleph_7=\nabla^2 p$	$\Sigma_3=R$, $\Sigma_7=R^{g}$
	$\aleph_4=\boldsymbol{\alpha}$, $\aleph_8=\nabla^2\boldsymbol{\alpha}$	$\Sigma_4=\mathrm{X}$, $\Sigma_8=\mathrm{X}^{g}$
	$\aleph_5=r$, $\aleph_9=\nabla^2 r$	$\Sigma_5=K$, $\Sigma_9=K^{g}$
	$\aleph_6=\boldsymbol{\Gamma}$, $\aleph_{10}=\nabla^2\boldsymbol{\Gamma}$	$\Sigma_6=\mathrm{H}$, $\Sigma_{10}=\mathrm{H}^{g}$

Table 2
The thermodynamic state laws

Plasticity	Isotropic hardening	$R=\rho\dfrac{\partial\Psi^{tvp}}{\partial p}$	$R^{g}=\rho\dfrac{\partial\Psi^{tvp}}{\partial\nabla^2 p}$
	Kinematic hardening	$X_{ij}=\rho\dfrac{\partial\Psi^{tvp}}{\partial\alpha_{ij}}$	$X^{g}_{ij}=\rho\dfrac{\partial\Psi^{tvp}}{\partial\nabla^2\alpha_{ij}}$
Damage	Isotropic hardening	$K=\rho\dfrac{\partial\Psi^{tvd}}{\partial r}$	$K^{g}=\rho\dfrac{\partial\Psi^{tvd}}{\partial\nabla^2 r}$
	Kinematic hardening	$H_{ij}=\rho\dfrac{\partial\Psi^{tvd}}{\partial\Gamma_{ij}}$	$H^{g}_{ij}=\rho\dfrac{\partial\Psi^{tvd}}{\partial\nabla^2\Gamma_{ij}}$
	Damage force	$-Y_{ij}=\rho\dfrac{\partial\Psi^{te}}{\partial\phi_{ij}}$	$-Y^{g}_{ij}=\rho\dfrac{\partial\Psi^{te}}{\partial\nabla^2\phi_{ij}}$

force corresponding to the second-order gradient or Laplacian of the assumed internal state variables.

Substituting Eqs. (26) into relation (25), one reduces the Clausius–Duhem inequality in order to express the fact that the dissipation energy, Π, is necessarily positive as follows:

$$\Pi=\sigma_{ij}\dot{\varepsilon}^{vi}_{ij}-\Pi_{\mathrm{int}}-q_i\left(\frac{\nabla_i T}{T}+\frac{\nabla_i\dot{T}}{\dot{T}}\right)\geqslant 0 \tag{27}$$

where the internal viscoinelastic dissipation energy, Π_{int}, can be written as:

$$\begin{aligned}\Pi_{\mathrm{int}}&=\sum_{k=1}^{10}\sum_{k}\dot{\aleph}_k\\&=R\dot{p}+R^{g}\nabla^2\dot{p}+X_{ij}\dot{\alpha}_{ij}+X^{g}_{ij}\nabla^2\dot{\alpha}_{ij}+K\dot{r}+K^{g}\nabla^2\dot{r}+H_{ij}\dot{\Gamma}_{ij}+H^{g}_{ij}\nabla^2\dot{\Gamma}_{ij}\\&\quad-Y_{ij}\dot{\phi}_{ij}-Y^{g}_{ij}\nabla^2\dot{\phi}_{ij}\geqslant 0\end{aligned} \tag{28}$$

To this end, the following split is proposed: $n_{\text{int}} = n_{\text{int}}^{vp} + n_{\text{int}}^{vd}$, where n_{int}^{vp} and n_{int}^{vd} refer to the number of internal state variables related to viscoplastic and viscodamage (irreversible that may occur in every material) effects, respectively. In this context, this assumption leads to rewriting the dissipation energy, Π, as the summation of dissipations due to mechanical (viscoplasticity and viscodamage) and thermal (heat conduction) effects as (e.g. Coleman and Gurtin, 1967):

$$\Pi = \Pi^{vp} + \Pi^{vd} + \Pi^{th} \geqslant 0, \tag{29}$$

where

$$\Pi^{vp} = \sigma_{ij}\dot{\varepsilon}_{ij}^{vi} - R\dot{p} - R^g\nabla^2\dot{p} - X_{ij}\dot{\alpha}_{ij} - X_{ij}^g\nabla^2\dot{\alpha}_{ij} \geqslant 0, \tag{30}$$

$$\Pi^{vd} = -K\dot{r} - K^g\nabla^2\dot{r} - H_{ij}\dot{\Gamma}_{ij} - H_{ij}^g\nabla^2\dot{\Gamma}_{ij} + Y_{ij}\dot{\phi}_{ij} + Y_{ij}^g\nabla^2\dot{\phi}_{ij} \geqslant 0, \tag{31}$$

$$\Pi^{th} = -q_i\left(\frac{\nabla_i T}{T} + \frac{\nabla_i \dot{T}}{\dot{T}}\right) \geqslant 0 \tag{32}$$

This result requires all viscoinelastic work to dissipate away as heat, except for that energy which is stored because of the rearrangement of the material internal structure. Although, we write the thermoviscoinelastic dissipation function Π in the decoupled form as shown by Eq. (29); however, this does not imply that the corresponding physical mechanisms are decoupled. Strong coupling does occur in the viscoplastic potential given by Eq. (30) between viscoplasticity and viscodamage since the conjugate forces and their associated fluxes are expressed in the current, deformed, and damaged configuration of the material. Hence, two additive damage mechanisms are introduced in the dissipation function, Eq. (29); one mechanism is coupled with viscoplasticity and the other occurs independent of viscoplastic deformation.

Complementary laws can be related to the dissipation processes given by Eqs. (30–32), which implies the existence of the dissipation potential expressed as a continuous and convex scalar valued function of the flux variables as shown below:

$$\Theta\left(\dot{\varepsilon}_{ij}^{vi}, \dot{\aleph}_k, q_i/T\right) = \Theta^I\left(\dot{\varepsilon}_{ij}^{vi}, \dot{\aleph}_k\right) + \Theta^{th}(T, \nabla_i T) \tag{33}$$

The superscript 'I' designates the inelastic deformation attributed to the viscoplasticity and viscodamage deformation processes. The complementary laws are then expressed by the normality property as follows:

$$\sigma_{ij} = \frac{\partial\Theta^I}{\partial\dot{\varepsilon}_{ij}^{vi}}; \sum_k = -\frac{\partial\Theta^I}{\partial\dot{\aleph}_k}; \frac{q_i}{\dot{T}} = -\frac{\partial\Theta^{th}}{\partial(\nabla_i T)} \tag{34}$$

By using the Legendre–Fenchel transformation of the dissipation potential, Θ, one can define the corresponding dual potential with respect to force variables as:

$$\Theta^*\left(\sigma_{ij}, \sum_k, \nabla_i T\right) = \Pi\left(\sigma_{ij}, \sum_k, q_i; \dot{\varepsilon}^{vi}_{ij}, \dot{\aleph}_k, T, \nabla_i T\right) - \Theta\left(\dot{\varepsilon}^{vi}_{ij}, \dot{\aleph}_k, T, \nabla_i T\right)$$
$$= \Theta^{*I}\left(\sigma_{ij}, \sum_k\right) + \Theta^{*th}(T, \nabla_i T) \tag{35}$$

from which the complementary laws in the form of the evolution laws of flux variables as function of the dual variables can then be written as follows:

$$\dot{\varepsilon}^{vi}_{ij} = \frac{\partial \Theta^{*I}}{\partial \sigma_{ij}}; \; -\dot{\aleph}_k = \frac{\partial \Theta^{*I}}{\partial \sum_k}; \; -\frac{\nabla_i T}{\dot{T}} = \frac{\partial \Theta^{*th}}{\partial q_i} \tag{36}$$

The intrinsic dissipation, Θ^{*I}, is attributed to two distinct mechanical processes: viscoplasticity and viscodamage.

It is clearly seen that the definition of Ψ, $\Theta^{*\mathrm{I}}$, and consequently of $\dot{\aleph}_k$ ($k = 1, ..., 10$) are essential features of the formulation in order to describe the thermomechanical/microstructural behavior of the material involved in the deformation process. The associative evolution laws of $\boldsymbol{\varepsilon}^{vi}$ and $\boldsymbol{\phi}$ can be obtained by utilizing the calculus of several variables with Lagrange multipliers $\dot{\lambda}^{vp}$ and $\dot{\lambda}^{vd}$. The inelastic dissipation function $\Pi^I = \Pi^{vp} + \Pi^{vd}$ [Eq. (29)] is subjected to the two constraints, namely $f = 0$ and $g = 0$ (e.g. Voyiadjis and Kattan, 1992a,b; 1999), such that:

$$\Omega = \Pi^{vp} + \Pi^{vd} - \dot{\lambda}^{vp} f - \dot{\lambda}^{vd} g \tag{37}$$

One now makes use of the maximum viscoinelastic dissipation principle (e.g. Simo and Honein, 1990; Simo and Hughes, 1998), which states that the actual state of the thermodynamic forces ($\boldsymbol{\sigma}$, $\boldsymbol{Y}$) is that which maximizes the inelastic dissipation function over all other possible admissible states. Therefore, we maximize the objective function Ω by using the necessary conditions as follows:

$$\frac{\partial \Omega}{\partial \sigma_{ij}} = 0 \;\text{ and }\; \frac{\partial \Omega}{\partial Y_{ij}} = 0 \tag{38}$$

Substitution of Eq. (37) into Eq. (38) along with Eqs. (30) and (31) yields the thermodynamic laws corresponding to the evolution of the viscoinelastic strain rate ($\dot{\boldsymbol{\varepsilon}}^{vi}$) and the viscodamage variable ($\dot{\boldsymbol{\phi}}$), where Eq. $(38)_1$ gives the viscoinelastic strain rate as follows:

$$\dot{\varepsilon}^{vi}_{ij} = {}^{(1)}\dot{\varepsilon}^{vi}_{ij} + {}^{(2)}\dot{\varepsilon}^{vi}_{ij} \tag{39}$$

where

$$ {}^{(1)}\dot{\varepsilon}_{ij}^{vi} = \dot{\lambda}^{vp}\frac{\partial f}{\partial \sigma_{ij}}, \; {}^{(2)}\dot{\varepsilon}_{ij}^{vi} = \dot{\lambda}^{vd}\frac{\partial g}{\partial \sigma_{ij}} \tag{40} $$

Similarly, Eq. (38)$_2$ gives the viscodamage evolution law as follows:

$$ \dot{\phi}_{ij} = {}^{(1)}\dot{\phi}_{ij} + {}^{(2)}\dot{\phi}_{ij} \tag{41} $$

where

$$ {}^{(1)}\dot{\phi}_{ij} = \dot{\lambda}^{vp}\frac{\partial f}{\partial Y_{ij}}, \quad {}^{(2)}\dot{\phi}_{ij} = \dot{\lambda}^{vd}\frac{\partial g}{\partial Y_{ij}} \tag{42} $$

where f and g are the dynamic viscoplastic and viscodamage loading surfaces that will be defined in Sections 3.6 and 3.7, respectively.

Eqs. (39–42) suggest that microdamage (i.e. microcracks and microvoids) growth enhances dislocation evolution as the dislocation movement enhances microdamage evolution. Eq. (40)$_1$ is the conventional associated flow rule that is widely used in classical plasticity theories, which describes the formation of "statistically stored dislocations", while Eq. (40)$_2$ describes the irreversible strain rate due to microdamage growth and plastic flow at the tip of the crack. Similarly, Eq. (42)$_1$ indicates that dislocation movement is responsible for microdamage growth, and Eq. (42)$_2$ describes the microdamage evolution due to hydrostatic stresses. Thus, if the material is damage-free, only ${}^{(1)}\dot{\boldsymbol{\varepsilon}}^{vi}$ is used to describe the irreversible thermodynamic process evolving in time. On the other hand, if the material is dislocation-free, only ${}^{(2)}\dot{\boldsymbol{\phi}}$ is used to describe the irreversible thermodynamics. As a result one can define, respectively, the conventional equivalent viscoplastic strain rate, $\dot{p}$, and the accumulative viscodamage rate, $\dot{r}$, as follows (Voyiadjis and Deliktas, 2000b):

$$ \dot{p} = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{vi}\dot{\varepsilon}_{ij}^{vi}}, \quad \dot{r} = \sqrt{\dot{\phi}_{ij}\dot{\phi}_{ij}} \tag{43} $$

Next in order to obtain non-associative rules for the viscoplasticity and viscodamage local hardening variables (p, $\boldsymbol{\alpha}$, r, $\boldsymbol{\Gamma}$), one can assume the existence of a viscoplastic potential F and a viscodamage potential G such that they are, respectively, not equal to f and g. This postulate is essential in order to obtain nonlinear evolutions of the viscoplastic and viscodamage hardening rules, which gives a more realistic characterization of the material response in the deformation process. The complementary laws for the evolution of the local internal state variables (Ξ_n, $n = 1, \ldots, 5$) can then be obtained directly from the generalized normality rules, which are summarized in Table 3. On the other hand, by adopting the assumption of isotropic influence, the complementary laws for the evolution of the second-order gradients of the assumed local internal state variables ($\nabla^2\Xi_n$, $n = 1, \ldots, 5$) can be directly obtained by operating on the local equations of Table 3 with the Laplacian. Similarly, $\nabla^2\dot{\phi}$ can be obtained from Eq. (42). The resulting evolution laws are listed in Table 3. By doing this we enhance the coupling between the evolutions of Ξ_n and $\nabla^2\Xi_n$ which will be thoroughly demonstrated in the subsequent sections.

Table 3
The thermodynamic laws for the evolution of the internal state variables

Plasticity	Isotropic hardening	$\dot{p} = -\dot{\lambda}^{vp}\dfrac{\partial F}{\partial R}$	$\nabla^2\dot{p} = -\nabla^2\dot{\lambda}^{vp}\partial_R F - \dot{\lambda}^{vp}\nabla^2\partial_R F$
	Kinematic hardening	$\dot{\alpha}_{ij} = -\dot{\lambda}^{vp}\dfrac{\partial F}{\partial X_{ij}}$	$\nabla^2\dot{\alpha}_{ij} = -\nabla^2\dot{\lambda}^{vp}\partial_{\boldsymbol{X}} F - \dot{\lambda}^{vp}\nabla^2\partial_{\boldsymbol{X}} F$
Damage	Isotropic hardening	$\dot{r} = -\dot{\lambda}^{vd}\dfrac{\partial G}{\partial K}$	$\nabla^2\dot{r} = -\nabla^2\dot{\lambda}^{vd}\partial_K G - \dot{\lambda}^{vd}\nabla^2\partial_K G$
	Kinematic hardening	$\dot{\Gamma}_{ij} = -\dot{\lambda}^{vd}\dfrac{\partial G}{\partial H_{ij}}$	$\nabla^2\dot{\Gamma}_{ij} = -\nabla^2\dot{\lambda}^{vd}\partial_{\boldsymbol{H}} G - \dot{\lambda}^{vd}\nabla^2\partial_{\boldsymbol{H}} G$
	Damage variable	$\nabla^2\dot{\phi}_{ij} = \nabla^{2(1)}\dot{\phi}_{ij} + \nabla^{2(2)}\dot{\phi}_{ij}$ $\nabla^{2(1)}\dot{\phi}_{ij} = \nabla^2\dot{\lambda}^{vp}\partial_{\boldsymbol{Y}} f + \dot{\lambda}^{vp}\nabla^2\partial_{\boldsymbol{Y}} f$ $\nabla^{2(2)}\dot{\phi}_{ij} = \nabla^2\dot{\lambda}^{vd}\partial_{\boldsymbol{Y}} g + \dot{\lambda}^{vd}\nabla^2\partial_{\boldsymbol{Y}} g$	

3.3. Specific free energy function

As it is clearly seen in the previous section, the complexity of a model is directly determined by the form of the Helmholtz free energy Ψ and by the number of conjugate pairs of variables. Therefore the definition of Ψ constitutes a crucial point of the formulation since it is the basis for the derivation of all the constitutive equations to be described in what follows. It is possible to decouple the Helmholtz free energy into a potential function for each of the internal state variable in such a way that an analytical expression for the thermodynamic potential is given as a quadratic form of its internal state variables. However, coupling is possible in the viscoplastic potential or the viscodamage potential if they depend on more than one variable (Henry and Haslach, 2002), which makes the evolution equations more complex. Moreover, Chaboche (1991) indicated that an energy Ψ with non-quadratic kinematic hardening variables leads to abnormal results; however, coupling with temperature was not discussed there. As a matter of fact under high strain rate loading a significant temperature rise is produced due to adiabatic heating, which should be considered in the constitutive modeling. Experimental results also indicate that plastic deformation is not the dominating source of heat generation during dynamic deformation, but thermodamage coupling must be considered in the simulation for more accurate comparisons with experiments (Bjerke et al., 2002). Thus, a necessary multiplicative temperature coupling term can be introduced in the viscoplasticity and viscodamage hardening state variables for more realistic description of their evolution. The thermoelastic energy, Ψ^{te}, can then be postulated as follows:

$$\Psi^{te} = \frac{1}{2\rho}\varepsilon^e_{ij}E_{ijkl}\left(\boldsymbol{\phi}, \nabla^2\boldsymbol{\phi}\right)\varepsilon^e_{kl} - \frac{1}{\rho}\beta_{ij}\varepsilon^e_{ij}(T\text{-}T_r) - \eta_r(T\text{-}T_r) - \frac{1}{2}c \times (T\text{-}T_r)^2 - \frac{1}{2\rho\dot{T}}k_{ij}\nabla_i T\nabla_j T \tag{44}$$

On the other hand, the thermoviscoplastic and thermoviscodamage energies, Ψ^{tvp} and Ψ^{tvd}, on the long-term manifold (neglecting the short-term manifolds) are assumed as follows:

$$\rho\Psi^{tvp} = \frac{1}{2}a_1 p^2\vartheta + \frac{1}{2}b_1\left(\nabla^2 p\right)^2\vartheta + \frac{1}{2}a_2\alpha_{ij}\alpha_{ij}\vartheta + \frac{1}{2}b_2\nabla^2\alpha_{ij}\nabla^2\alpha_{ij}\vartheta \tag{45}$$

$$\rho\Psi^{tvd} = \frac{1}{2}a_3 r^2\vartheta + \frac{1}{2}b_3\left(\nabla^2 r\right)^2\vartheta + \frac{1}{2}a_4\Gamma_{ij}\Gamma_{ij}\vartheta + \frac{1}{2}b_4\nabla^2\Gamma_{ij}\nabla^2\Gamma_{ij}\vartheta \tag{46}$$

where $\mathbf{E}(\boldsymbol{\phi},\nabla^2\boldsymbol{\phi})$ is the fourth-order damage elastic tensor, $\boldsymbol{\beta}$ is the second-order tensor of the thermo-mechanical coefficients or sometimes referred to as the tangent conjugate of thermal dilatation (see e.g. Lubliner, 1990), c is the coefficient of thermal expansion, η_r is the reference entropy, a_k and $b_k (k = 1, \ldots, 4)$ are the material-dependent constants which are considered independent of temperature, T_r is the reference temperature, $\mathbf{k} = \mathrm{k}\boldsymbol{\delta}$ is the heat conductivity second-rank tensor (k being the conductivity coefficient and $\boldsymbol{\delta}$ is the Kronecker delta), and ϑ is the homologous temperature defined as $\vartheta = 1 - (T/T_m)^n$, where T_m is the melting temperature. n is the temperature softening component, which might be assumed different for each hardening mechanism.

The proposed definition of Ψ allows the derivation of the constitutive equations and the internal dissipation described next. The constitutive equations for stress, Eq. $(26)_1$, can be written from the thermodynamic potential Eq. (44) as follows:

$$\sigma_{ij} = E_{ijkl}\left(\varepsilon_{kl} - \varepsilon^{vi}_{kl}\right) - \beta_{ij}(T - T_r) \tag{47}$$

where it can be directly concluded that

$$E_{ijkl} = \rho\frac{\partial^2\Psi}{\partial\varepsilon^e_{ij}\partial\varepsilon^e_{kl}}, \quad \beta_{ij} = -\rho\frac{\partial^2\Psi}{\partial\varepsilon^e_{ij}\partial T} \tag{48}$$

Furthermore, by using Eqs. (3),(5) and (8), the nominal stress relation Eq. (47) can be written in the effective configuration as follows:

$$\bar{\sigma}_{ij} = \bar{E}_{ijkl}\bar{\varepsilon}^e_{kl} - \bar{\beta}_{ij}(T\text{-}T_r) \tag{49}$$

such that $\bar{\beta}$ is defined as:

$$\bar{\beta}_{ij} = \hat{M}_{ikjl}\beta_{kl} \tag{50}$$

The constitutive equation for entropy, Eq. $(26)_2$, can be written from the thermodynamic potential, Eqs. (44–46), assuming the decoupling between the thermal effects induced through elasticity, viscoplasticity, and viscodamage such that:

$$\eta = \eta^{te} + \eta^{tvp} + \eta^{tvd} \tag{51}$$

where

$$\eta^{te} = \eta_r + c(T - T_r) + \frac{1}{\rho}\beta_{ij}\left(\varepsilon_{ij} - \varepsilon_{ij}^{vi}\right) \tag{52}$$

$$\eta^{tvp} = \frac{1}{2\rho}\left[a_1 p^2 + b_1\left(\nabla^2 p\right)^2 + a_2\alpha_{ij}\alpha_{ij} + b_2\nabla^2\alpha_{ij}\nabla^2\alpha_{ij}\right]Z \tag{53}$$

$$\eta^{tvd} = \frac{1}{2\rho}\left[a_3 r^2 + b_3\left(\nabla^2 r\right)^2 + a_4\Gamma_{ij}\Gamma_{ij} + b_4\nabla^2\Gamma_{ij}\nabla^2\Gamma_{ij}\right]Z \tag{54}$$

In the above equations Z is given as:

$$Z = -\frac{\partial\vartheta}{\partial T} = \frac{n}{T_m}\left(\frac{T}{T_m}\right)^{n-1} \tag{55}$$

The constitutive equation for the heat flux vector $\mathbf{q}$ can be obtained from Eq. $(26)_3$ as follows:

$$q_i = -k_{ij}\nabla_j T \tag{56}$$

which is the well-known Fourier heat conduction law. The negative sign indicates the heat flow is opposite to the direction of temperature increase.

The state laws of the assumed internal state variables, Eq. $(26)_4$, are obtained using the equations outlined in Table 2 along with the thermodynamic potentials, Eqs. (44–46), and are listed in Table 4. Those conjugates are linear relations in terms of their associated internal state variables due to the particular definition of Ψ^{tvp} and Ψ^{tvd} given by Eqs. (45) and (46), respectively. Furthermore, the specific internal energy e can be found by substituting the specific free energy potentials given by Eqs. (44–46), and the specific entropy given by Eqs. (51–54) into Eq. (19). It should be noted that the specific entropy and the specific internal energy functions preserve the additive decomposition previously assumed for Ψ.

A systematic way to show the dependency between the derived local state laws and the state laws associated with their Laplacian is outlined in the subsequent section.

3.4. Weak non-local formulation using the gradient approach

As we mentioned in Section 3.1, the assumed internal state variables in the current work are no longer independent and special care must be taken to properly account

Table 4
The thermodynamic conjugate forces expressions derived from specific free energy function

Plasticity	Isotropic hardening	$R = a_1 p\vartheta$	$R^g = b_1\left(\nabla^2 p\right)\vartheta$
	Kinematic hardening	$X_{ij} = a_2\alpha_{ij}\vartheta$	$X_{ij}^g = b_2\left(\nabla^2\alpha_{ij}\right)\vartheta$
Damage	Isotropic hardening	$K = a_3 r\vartheta$	$K^g = b_3\left(\nabla^2 r\right)\vartheta$
	Kinematic hardening	$H_{ij} = a_4\Gamma_{ij}\vartheta$	$H_{ij}^g = b_4\left(\nabla^2\Gamma_{ij}\right)\vartheta$
	Damage force	$-Y_{ij} = \frac{\partial}{\partial\phi_{ij}}\begin{pmatrix}\varepsilon_{mn}^e E_{mnpq}\varepsilon_{pq}^e \\ -\beta_{mn}\varepsilon_{mn}^e(T - T_r)\end{pmatrix}$	$-Y_{ij}^g = \frac{\partial}{\partial\nabla^2\phi_{ij}}\begin{pmatrix}\varepsilon_{mn}^e E_{mnpq}\varepsilon_{pq}^e \\ -\beta_{mn}\varepsilon_{mn}^e(T - T_r)\end{pmatrix}$

for state variable coupling between Ξ_n and $\nabla^2\Xi_n$. In order to enhance this coupling, one can start by defining the evolution of the pure non-local variable $\hat{\dot{\Xi}}_n$ at position $\boldsymbol{x}$ as the weighted average of its local counterpart $\dot{\Xi}_n$ over a surrounding volume V at a small distance $|\zeta| \leqslant l_C$ from the considered point (Kroner, 1967; Pijaudier-Cabot and Bazant, 1987), such that:

$$\hat{\dot{\Xi}}_n(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})}\int_V \mathbf{h}(\zeta)\dot{\Xi}_n(\mathbf{x}+\zeta)\,\mathrm{d}V \tag{57}$$

in which

$$V_r(\mathbf{x}) = \int_V \mathbf{h}(\zeta)\,dV \tag{58}$$

Superimposed hat denotes the spatial non-local operator, l_c is an intrinsic characteristic length and $\mathbf{h}(\zeta)$ is a weight function that decays smoothly with distance and in this work is given by $\mathbf{h}(\zeta) = \mathbf{I}\ h(\zeta)$, where $\mathbf{I}$ is an identity tensor. However, the identity tensor $\mathbf{I}$ may be suitably substituted by another tensor in order to induce further anisotropic behavior of the material (Voyiadjis and Dorgan, 2001). The evolution of the local variable Ξ_n in Eq. (57) can be approximated by a Taylor expansion at $\zeta = 0$, such that:

$$\dot{\Xi}_n(\mathbf{x}+\zeta)|_{\zeta=0} = \dot{\Xi}_n(\mathbf{x}) + \nabla\dot{\Xi}_n(\mathbf{x})\zeta + \frac{1}{2}\nabla^2\dot{\Xi}_n(\mathbf{x})\zeta\zeta + \frac{1}{3!}\nabla^3\dot{\Xi}_n(\mathbf{x})\zeta\zeta\zeta + \cdots \tag{59}$$

where ∇^i denotes the i-th order gradient operator. Assuming only an isotropic influence of the averaging equation (i.e. the non-local weighting function $\mathbf{h}$ is isotropic), as we stated previously, the integrals of the odd terms in Eq. (59) vanish. Moreover, Fleck and Hutchinson (2001) showed that the inclusion of the odd gradient terms in the plastic yield function is not physically sound since a term of this type could not arise from a variational principle and, fundamentally, is inherently inconsistent with the boundary condition requirements for solid bodies. If one assumes a more general tensorial character for $\mathbf{h}$ not necessarily confined to the expression in terms of an identity tensor, then one obtains a different weighting of the individual coefficients.

Furthermore, making use of Eqs. (57) and (59) and truncating the Taylor series after the quadratic term leads to the following expression for the non-local variable $\hat{\dot{\Xi}}_n$:

$$\hat{\dot{\Xi}}_n = \frac{1}{V_r}\int_V h(\zeta)\dot{\Xi}_n(\mathbf{x})\,\mathrm{d}V + \frac{1}{2V_r}\int_V \mathrm{h}(\zeta)\nabla^2\dot{\Xi}_n(\mathbf{x})\zeta\zeta\,\mathrm{d}V \tag{60}$$

This relation can be expressed as a partial differential equation such that:

$$\hat{\dot{\Xi}}_n = \dot{\Xi}_n + \left(\frac{1}{2V_r}\int_V [h(\zeta)]\zeta\zeta\,\mathrm{d}V\right)\nabla^2\dot{\Xi}_n \tag{61}$$

However, setting $\dfrac{1}{V_r}\displaystyle\int_V [h(\zeta)\,dV = 1$, gives the weak form of the non-local evolution of $\hat{\Xi}_n$ as compared to the corresponding strong form presented in Eq. (57) as follows (note that in all equations that follow there is no sum in n index when n is repeated):

$$\dot{\hat{\Xi}}_n = \dot{\Xi}_n + \frac{1}{2} l_n^2 \nabla^2 \dot{\Xi}_n, \quad (n = 1, \ldots, 4) \tag{62}$$

such that:

$$\dot{\hat{\mathrm{p}}} = \dot{p} + \frac{1}{2} l_1^2 \nabla^2 \dot{p} \tag{63}$$

$$\dot{\hat{\alpha}}_{ij} = \dot{\alpha}_{ij} + \frac{1}{2} l_2^2 \nabla^2 \dot{\alpha}_{ij} \tag{64}$$

$$\dot{\hat{\mathrm{r}}} = \dot{r} + \frac{1}{2} l_3^2 \nabla^2 \dot{r} \tag{65}$$

$$\dot{\hat{\Gamma}}_{ij} = \dot{\Gamma}_{ij} + \frac{1}{2} l_4^2 \nabla^2 \dot{\Gamma}_{ij} \tag{66}$$

In Eqs. (62–66), the gradient parameters l_n $(n = 1, ..., 4)$ are length parameters required for dimensional consistency and they set the scales at which the gradients become important. They give rise to explicit length-scale measures. Those length measures are treated as constants in this study. However, those length measures can be functions of temperature and strain rate, which is not the subject of this study. Fleck and Hutchinson (2001) showed that the variety of plasticity phenomena at small-scale levels dictates the necessity of more than one length parameter in the gradient description. Hence, different length-scales that characterize the plasticity and damage mechanisms are presented in this work. The evolution laws of $\dot{\Xi}_n$ $(n = 1, ..., 4)$ and coresponding Laplacian $\nabla^2 \dot{\Xi}_n$ are given in Table 3.

Note that the following equality holds (Ganghoffer et al., 1999):

$$\dot{\hat{\Xi}}_n = \dot{\hat{\Xi}}_n \tag{67}$$

that is the rate of any non-local quantity is equal to its non-local rate. This Lemma is imperative for the subsequent derivations.

Following the equations listed in Table 4 for the viscoplasticity and viscodamage hardening conjugate forces, one can write the evolution equations of the corresponding non-local conjugate forces at a given temperature as follows:

$$\dot{\hat{\Sigma}}_n = c_n \dot{\hat{\Xi}}_n \vartheta \quad (n = 1, ..., 4) \tag{68}$$

where c_n $(n = 1, ..., 4)$ are material parameters that are characterized as hardening moduli. Moreover, $\dot{\hat{\Sigma}}_n$ is additively decomposed into two parts: one corresponding to the local counterpart of $\dot{\hat{\Xi}}_n$ and the other is corresponding to the Laplacian counterpart, $\nabla^2 \dot{\Xi}_n$, such that (see Table 4):

$$\dot{\hat{\Sigma}}_n = \dot{\Sigma}_n + \dot{\Sigma}_n^g \quad (n = 1, \ldots, 4) \tag{69}$$

$\dot{\Sigma}_n$ and $\dot{\Sigma}_n^g$ $(n = 1, ..., 4)$ are given at constant temperature as follows (see Table 4):

$$\dot{\Sigma}_n = a_n \dot{\Xi}_n \vartheta \quad (n = 1, \ldots, 4) \tag{70}$$

$$\dot{\Sigma}_n^g = b_n \nabla^2 \dot{\Xi}_n \vartheta \quad (n = 1, \ldots, 4) \tag{71}$$

However, substituting Eq. (62) into Eq. (68) yields:

$$\dot{\hat{\Sigma}}_n = c_n \dot{\Xi}_n \vartheta + \frac{1}{2} c_n l_n^2 \nabla^2 \dot{\Xi}_n \vartheta \quad (n = 1, \ldots, 4) \tag{72}$$

Comparing Eq. (72) with Eqs. (69–71) yields:

$$l_n = \sqrt{\frac{2b_n}{a_n}} \text{ where } c_n = a_n \ (n = 1, \ldots, 4) \tag{73}$$

Taking the Laplacian of Eq. (70) and neglecting temperature gradients one obtains:

$$\nabla^2 \dot{\Sigma}_n = a_n \nabla^2 \dot{\Xi}_n \vartheta \quad (n = 1, \ldots, 4) \tag{74}$$

Comparing the above equation with Eq. (71) yields

$$\dot{\Sigma}_n^g = \frac{b_n}{a_n} \nabla^2 \dot{\Sigma}_n \quad (n = 1, \ldots, 4) \tag{75}$$

Hence, Eq. (69) can be rewritten as follows:

$$\dot{\Sigma}_n = \dot{\Sigma}_n + \frac{b_n}{a_n} \nabla^2 \dot{\Sigma}_n \quad (n = 1, \ldots, 4) \tag{76}$$

or substituting for b_n/a_n from Eq. (73) one can rewrite the above equation as follows:

$$\dot{\hat{\Sigma}}_n = \dot{\Sigma}_n + \frac{1}{2} l_n^2 \nabla^2 \dot{\Sigma}_n \quad (n = 1, \ldots, 4) \tag{77}$$

which shows a similar relation to that of their corresponding fluxes given by Eq. (62). The above results are summarized in Table 5.

The non-local evolution equation of the damage variable $\boldsymbol{\Phi}$ can also be written as follows:

$$\dot{\hat{\phi}}_{ij} = \dot{\phi}_{ij} + a \nabla^2 \dot{\phi}_{ij} \tag{78}$$

where $a = \frac{1}{2} l_5^2$ and the evolution equations of $\boldsymbol{\Phi}$ and $\nabla^2 \phi$ are given in Table 3. Using Eq. (6) one can then write:

$$\hat{M}_{ikjl}^{-1} = M_{ikjl}^{-1} + a \nabla^2 M_{ikjl}^{-1} \tag{79}$$

where

$$M^{-1}_{ikjl} = \frac{1}{2}\left[(\delta_{ik} - \phi_{ik})\delta_{jl} + \delta_{ik}\left(\delta_{jl} - \phi_{jl}\right)\right] \tag{80}$$

and

$$\nabla^2 M^{-1}_{ikjl} = -\frac{1}{2}\left(\nabla^2\phi_{ik}\delta_{jl} + \delta_{ik}\nabla^2\phi_{jl}\right) \tag{81}$$

Table 5 shows that the variety of plasticity and damage phenomena dictates the necessity of more than one length parameter in the gradient description. However, the full utility of the proposed gradient-type theory hinges on one's ability to determine the constitutive length-scale parameters ℓ_n $(n = 1, \ldots, 5)$ that scale the gradient effects. We tend to express these material length-scales in terms of macroscopic measurable material parameters, Eq. (73). Tsagrakis et al. (in press) have done similar work, where they tend to express the plasticity isotropic hardening length parameter as $l_1 = \sqrt{2b_1/\mu}$ where μ is the shear modulus which is on the order of the hardening coefficient a_1. However, the material parameters b_n must be calibrated from micromechanical tests where size effects and plastic and damage heterogeneity are encountered. Examples of such tests are micro- and/or nano-indentation tests (Stelmashenko et al., 1993), micro-bending tests (Stolken and Evans, 1998), and micro-torsion tests (Fleck et al., 1994). Whereas the hardening parameters a_n can be identified from macroscopic tests such as the typical tension test.

Based on the Taylor dislocation model, Nix and Gao (1998) have identified the plasticity isotropic hardening intrinsic length-scale parameter (l_1) as the square of dislocation spacing over the Burgers vector, i.e. $\ell_1 = \lambda^2/b$. In terms of the macroscopic quantities, this length is given as $l_1 = b\left(\mu/\sigma_y\right)^2$ where σ_y is the flow stress which can be a function of strain rate and temperature (see Eq. (106)). In a recent work by the authors (Abu Al-Rub and Voyiadjis, 2003, in press; Voyiadjis and Abu Al-Rub, 2002), a micromechanical model that nonlinearly couples statistically stored dislocations (SSDs) and geometrically necessary dislocations (GNDs) is used to derive an analytical form for the length-scale parameter l_1 in terms of measurable

Table 5
The non-local thermodynamic laws and the corresponding length parameters at a given temperature

		Local	Gradient	Non-local	Length paramaeter
Plasticity	Isotropic hardening	$\dot{R} = a_1\dot{p}\vartheta$	$\dot{R}^g = b_1\nabla^2\dot{p}\vartheta$	$\dot{\tilde{R}} = \dot{R} + \frac{b_1}{a_1}\nabla^2\dot{R}$	$l_1 = \sqrt{2b_1/a_1}$
	Kinematic hardening	$\dot{X}_{ij} = a_2\dot{\alpha}_{ij}\vartheta$	$\dot{X}^g_{ij} = b_2\nabla^2\dot{\alpha}_{ij}\vartheta$	$\dot{\tilde{X}}_{ij} = \dot{X}_{ij} + \frac{b_2}{a_2}\nabla^2\dot{X}_{ij}$	$l_2 = \sqrt{2b_2/a_2}$
Damage	Isotropic hardening	$\dot{K} = a_3\dot{r}\vartheta$	$\dot{K}^g = b_3\nabla^2\dot{r}\vartheta$	$\dot{\tilde{K}} = \dot{K} + \frac{b_3}{a_3}\nabla^2\dot{K}$	$l_3 = \sqrt{2b_3/a_3}$
	Kinematic hardening	$\dot{H}_{ij} = a_4\dot{\Gamma}_{ij}\vartheta$	$\dot{H}^g_{ij} = b_4\nabla^2\dot{\Gamma}_{ij}\vartheta$	$\dot{\tilde{H}}_{ij} = \dot{H}_{ij} + \frac{b_4}{a_4}\nabla^2\dot{H}_{ij}$	$l_4 = \sqrt{2b_4/a_4}$

microstructural physical parameters. It was shown that l_1 is proportional to the mean free-path distance between dislocations. Abu Al-Rub and Voyiadjis (2003, in press), and Voyiadjis and Abu Al-Rub (2002) also presented a method for identifying l_1 from micro- and nano-indentation experiments. That work, therefore, provides an initial effort for identification of the current material intrinsic length parameters from micro or nano experiments.

Moreover, many authors tend to use a constant value for the length-scale parameter and neglecting its variation with the state of loading. For example, the damage zone ahead of the crack tip or the mean dislocation spacing will decrease with increasing strain rate and increase with decreasing temperature, which for small scale yielding is of the order of microns. This causes the intrinsic material length-scale to decrease with increasing strain-rates and to increase with temperature decrease. However, opposite behavior is anticipated for the gradient term; that is, gradients are inversely proportional to the length scale over which plastic and/or damage deformations occur. Therefore, the strain-rate effect and temperature variation are crucial to the reliability of the proposed length-scale parameters. Particularly, in dynamic problems their inclusion becomes more necessary. To the authors' best knowledge, very limited numerical investigations and experimental studies have been carried out that incorporate the influence of strain-rate and temperature variation on the gradient plasticity and damage, or more specifically, on the size effect. Motivated by this crucial observation, Abu Al-Rub and Voyiadjis (2003, in press) proposed the following evolution of the length scale associated with plasticity hardening, such that:

$$\dot{\ell}_1 = \ell_1 \nu_o \exp\left[-(U_o/kT)\{1 - \left(\sigma^*/\sigma_o^*\right)^p\}^q\right] \tag{82}$$

where ℓ_1 as given by Nix and Gao (1998) or equivalently by $\ell_1 = A\lambda$ (Abu Al-Rub and Voyiadjis, 2003, in press), where A is a material constant and λ is the mean free path distance between dislocations. ν_o is the fundamental vibrational frequency of the dislocation, p and q are material constants, k is the Boltzmann's constant, U_o is the referential activation energy, σ^* is the thermal stress, and σ_o^* is the reference thermal stress.

Similar correlation of the other material length parameters with measurable physical parameters is under investigation. Models that are based on fundamental physical laws that govern dislocation motion, micro-cracks propagation, and voids growth, nucleation, and coalescence and their interaction with various defects and interfaces could be used to shed some ensight on how the evolution of the damage and plasticity length-scales may look like. The work of Zbib and his co-authors falls within these desired models (e.g. Zbib and Diaz de la Rubia, 2002; Taylor et al., 2002; and the references quoted therein).

3.5. Viscoplasticity and viscodamage potentials

The next important step is the selection of the appropriate form of the viscoplastic potential function F and the viscodamage potential function G in order to establish

the desired constitutive equations that describe the mechanical behavior of the material. It is clearly seen in the previous part of this work that the viscodamage evolution laws are strongly coupled with viscoplasticity. To maintain this strong coupling, two independent damage mechanisms are distinguished. One mechanism is coupled with viscoplasticity, while the other occurs independent of viscoplastic deformation. The first mechanism is dominant in the case of shear stresses and the second is due to hydrostatic stresses. In order to be consistent and satisfy the generalized normality rule of thermodynamics, a proper analytical form for the viscoplastic and the viscodamage potentials need to be postulated in order to obtain the evolution equations of the assumed flux variables, such that:

$$F = f + \frac{1}{2}k_1\bar{\bar{R}}^2 + \frac{1}{2}k_2\bar{\bar{X}}_{ij}\bar{\bar{X}}_{ij} \tag{83}$$

$$G = g + \frac{1}{2}h_1\widehat{K}^2 + \frac{1}{2}h_2\widehat{H}_{ij}\widehat{H}_{ij} \tag{84}$$

where k_i and h_i $(i = 1, 2)$ are material constants used to adjust the units of the terms comprising the above equations. $\mathbf{X} = \bar{\mathbf{X}} + \bar{\mathbf{X}}^g$ is the non-local effective backstress tensor associated with the kinematic hardening and is expressed as follows:

$$\bar{\bar{X}}_{ij} = \widehat{M}_{ikjl}\widehat{X}_{kl} \tag{85}$$

where the evolution of $\widehat{\mathbf{X}}$ is given by Eq. (69) (see Table 5), from which one can write the following relations:

$$\bar{X}_{ij} = \widehat{M}_{ikjl}X_{kl} \text{ and } \bar{X}^g_{ij} = \widehat{M}_{ikjl}X^g_{kl} \tag{86}$$

$\widehat{\mathbf{M}}$ is the non-local fourth-order damage tensor and is given in terms of the non-local damage variable $\widehat{\boldsymbol{\phi}}$ (Eq. (78)) as shown in Eq. (4).

The isotropic hardening represents a global expansion in the size of the yield surface with no change in shape. Thus for a given yield criterion and flow rule, isotropic hardening in any process can be predicted from the knowledge of the function $\bar{R} = \bar{R} + \bar{R}^g$ and this function may, in principle, be determined from a single test (e.g. the tension test). Therefore, the effective isotropic hardening functions R, $\bar{R}$, and $\bar{\mathbf{R}}^g$ are related to the nominal isotropic hardening functions $\widehat{R}$, R, and R^g, respectively, as follows:

$$\bar{\bar{R}} = \frac{\widehat{R}}{1 - \widehat{r}} \tag{87}$$

or from Eq. (69)

$$\bar{R} = \frac{R}{1 - r}, \qquad \bar{R}^g = \frac{R^g}{1 - r} \tag{88}$$

where $\widehat{r}$ is defined as the non-local accumulative viscodamage with $\dot{r}$ given by Eq. (43)$_2$, such that one can write:

$$\widehat{r} = \int_0^t \sqrt{\dot{\widehat{\phi}}_{ij}\dot{\widehat{\phi}}_{ij}}\,\mathrm{d}t \tag{89}$$

Note that if $\bar{\mathrm{A}}$ is a quantity in the effective configuration, the following equality holds true:

$$\bar{\widehat{\mathrm{A}}} = \bar{\widehat{\mathrm{A}}} \tag{90}$$

This Lemma simply implies that the non-local of any quantity in the effective configuration is equal to its effective non-local quantity, thus we get:

$$\bar{\widehat{R}} = \bar{\widehat{R}}, \quad \bar{\widehat{X}}_{ij} = \bar{\widehat{X}}_{ij} \tag{91}$$

The assumed potential functions, F and G, indicate the need for two loading surfaces f and g, one for viscoplasticity and another for viscodamage, respectively. Thus, the coupled anisotropic viscodamage and viscoplasticity formulation is a two-surface model whereby anisotropic viscodamage is formulated in the spirit of viscoplasticity, complete with a viscodamage criterion and flow rules. However, it is possible to activate one or both of the surfaces depending on the corresponding criteria for viscoplasticity and viscodamage.

3.6. Viscoplasticity surface and corresponding hardening rules

3.6.1. Viscoplasticity surface

Once a material is damaged, further loading can only affect the undamaged region. Thus, the damage function g is defined in terms of the effective stresses and strains. By combining viscoplasticity with damage, it seems natural that viscoplasticity can only affect the undamaged material skeleton. Therefore, the yield function f is also defined in terms of the effective stresses and strains. For the classical J_2 rate-independent plasticity, the static yield surface (at negligible plastic strain rate) f_s is assumed to be of a von Mises type with isotropic and kinematic hardening, which is defined as follows:

$$f_s = \sqrt{3J_2} - \bar{\sigma}_{yp}(T) - \bar{\widehat{R}}\left(\bar{p}, \nabla^2\bar{p}, T\right) \leqslant 0 \tag{92}$$

where $J_2 = 1/2(\bar{\boldsymbol{\tau}} - \bar{\widehat{\mathbf{X}}}) : (\bar{\boldsymbol{\tau}} - \bar{\widehat{\mathbf{X}}})$ is the second invariant of the resultant deviatoric stress tensor $(\bar{\boldsymbol{\tau}} - \bar{\widehat{\mathbf{X}}})$, $\bar{\sigma}_{yp}$ is the initial yield strength (note that the subscript 'yp' does not indicate tensorial indices), $\bar{p}$ is the effective accumulative viscoplastic strain, and $\bar{\boldsymbol{\tau}}$ is the effective deviatoric stress tensor and is expressed in terms of the damage tensor $\widehat{\mathbf{M}}$ (given by Eq. (4)) and the corresponding damage state as follows (Voyiadjis and Kattan, 1999):

$$\bar{\tau}_{ij} = M'_{ikjl}\sigma_{kl} \quad \text{and} \quad M'_{ikjl} = \widehat{M}_{ikjl} - \frac{1}{3}\widehat{M}_{rkrl}\delta_{ij} \tag{93}$$

One can define the initial yield strength $\bar{\sigma}_{yp}$ as a function of temperature as follows:

$$\bar{\sigma}_{yp}(T) = \bar{Y}_o \vartheta \tag{94}$$

where $\vartheta = 1 - (T/T_m)^n$ and $\bar{Y}_o$ is the initial yield stress in the undamaged state at zero absolute temperature, zero viscoplastic strain, and static strain rate.

The extension of Eq. (92) to include the rate-dependent plasticity (viscoplasticity) implies that the stress state is no longer constrained to remain on the yield surface but one can have $f_s \geqslant 0$. Therefore, we define the overstress as:

$$\bar{\sigma}_v = \langle \sqrt{3J_2} - \left(\bar{\sigma}_{yp} + \bar{\bar{R}} \right) \rangle \tag{95}$$

where $\bar{\sigma}_v$ is the viscous stress in the effective configuration (or the overstress, i.e. the difference between the dynamic stress and its static counterpart) and $\langle\rangle$ denotes the MacAuley brackets defined by $\langle x \rangle = (x + |x|)/2$. $\bar{\sigma}_v$ is the common notion of viscoplasticity (Perzyna, 1966), which implies that an inelastic process can only take place if, and only if, $\bar{\sigma}_v$ is positive. In that case, $f_s \geqslant 0$. Therefore, we define the dynamic yield surface, f, as follows:

$$f = \sqrt{3J_2} - \bar{\sigma}_{yp}(T) - \bar{\bar{R}}\left(\bar{p}, \nabla^2 \bar{p}, T\right) - \bar{\sigma}_v\left(\dot{\bar{p}}, \nabla^2 \dot{\bar{p}}, \bar{p}, \nabla^2 \bar{p}, T\right) \equiv 0 \tag{96}$$

The effective rate of the accumulative viscoplastic strain, $\dot{\bar{p}}$, is defined by:

$$\dot{\bar{p}} = \sqrt{\frac{2}{3} \dot{\bar{\varepsilon}}_{ij}^{vp} \dot{\bar{\varepsilon}}_{ij}^{vp}} \tag{97}$$

where $\dot{\bar{\varepsilon}}^{vp}$ is the viscoplastic strain rate in the effective configuration. Moreover, one can adopt the energy-correlating hypothesis; each type of energy process in the damaged state is equal to the corresponding in the effective configuration. Thus, $\dot{\bar{\varepsilon}}^{vp}$ can be related to $\dot{\varepsilon}^{vp} = {}^{(1)}\dot{\varepsilon}^{vi}$ [Eq. $(40)_1$] through the use of the viscoplastic strain energy equivalence hypothesis (e.g. Lee et al., 1985; Voyiadjis and Thiagarajan, 1997; Voyiadjis and Deliktas, 2000b), which states that the viscoplastic energy in terms of the effective and nominal stress and strain quantities must be equal. This gives the following relation, such that:

$$\dot{\bar{\varepsilon}}_{ij}^{vp} = \widehat{M}_{ikjl}^{-1} \, {}^{(1)}\dot{\varepsilon}_{kl}^{vi} \tag{98}$$

where $\widehat{\mathbf{M}}^{-1}$ is the inverse of the fourth-order damage tensor given by Eq. (6).

Substituting Eq. $(40)_1$ into Eq. (98) and by making use of the effective stress definition [Eq. (3)] and the chain rule, we can write the viscoplastic strain rate in the effective configuration, $\dot{\bar{\varepsilon}}^{vp}$, as follows:

$$\dot{\bar{\varepsilon}}_{ij}^{vp} = \dot{\lambda}^{vp} \frac{\partial f}{\partial \bar{\sigma}_{ij}} \tag{99}$$

Since $\partial f/\partial \boldsymbol{\sigma} : \partial f/\partial \boldsymbol{\sigma} = 1.5$, it can be easily shown that $\dot{\bar{p}}$ defined by Eq. (97) is related to $\dot{\lambda}^{vp}$ by:

$$\dot{\bar{p}} = \dot{\lambda}^{vp} \tag{100}$$

Substituting Eq. (83) into the evolution law of $\dot{p}$ from Table 3, making use of the chain rule with Eq. $(88)_1$, and finally comparing the result with Eq. (100) yields the following expression:

$$\dot{p} = \frac{\dot{\bar{p}}}{1-r}\left(1 - k_1 \bar{\bar{R}}\right) \tag{101}$$

Note that in obtaining the above equation we set $\partial f/\partial R = -1$ since the evolution of R is independent of the overstress function $\bar{\sigma}_v$, where R characterizes the radius of the yield surface in the absence of the strain rate effect. Making use of the assumption of isotropy (i.e. neglecting the odd gradient terms) and neglecting higher-order terms, one obtains the following relation by either substituting Eq. $(88)_1$ and $\dot{\lambda}^{vp}$ from Eq. (100) into the definition of $\nabla^2 \dot{p}$ outlined in Table 3 or by simply taking the Laplacian of Eq. (101), such that:

$$\nabla^2 \dot{p} = \left(\frac{\nabla^2 \dot{\bar{p}}}{1-\widehat{r}} + \frac{\nabla^2 r}{(1-\widehat{r})^2}\dot{\bar{p}}\right)\left(1 - k_1 \bar{\bar{R}}\right) - \frac{\dot{\bar{p}}}{1-\widehat{r}} k_1 \nabla^2 \bar{R} \tag{102}$$

$\nabla^2 \bar{R}$ is obtained by substituting Eq. $(88)_1$ into Eq. (75) and making use of the Lemma presented in Eq. (67), such that:

$$\nabla^2 \bar{R} = \frac{a_1}{b_1}\bar{R}^g + \frac{\nabla^2 r}{1-\widehat{r}}\bar{R} \tag{103}$$

In classical viscoplastic models of the Perzyna-type (Perzyna, 1963, 1966), which are considered as penalty regularization of rate-dependent plasticity (viscoplasticity), the non-local consistency parameter $\dot{\lambda}^{vp} = \dot{\lambda}^{vp} + \frac{1}{2} l_1^2 \nabla^2 \dot{\lambda}^{vp}$ (note that the associated length parameter is the same as for $\dot{p}$ since Eq. (100) holds true) in the effective configuration can be replaced by an increasing function of the overstress. For example:

$$\dot{\bar{\lambda}}^{vp} = \frac{1}{\eta^{vp}}\left\langle \frac{\bar{\sigma}_v}{\bar{\sigma}_{yp} + \bar{\bar{R}}} \right\rangle^{m_1} \tag{104}$$

where m_1 is the viscoplastic rate sensitivity parameter and η^{vp} is the viscosity or fluidity parameter, which is referred to as the relaxation time according to the notation given by Perzyna (1988).

By making use of Eqs. (104), (100), and (63), one can write an expression for the overstress function $\bar{\sigma}_v$ as follows:

$$\bar{\sigma}_v = [\eta^{vp}\dot{\bar{\bar{p}}}]^{1/m_1}[\bar{\sigma}_{yp} + \bar{\bar{R}}] \tag{105}$$

Substituting $\bar{\sigma}_v$ into Eq. (96) gives the following expression for the dynamic yield surface f in the effective configuration:

$$f = \sqrt{3J_2} - [\bar{\sigma}_{yp} + \bar{\bar{R}}][1 + (\eta^{vp}\dot{\bar{\bar{p}}})^{1/m_1}] \equiv 0 \tag{106}$$

This criterion is a generalization of the classical von-Mises yield criterion $f_s = 0$, Eq. (92), for rate-dependent materials. The latter can be simply recovered by imposing $\eta^{vp} = 0$(no viscosity effect), so that one has the plasticity case $f = f_s \leqslant 0$. In the elastic domain, both f_s and f are equivalent since, in that case, $\dot{\bar{p}} = 0$. Therefore, the admissible stress states are constrained to remain on or within the elastic domain, so that one has similar to rate-independent plasticity $f \leqslant 0$. However, during the unloading process for rate dependent behavior, $f < 0$ and for a particular strain-rate does not imply that the material is in the elastic domain, but it may also be in a viscoplastic state with a smaller strain-rate. Moreover, the well-known fact that, from the relation in Eq. (104), it can be noted that as the viscosity parameter η^{vp} goes to zero (rate-independent case), the consistency parameter $\dot{\bar{\lambda}}^{vp}$ remains finite and positive (though indeterminate) since $\bar{\sigma}_v$ also goes to zero. The extended criterion given by Eq. (106) will play a crucial rule in the dynamic finite element formulation described hereafter. It also allows a generalization of the standard Kuhn–Tucker loading/unloading conditions:

$$f \leqslant 0, \quad \dot{\bar{\lambda}}^{vp} \geqslant 0, \quad \dot{\bar{\lambda}}^{vp} f = 0 \tag{107}$$

Thus, f still satisfies the constraint equation, Eq. (37), and the maximum dissipation principle, Eq. $(38)_1$. For this reason, the dynamic yield surface can expand and shrink not only by softening or hardening effects, but also due to softening/ hardening rate effects.

Furthermore, it is noteworthy that $\bar{\sigma}_{yp}$ increases with strain-rate increase, Eq. (106), and decreases with temperature increase, Eq. (94). This agrees well with the experimental observations that show that the onset of yielding depends on both strain-rate and temperature (see e.g. Johnson and Cook, 1985; Zerilli and Armstrong, 1987). Thus, the assumed expression in Eq. (94) is valid for different types of loading rates. Moreover, the right-hand-side of Eq. (106) defines the flow stress as a function of strain, strain-rate, and temperature and then converges to a great extent to the constitutive laws of Johnson and Cook (1985) and Zerilli and Armstrong (1987).

3.6.2. *Viscoplasticity hardening rules*

The evolution of the viscoplasticity driving forces associated with their corresponding internal state variables chosen for the description of isotropic and kinematic hardening are derived next. The evolution equations are derived in the effective configuration to be directly substituted in the viscoplasticity surface function f, Eq. (106).

Making use of $\dot{R}$ and $\dot{R}^g$ relations in Table 5 and the time derivative of Eq. (88) for a given temperature while maintaining the damage history constant (i.e. the damage internal state variable $\boldsymbol{\Phi}$ and temperature T are kept constant) and substituting $\dot{\bar{p}}$ for $\dot{p}$ from Eq. (101) and $\nabla^2\dot{\bar{p}}$ for $\nabla^2\dot{p}$ from Eq. (102) along with Eq. (103), one can write the following evolution equations for $\bar{R}$ and $\bar{R}^g$, respectively, as follows:

$$\dot{\bar{R}} = \frac{a_1\vartheta}{(1-\hat{r})^2}\left(1 - k_1\bar{\bar{R}}\right)\dot{\bar{p}} \tag{108}$$

$$\dot{\bar{R}}^g = S_1^R\dot{\bar{p}} + S_2^R\nabla^2\dot{\bar{p}} \tag{109}$$

where

$$S_1^R = \frac{\vartheta}{(1-\hat{r})^3}\left[b_1\left(1 - 2k_1\bar{R} - k_1\bar{R}^g\right)\nabla^2 r - k_1 a_1(1-\hat{r})\bar{R}^g\right] \tag{110}$$

$$S_2^R = \frac{b_1\vartheta}{(1-\hat{r})^2}\left(1 - k_1\bar{\bar{R}}\right) \tag{111}$$

The non-local evolution equation for the isotropic hardening in the effective configuration, $\dot{\bar{\bar{R}}}$, can then be written according to Eq. (69) or from Table 5 as follows:

$$\dot{\bar{\bar{R}}} = \dot{\bar{R}} + \dot{\bar{R}}^g \tag{112}$$

It is worthy to mention that when the rate and temperature independent response of a material is assumed, the micro-damage effects are neglected, and an associative hardening rule is used then the isotropic hardening law given by Eq. (111) reduces to that proposed by Mühlhaus and Aifantis (1991) and de Borst and Mühlhaus (1992).

In order to derive the non-local kinematic hardening evolution equation associated with viscoplasticity, we first make use of the $\dot{\boldsymbol{\alpha}}$ law in Table 3 along with the chain rule and Eq. $(86)_1$, such that we can write the following relation:

$$\dot{\alpha}_{ij} = -\dot{\lambda}^{vp}\widehat{M}_{minj}\frac{\partial F}{\partial\bar{X}_{mn}} \tag{113}$$

Substituting Eq. (83) into the above equation yields:

$$\dot{\alpha}_{ij} = -\dot{\lambda}^{vp}\widehat{M}_{minj}\left(\frac{\partial f}{\partial\bar{X}_{mn}} + k_2\bar{\bar{X}}_{mn}\right) \tag{114}$$

Since $\partial f/\partial\bar{\mathbf{X}} = -\partial f/\partial\bar{\boldsymbol{\tau}} = -\partial f/\partial\bar{\boldsymbol{\sigma}}$ as it is clear from Eq. (106), it can be easily shown by using Eq. (99) and (100) that Eq. (114) can be rewritten as follows:

$$\dot{\alpha}_{ij} = \widehat{M}_{minj}\left(\dot{\bar{\varepsilon}}^{vp}_{mn} - k_2\bar{\bar{X}}_{mn}\dot{\bar{p}}\right) \tag{115}$$

However, using the $\dot{\mathbf{X}}$ relation in Table 5 and operating on the $\bar{\mathbf{X}}$ relation of Eq. (86)$_1$ with the time derivative for a given temperature and keeping the damage history constant (i.e. the viscodamage effective tensor $\widehat{\mathbf{M}}$ and the temperature T are kept constant), we can write the following evolution equation for $\bar{\mathbf{X}}$ as follows:

$$\dot{\bar{X}}_{ij} = \widehat{M}_{ikjl}\widehat{M}_{mknl}\left(a_2\dot{\bar{\varepsilon}}^{vp}_{mn} - k_2 a_2 \dot{\bar{p}}\,\bar{\bar{X}}_{mn}\right)\vartheta \tag{116}$$

When the rate and temperature independent response of a material is assumed and the micro-damage and non-local effects are neglected then the kinematic hardening law given by Eq. (116) reduces to that proposed by Armstrong and Frederick (1966).

Again, with the assumption of isotropy (i.e. neglecting the odd gradient terms) and neglecting higher-order terms, either by utilizing Eqs. (83), (99), and (100) into the evolution law of $\nabla^2\dot{\boldsymbol{\alpha}}$ from Table 3 or by taking the Laplacian of Eq. (115), the following expression can be obtained:

$$\nabla^2\dot{\alpha}_{ij} = \widehat{M}_{minj}\left[\nabla^2\dot{\bar{\varepsilon}}^{vp}_{mn} - k_2\left(\bar{\bar{X}}_{mn}\nabla^2\dot{\bar{p}} + \dot{\bar{p}}\nabla^2\bar{X}_{mn}\right)\right] + \nabla^2\widehat{M}_{minj}\left(\dot{\bar{\varepsilon}}^{vp}_{mn} - k_2\bar{\bar{X}}_{mn}\dot{\bar{p}}\right) \tag{117}$$

$\nabla^2\bar{\mathbf{X}}$ is obtained by substituting Eqs. (86) into Eq. (75) with consideration of the Lemma presented by Eq. (67) such that:

$$\nabla^2\bar{X}_{ij} = \frac{a_2}{b_2}\bar{X}^g_{ij} - \widehat{M}_{kilj}\nabla^2 M^{-1}_{krls}\bar{X}_{rs} \tag{118}$$

where $\widehat{\mathbf{M}}$ and $\nabla^2\mathbf{M}^{-1}$are given by Eqs. (4) and (81), respectively.

Making use of Eq. (117) into the $\dot{\mathbf{X}}^g$ relation from Table 5 along with Eq. (118) and the time derivative of $\bar{\mathbf{X}}^g$ relation of Eq. (86)$_2$ for a given temperature and keeping the damage history constant, one can then write the gradient-dependent evolution equation of viscoplasticity kinematic hardening in the effective configuration, $\bar{\mathbf{X}}^{\mathrm{g}}$, as follows:

$$\dot{\bar{X}}^g_{ij} = {}^{(1)}\dot{\bar{X}}^g_{ij} + {}^{(2)}\dot{\bar{X}}^g_{ij} \tag{119}$$

with

$${}^{(1)}\dot{\bar{X}}^g_{ij} = \widehat{M}_{ikjl}\widehat{M}_{mknl}\left[b_2\nabla^2\dot{\bar{\varepsilon}}^{vp}_{mn} - k_2 b_2\bar{\bar{X}}_{mn}\nabla^2\dot{\bar{p}} - k_2\left(a_2\bar{X}^g_{mn} - b_2\widehat{M}_{pmqn}\nabla^2 M^{-1}_{prqs}\bar{X}_{rs}\right)\dot{\bar{p}}\right]\vartheta \tag{120}$$

$${}^{(2)}\dot{\bar{X}}^g_{ij} = \widehat{M}_{ikjl}\nabla^2\widehat{M}_{mknl}\left(b_2\dot{\bar{\varepsilon}}^{vp}_{mn} - k_2 b_2\bar{\bar{X}}_{mn}\dot{\bar{p}}\right)\vartheta \tag{121}$$

where

$$\nabla^2\dot{\bar{\varepsilon}}^{vp}_{ij} = \nabla^2\dot{\bar{p}}\frac{\partial f}{\partial\bar{\sigma}_{ij}} \tag{122}$$

Note that since $f=0$, Eqs. (119)–(122) are obtained by assuming that $\nabla^2(\partial_{\bar{\sigma}} f)=\partial_{\bar{\sigma}}(\nabla^2 f)=0$. This assumption implies that the considered point has yielded as well as the surrounding volume of a sphere of diameter l_c (i.e. the length parameter) and there is no change in the flow direction with position within l_c.

By taking the Laplacian of the identity $\widehat{\mathbf{M}}:\widehat{\mathbf{M}}^{-1}=\mathbf{I}$, one can write the following relation:

$$\nabla^2 \widehat{M}_{ikjl} = -\widehat{M}_{kmln}\widehat{M}_{irjs}\nabla^2 M^{-1}_{rmsn} \tag{123}$$

where $\nabla^2\mathbf{M}^{-1}$, $\widehat{\mathbf{M}}$, and $\widehat{\mathbf{M}}^{-1}$ are given by Eqs. (81), (4), and (6), respectively.

The non-local evolution equation for the kinematic hardening in the effective configuration, $\dot{\widehat{\mathbf{X}}}$, can now be expressed according to Eq. (69) or from Table 5 as follows:

$$\dot{\widehat{X}}_{ij} = \dot{\widehat{X}}_{ij} + \dot{\widehat{X}}^{g}_{ij} \tag{124}$$

It is worthy to point out here that the derived evolution equations of the non-local viscoplasticity hardening laws ($\dot{\widehat{R}}$ and $\dot{\widehat{\mathbf{X}}}$) contain both hardening terms that represent the strengthening mechanism and recovery terms that represent the softening mechanism. Both the hardening and recovery terms are affected by the static (thermal) recovery term ϑ in such a way that the functional dependence of the hardening of materials could be multiplicatively decomposed into two functions, thermal (static) and stress (dynamic). Therefore, those evolution equations characterize the time and thermal effects due to the rate and temperature dependency. Freed and his co-authors (1990, 1991, 1993) showed that, at elevated temperature, thermal recovery of materials usually plays an important rule in the deformation process. In impact dynamic related problems, the thermomechanical response of the assumed internal state variables is strongly dependent on the temperature history. This requires existence of thermal recovery (softening) terms in the evolution equations of the internal state variables.

3.7. Damage evolution criterion and corresponding hardening rules

3.7.1. Damage evolution criterion

The anisotropic viscodamage governing equations are viscoplasticity-like; analogous to the dynamic viscoplastic surface presented in the previous section. The dynamic viscodamage surface g is postulated as follows:

$$g = \sqrt{\left(\widehat{Y}_{ij}-\widehat{H}_{ij}\right)\left(\widehat{Y}_{ij}-\widehat{H}_{ij}\right)} - l - \widehat{K} - L = 0 \tag{125}$$

where the non-local damage forces $\widehat{\mathbf{Y}}$ and $\widehat{\mathbf{H}}$ are, respectively, characterizing the viscodamage evolution and the viscodamage kinematic hardening laws, l is the initial damage threshold as a function of temperature, which can have a form similar to

Eq. (94) (i.e. $l = l_o\vartheta$, where l_o is the initial damage threshold at zero absolute temperature, zero damage strain, and static strain rate), $\widehat{K}$ is the non-local damage isotropic hardening function, and L is the viscoplasticity overstress-like function which will be referred to as the overforce damage function in the following.

At negligible strain rates, one retains the rate-independent case with g_s defined as the static damage surface, such that:

$$g_s = \sqrt{\left(\widehat{Y}_{ij} - \widehat{H}_{ij}\right)\left(\widehat{Y}_{ij} - \widehat{H}_{ij}\right)} - l - \widehat{K} \leqslant 0 \tag{126}$$

Moreover, analogous to the expression for $\dot{\widehat{\lambda}}^{vp}$ [Eq. (104)], one can postulate $\dot{\widehat{\lambda}}^{vd} = \dot{\lambda}^{vd} + a\nabla^2\dot{\lambda}^{vd}$ to have the following form:

$$\dot{\widehat{\lambda}}^{vd} = \frac{1}{\eta^{vd}}\left\langle \frac{L}{l + \widehat{K}} \right\rangle^{m_2} \tag{127}$$

where m_2 is the viscodamage rate sensitivity parameter and η^{vd} is the relaxation time that corresponds to the damage growth. For generality, we assume here that the time-dependent behavior of both viscoplasticity and viscodamage mechanisms are controlled by different relaxation times associated with $\dot{\widehat{\lambda}}^{vp}$ and $\dot{\widehat{\lambda}}^{vd}$, which may not generally be the case. This is suggested to occur only for material behavior above a certain material threshold of wave speed. This is not the subject of the present work, but it will be discussed thoroughly in a forthcoming paper.

By making use of Eq. (127), one can write an expression for the overforce damage function L as follows:

$$L = \left(\eta^{vd}\dot{\widehat{\lambda}}^{vd}\right)^{1/m_2}\left(l + \widehat{K}\right) \tag{128}$$

Substituting L into Eq. (125) gives the following expression for the dynamic damage surface g:

$$g = \sqrt{\left(\widehat{Y}_{ij} - \widehat{H}_{ij}\right)\left(\widehat{Y}_{ij} - \widehat{H}_{ij}\right)} - [l + \widehat{K}][1 + (\eta^{vd}\dot{\widehat{\lambda}}^{vd})^{1/m_2}] \equiv 0 \tag{129}$$

The above postulated dynamic viscodamage function $g = 0$ is a generalization of the static damage surface $g_s \leqslant 0$, Eq. (126), for rate-dependent materials. In the undamaged domain, both g_s and g are equivalent since, in that case, $\dot{\lambda}^{vd} = 0$. Therefore, the admissible damage forces are constrained to remain on or within the undamaged domain ($g \leqslant 0$). Similar to the viscoplastic surface, the static damage surface can be simply recovered by imposing $\eta^{vd} = 0$ (no viscous effect), so that one has the rate-independent damage case $g \leqslant 0$. The model response in the viscodamage domain is then characterized by the Kuhn-Tucker complementary conditions as follows:

$$g \leqslant 0, \quad \dot{\lambda}^{vd} \geqslant 0, \quad \dot{\lambda}^{vd} g = 0 \tag{130}$$

3.7.2. Damage Hardening Rules

In order to derive the hardening evolution equations associated with the visco-damage process, we follow the same procedure presented in the former section for viscoplasticity. By substituting Eq. (84) into the evolution law of $\dot{r}$ form Table 3, the following relation is obtained:

$$\dot{r} = \dot{\lambda}^{vd}\Big(1 - h_1\widehat{K}\Big) \tag{131}$$

Note that analogous to the derivation of Eq. (101) for the viscoplasticity case, one can set $\partial g/\partial K = -1$ since the evolution of K is independent of the overforce damage function L. K characterizes the radius of the damage surface in absence of the strain rate effect. Making use of the assumption of isotropy and by either taking the Laplacian of Eq. (131) or substituting Eq. (84) into the definition of $\nabla^2\dot{r}$ from Table 3 and realizing that $\nabla^2\partial_K g = 0$ with neglecting higher-order terms, one obtains:

$$\nabla^2\dot{r} = \nabla^2\dot{\lambda}^{vd}\Big(1 - h_1\widehat{K}\Big) - h_1\dot{\lambda}^{vd}\nabla^2 K \tag{132}$$

where $\nabla^2 K$ is obtained from Eq. (75) with consideration of the Lemma presented by Eq. (67) as follows:

$$\nabla^2 K = \frac{a_3}{b_3}K^g \tag{133}$$

The evolution equations for the non-local viscodamage isotropic hardening function $\widehat{K}$ can be obtained by first making use of Eqs. (131) and (132) into $\dot{K}$ and $\dot{K}^g$ relations in Table 5 for a given temperature, such that the following expressions are obtained:

$$\dot{K} = a_3\Big(1 - h_1\widehat{K}\Big)\dot{\lambda}^{vd}\vartheta \tag{134}$$

$$\dot{K}^g = \Big[b_3\Big(1 - h_1\widehat{K}\Big)\nabla^2\dot{\lambda}^{vd} - h_1 a_3 K^g\dot{\lambda}^{vd}\Big]\vartheta \tag{135}$$

The non-local evolution equation for the viscodamage isotropic hardening law, $\dot{\widehat{K}}$, can then be written according to Eq. (69) or from Table 5 as follows:

$$\dot{\widehat{K}} = \dot{K} + \dot{K}^g \tag{136}$$

Furthermore, the evolution equation for the non-local viscodamage kinematic hardening can be obtained by utilizing Eq. (84) into $\dot{\mathbf{\Gamma}}$ and $\nabla^2\dot{\mathbf{\Gamma}}$ relations from Table 3, realizing that $\partial g/\partial\mathbf{H} = \partial g/\partial\mathbf{H}^g = -\partial g/\partial\mathbf{Y}$ as it is clear from Eq. (129), and along with Eq. $(42)_2$, such that:

$$\dot{\Gamma}_{ij} = {}^{(2)}\dot{\phi}_{ij} - h_2\dot{\lambda}^{vd}\widehat{H}_{ij} \tag{137}$$

$$\nabla^2\dot{\Gamma}_{ij} = \nabla^{2(2)}\dot{\phi}_{ij} - h_2\left(\widehat{H}_{ij}\nabla^2\dot{\lambda}^{vd} + \dot{\lambda}^{vd}\nabla^2 H_{ij}\right) \tag{138}$$

where $\nabla^2\mathbf{H}$ is obtained from Eq. (75) with consideration of the Lemma presented by Eq. (67) as follows:

$$\nabla^2 H_{ij} = \frac{a_4}{b_4} H^g_{ij} \tag{139}$$

It can then be easily shown by utilizing, respectively, Eqs. (137) and (138) into $\dot{\mathbf{H}}$ and $\dot{\mathbf{H}}^g$ laws from Table 5 for a given temperature that the following evolution equations are obtained:

$$\dot{H}_{ij} = \left(a_4{}^{(2)}\dot{\phi}_{ij} - h_2 a_4\dot{\lambda}^{vd}\widehat{H}_{ij}\right)\vartheta \tag{140}$$

$$\dot{H}^g_{ij} = \left[b_4\nabla^{2(2)}\dot{\phi}_{ij} - h_2 b_4\widehat{H}_{ij}\nabla^2\dot{\lambda}^{vd} - h_2 a_4 H^g_{ij}\dot{\lambda}^{vd}\right]\vartheta \tag{141}$$

where $\nabla^{2(2)}\phi$ is given by:

$$\nabla^{2(2)}\dot{\phi}_{ij} = \nabla^2\dot{\lambda}^{vd}\frac{\partial g}{\partial Y_{ij}} \tag{142}$$

The non-local evolution equation for the viscodamage kinematic hardening law, $\dot{\hat{\mathbf{H}}}$, can then be written according to Eq. (69) or from Table 5 as follows:

$$\dot{\widehat{H}}_{ij} = \dot{H}_{ij} + \dot{H}^g_{ij} \tag{143}$$

It is noteworthy to mention again that since $g = 0$, Eqs. (140)–(142) are obtained by assuming that $\nabla^2(\partial_Y g) = \partial_Y\left(\nabla^2 g\right) = 0$. As was previously pointed out for the viscoplasticity case that this assumption implies that the considered point is damaged as well as the surrounding volume of a sphere of diameter l_c (i.e. the length parameter) and the damage flow direction does not change with position within l_c.

Similar to the viscoplastic hardening evolution equations, the derived viscodamage evolution equations consider the dynamic recovery as well as the static recovery (thermal recovery). Besides, the static recovery occurs in both the hardening and the dynamic recovery terms. The hardening term in each of the assumed internal state variable accounts for strengthening mechanisms, while the recovery term accounts for softening mechanisms.

Finally in order to obtain the non-local viscodamage force, first one expands the state laws $\mathbf{Y}$ and $\mathbf{Y}^g$ from Table 4 such that the following expressions are obtained after some algebra:

$$Y_{ij} = -\left(\frac{1}{2}\varepsilon^e_{rs}E_{mnkl}\varepsilon^e_{kl} - \varepsilon^e_{rs}\beta_{mn}(T - T_r)\right)\widehat{M}_{manb}J_{arbsij} \tag{144}$$

$$Y_{ij}^{g} = -a\left(\frac{1}{2}\varepsilon_{rs}^{e}E_{mnkl}\varepsilon_{kl}^{e} - \varepsilon_{rs}^{e}\beta_{mn}(T-T_r)\right)\widehat{M}_{manb}J_{arbsij} \tag{145}$$

where $\mathbf{J}$ is a sixth-order constant tensor and is given by:

$$J_{arbsij} = -\frac{\partial \widehat{M}_{arbs}^{-1}}{\partial \widehat{\phi}_{ij}} = \frac{1}{2}\left(\delta_{ar}\delta_{bi}\delta_{sj} + \delta_{ai}\delta_{rj}\delta_{bs}\right) \tag{146}$$

Hence the non-local viscodamage force is obtained as the sum of Eqs. (144) and (145), such that:

$$\widehat{Y}_{ij} = -(1+a)\left(\frac{1}{2}\varepsilon_{rs}^{e}E_{mnkl}\varepsilon_{kl}^{e} - \varepsilon_{rs}^{e}\beta_{mn}(T-T_r)\right)\widehat{M}_{manb}J_{arbsij} \tag{147}$$

It is worthy to emphasize that the assumed constitutive nature of the viscoplastic and viscodamage surfaces, f and g, and their dependence on the internal variables and the second-order gradients of the internal variables is imperative, since they dictate, among other things, the length-scale of the problem and the phenomena that can be predicted by the proposed model.

4. Rate-type constitutive relation

Operating on the stress relation Eq. (47) with the time derivative yields:

$$\dot{\sigma}_{ij} = E_{ijkl}\left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^{vi}\right) + \dot{E}_{ijkl}\varepsilon_{kl}^{e} - \beta_{ij}\dot{T} - \dot{\beta}_{ij}(T-T_r) \tag{148}$$

where $\dot{\mathbf{E}}$ and $\dot{\boldsymbol{\beta}}$ are given by:

$$\dot{E}_{ijkl} = \frac{\partial E_{ijkl}}{\partial \widehat{M}_{abuv}^{-1}}\dot{\widehat{M}}_{abuv}^{-1} = \frac{\partial E_{ijkl}}{\partial \widehat{M}_{abuv}^{-1}}\frac{\partial \widehat{M}_{abuv}^{-1}}{\partial \widehat{\phi}_{mn}}\dot{\widehat{\phi}}_{mn} \tag{149}$$

$$\dot{\beta}_{ij} = \frac{\partial \beta_{ij}}{\partial \widehat{M}_{abuv}^{-1}}\dot{\widehat{M}}_{abuv}^{-1} = \frac{\partial \beta_{ij}}{\partial \widehat{M}_{abuv}^{-1}}\frac{\partial \widehat{M}_{abuv}^{-1}}{\partial \widehat{\phi}_{mn}}\dot{\widehat{\phi}}_{mn} \tag{150}$$

Making use of Eqs. (5), (9), (49), (50), and (146), one can then obtain, after some lengthy manipulations, for a general themo- elasto- viscoplastic and viscodamage flow processes the following relation:

$$\dot{\sigma}_{ij} = E_{ijkl}\left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^{vi}\right) - A_{ijkl}\dot{\widehat{\phi}}_{kl} - \beta_{ij}\dot{T} \tag{151}$$

where

$$A_{ijkl} = \left[-\frac{\partial E_{ijmn}}{\partial \bar{M}^{-1}_{abuv}} \varepsilon^{e}_{mn} + \frac{\partial \beta_{ij}}{\partial \bar{M}^{-1}_{abuv}} (T - T_r) \mid \right] \frac{\partial \widehat{M}^{-1}_{abuv}}{\partial \widehat{\phi}_{kl}}$$
$$= J_{ibjvkl} \bar{\sigma}_{bv} + \bar{M}^{-1}_{ipjq} \bar{E}_{pqau} \varepsilon^{e}_{bv} J_{abuvkl} \tag{152}$$

It in noteworthy that the above incremental stress–strain relation can be found similarly using the stress–strain relation in the effective configuration (Eq. (49)). Furthermore, substituting into Eq. (151) the evolution equations of $\boldsymbol{\varepsilon}^{vi}$ given by Eqs. (39) and (40) and the evolution equations of $\widehat{\phi}$ given by Eq. (78) along with Eqs. (41), (42), and the relations from Table 3 yields the following incremental relation:

$$\dot{\sigma}_{ij} = E_{ijkl}\left(\dot{\varepsilon}_{kl} - \dot{\lambda}^{vp} \frac{\partial f}{\partial \sigma_{kl}} - \dot{\lambda}^{vd} \frac{\partial g}{\partial \sigma_{kl}} \right) - A_{ijkl}\left(\dot{\bar{\lambda}}^{vp} \frac{\partial f}{\partial Y_{kl}} + \dot{\bar{\lambda}}^{vd} \frac{\partial g}{\partial Y_{kl}} \right) - \beta_{ij} \dot{T} \tag{153}$$

where $\dot{\bar{\lambda}}^{vp}$ and $\dot{\bar{\lambda}}^{vd}$ are given as:

$$\dot{\bar{\lambda}}^{vp} = \dot{\lambda}^{vp} + a \nabla^2 \dot{\lambda}^{vp}, \; \dot{\bar{\lambda}}^{vd} = \dot{\lambda}^{vd} + a \nabla^2 \dot{\lambda}^{vd} \tag{154}$$

5. Thermomechanical coupling

A local increase in temperature may influence the material behavior during deformation; necessitating the inclusion of temperature in the constitutive modeling of the material. Substituting the internal energy density, e, from Eq. (19) into the first law of thermodynamics, Eq. (17), yields the following energy balance equation:

$$\rho\left(\dot{\Psi} + \eta \dot{T} + \dot{\eta} T \right) - \sigma_{ij} \dot{\varepsilon}_{ij} - \rho r_{ext} + q_{i,i} = 0 \tag{155}$$

Substituting $\dot{\Psi}$ from Eq. (21) into the above equality and taking into account the results presented by Eqs. (26), (28), and (56) yields the following:

$$\rho \dot{\eta} T = \sigma_{ij} \dot{\varepsilon}^{vi}_{ij} - \Pi_{\text{int}} + \rho r_{ext} + k_{ij} \nabla_i T \nabla_j T + \frac{1}{\dot{T}} k_{ij} \nabla_i \dot{T} \nabla_j T \tag{156}$$

According to the definition given in Section 3.3 for the specific entropy $\eta = \tilde{\eta}(\boldsymbol{\varepsilon}^{e}, T, \aleph_k)$ $(k = 1, ..., 10)$; operating on the entropy relation, Eq. $(26)_2$, with the time derivative and substituting the result into Eq. (156) gives the thermomechanical heat balance equation as follows:

$$\rho c_p \dot{T} = \Upsilon \sigma_{ij} \dot{\varepsilon}^{vi}_{ij} + \sum_{k=3}^{10} \left(\frac{\partial \Sigma_k}{\partial T} T - \Sigma_k \right) \dot{\aleph}_k - \beta_{ij} \dot{\varepsilon}^{e}_{ij} T + \rho r_{ext} + k_{ij} \nabla_i T \nabla_j T \tag{157}$$

where $c_p = T \partial \eta / \partial T$ is the tangent specific heat capacity at constant pressure used to approximate the specific heat capacity at constant stress (which an expression can be found for it by using the specific entropy expression outlined in Section 3.3), and Υ is the fraction of the viscoinelastic work rate converted to heat which is incorporated

into the heat balance equation for more accurate comparisons with experiments. Furthermore, the internal dissipation terms due to $\eta = \tilde{\eta}(\phi, \nabla^2\phi)$, i.e. $(T\partial\mathbf{Y}/\partial T - \mathbf{Y})\dot{\phi}$ and $(T\partial\mathbf{Y}^g/\partial T - \mathbf{Y}^g)\nabla^2\dot{\phi}$ are implicitly included in the definition of entropy given in Section 3.3. Therefore, it can be demonstrated that these terms should not be considered in Eq. (157).

The left-hand side of Eq. (157) represents the total thermal dissipation. The first term on the right-hand side represents the rate of viscoinelastic work converted to heat; the second term represents the rate of internal heat generation that encompasses all dissipative processes excluding the gross viscoinelastic deformation that is present during viscoplastic and viscodamage deformation; the third term is the reversible thermoelastic effect; the fourth term represents an external heat source; and the last term is due to the parabolic heat conduction. If the duration of the event is sufficiently short, the thermal conduction term becomes negligible, i.e. adiabatic conditions prevail.

The rate type-equations, Eq. (153) and (157), take into account effects of the viscoplastic and viscodamage strain-induced anisotropy (i.e. kinematic hardening), flow stress (i.e. isotropic hardening), temperature and strain-rate sensitivity, anisotropic damage (i.e. softening generated by damage defects nucleation and growth mechanisms), thermomechanical coupling (i.e. thermal viscoplastic softening and thermal expansion), strong viscoplasticity and viscodamage coupling, and heterogeneity in the material behavior (i.e. the non-local influence).

6. Computational issues of the gradient approach

Let t_o, t_1, …, t_n, $t_{n+1} = t_n + \Delta t$, … be convenient time instances along the time interval over which the dynamic response of the body is sought. Consider the time step $\Delta t = t_{n+1} - t_n$: at $t = t_n$ where all quantities are known, which are the converged values of the previous step, and the solution must be computed at t_{n+1} for a given body load increment, $\Delta\mathbf{b}$, and surface load increment, $\Delta\mathbf{t}$.

Let the dynamic evolution of an elasto-viscoplastic damaged body of volume V and surface S be governed at time step $n+1$, in the small strain range, by the above derived constitutive relations and by the following momentum, initial, and linear compatibility relations:

$$\mathbf{C}^T\boldsymbol{\sigma}_{n+1} + \rho\mathbf{b}_{n+1} = \rho\ddot{\mathbf{u}}_{n+1} \text{ in } V;\ \mathbf{t}_{n+1} = \boldsymbol{\sigma}_{n+1}\mathbf{n} \text{ on } S_t \tag{158}$$

$$\mathbf{u} = \mathbf{u}_o, \dot{\mathbf{u}} = \dot{\mathbf{u}}_o \text{ at } t = t_o \tag{159}$$

$$\boldsymbol{\varepsilon}_{n+1} = \mathbf{C}\,\mathbf{u}_{n+1} \text{ in } V;\ \mathbf{u}_{n+1} = \mathbf{U}_{n+1} \text{ on } S_u \tag{160}$$

where, for algorithmic convenience, we have shifted to matrix vector notation. Eqs. (158) express the discrete dynamic motion in the volume V and equilibrium on the

free part of the boundary S_t at time step $n+1$. The superscript T denotes the transpose symbol. Viscoelasticity is not considered in this study; viscous damping effects are neglected. $\mathbf{C}^T$ is the differential operator, $\mathbf{b}$ and $\mathbf{t}$ are the body force and the surface traction vectors, respectively, ρ is the mass density, $\mathbf{u}$ is the three-component displacement vector, a superimposed dot implies the derivative with respect to time t, and $\mathbf{n}$ denotes the outward normal to the surface S. The initial conditions on displacements and velocities are given by Eqs. (159). Linear compatibility in volume V and on the constrained part of the boundary S_u is described by Eqs. (160), where $\mathbf{U}$ being the assigned displacement vector, such that $S_t \bigcup S_u = S$ and $S_t \bigcap S_u = 0$.

Moreover, in order to devise a proper algorithm at the end of time step $n+1$, we require that the viscoplasticity condition:

$$f\left(\boldsymbol{\sigma}_{n+1}, p_{n+1}, \nabla^2 p_{n+1}, \dot{p}_{n+1}, \nabla^2 \dot{p}_{n+1}, \boldsymbol{\alpha}_{n+1}, \nabla^2 \boldsymbol{\alpha}_{n+1}, \boldsymbol{\Phi}_{n+1}, \nabla^2 \boldsymbol{\Phi}_{n+1}, T_{n+1}\right) = 0 \tag{161}$$

and the viscodamage growth condition

$$g\left(\mathbf{Y}_{n+1}, \mathbf{Y}^g_{n+1}, r_{n+1}, \nabla^2 r_{n+1}, \dot{\lambda}^{vd}_{n+1}, \nabla^2 \dot{\lambda}^{vd}_{n+1}, \boldsymbol{\Gamma}_{n+1}, \nabla^2 \boldsymbol{\Gamma}_{n+1}, \boldsymbol{\Phi}_{n+1}, \nabla^2 \boldsymbol{\Phi}_{n+1}, T_{n+1}\right) = 0 \tag{162}$$

are satisfied (e.g. de Borst and Mühlhaus, 1992; de Borst et al., 1999, Voyiadjis et al., 2001).

In the context of the finite element method, the discrete problem can be obtained via a spatial Galerkin projection of the simidiscrete (i.e. discrete in space and continuous in time) problem into a finite dimensional subspace of admissible continuous shape functions. Consequently, in the following sections we outline the procedure for solving the derived set of governing equations using the finite element method.

6.1. Discretization of the motion equation

6.1.1. Backward-difference time discretization

In the proposed formulation the update of rate-dependent constitutive equations largely follows the time integration algorithm proposed by Peirce et al. (1984), which has been used in the context of viscoplasticity by LeMonds and Needleman (1986a, b) and Voyiadjis and Mohammad (1988, 1991). The essence of the rate tangent modulus method is to approximate a function of time in the interval, $t_{n+\theta} \in [t_n, t_{n+1}]$ where $\theta \in [0, 1]$ as follows:

$$Z_{n+\theta} = (1-\theta) Z_n + \theta Z_{n+1} \tag{163}$$

In the above equation, setting $\theta = 0$ one obtains the fully explicit Euler scheme, whereas setting $\theta = 1$ one gets the fully implicit scheme. The trapezoidal method is obtained by setting $\theta = 1/2$. Then it follows that at $t_{n+\theta} = (n+\theta)\Delta t$ the velocity and

the acceleration, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$, respectively, can be written as:

$$\dot{\mathbf{u}}_{n+\theta} = (1-\theta)\dot{\mathbf{u}}_n + \theta\dot{\mathbf{u}}_{n+1}, \ddot{\mathbf{u}}_{n+\theta} = (1-\theta)\ddot{\mathbf{u}}_n + \theta\ddot{\mathbf{u}}_{n+1} \tag{164}$$

By integrating according to the backward difference scheme and solving for $\ddot{\mathbf{u}}_{n+1}$ in Eqs. (164), one obtains:

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\theta^2 \Delta t^2}(\mathbf{u}_{n+1} - \mathbf{u}_n) - \frac{1}{\theta^2 \Delta t}\dot{\mathbf{u}}_n - \frac{1-\theta}{\theta}\ddot{\mathbf{u}}_n \tag{165}$$

Substituting $\ddot{\mathbf{u}}_{n+1}$ into the momentum equation, Eq. $(158)_1$, one obtains:

$$\mathbf{C}^T\boldsymbol{\sigma}_{n+1} + \rho\mathbf{b}^*_{n+1} = \rho^*\mathbf{u}_{n+1} \tag{166}$$

having set

$$\rho^* = \frac{\rho}{\theta^2\Delta t^2}, \mathbf{b}^*_{n+1} = \mathbf{b}_{n+1} + \left(\frac{1-\theta}{\theta}\ddot{\mathbf{u}}_n + \frac{1}{\theta^2\Delta t}\dot{\mathbf{u}}_n + \frac{1}{\theta^2\Delta t^2}\mathbf{u}_n\right) \tag{167}$$

The above integration algorithm belongs to the popular class of time integrators termed the Newmark β-method using the generalized mid-point rule algorithm (e.g. Houghs, 1987; Zienkiewicz and Taylor, 1989).

6.1.2. Weak form for the momentum equation

The non-linear initial boundary value problem concerning the finite step Δt must be solved with an iterative scheme. For simplicity in the notation, in what follows the index $n+1$ of the time step has been omitted and the subscripts i and $i+1$ refer to the iteration. Eqs. (166), (160), and (153) can be written at the $i+1$ iteration of the current time step as follows:

$$\mathbf{C}^T\boldsymbol{\sigma}_{i+1} + \rho b^*_{i+1} = \rho^*\mathbf{u}_{i+1} \text{ in } V;\ \mathbf{t}_{i+1} = \boldsymbol{\sigma}_{i+1}\mathbf{n} \text{ on } S_t \tag{168}$$

$$\boldsymbol{\varepsilon}_{i+1} = \mathbf{C}\,\mathbf{u}_{i+1} \text{ in } V;\ \mathbf{u}_{i+1} = \mathbf{U} \text{ on } S_u \tag{169}$$

$$\Delta\boldsymbol{\sigma} = \mathbf{E}_i\left(\Delta\boldsymbol{\varepsilon} - \Delta\lambda^{vp}\left.\frac{\partial f}{\partial\boldsymbol{\sigma}}\right|_i - \Delta\lambda^{vd}\left.\frac{\partial g}{\partial\boldsymbol{\sigma}}\right|_i\right) - \mathbf{A}_i\left(\Delta\bar{\lambda}^{vp}\frac{\partial f}{\partial\mathbf{Y}}|_i + \Delta\bar{\lambda}^{vd}\frac{\partial g}{\partial\mathbf{Y}}|_i\right) - \boldsymbol{\beta}_i\Delta T \tag{170}$$

where the Δ-symbol indicates the difference between the values of a variable at the end of $i+1$ iteration and i iteration, e.g. $\Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}_{i+1} - \boldsymbol{\sigma}_i$. It has been assumed that the viscoplastic flow direction and the viscodamage growth direction is determined from the previous i iteration. Moreover, the degraded elastic-damage moduli, $\mathbf{E}$, and thermo-mechanical second-order tensor, $\boldsymbol{\beta}$, are assumed to equal their values at the end of the previous iteration; i.e. at $\boldsymbol{\phi} = \boldsymbol{\phi}_i$.

A weak form of the equation of motion, Eq. (168)$_1$, is obtained by setting:

$$\int_V \delta\mathbf{u}^T\left(\mathbf{C}^T\boldsymbol{\sigma}_{i+1} + \rho\mathbf{b}^*_{i+1} - \rho^*\mathbf{u}_{i+1}\right)\mathrm{d}V = 0 \tag{171}$$

where δ-symbol denotes the variation of a quantity.

Making use of integration by parts, the divergence theorem, and the substitution of the natural boundary conditions [Eq. (168)$_2$], the linear compatibility relation [Eq. (169)$_1$], along with the decomposition $\boldsymbol{\sigma}_{i+1} = \boldsymbol{\sigma}_i + \Delta\boldsymbol{\sigma}$ and $\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta\mathbf{u}$, Eq. (171)can be expressed as follows:

$$\begin{aligned}\int_V \delta\boldsymbol{\varepsilon}^T\,\Delta\boldsymbol{\sigma}\,\mathrm{d}V + \int_V \delta\mathbf{u}^T\,\rho^*\,\Delta\mathbf{u}\,\mathrm{d}V = \int_{S_t} \delta\mathbf{u}^T\,\mathbf{t}_{i+1}\,\mathrm{d}S + \int_V \delta\mathbf{u}^T\rho\mathbf{b}^*_{i+1}\,\mathrm{d}V \\ - \int_V \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}_i\,\mathrm{d}V - \int_V \delta\mathbf{u}^T\,\rho^*\,\mathbf{u}_i\,\mathrm{d}V\end{aligned} \tag{172}$$

Substituting the incremental form of $\boldsymbol{\sigma}$ from Eq. (170) into Eq. (172), one obtains the following relation:

$$\begin{aligned}&\int_V \delta\boldsymbol{\varepsilon}^T\,\mathbf{E}_i\left(\Delta\boldsymbol{\varepsilon} - \Delta\lambda^{vp}\left.\frac{\partial f}{\partial\boldsymbol{\sigma}}\right|_i - \Delta\lambda^{vd}\left.\frac{\partial g}{\partial\boldsymbol{\sigma}}\right|_i\right)\mathrm{d}V - \int_V \mathbf{A}_i \\ &\quad\times\left(\Delta\bar{\lambda}^{vp}\left.\frac{\partial f}{\partial\mathbf{Y}}\right|_i + \Delta\bar{\lambda}^{vd}\left.\frac{\partial g}{\partial\mathbf{Y}}\right|_i\right)\mathrm{d}V + \int_V \delta\mathbf{u}^T\,\rho^*\,\Delta\mathbf{u}\,\mathrm{d}V - \int_V \delta\mathbf{u}^T\,\boldsymbol{\beta}_i\,\Delta T\,\mathrm{d}V \\ &\quad= \int_{S_t} \delta\mathbf{u}^T\,\mathbf{t}_{i+1}\,\mathrm{d}S + \int_V \delta\mathbf{u}^T\rho\mathbf{b}^*_{i+1}\,\mathrm{d}V - \int_V \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}_i\,\mathrm{d}V - \int_V \delta\mathbf{u}^T\,\rho^*\,\mathbf{u}_i\,\mathrm{d}V\end{aligned} \tag{173}$$

One can then proceed as in the classical finite element method discretizing the displacement field $\mathbf{u}$ and the temperature field T as follows:

$$\mathbf{u} = \mathbf{N}\,\mathbf{a},\ T = \boldsymbol{W}^T\,\boldsymbol{T}^e \tag{174}$$

where $\mathbf{N} = [\mathbf{N}_1, \ldots, \mathbf{N}_{n_{node}}]$ and $\mathbf{W} = [\mathbf{W}_1, \ldots, \mathbf{W}_{n_{node}}]$ are both for $k = 1, \ldots, n_{node}$ and $e = 1, \ldots, n_{elem}$.

In the above, $\mathbf{N}$ and $\mathbf{W}$ are the displacement and temperature shape functions, respectively, and contain continuous interpolation functions of order C^o. $\mathbf{a}$ is the nodal displacement vector and $\mathbf{T}^e$ is the nodal temperature vector (the superscript e denotes element values). Consequently, one obtains from the linear kinematic relation, Eq. (169)$_1$, the discretized form for the strains $\boldsymbol{\varepsilon}$:

$$\begin{aligned}&\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{a}, \\ &\mathbf{B} = \mathbf{C}\mathbf{N}\end{aligned} \tag{175}$$

where $\mathbf{B}$ is the strain-displacement matrix.

In a similar fashion the viscoplastic and viscodamage multipliers, λ^{vp} and λ^{vd}, can be discretized following de Borst approach for gradient-dependent models (cf. de

Borst and Mühlhaus, 1992; de Borst et al., 1993, 1999), where λ^{vp} and λ^{vd} are taken as independent variables. However, the discretization of λ^{vp} and λ^{vd} requires C^1 continuous interpolation functions contained in $\mathbf{h} = [\mathbf{h}_1, ..., \mathbf{h}_{n_{node}}]$ since the Laplacian of λ^{vp} and λ^{vd} should be computed, such that:

$$\lambda^{vp} = \mathbf{h}^T \boldsymbol{\Lambda}^{vp}, \nabla^2 \lambda^{vp} = \mathbf{q}^T \boldsymbol{\Lambda}^{vp} \tag{176}$$

$$\lambda^{vp} = \mathbf{h}^T \boldsymbol{\Lambda}^{vp}, \nabla^2 \lambda^{vd} = \mathbf{q}^T \boldsymbol{\Lambda}^{vd} \tag{177}$$

where $\mathbf{q}^T = \nabla^2 \mathbf{h}$, and $\boldsymbol{\Lambda}^{vp}$ and $\boldsymbol{\Lambda}^{vd}$ denote the additional nodal degrees of freedom associated with the viscoplastic and viscodamage multipliers, respectively.

To the authors' best knowledge, no attempt has been made until now to use the de Borst approach to discretize rate-dependent problems. Furthermore, a combined gradient- and rate- dependent plasticity and damage model has not been proposed in the literature sofar. Failure to do so is due to the difficulty in enforcing the consistency condition for rate-dependent problems. The rate relation for the viscoplastic and viscodamage multipliers for rate-dependent problems can be expressed as an increment by:

$$\Delta\lambda^{vp} = \mathbf{h}^T \Delta\boldsymbol{\Lambda}^{vp}, \nabla^2 \Delta\lambda^{vp} = \mathbf{q}^T \Delta\boldsymbol{\Lambda}^{vp} \tag{178}$$

$$\Delta\lambda^{vd} = \mathbf{h}^T \Delta\boldsymbol{\Lambda}^{vd}, \nabla^2 \Delta\lambda^{vd} = \mathbf{q}^T \Delta\boldsymbol{\Lambda}^{vd} \tag{179}$$

Using Eqs. (174)–(179) in Eq. (173) and requiring that the result is valid for any admissible variation $\delta\mathbf{a}$ yields the following relation:

$$\begin{aligned} &\int_V \left(\mathbf{B}^T \mathbf{E}_i \mathbf{B} - \rho^* \mathbf{N}^T \mathbf{N}\right) \Delta\mathbf{a}\, \mathrm{d}V - \int_V \mathbf{B}^T \left(\mathbf{E}_i \mathbf{h}^T \frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|_i + \mathbf{A}_i \hat{\mathbf{h}}^T \frac{\partial f}{\partial \mathbf{Y}}\bigg|_i x \right) \Delta\boldsymbol{\Lambda}^{vp}\, \mathrm{d}V \\ &\quad - \int_V \mathbf{B}^T \left(\mathbf{E}_i \mathbf{h}^T \frac{\partial g}{\partial \boldsymbol{\sigma}}\bigg|_i + \mathbf{A}_i \hat{\mathbf{h}}^T \frac{\partial g}{\partial \mathbf{Y}}\bigg|_i \right) \Delta\boldsymbol{\Lambda}^{vd} \mathrm{d}V - \int_V \mathbf{N}^T \boldsymbol{\beta}_i \mathbf{W}^T \Delta\mathbf{T}^e \mathrm{d}V \end{aligned} \tag{180}$$

$$= \int_{S_t} \mathbf{N}^T \mathbf{t}_{i+1} \mathrm{d}S + \int_V \rho \mathbf{N}^T \mathbf{b}^*_{i+1} \mathrm{d}V - \int_V \mathbf{B}^T \boldsymbol{\sigma}_i \mathrm{d}V - \int_V \rho^* \mathbf{N}^T \mathbf{N}\, \mathbf{a}_i \mathrm{d}V$$

where $\hat{\mathbf{h}}$ is given by:

$$\hat{\mathbf{h}} = \mathbf{h} + a\mathbf{q} \tag{181}$$

6.2. Discretization of the viscoplastic condition

In the same way as in the proceeding section we shall formulate the weak form of the viscoplasticity surface function, Eq. (161), at $i+1$ iteration such that the viscoplasticity condition is satisfied in a distributed sense. However, the test function used in discretizing the viscoplastic surface function is $\delta\lambda^{vp}$ (e.g. de Borst and Mühlhaus, 1992), such that:

$$\int_V \delta\lambda^{vp} f\Big(\boldsymbol{\sigma}_{i+1}, p_{i+1}, \nabla^2 p_{i+1}, \dot{p}_{i+1}, \nabla^2\dot{p}_{i+1}, \boldsymbol{\alpha}_{i+1}, \nabla^2\boldsymbol{\alpha}_{i+1}, \boldsymbol{\phi}_{i+1}, \nabla^2\boldsymbol{\phi}_{i+1}, T_{i+1}\Big)\mathrm{d}V = 0 \tag{182}$$

where $(\bullet)_{i+1} = (\bullet)_i + \Delta(\bullet)$ is the additive decomposition of each of the internal variables.

Furthermore, the yield function f_{i+1} can be approximated in $(\boldsymbol{\sigma}_i, p_i, \nabla^2 p_i, \dot{p}_i, \nabla^2\dot{p}_i, \boldsymbol{\alpha}_i, \nabla^2\boldsymbol{\alpha}_i, \boldsymbol{\phi}_i, \nabla^2\boldsymbol{\phi}_i, T_i)$ using the first order Taylor expansion series, such that:

$$\begin{aligned} f_{i+1} \approx f_i &+ \frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|_i^T \Delta\boldsymbol{\sigma} + \frac{\partial f}{\partial p}\bigg|_i \Delta p + \frac{\partial f}{\partial \nabla^2 p}\bigg|_i \nabla^2\Delta p + \frac{\partial f}{\partial \dot{p}}\bigg|_i \Delta\dot{p} + \frac{\partial f}{\partial \nabla^2\dot{p}}\bigg|_i \nabla^2\Delta\dot{p} \\ &\frac{\partial f}{\partial \boldsymbol{\alpha}}\bigg|_i^T \Delta\boldsymbol{\alpha} + \frac{\partial f}{\partial \nabla^2\boldsymbol{\alpha}}\bigg|_i^T \nabla^2\Delta\boldsymbol{\alpha} + \frac{\partial f}{\partial \boldsymbol{\phi}}\bigg|_i^T \Delta\boldsymbol{\phi} + \frac{\partial f}{\partial \nabla^2\boldsymbol{\phi}}\bigg|_i^T \nabla^2\Delta\boldsymbol{\phi} + \frac{\partial f}{\partial T}\bigg|_i \Delta T = 0 \end{aligned} \tag{183}$$

where f_i is the i-th residual of the viscoplastic function. However, emphasizing that we are applying the local iteration process within the time step $t+\Delta t$ (i.e. at step $n+1$), one can then write Δp and $\nabla^2\Delta p$ as follows:

$$\Delta p = \Delta\dot{p}\,\Delta t; \quad \nabla^2\Delta p = \nabla^2\Delta\dot{p}\,\Delta t \tag{184}$$

Substituting for $\Delta\dot{p}$ and $\nabla^2\Delta\dot{p}$ from the above equations into Eq. (183) yields:

$$\begin{aligned} f_{i+1} \approx f_i &+ \frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|_i^T \Delta\boldsymbol{\sigma} + \left[\frac{\partial f}{\partial p}\bigg|_i + \frac{1}{\Delta t}\frac{\partial f}{\partial \dot{p}}\bigg|_i\right]\Delta p + \left[\frac{\partial f}{\partial \nabla^2 p}\bigg|_i + \frac{1}{\Delta t}\frac{\partial f}{\partial \nabla^2\dot{p}}\bigg|_i\right]\nabla^2\Delta p \\ &+ \frac{\partial f}{\partial \boldsymbol{\alpha}}\bigg|_i^T \Delta\boldsymbol{\alpha} + \frac{\partial f}{\partial \nabla^2\boldsymbol{\alpha}}\bigg|_i^T \nabla^2\Delta\boldsymbol{\alpha} + \frac{\partial f}{\partial \boldsymbol{\phi}}\bigg|_i^T \Delta\boldsymbol{\phi} + \frac{\partial f}{\partial \nabla^2\boldsymbol{\phi}}\bigg|_i^T \nabla^2\Delta\boldsymbol{\phi} + \frac{\partial f}{\partial T}\bigg|_i \Delta T = 0 \end{aligned} \tag{185}$$

The evolution equations of the isotropic and kinematic hardening fluxes and their corresponding gradients given by Eqs. (101), (102), (115) and (117) are then substituted into Eq. (183), such that f_{i+1} is reduced as follows:

$$\begin{aligned} f_{i+1} = f_i &+ \frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|_i^T \mathbf{E}_i\Delta\boldsymbol{\varepsilon} + Q_1^p\Delta\lambda^{vp} + Q_2^p\nabla^2\Delta\lambda^{vp} + Q_3^p\Delta\lambda^{vd} + Q_4^p\nabla^2\Delta\lambda^{vd} \\ &+ \left(\frac{\partial f}{\partial T}\bigg|_i - \frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|_i^T \boldsymbol{\beta}_i\right)\Delta T \\ &= 0 \end{aligned} \tag{186}$$

where Q_k^p $(k = 1, ..., 4)$ are obtained from the previous i iteration and are given as follows:

$$Q_1^p = -\frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|^T \left(\mathbf{E}\frac{\partial f}{\partial \boldsymbol{\sigma}} + \mathbf{A}\frac{\partial f}{\partial \mathbf{Y}}\right) + \left[\frac{\partial f}{\partial p} + \frac{1}{\Delta t}\frac{\partial f}{\partial \dot{p}}\right]\frac{\left(1 - k_1\bar{\bar{R}}\right)}{1 - \hat{r}} + \left[\frac{\partial f}{\partial \nabla^2 p} + \frac{1}{\Delta t}\frac{\partial f}{\partial \nabla^2 \dot{p}}\right] \times \left[\frac{\nabla^2 r}{(1-\hat{r})^2}\left(1 - k_1\bar{\bar{R}}\right) - \frac{k_1}{1-\hat{r}}\nabla^2\bar{R}\right] - \frac{\partial f}{\partial \boldsymbol{\alpha}}\bigg|^T \widehat{\mathbf{M}}\left(\frac{\partial \mathrm{f}}{\partial \mathbf{X}} + \mathrm{k}_2\hat{\mathbf{X}}\right) - \frac{\partial \mathrm{f}}{\partial \nabla^2 \boldsymbol{\alpha}}\bigg|^{\mathrm{T}}\left[\nabla^2\widehat{\mathbf{M}}\left(\frac{\partial \mathrm{f}}{\partial \mathbf{X}} + \mathrm{k}_2\hat{\mathbf{X}}\right) + \mathrm{k}_2\widehat{\mathbf{M}}\nabla^2\mathbf{X}\right] + \frac{\partial \mathrm{f}}{\partial \boldsymbol{\phi}}\bigg|^{\mathrm{T}}\frac{\partial \mathrm{f}}{\partial \mathbf{Y}} \tag{187}$$

$$Q_2^p = -a\frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|^T \mathbf{A}\frac{\partial f}{\partial \mathbf{Y}} + \left[\frac{\partial f}{\partial \nabla^2 p} + \frac{1}{\Delta t}\frac{\partial f}{\partial \nabla^2 \dot{p}}\right]\frac{\left(1 - k_1\bar{\bar{R}}\right)}{1 - \hat{r}} - \frac{\partial f}{\partial \nabla^2 \boldsymbol{\alpha}}\widehat{\mathbf{M}}\left(\frac{\partial \mathrm{f}}{\partial \mathbf{X}} + \mathrm{k}_2\hat{\mathbf{X}}\right) + \frac{\partial \mathrm{f}}{\partial \nabla^2 \boldsymbol{\phi}}\bigg|^{\mathrm{T}}\frac{\partial \mathrm{f}}{\partial \mathbf{Y}} \tag{188}$$

$$Q_3^p = -\frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|^T \left(\mathbf{E}\frac{\partial g}{\partial \boldsymbol{\sigma}} + \mathbf{A}\frac{\partial g}{\partial \mathbf{Y}}\right) + \frac{\partial f}{\partial \boldsymbol{\phi}}\bigg|^T \frac{\partial g}{\partial \mathbf{Y}} \tag{189}$$

$$Q_4^p = -a\frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|^T \mathbf{A}\frac{\partial g}{\partial \mathbf{Y}} + \frac{\partial f}{\partial \nabla^2 \boldsymbol{\phi}}\bigg|^T \frac{\partial g}{\partial \mathbf{Y}} \tag{190}$$

Upon substitution of Eq. (186) along with the finite element discretization equations of $\Delta\mathbf{u}$, ΔT, $\Delta\lambda^{vp}$, and $\Delta\lambda^{vd}$, Eqs. (174)–(179), into Eq. (182) and requiring that the result is valid for any admissible variation $\delta\boldsymbol{\Lambda}^{vp}$, one can obtain:

$$\int_V \mathbf{h}\frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|_i^T \mathbf{E}_i\mathbf{B}\,\Delta\mathbf{a}\,\mathrm{d}V + \int_V \mathbf{h}\left(Q_1^p\mathbf{h}^T + Q_2^p\mathbf{q}^T\right)\Delta\boldsymbol{\Lambda}^{vp}\,\mathrm{d}V + \int_V \mathbf{h}\left(Q_3^p\mathbf{h}^T + Q_4^p\mathbf{q}^T\right)\Delta\boldsymbol{\Lambda}^{vd}\,\mathrm{d}V + \int_V \mathbf{h}\left(\frac{\partial f}{\partial T}\bigg|_i - \frac{\partial f}{\partial \boldsymbol{\sigma}}\bigg|_i^T \boldsymbol{\beta}_i\right)\mathbf{W}^T\Delta\mathbf{T}^e\,\mathrm{d}V = -\int_V \mathbf{h}f_i\,\mathrm{d}V \tag{191}$$

It is noteworthy that the current formulation does not satisfy the viscoplastic condition $f = 0$ in a point wise manner, but merely in a distributed sense (de Borst and Mühlhaus, 1992). Therefore, the later condition $\int_V \mathbf{h}f_i\,\mathrm{d}V = 0$ is not satisfied at each iteration, but only until convergence is achieved at the end of the loading step. However, this approach is inevitably required in the gradient-enhanced models since the difficulty in calculating the Laplacian using the conventional return-mapping algorithms which are probably as good as the current formulation. Moreover, the

non-standard boundary conditions at the elasto-viscoplastic boundary $S_{\lambda^{vp}}$: $\dot{\lambda}^{vp}=0$ or $\left(\nabla\dot{\lambda}^{vp}\right)^{T}\mathbf{n}_{\lambda^{vp}}=0$, where $\mathbf{n}_{\lambda^{vp}}$ is the outward normal at $S_{\lambda^{vp}}$. These results can also be derived directly from a variational principle (e.g. Mühlhaus and Aifantis, 1991; de Borst and Mühlhaus, 1992).

6.3. Discretization of the damage condition

Analogous to the discretization of the yield function, the finite element implementation of the viscodamage criterion is based on the following weak-form governing the generalized viscodamage consistency, Eq. (130), where the test function used is $\delta\lambda^{vd}$, such that:

$$\int_{V}\delta\lambda^{vd}g\Big(\mathbf{Y}_{i+1},\mathbf{Y}^{g}_{i+1},r_{i+1},\nabla^{2}r_{i+1},\dot{\lambda}^{vd}_{i+1},\nabla^{2}\dot{\lambda}^{vd}_{i+1},\mathbf{\Gamma}_{i+1},\nabla^{2}\mathbf{\Gamma}_{i+1},\boldsymbol{\phi}_{i+1},\nabla^{2}\boldsymbol{\phi}_{i+1},T_{i+1}\Big)\mathrm{d}V=0 \tag{192}$$

where $(\bullet)_{i+1}=(\bullet)_{i}+\Delta(\bullet)$ is the additive decomposition of each of the internal variables. Since the viscodamage driving forces $\mathbf{Y}$ is a function of $\boldsymbol{\sigma}$ and $\boldsymbol{\phi}$ [see Eq. (147)], one can expand the viscodamage condition g_{i+1} around $(\mathbf{Y}_i,\ r_i,\ \nabla^2 r_i,\ \dot{\lambda}^{vd}_i,\ \nabla^2\dot{\lambda}^{vd}_i,\ \mathbf{\Gamma}_i,\ \nabla^2\mathbf{\Gamma}_i,\ \boldsymbol{\phi}_i,\ \nabla^2\boldsymbol{\phi}_i,\ T_i)$ using the Taylor series as follows:

$$\begin{aligned}g_{i+1}\approx g_i&+\frac{\partial g}{\partial\boldsymbol{\sigma}}\bigg|_i^{T}\Delta\boldsymbol{\sigma}+\frac{\partial g}{\partial r}\bigg|_i\Delta r+\frac{\partial g}{\partial\nabla^2 r}\bigg|_i\nabla^2\Delta r+\frac{\partial g}{\partial\dot{\lambda}^{vd}}\bigg|_i\Delta\dot{\lambda}^{vd}+\frac{\partial g}{\partial\nabla^2\dot{\lambda}^{vd}}\bigg|_i\nabla^2\Delta\dot{\lambda}^{vd}\\&+\frac{\partial g}{\partial\mathbf{\Gamma}}\bigg|_i^{T}\Delta\mathbf{\Gamma}+\frac{\partial g}{\partial\nabla^2\mathbf{\Gamma}}\bigg|_i^{T}\nabla^2\Delta\mathbf{\Gamma}+\frac{\partial g}{\partial\boldsymbol{\phi}}\bigg|_i^{T}\Delta\boldsymbol{\phi}+\frac{\partial g}{\partial\nabla^2\boldsymbol{\phi}}\bigg|_i^{T}\nabla^2\Delta\boldsymbol{\phi}+\frac{\partial g}{\partial T}\bigg|_i\Delta T=0\end{aligned} \tag{193}$$

where g_i is the i-th residual of the viscodamage function. We emphasizing again that we are applying the local iteration process within the time step $t+\Delta t$ (i.e. at step $n+1$). One can then write $\Delta\dot{\lambda}^{vd}$ and $\nabla^2\Delta\dot{\lambda}^{vd}$ as follows:

$$\Delta\lambda^{vd}=\Delta\dot{\lambda}^{vd}\Delta t;\quad \nabla^2\Delta\lambda^{vd}=\nabla^2\Delta\dot{\lambda}^{vd}\Delta t \tag{194}$$

Substituting for $\Delta\dot{\lambda}^{vd}$ and $\nabla^2\Delta\dot{\lambda}^{vd}$ from the above equation into Eq. (193) yields:

$$\begin{aligned}g_{i+1}\approx g_i&+\frac{\partial g}{\partial\boldsymbol{\sigma}}\bigg|_i^{T}\Delta\boldsymbol{\sigma}+\frac{\partial g}{\partial r}\bigg|_i\Delta r+\frac{\partial g}{\partial\nabla^2 r}\bigg|_i\nabla^2\Delta r+\frac{1}{\Delta t}\frac{\partial g}{\partial\dot{\lambda}^{vd}}\bigg|_i\Delta\lambda^{vd}+\frac{1}{\Delta t}\frac{\partial g}{\partial\nabla^2\dot{\lambda}^{vd}}\bigg|_i\nabla^2\Delta\lambda^{vd}\\&+\frac{\partial g}{\partial\mathbf{\Gamma}}\bigg|_i^{T}\Delta\mathbf{\Gamma}+\frac{\partial g}{\partial\nabla^2\mathbf{\Gamma}}\bigg|_i^{T}\nabla^2\Delta\mathbf{\Gamma}+\frac{\partial g}{\partial\boldsymbol{\phi}}\bigg|_i^{T}\Delta\boldsymbol{\phi}+\frac{\partial g}{\partial\nabla^2\boldsymbol{\phi}}\bigg|_i^{T}\nabla^2\Delta\boldsymbol{\phi}+\frac{\partial g}{\partial T}\bigg|_i\Delta T=0\end{aligned} \tag{195}$$

The evolution equations of the isotropic and kinematic hardening associated with the viscodamage fluxes and their corresponding gradients given by Eqs. (131), (132),

(137), and (138) along with Eqs. (41) and (25) from Table 3 are then substituted into Eq. (193), such that g_{i+1} is reduced to:

$$g_{i+1} = g_i + \left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|_i^T \mathbf{E}_i \Delta\boldsymbol{\varepsilon} + Q_1^d vp + Q_2^d \nabla^2 vp + Q_3^d \Delta\lambda^{vd} + Q_4^d \nabla^2 \Delta\lambda^{vd} + \left(\left.\frac{\partial g}{\partial T}\right|_i - \left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|_i^T \boldsymbol{\beta}_i\right)\Delta T = 0 \tag{196}$$

where Q_k^d $(k = 1, \ldots, 4)$ are obtained from the previous i iteration and are given as follows:

$$Q_1^d = -\left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|^T \left(\mathbf{E}\frac{\partial f}{\partial \boldsymbol{\sigma}} + \mathbf{A}\frac{\partial f}{\partial \mathbf{Y}}\right) + \left.\frac{\partial g}{\partial \boldsymbol{\phi}}\right|^T \frac{\partial f}{\partial \mathbf{Y}} \tag{197}$$

$$Q_2^d = -a\left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|^T \mathbf{A}\frac{\partial f}{\partial \mathbf{Y}} + \left.\frac{\partial g}{\partial \nabla^2 \boldsymbol{\phi}}\right|^T \frac{\partial f}{\partial \mathbf{Y}} \tag{198}$$

$$Q_3^d = -\left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|^T \left(\mathbf{E}\frac{\partial g}{\partial \boldsymbol{\sigma}} + \mathbf{A}\frac{\partial g}{\partial \mathbf{Y}}\right) + \frac{\partial g}{\partial r}\left(1 - h_1\widehat{K}\right) - h_1\frac{\partial g}{\partial \nabla^2 r}\nabla^2 K + \frac{1}{\Delta t}\frac{\partial g}{\partial \dot{\lambda}^{vd}} - \left.\frac{\partial g}{\partial \boldsymbol{\Gamma}}\right|^T \left(\frac{\partial g}{\partial \mathbf{H}} + h_2\widehat{\mathbf{H}}\right) - h_2\left.\frac{\partial g}{\partial \nabla^2 \boldsymbol{\Gamma}}\right|^T \nabla^2\mathbf{H} + \left.\frac{\partial g}{\partial \boldsymbol{\phi}}\right|^T \frac{\partial g}{\partial \mathbf{Y}} \tag{199}$$

$$Q_4^d = -a\left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|^T \mathbf{A}\frac{\partial g}{\partial \mathbf{Y}} + \frac{\partial g}{\partial \nabla^2 r}\left(1 - h_1\widehat{K}\right) + \frac{1}{\Delta t}\frac{\partial g}{\partial \nabla^2 \dot{\lambda}^{vd}} - \frac{\partial g}{\partial \nabla^2 \boldsymbol{\Gamma}}\left(\frac{\partial g}{\partial \mathbf{H}} + h_2\widehat{\mathbf{H}}\right) + \left.\frac{\partial g}{\partial \nabla^2 \boldsymbol{\phi}}\right|^T \frac{\partial g}{\partial \mathbf{Y}} \tag{200}$$

Upon substitution of Eq. (196) along with the finite element discretization equations of $\Delta\mathbf{u}$, ΔT, $\Delta\lambda^{vp}$, and $\Delta\lambda^{vd}$ [Eqs. (174)–(179)] into Eq. (192) and requiring that the result is valid for any admissible variation $\delta\boldsymbol{\Lambda}^{vd}$, one obtains:

$$\int_V \mathbf{h}\left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|_i^T \mathbf{E}_i\mathbf{B}\,\Delta\mathbf{a}\,\mathrm{d}V + \int_V \mathbf{h}\left(Q_1^d\mathbf{h}^T + Q_2^d\mathbf{q}^T\right)\Delta\boldsymbol{\Lambda}^{vp}\,\mathrm{d}V + \int_V \mathbf{h}\left(Q_3^d\mathbf{h}^T + Q_4^d\mathbf{q}^T\right)\Delta\boldsymbol{\Lambda}^{vd}\,\mathrm{d}V + \int_V \mathbf{h}\left(\left.\frac{\partial g}{\partial T}\right|_i - \left.\frac{\partial g}{\partial \boldsymbol{\sigma}}\right|_i^T \boldsymbol{\beta}_i\right)\mathbf{W}^T\Delta\mathbf{T}^e\,\mathrm{d}V = -\int_V \mathbf{h}\,g_i\,\mathrm{d}V \tag{201}$$

Analogous to the yield condition, the viscodamage condition $g = 0$ is not satisfied in a point wise manner, but merely in a distributed sense. Therefore, the later

condition $\int_V \mathbf{h}\, g_i \,\mathrm{d}V = 0$ is not satisfied at each iteration, but only at the end of the loading step. Moreover, the non-standard boundary conditions at the viscodamage boundary $S_{\lambda^{vd}}$: $\dot{\lambda}^{vd} = 0$ or $\left(\nabla\dot{\lambda}^{vd}\right)^T \mathbf{n}_{\lambda^{vd}} = 0$, where $\mathbf{n}_{\lambda^{vd}}$ is the outward normal at $S_{\lambda^{vd}}$. However, more detailed explanation for the non-standard boundary conditions of viscoplasticity and viscodamage will be the subject of a forthcoming paper.

6.4. Discretization of the heat equation

To establish the actual heat generation that occurs during the highly transient dynamic events of the thermomechanically coupled finite element, discretization of the heat equation, Eq. (157), is imperative. The heat balance equation, Eq. (157), can then be discretized following the standard procedure as follows:

$$\Upsilon\boldsymbol{\sigma}_i \Delta\boldsymbol{\varepsilon}^{vi} + \sum_{k=3}^{10} V_k \Big|_i \Delta\aleph_k - \boldsymbol{\beta}\,\Delta\boldsymbol{\varepsilon}^e T_i + \rho\mathbf{r}_{ext} + \mathbf{k}\,\nabla T_i\,\nabla T_i - \rho c_p \Delta T = 0 \tag{202}$$

where $V_k = T\partial\Sigma_k/\partial T - \Sigma_k$ $(k = 3, \ldots, 10)$ and their expressions are similar to those outlined in Table 4, but instead of ϑ we substitute $\gamma = T(\partial\vartheta/\partial T) - \vartheta$, and their magnitudes are obtained from the previous iteration i.

Substituting the strain rate decomposition relation, Eq. (9), for $\Delta\boldsymbol{\varepsilon}^e$ into Eq. (202) yields the following expression:

$$(\Upsilon\boldsymbol{\sigma}_i + \boldsymbol{\beta}\,T_i)\Delta\boldsymbol{\varepsilon}^{vi} - \boldsymbol{\beta}\,T_i\Delta\boldsymbol{\varepsilon} + \sum_{k=3}^{10} V_k \Big|_i \Delta\aleph_k + \rho\mathbf{r}_{ext} + \mathbf{k}\,\nabla T_i\,\nabla T_i - \rho c_p \Delta T = 0 \tag{203}$$

A finite element formulation can then be developed from the following functional:

$$\int_V \delta T \left((\Upsilon\boldsymbol{\sigma}_i + \boldsymbol{\beta}\,T_i)\Delta\boldsymbol{\varepsilon}^{vi} - \boldsymbol{\beta}\,T_i\Delta\boldsymbol{\varepsilon} + \sum_{k=3}^{10} V_k \Big|_i \Delta\aleph_k + \rho\mathbf{r}_{\mathrm{ext}} + \mathbf{k}\,\nabla T_i\,\nabla T_i - \rho c_p \Delta T \right) \mathrm{d}V = 0 \tag{204}$$

By making use of the evolution equation of $\boldsymbol{\varepsilon}^{vi}$ given by Eqs. (30) and (40), and the evolution equations of $\aleph_k$ $(k = 3, \ldots, 10)$ given in Sections 3.6 and 3.7, (i.e. $\dot{p}$, $\nabla^2\dot{p}$, $\dot{\boldsymbol{\alpha}}$, $\nabla^2\dot{\boldsymbol{\alpha}}$, $\dot{r}$, $\nabla^2\dot{r}$, $\dot{\boldsymbol{\Gamma}}$, and $\nabla^2\dot{\boldsymbol{\Gamma}}$), Eq. (204) reduces to:

$$\begin{aligned} &\int_V \delta T \left(-\boldsymbol{\beta}_i T_i \Delta\boldsymbol{\varepsilon} + Q_1^{tp}\, {}^{vp} + Q_2^{tp}\,\nabla^2 vp + Q_1^{td}\,\Delta\lambda^{vd} + Q_2^{td}\,\nabla^2\Delta\lambda^{vd} - \rho c_p \Delta T \right) \mathrm{d}V \\ &= -\int_V \delta T (\rho\mathbf{r}_{\mathrm{ext}} + \mathbf{k}\,\nabla T_i\,\nabla T_i)\mathrm{d}V \end{aligned} \tag{205}$$

where Q_k^{tp} and Q_k^{td} $(k = 1, 2)$ are obtained from the previous iteration i and are given as follows:

$$Q_1^{tp} = (\Upsilon\boldsymbol{\sigma} + \boldsymbol{\beta} T)\frac{\partial f}{\partial \boldsymbol{\sigma}} + \left(\frac{V_3}{1-\hat{r}} + \frac{V_7\nabla^2 r}{(1-\hat{r})^2}\right)\left(1 - k_1\bar{\bar{R}}\right) - \frac{k_1 V_7}{1-\hat{r}}\nabla^2\bar{R} - \left(\mathbf{V}_4\hat{\mathbf{M}} + \mathbf{V}_8\nabla^2\hat{\mathbf{M}}\right)\left(\frac{\partial \mathrm{f}}{\partial \mathbf{X}} + \mathrm{k}_2\hat{\mathbf{X}}\right) - \mathrm{k}_2\mathbf{V}_8\,\hat{\mathbf{M}}\nabla^2\mathbf{X} \tag{206}$$

$$Q_2^{tp} = \frac{V_7}{1-\hat{r}}\left(1 - k_1\bar{\bar{R}}\right) - \mathbf{V}_8\hat{\mathbf{M}}\left(\frac{\partial \mathrm{f}}{\partial \bar{\bar{\mathbf{X}}}} + \mathrm{k}_2\bar{\bar{\mathbf{X}}}\right) \tag{207}$$

$$Q_1^{td} = (\Upsilon\boldsymbol{\sigma} + \boldsymbol{\beta} T)\frac{\partial g}{\partial \boldsymbol{\sigma}} + V_5 (1 - h_1\hat{K}) - h_1 V_9\nabla^2 K - \mathbf{V}_6\left(\frac{\partial g}{\partial \mathbf{H}} + h_2\hat{\mathbf{H}}\right) - \mathrm{h}_2\mathbf{V}_{10}\nabla^2\mathbf{H} \tag{208}$$

$$Q_2^{td} = V_9\left(1 - h_1\hat{K}\right) - \mathbf{V}_{10}\left(\frac{\partial g}{\partial \mathbf{H}} + h_2\hat{\mathbf{H}}\right) \tag{209}$$

Upon substitution the finite element discretization equations of ΔT, $\Delta\lambda^{vp}$, and $\Delta\lambda^{vd}$ [Eqs. (174) and (179)] into Eq. (205) and requiring that the result is valid for any admissible variation $\delta\mathbf{T}^e$, one obtains:

$$\begin{aligned} &-\int_V \mathbf{W}\boldsymbol{\beta}_i T_i \mathbf{B}\,\Delta\mathbf{a}\,\mathrm{d}V + \int_V \mathbf{W}\left(Q_1^{tp}\mathbf{h}^T + Q_2^{tp}\mathbf{q}^T\right)\Delta\boldsymbol{\Lambda}^{vp}\mathrm{d}V \\ &+\int_V \mathbf{W}\left(Q_1^{td}\mathbf{h}^T + Q_2^{td}\mathbf{q}^T\right)\Delta\boldsymbol{\Lambda}^{vd}\mathrm{d}V - \int_V \mathbf{W}\rho c_p\mathbf{h}^T\Delta\mathbf{T}^e\,\mathrm{d}V \\ &= -\int_V \mathbf{W}\rho\mathbf{r}_{\mathrm{ext}}\,\mathrm{d}V - \int_V \nabla\mathbf{W}\,\mathbf{k}\,\nabla\mathbf{W}^T\mathbf{T}_i^e\,\mathrm{d}V \end{aligned} \tag{210}$$

6.5. Combined discretization equations

Combining the discretized equations of linear momentum [Eq. (180)], viscoplasticity condition [Eq. (191)], viscodamage condition [Eq. (201)], and thermomechanical coupling equation[Eq. (210)] augments us with a set of linear algebraic equations in terms of the variations $\Delta\mathbf{a}$, $\Delta\boldsymbol{\Lambda}^{vp}$, $\Delta\boldsymbol{\Lambda}^{vd}$, and $\Delta\mathbf{T}^e$, such that:

$$\begin{bmatrix} \mathbf{K}_{aa} - \mathbf{M}^* & \mathbf{K}_{a\lambda^{vp}} & \mathbf{K}_{a\lambda^{vd}} & \mathbf{K}_{aT} \\ \mathbf{K}_{\lambda^{vp}a} & \mathbf{K}_{\lambda^{vp}\lambda^{vp}} & \mathbf{K}_{\lambda^{vp}\lambda^{vd}} & \mathbf{K}_{\lambda^{vp}T} \\ \mathbf{K}_{\lambda^{vd}a} & \mathbf{K}_{\lambda^{vd}\lambda^{vp}} & \mathbf{K}_{\lambda^{vd}\lambda^{vd}} & \mathbf{K}_{\lambda^{vd}T} \\ \mathbf{K}_{Ta} & \mathbf{K}_{T\lambda^{vp}} & \mathbf{K}_{T\lambda^{vd}} & \mathbf{K}_{TT} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{a} \\ \Delta\boldsymbol{\Lambda}^{vp} \\ \Delta\boldsymbol{\Lambda}^{vd} \\ \Delta\mathbf{T}^e \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{ext} + \mathbf{f}^{\mathrm{int}} \\ \mathbf{f}_{\lambda^{vp}} \\ \mathbf{f}_{\lambda^{vd}} \\ \mathbf{f}_T \end{Bmatrix} \tag{211}$$

where the diagonal matrices are defined as follows:

$$\mathbf{K}_{aa} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\,\mathrm{d}V \tag{212}$$

$\mathbf{M}^*$ is the modified consistent mass matrix given as

$$\mathbf{M}^* = \int_V \rho^* \mathbf{N}^T \mathbf{N} \, \mathrm{d}V \tag{213}$$

while,

$$\mathbf{K}_{\lambda^{vp}\lambda^{vp}} = \int_V \mathbf{h}\left(Q_1^p \mathbf{h}^T + Q_2^p \mathbf{q}^T\right) \mathrm{d}V, \quad \mathbf{K}_{\lambda^{vd}\lambda^{vd}} = \int_V \mathbf{h}\left(Q_3^d \mathbf{h}^T + Q_4^d \mathbf{q}^T\right) \mathrm{d}V \tag{214}$$

$$\mathbf{K}_{TT} = -\int_V \mathbf{h} \rho c_p \mathbf{h}^T \, \mathrm{d}V \tag{215}$$

and the off diagonal matrices are given by:

$$\mathbf{K}_{a\lambda^{vp}} = -\int_V \mathbf{B}^T \left(\mathbf{E}\mathbf{h}^T \frac{\partial f}{\partial \boldsymbol{\sigma}} + \mathbf{A}\widehat{\mathbf{h}}^{\mathrm{T}} \frac{\partial \mathrm{f}}{\partial \mathbf{Y}} \right) \mathrm{dV}, \quad \mathbf{K}_{\lambda^{\mathrm{vp}}\mathrm{a}} = \int_{\mathrm{V}} \mathbf{h} \frac{\partial \mathrm{f}}{\partial \boldsymbol{\sigma}} \Big|^{\mathrm{T}} \mathbf{E}\mathbf{B} \, \mathrm{dV} \tag{216}$$

$$\mathbf{K}_{a\lambda^{vd}} = -\int_V \mathbf{B}^T \left(\mathbf{E}\mathbf{h}^T \frac{\partial g}{\partial \boldsymbol{\sigma}} + \mathbf{A}\widehat{\mathbf{h}}^{\mathrm{T}} \frac{\partial \mathrm{g}}{\partial \mathbf{Y}} \right) \mathrm{dV}, \quad \mathbf{K}_{\lambda^{\mathrm{vd}}\mathrm{a}} = \int_{\mathrm{V}} \mathbf{h} \frac{\partial \mathrm{g}}{\partial \boldsymbol{\sigma}} \Big|^{\mathrm{T}} \mathbf{E}\mathbf{B} \, \mathrm{dV} \tag{217}$$

$$\mathbf{K}_{\lambda^{vp}\lambda^{vd}} = \int_V \mathbf{h}\left(Q_3^p \mathbf{h}^T + Q_4^p \mathbf{q}^T\right) \mathrm{d}V, \quad \mathbf{K}_{\lambda^{vd}\lambda^{vp}} = \int_V \mathbf{h}\left(Q_1^d \mathbf{h}^T + Q_2^d \mathbf{q}^T\right) \mathrm{d}V \tag{218}$$

$$\mathbf{K}_{\lambda^{vp}T} = \int_V \mathbf{h}\left(\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \boldsymbol{\sigma}} \Big|^T \boldsymbol{\beta} \right) \mathbf{W}^T \, \mathrm{d}V, \quad \mathbf{K}_{T\lambda^{vp}} = \int_V \mathbf{W}\left(Q_1^{tp} \mathbf{h}^T + Q_2^{tp} \mathbf{q}^T\right) \mathrm{d}V \tag{219}$$

$$\mathbf{K}_{\lambda^{vd}T} = \int_V \mathbf{h}\left(\frac{\partial g}{\partial T} - \frac{\partial g}{\partial \boldsymbol{\sigma}} \Big|^T \boldsymbol{\beta} \right) \mathbf{W}^T \, \mathrm{d}V, \quad \mathbf{K}_{T\lambda^{vd}} = \int_V \mathbf{W}\left(Q_1^{td} \mathbf{h}^T + Q_2^{td} \mathbf{q}^T\right) \mathrm{d}V \tag{220}$$

$$\mathbf{K}_{aT} = -\int_V \mathbf{N}^T \boldsymbol{\beta} \, \mathbf{W}^T \, \mathrm{d}V, \quad \mathbf{K}_{Ta} = -\int_V \mathbf{W}\boldsymbol{\beta} \, T_i \mathbf{B} \, \mathrm{d}V \tag{221}$$

The external force and internal force vectors appear on the right-hand side of Eq. (211) are defined as follows:

$$\mathbf{f}^{\mathrm{ext}} = \int_{S_t} \mathbf{N}^T \mathbf{t}_{i+1} \, \mathrm{d}S + \int_V \rho \mathbf{N}^T \mathbf{b}_{i+1}^* \, \mathrm{d}V, \quad \mathbf{f}^{\mathrm{int}} = -\int_V \mathbf{B}^T \boldsymbol{\sigma}_i \, \mathrm{d}V - \mathbf{M}^* \mathbf{a}_i \tag{222}$$

$$\mathbf{f}_{\lambda^{vp}} = -\int_V \mathbf{h} f_i \, \mathrm{d}V, \quad \mathbf{f}_{\lambda^{vd}} = -\int_V \mathbf{h} \, g_i \, \mathrm{d}V \tag{223}$$

$$\mathbf{f}_T = -\int_V \mathbf{W}\rho\mathbf{r}_{\text{ext}}\,\mathrm{d}V - \int_V \nabla\mathbf{W}\,\mathbf{k}\,\nabla\mathbf{W}^T\,\mathbf{T}_i^e\,\mathrm{d}V \tag{224}$$

As usual, all vectors and matrices are assembled from the element contributions in the standard manner. A detailed description of ways of solving the present coupled system of equations with different implementations is presented by de Borst and Mühlhaus (1992), de Borst and Pamin (1996), de Borst et al. (1999), and Bammann et al. (1999).

7. Conclusions

In the current paper the systematic construction of a thermodynamic consistent gradient-enhanced framework for heterogeneous materials, which provides a strong coupling between viscoplasticity and anisotropic viscodamage, is presented for dynamic problems. Thermodynamic consistency is restored through the derived constitutive equations in order to introduce and address issues such as the statistical inhomogeneity in the evolution related viscoinelasticity macroscopic variables, and temperature and strain-rate sensitivity. This framework is general enough to describe the evolution of viscoinelasticity in a material body accounting for physical discontinuities through the use of a non-local approach based on the gradient-dependent theory of viscoplasticity and viscodamage. The interaction of the length-scales is a crucial factor in understanding and controlling the distribution and size of the material defects such as statistically stored and geometrically necessary dislocations, voids, and cracks influence on the macroscopic response. The behavior of these defects is captured not only individually, but also the enhanced strong coupling between the two dissipative processes takes into account the interaction between these defects and their ability to create spatial-temporal patterns under different loading conditions.

The proposed gradient approach introduces second-order gradients in the hardening variables (isotropic and kinematic) and in the damage variable. These higher-order gradients are considered physically and mathematically related to their local counterparts. Special care is used to properly account for the coupling between the state variable and its corresponding higher-order gradient.

Length-scale parameters are implicitly and explicitly introduced into the present dynamical formulism. Implicit length-scale measure is introduced through the use of the rate-dependent theory, while explicit length-scale measures are introduced through the use of the gradient-dependent theory.

A strong coupling between the two dissipative processes, viscoplasticity and viscodamage, is implemented. This strong coupling is assessed by using two separate viscoplasticity and viscodamage surfaces with coupled non-associated flow rules in such a way that both viscoplastic and viscodamage rules are dependent on the viscoplastic and viscodamage potentials. Two viscodamage mechanisms have been considered, one mechanism is coupled with viscoplasti-

city, and while the other occurs independent of viscoplastic deformation. The dissipation function of the latter occurs in both the elastic and viscoplastic domains. The constitutive equations for the damaged material are written according to the principle of strain energy equivalence between the virgin material and the damaged material.

Computational issues of the current gradient-dependent formulation are introduced in a finite element context. A weak (virtual work) formulation of the nonlocal dynamic viscoplastic and viscodamage conditions is derived, which can serve as a basis for the numerical solution of initial boundary value problems in the sense of the finite element method. Explicit expressions for the generalized tangent stiffness matrix and the generalized nodal forces are given. In addition, issues concerned with the formulation of initial-boundary value problems by the finite element analysis in relation to the formulation and the non-standard boundary conditions are presented.

Although there has been a tremendous work to understand the physical role of the gradient theory, this research area is still in a critical state with controversy. This is to some extent due to the difficulty in calibration of the different material properties associated with the gradient-dependent models, which are impossible to obtain for certain cases. But more important have been the difficulty of carrying out truly definitive experiments on critical aspects of the evolution of the dislocation, crack, and void structures. However, the proposed framework is generalized to that of viscoplasticity coupled with viscodamage, and one needs more studies to be performed in order to effectively assess the potential applications for this framework. Furthermore, it is believed that the calibration of a gradient-dependent model of viscoplasticity and viscodamage should not only be based on stress-strain behavior obtained from macroscopic mechanical tests, but should also draw information from micromechanical, gradient-dominant tests such as micro-indentation and/or nano-indentation tests, micro-bending tests, mirco-torsion tests, accompained by metallographic studies and stereology based quantification methods using tomography images.

Additional aspects for future studies is the consideration of the dependency of the intrinsic material length-scale on the strain-rate and temperature variation, testing the viscoplasticity and viscodamage consistency conditions, and detailed description of solving the present coupled system of equations.

Acknowledgements

The authors gratefully acknowledge the financial support of Dr Len Sakell of AFOSR.

References

Abu Al-Rub, R.K., Voyiadjis, G.Z., 2003. On the coupling of anisotropic damage and plasticity models for ductile materials. Int. J. Solids and Structures 40, 2611–2643.

Abu Al-Rub, R.K., Voyiadjis, G.Z. Determination of the material intrinsic length scale of gradient plasticity theory. Int. J. Multiscale Computational Eng. (in press).
Acharya, A., Bassani, J.L., 2000. Lattice incompatibility and a gradient theory of crystal plasticity. J. Mech. Physics of Solids 48, 1595.
Aifantis, E.C., 1984. On the microstructural origin of certain inelastic models. J. Eng. Mater. Technol 106, 326–330.
Aifantis, E.C., 1992. On the role of gradients in the localization of deformation and fracture. Int. J. Eng. Sci. 30, 1279–1299.
Aifantis, E.C., 1995. From micro-plasticity to macro-plasticity: the scale-invariance approach. J. Eng. Mater. Tech., Transactions of ASME 117 (4), 352–355.
Aifantis, E.C., Oka, F., Yashima, A., Adachi, T., 1999. Instability of gradient dependent elastoviscoplasticity for clay. Int. J. Numerical and Analytical Methods in Geomechanics 23 (10), 973–994.
Armstrong, P.J., Frederick, C.O., 1966. A Mathematical Representation of the Multiaxial Bauschinger effect. CEGB Report RD/B/N/731, Berkeley Laboratories, R&D Department, CA.
Armero, F., Oller, S., 2000. A general framework for continuum damage models: I. Infinitesimal plastic damage models in stress space. II. Integration algorithms, with applications to the numerical simulation of porous metals. Int. J. Solids and Structures 37, 7409–7464.
Askes, H., Pamin, J., de Borst, R., 2000. Dispersion analysis and element-free galerkin solutions of second- and fourth-order gradient enhanced damage models. Int. J. Numer. Meth. Eng. 49, 811–832.
Ashby, M.F., 1970. The deformation of plastically non-homogenous alloys. Philos Magazine 21, 399–424.
Bazant, Z.P., Belytschko, T., Chang, T.P., 1984. Continuum theory for strain softening. ASCM J. Eng. Mech. 110, 1666–1691.
Bazant, Z.P., Pijaudier-Cobot, G., 1988. Nonlocal continuum damage, localization instability and convergence. J. Appl. Mech. 55, 287–293.
Bazant, Z.P., Ozbolt, J., 1990. Nonlocal microplane model for fracture, damage, and size effect in structures. J. Eng. Mech., ASCE 116 (11), 2485–2505.
Bammann, D.J., Mosher, D., Hughes, D.A., Moody, N.R., Dawson, P.R., 1999. Using Spatial Gradients to Model Localization Phenomena. Sandia National Laboratories Report, SAND99- 8588, Albuquerque, New Mexico 87185 and Livermore, CA.
Bassani, J., 2001. L Incompatibility and a simple gradient theory of plasticity. J. Mech. and Physics of Solids 49, 1983–1996.
Bjerke, T., Zhouhua, L., Lambros, J., 2002. Role of plasticity in heat generation during high rate deformation and fracture of polycarbonate. Int. J. Plasticity 18, 549–567.
Chaboche, J.-L., 1991. Thermodynamically based viscoplastic constitutive equations: theory versus experiment. In: High Temperature Constitutive Modeling—Theory and Application, MD-Vol.26/AMD-Vol. 121, ASME, pp. 207–226.
Chen, S.H., Wang, T.C., 2002. A new deformation theory with strain gradient effects. Int. J. Plasticity 18, 971–995.
Chow, C.L., Wang, J., 1987. An anisotropic theory of continuum damage mechanics for ductile fracture. Eng. Fracture Mech. 27 (5), 547–558.
Coleman, B., Gurtin, M., 1967. Thermodynamics with internal state variables. The J. Chemical Physics 47 (2), 597–613.
Cordebois, J.P., Sidoroff, F., 1979. Anisotropic damage in elasticity and plasticity. J. Mécanique Théorique et Appliquée, Numéro Spécial 45–60.
de Borst, R., Mühlhaus, H.-B., 1992. Gradient-dependent plasticity formulation and algorithmic aspects. Int. J. Numer. Methods Eng. 35, 521–539.
de Borst, R., Pamin, J., 1996. Some novel developments in finite element procedures for gradient dependent plasticity. Int. J. Numer. Methods Eng. 39, 2477–2505.
de Borst, R., Pamin, J., Geers, M., 1999. On coupled gradient-dependent plastcity and damage theories with a view to localization. Eur. J. Mech. A/Solids 18, 939–962.
de Borst, R., Sluys, L.J., Mühlhaus, H.-B., Pamin, J., 1993. Fundamental issues in finite element analysis of localization of deformation. Eng. Computations 10, 99–121.
Di Prisco, C., Imposimato, S., Aifantis, E.C., 2002. A visco-plastic constitutive model for granular soils

modified according to non-local and gradient approaches. Int. J. Num. Anal. Meth. Geomechanics 26 (2), 121–138.
Doghri, I., 2000. Mechanics of Deformable Solids: Linear and Nonlinear. Analytical and Computational Aspects, Springer, Germany.
Dornowski, W., Perzyna, P., 2000. Localization phenomena in thermo-viscoplastic flow processes under cyclic dynamic loadings. Comp. Assisted Mech. and Eng. Sci. 7, 117–160.
Ehlers, W., 1995. A single-surface yield function for geomaterials. Archive of Appl. Mech. 65, 246–259.
Eringen, A.C., Edelen, D.G.B., 1972. On non-local elasticity. Int. J. Eng. Sci. 10, 233–248.
Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W., 1994. Strain gradient plasticity: theory and experiment. Acta Metallurgica et Materialia 42, 475–487.
Fleck, N.A., Hutchinson, J.W., 1997. Strain gradient plasticity. Adv. Appl. Mech 33, 295–361.
Fleck, N.A., Hutchinson, J.W., 2001. A reformulation of strain gradient plasticity. J. Mech. Phys. Solids 49, 2245–2271.
Freed, A.D., Chaboche, J.L., Walker, K.P., 1991. A viscoplastic theory with thermodynamic considerations. Acta Mechanica 90, 155–174.
Freed, A.D., Walker, K.P., 1990. Steady state and transient zener parameters in viscoplasticity: drag strength versus yield strength. App. Mech. Rev. 43 (5), S328–S337.
Freed, A.D., Walker, K.P., 1993. Viscoplasticity with creep and plasticity bounds. Int. J. Plasticity 9, 213.
Fremond, M., Nedjar, B., 1996. Damage, gradient of damage and principle of virtual power. Int. J. Solids Structures 33 (8), 1083–1103.
Gao, H., Huang, Y., Nix, W.D., 1999a. Modeling plasticity at the micrometer scale. Naturwissenschaften 86, 507–515.
Gao, H., Huang, Y., Nix, W.D., Hutchinson, J.W., 1999b. Mechanism-based strain gradient plasticity—I. Theory. J. Mech. Physics of Solids 47 (6), 1239–1263.
Ganghoffer, J.F., Sluys, L.J., de Borst, R., 1999. A reappraisal of nonlocal mechanics. Eur. J. Mech. A/Solids 18, 17–46.
Glema, A., Lodygowski, T., Perzyna, P., 2000. Interaction of deformation waves and localization phenomena in inelastic solids. Comput. Methods Appl. Mech. Eng. 183, 123–140.
Gurson, A.L., 1977. Continuum theory of ductile rupture by void nucleation and growth. Part I: yield criteria and flow rules for porous ductile media. J. Eng. Mater. Tech. 99, 2–15.
Gurtin, M.E., 2002. A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations. J. Mech. Phys. Solids 50 (1), 5–32.
Gurtin, M.E., 2003. On a framework for small-deformation viscoplasticity: free energy, microforces, strain gradients. Int. J. Plasticity 19, 47–90.
Hansen, N.R., Schreyer, H.L., 1994. Thermodynamically consistent framework for theories of elastoplasticity coupled with damage. Int. J. Solids Structures 31 (3), 359–389.
Henry, W., Haslach Jr., X., 2002. A non-equilibrium thermodynamic geometric structure for thermoviscoplastic with maximum dissipation. Int. J. Plasticity 18, 127–153.
Hesebeck, O., 2001. On an isotropic damage mechanics model for ductile materials. Int. J. Damage Mech. 10, 325–346.
Houghs, T., 1987. The Finite Element Method. Prentice-Hall International.
Hwang, K.C., Jiang, H., Huang, Y., Gao, H., 2003. Finite deformation analysis of mechanism-based strain gradient plasticity: torsion and crack tip field. Int. J. Plasticity 19, 235–251.
Johnson, G.R., Cook, H.W., 1985. Fracture characteristics of three metals subjected to various strains, strain rates, temperature and pressures. Eng. Fract. Mech. 21 (1), 31–48.
Kachanov, L.M., 1958. On the creep fracture time. Izv Akad, Nauk USSR Otd. Tech. 8, 26–31. (in Russian).
Kachanov, L.M., 1986. Introduction to Continuum Damage Mechanics. Martinus Nijhoff Publishers, The Netherlands.
Kattan, P.I., Voyiadjis, G.Z., 2001. Decomposition of damage tensor in continuum damage mechanics. J. Eng. Mech., ASCE 127 (9), 940–944.
Kroner, E., 1967. Elasticity theory of materials with long range cohesive forces. Int. J. Solids Structures 24, 581–597.

Krajcinovic, D., 1983. Constitutive equations for damaging materials. J. Appl. Mech. 50, 355–360.
Krajcinovic, D., 1996. Damage Mechanics. Elsevier, Amsterdam.
Krajcinovic, D., Foneska, G.U., 1981. The continuum damage theory of brittle materials. J. Appl. Mech. 48, 809–824.
Kuhl, E., Ramm, E., de Borst, R., 2000. An anistropic gradient damage model for quasi-brittle materials. Comput. Methods Appl. Mech. Eng. 183, 87–103.
Lasry, D., Belytschko, T., 1988. Localization limiters in transient problems. Int. J. Solids Structures 24, 581–597.
Lacey, T.E., McDowell, D.L., Talreja, R., 2000. Non-local concepts for evolution of damage. Mech. Mater. 31 (12), 831–860.
Lemaitre, J., 1992. A Short Course in Damage Mechanics. Springer-Verlag, New York.
Lemaitre, J., Chaboche, J.-L., 1990. Mechanics of Solid Materials. Cambridge University Press, London.
Lee, H., Peng, K., Wang, J., 1985. An anisotropic damage criterion for deformation instability and its application to forming limit analysis of metal plates. Eng. Fract. Mech 21, 1031–1054.
LeMonds, J., Needleman, A., 1986a. Finite element analysis of shear localization in rate and temperature dependent solids. Mech. Mater. 5, 339–361.
LeMonds, J., Needleman, A., 1986b. An analysis of shear band development incorporating heat conduction. Mech. Mater. 5, 363.
Li, S., Liu, W.K., Rosakis, A.J., Belystchko, T., Hao, W., 2002. Mesh-free galerkin simulations of dynamic shear band propagation and failure mode transition. Int. J. of Solids and Structures 39, 1213–1240.
Lubliner, J., 1990. Plasticity Theory. Macmillan Publishing Company, New York.
Lubarda, V.A., Krajcinovic, D., 1995. Some fundemental issues in rate theory of damage-elastocplasticity. Int. J. Plasticity 11, 763–797.
Mahnken, R., 2002. Theoretical, numerical and identification aspects of a new model class for ductile damage. Int. J. Plasticity 18, 801–831.
Menzel, A., Ekh, M., Steinmann, P., Runesson, K., 2002. Anisotropic damage coupled to plasticity: modeling based on the effective configuration concept. Int. J. Numer. Meth. Eng. 54, 1409–1430.
Murakami, S., Hayakawa, K., Liu, Y., 1998. Damage evolution and damage surface of elastic–plastic-damage materials under multiaxial loading. Int. J. Damage Mech. 7, 103–128.
Murakami, S. and Ohno, N. (1981). A continuum theory of creep and creep damage. In: Proc. 3rd IUTAM Symposium on Creep in Structures, Springer, Berlin, pp. 422–444.
Mühlhaus, H.B., Aifantis, E.C., 1991. A variational principle for gradient principle. Int. J. Solids Structures 28, 845–857.
Needleman, A., 1988. Material rate dependent and mesh sensitivity in localization problems. Comp. Meth. Appl. Mech. Eng. 67, 68–85.
Nesnas, K., 2002. and Saanouni, K Integral formulation of coupled damage and viscoplastic constitutive equations: formulation and computational issues. Int. J. Damage Mech 11, 367–397.
Nix, W.D., Gao, H., 1998. Indentation size effects in crystalline materials: a law for strain gradient plasticity. J. Mech. Phys. Solids 46 (3), 411–425.
Oka, F., Yashima, A., Sawada, K., Aifantis, E.C., 2000. Instability of gradient-dependent elastoviscoplastic model for clay and strain localization analysis. Comp. Meth. Appl. Mech. Eng. 183 (1), 67–86.
Perzyna, P., 1963. The constitutive equations for rate-sensitive materials. Quart. Appl. Math 20, 321–332.
Perzyna, P. (1966). Fundamental problems visco-plasticity. In: Kuerti, H. (Ed.), Advances in Applied Mechanics, vol. 9. Academic Press, pp. 243–377.
Perzyna, P., 1971. Thermodynaic theory of viscoplastcity. Advances in Appl. Mech. 11, 313–354.
Perzyna, P., 1986. Internal state variable description of dynamic fracture of ductile solids. Int. J. Solids Structures 22, 797–818.
Perzyna, P., 1988. Temperature and rate dependent theory of plasticity of crystalline solids. Rev. Phys. Applique 23, 445–459.
Peirce, D., Shih, C.F., Needleman, A., 1984. A tangent modulus method for rate dependent solids. Computer & Structures 5, 875–887.

Peerlings, R.H.J., de Borst, R., Brekelmans, W.A.M., de Vree, J.H.P., 1996. Gradient enhanced damage for quasi-brittle materials. Int. J. Numer. Methods in Eng., Wiley, London 39, 3391–3403.

Pijaudier-Cabot, T.G.P., Bazant, Z.P., 1987. Nonlocal damage theory. ASCE J. Eng. Mech. 113, 1512–1533.

Rabotnov, YU.N., 1968. Creep rupture. In: Proceedings of the XII International Congress on Applied Mechanics, Stanford-Springer, pp. 342–349.

Rousselier, G., 1987. Ductile fracture models and their potential in local approach of fracture. Nuclear Eng. and Design 105, 97–111.

Saczuk, J., Hackl, K., Stumpf, H., 2003. Rate theory of nonlocal gradient damage-gradient viscoinelasticity. Int. J. Plasticity 19, 675–706.

Simo, J.C., Honein, T., 1990. Variational formulation, discrete conservation laws, and path domain independent integrals for elasto-viscoplasticity. J. Appl. Mech.

Simo, J.C., Hughes, T.J.R., 1998. Computational Inelasticity. Interdisciplinary Applied Mathematics, Springer, New York.

Simo, J.C., Ju, J.W., 1989. Finite deformation damage-elastoplasticity: a non-conventional framework. Int. J. Computational Mech. 5, 375–400.

Sidoroff, F. (1981). Description of anisotiopic damage application to elasticity. In: IUTAM Colloqium on Physical Nonlinearities in Structural Analysis, Springer-Verlag, Berlin, pp. 237–244.

Stelmashenko, N.A., Walls, M.G., Brown, L.M., Milman, Y.V., 1993. Microindentation on W and Mo oriented single crystals: an STM study. Acta Metallurgica et Materialia 41, 2855–2865.

Stolken, J.S., Evans, A.G., 1998. A microbend test method for measuring the plasticity length-scale. Acta Materialia 46, 5109–5115.

Taylor, M.B., Zbib, H.M., Khaleel, M.A., 2002. Damage and size effect during superplastic deformation. Int. J. Plasticity 18, 415–442.

Tsagrakis, I., Konstantinidis, A., Aifantis, E.C. Strain gradient and wavelet interpretation of size effects in yield and strength. Mech. Mater. (in press).

Tvergaard, V., 1982. Material failure by void coalescence in localized shear bands. Int. J. Solids Structures 18, 659–672.

Tvergaard, V., Needleman, A., 1984. Analysis of cup-cone fracture in a round tensile bar. Acta Metallurgica 32, 157–169.

Voyiadjis, G.Z., Abu Al-Rub, R.K., 2002. Length scales in gradient plasticity. In: Ahzi, S., Cherkaoui, M. (Eds.), Proceedings of the IUTAM Symposium on Multiscale Modeling and Characterization of Elastic-Inelastic Behavior of Engineering Materials, Edited by, Morocco, October 2002. Kluwer Academic Publishers.

Voyiadjis, G.Z., Abu-Lebdeh, X., 1993. Damage model for concrete using bounding surface concept. J. Eng. Mech. 119 (9), 1865–1885.

Voyiadjis, G.Z., Deliktas, A., 2000a. Multi-scale analysis of multiple damage mechanics coupled with inelastic behavior of composite materials. Mech. Res. Comm. 27 (3), 295–300.

Voyiadjis, G.Z., Deliktas, B., 2000b. A coupled anisotropic damage model for the inelastic response of composite materials. Comput. Methods Appl. Mech. Eng. 183, 159–199.

Voyiadjis, G.Z., Deliktas, B., Aifantis, E.C., 2001. Multiscale analysis of multiple damage mechanics coupled with inelastic behavior of composite materials. J. Eng. Mech. 127 (7), 636–645.

Voyiadjis, G.Z., Dorgan, R.J., 2001. Gradient formulation in coupled damage-plasticity. Archives of Mech. 53 (4-5), 565–597.

Voyiadjis, G.Z., Kattan, P.I., 1992a. A plasticity-damage theory for large deformations of solids. Part I: theoretical formulation. Int. J. Eng. Sci. 30, 1089–1108.

Voyiadjis, G.Z., Kattan, P.I., 1992b. Finite strain plasticity and damage in constitutive modeling of metals with spin tensors. Appl. Mech. Rev. 45, S95–S109.

Voyiadjis, G.Z., Kattan, P.I., 1999. Advances in Damage Mechanics: Metals and Metals Matrix Composites. Elsevier, Oxford.

Voyiadjis, G.Z., Mohammad, L.N., 1988. Rate equations for viscoplastic materials subjected to finite strains. Int. J. Plasticity 4, 215–229.

Voyiadjis, G.Z., Mohammad, L.N., 1991. Theory vs. experiment for finite strain viscoplastic, Lagrangian constitutive model. Int. J. Plasticity 7, 329–350.
Voyiadjis, G.Z., Park, T., 1997. Local and interfacial damage analysis of metal matrix composites using finite element method. Eng. Fract. Mech. 56 (4), 483–511.
Voyiadjis, G.Z., Park, T., 1999. Kinematics of damage for finite strain plasticity. Int. J. Eng. Sci. 1–28.
Voyiadjis, G.Z., Thiagarajan, G., 1997. Micro and macro anisotropic cyclic damage-plasticity models for MMCS. Int. J. Eng. Sci. 35 (5), 467–484.
Voyiadjis, G.Z., Venson, A.R., 1995. Experimental damage investigation of a SiC-Ti aluminide metal matrix composite. Int. J. Damage Mech 4, 338–361.
Wang, W.M., Askes, H., Sluys, L.J., 1998. Gradient viscoplastic modeling of material instabilities in metals. Metals and Materials-Korea 4 (3), 537–542.
Wang, W., Huang, Y., Hsia, K.J., Hu, K.X., Chandra, A., 2003. A study of microbend test by strain gradient plasticity. Int. J. Plasticity 19, 365–382.
Wang, W.M., Sluys, L.J., 2000. Formulation of an implicit algorithm for finite deformation viscoplasticity. Int. J. Solids Structures 37 (48-50), 7329–7348.
Wang, W.M., Sluys, L.J., de Borst, R., 1996. Interaction between material length scale and imperfection size for localization phenomena in viscoplastic media. Eur. J. Mech., A/Solids 15 (3), 447–464.
Zbib, H.M., Aifantis, E.C., 1992. On the gradient-dependent theory of plasticity and shear banding. Acta Mechanica 92, 209–225.
Zbib, H.M., Diaz de la Rubia, T., 2002. A multiscale model of plasticity. Int. J. Plasticity 18, 1133–1163.
Zerilli, F.J., Armstrong, R.W., 1987. Dislocation-mechanics-based constitutive relations for material dynamics calculations. J. Appl. Physics 61 (5), 445–459.
Zhu, Y.Y., Cescetto, S., 1995. Fully coupled elasto-visco-plastic damage theory for anisotropic materials. Int. J. Solids Structures 32 (11), 1607–1641.
Zienkiewicz, O., Taylor, R., 1989. The Finite Element Method, fourth ed. McGraw-Hill, London.