UNIVERZITET U BEOGRADU

MATEMATIČKI FAKULTET

Ali Ababoub

Lipšicov prostor i kvazikonformna preslikavanja

DOKTORSKA DISERTACIJA

BEOGRAD, 2013

UNIVERSITY OF BELGRADE

FACULTY OF MATHEMATICS

Ali Ababaoub

Lipschitz Space and Quasiconformal Mappings

DOCTORAL DISSERTATION

BELGRADE, 2013

Abstract

This thesis has been written under the supervision of my mentor Prof. Miodrag Mateljević, and my co-mentor dr. Vladimir Božin at the University of Belgrade in the academic year 2012-2013. The topic of this thesis is Complex analysis related with geometric function theory, more precisely the theory of quasiconformal mappings in the Euclidean n-dimensional space. For good survey of the field, see F. W. Gehring [20] in the handbook of Kühnau [33] which also contains many other surveys on quasiconformal mappings and related topics. The main source in this dissertation is J. Väisälä [67]. The thesis is divided into three chapters. Chapter 1 is divided into 5 sections. In this chapter, we focus on quasiconformal mappings in $\mathbb{R}^n$ *and discuss various equivalent definitions. We give The Modulus of family of curves in the first section, geometric definition of quasiconformal space mappings in second section, analytic definition of quasiconformal space mappings in third section, equivalence of the definitions in fourth section, and the Beltrami equation in fifth section. Chapter 2 is divided into 5 sections. We begin by generalizing the class of* $Lip_\alpha(\Omega)$, $0 < \alpha \leq 1$, *and some properties of that class. Chapter 2 is devoted to understanding the properties by introducing the notion of Linearity, Differentiability, and majorants. A majorant function is a certain generalization of the power functions* t^α, *this is done in the first section. In the second section we introducing the notion of moduli of continuity with its Some Properties which gotten from I.M. Kolodiy, F. Hildebrand paper [39]. In third section we produced harmonic mapping as preliminary for the fourth section which including subharmonicity of* $|f|^q$ *of harmonic quasiregular mapping in space. In the last section we introducing estimation of the Poisson kernel which were extracted from Krantz paper [42]. Chapter 3 is divided into 3 sections. This chapter is include the main result in this dissertation. In this chapter we prove that* $\omega_u(\delta) \leq C\omega_f(\delta)$, *where* $u : \overline{\Omega} \to \mathbb{R}^n$ *is the harmonic extension of a continuous map* $f : \partial\Omega \to \mathbb{R}^n$, *if* u *is a* K*-quasiregular map and* Ω *is bounded in* $\mathbb{R}^n$ *with* C^2 *boundary. Here* C *is a constant depending only on* n, ω_f *and* K *and* ω_h *denotes the modulus of continuity of* h. *We also prove a version of this result for* Λ_ω*-extension domains with c-uniformly perfect boundary and quasiconformal mappings, and we state some results regarding HQC self maps of the quadrant* $Q = \{z : z = x + iy, x, y > 0\}$.

Scientific field (naučna oblast): Mathematics (matematika)
Narrow scientific field (uža naučna oblast): Complex Analysis (kompleksna analiza)
UDC: 517.984+517.54(043.3)

Podaci o mentoru i članovima komisije:

MENTOR:
redovni profesor dr Miodrag Mateljević, akademik
Matematički fakultet,
Univerzitet u Beogradu

____

ČLANOVI KOMISIJE :
docent dr Vladimir Božin
Matematički fakultet,
Univerzitet u Beogradu

____

vanredni profesor dr Miloš Arsenović
Matematički fakultet,
Univerzitet u Beogradu

____

docent dr Vesna Manojlović
Fakultet organizacionih nauka,
Univerzitet u Beogradu

____

Datum odbrane:

Acknowledgements

I consider myself incredibly lucky to have had Prof. Miodrag Mateljević, as my advisor and teacher. I consider myself incredibly lucky to have known Prof. Miloš Arsenović and dr. Vladimir Božn and working with them. I am very grateful for the interest they have taken in my professional development and personal well-being. I have learnt a great deal from them and I am deeply indebted to them for everything they have done for me. I would also like to thank Prof. Miloš Arsenović and dr. Vladimir Božn for correcting my dissertation and for carefully reading a draft of this thesis and providing me with very useful feedback, and finding improvements and corrections to substantially improve the accuracy of this thesis. I am grateful to the members of reading committee to dr. Vesna Manojlović for spending their valuable time on reading my thesis carefully. The exposition here has significantly benefited from their input, and I would like to thank my friends specially who helps me to finish this project. My completion of this project could not have been accomplished without the support from all of them. Finally, I am indebted to my parents, my wife, my son, and my Daughters for providing fantastic support throughout my education, the culmination of which is represented by this thesis.

Ali ABABOUB

Contents

Chapter 1

Quasiconformal Space Mappings

1.1 The Modulus of family of curves

1.1.1 The Geometry of Curves

This section devoted to some preliminary material in this thesis which is basic for the sequel. Here we will explore the notion of curves.

Definition 1. A curve in the complex plane is defined by a complex valued continuous function $\gamma(t)$ on an interval $[\alpha, \beta]$, where $\gamma(t) = x(t) + iy(t)$, $\alpha \leq t \leq \beta$. The functions $x(t)$, $y(t)$ are real valued continuous functions of t. The complex valued function of the real variable t, is called a parametrization of γ.

The curve is said to be closed if $\gamma(\alpha) = \gamma(\beta)$, and called simple if it is not self-intersecting: if $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$ or $t_1, t_2 \in \{\alpha, \beta\}$.

Definition 2. A simple closed curve is called Jordan curve.

A simple closed curve is said to be positively oriented if the region interior to the curve is to the left of the curve while it is being traversed from $t = \alpha$ to $t = \beta$.

Theorem 1. ***(Jordan curve theorem).*** *Suppose γ is a Jordan closed curve in $\mathbb{C}$. Then there exists two disjoint domains Ω_1 and Ω_2 satisfying:*
1. $\mathbb{C} - \gamma = \Omega_1 \cup \Omega_2$,
2. exactly one of Ω_1 and Ω_2 is a bounded set, and
3. $\partial\Omega_1 = \partial\Omega_2 = \gamma$.

Definition 3. A curve $\gamma(t)$:$[\alpha, \beta] \to \mathbb{R}$ is called a C^1 (or continuously differentiable) if $\gamma'(t)$ exists on (α, β) is continuous, and has a continuous extension to $[\alpha, \beta]$.

Consider a partition of $[a, b]$ such that $a = t_0 \leq t_1 \leq ... \leq t_n = b$. We denote the length of $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ by $\ell(\gamma)$ such that

$$\ell(\gamma) = \sup\Big(\sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|\Big).$$

Hence, $0 \leq \ell(\gamma) \leq \infty$ for all $\gamma \subset \overline{\mathbb{R}^n}$. Clearly $\ell(\gamma) = 0$ if and only if γ is a constant curve.

Definition 4. We say the curve γ is rectifiable if $l(\gamma) < \infty$, and a curve γ is locally rectifiable if all of its closed subcurves are rectifiable

If $\rho(z) \equiv 1$ almost everywhere in Ω, then the 1-length of any rectifiable $\gamma \in \Omega$ coincides with its Euclidian length. In the non-rectifiable case nevertheless, the integral exists and the definition is valid.

Theorem 2. *If $\gamma : [a,b] \to \overline{\mathbb{R}}^n$ is a rectifiable parametrization curve, then there exists a unique parametrization curve $\gamma_0 : [0,c] \to \overline{\mathbb{R}}^n$ that satisfies the following properties:*
1. γ is obtained from γ_0 by an increasing change of parameter;
2. $\ell(\gamma_0 \mid_{[0,t]}) = t$ for $0 \le t \le c$, i.e., $s_{\gamma_0}(t) = t$.
Where $\ell(\gamma)$ is the length of γ. Moreover, $c = \ell(\gamma)$ and $\gamma = \gamma_0 \circ s_\gamma$.

proof. Take γ_0 to be a curve that satisfies conditions 1 and 2. Then $\gamma = \gamma_0 \circ h$ where $h : [a,b] \to [0,c]$ is increasing. Now if $a \le t \le b$, Lemma 1 implies that $\ell(\gamma \mid_{[0,t]}) = \ell(\gamma_0 \mid_{[0,h(t)]}) = h(t)$. Therefore, $h = s_\gamma$ and so γ_0 is unique. Now, if $s_{\gamma(t_1)} = s_\gamma(t_2)$, then $\gamma \mid_{[t_1,t_2]}$ is constant. Therefore there exists a well-defined mapping $\gamma_0 : [0, \ell(\gamma)] \to \overline{\mathbb{R}}^n$ such that $\gamma = \gamma_0 \circ s_\gamma$. □

Definition 5. Let γ be a curve in Ω. The integral

$$l_\rho(\gamma) = \int_\gamma \rho(z)|dz|$$

is said to be the ρ-length of γ.

Definition 6. (Smooth curve). A curve $\gamma(t)$ is said to be smooth if the function $\gamma(t)$ has a continuous derivative on its interval $[\alpha, \beta]$, and nonzero on its interval.

If γ is a smooth curve, then γ has a nonzero tangent vector at each point $z(t)$, which it given by $z'(t)$. thus a smooth curve has no corners or cups.

Definition 7. (Jordan Domain). We say that a bounded set $\Omega \subset \overline{\mathbb{R}}^n$ is a Jordan domain if its boundary $\partial\Omega$ is homeomorphic to $\partial\mathbb{B}^n$.

A Jordan domain need not be homeomorphic to $\mathbb{B}^n$, if $n \ge 3$. The interior of a Jordan curve in complex plane is a domain called a Jordan domain. A Jordan domains are simply connected.

1.1.2 Möbius transformation

I get this paragraph from Matti Vuorinen book [68]. For $x \in \mathbb{R}^n$ and $r > 0$ let

$$B^n(x,r) = \{z \in \mathbb{R}^n : |x - z| < r\}$$
$$S^{n-1}(x,r) = \{z \in \mathbb{R}^n : |x - z| = r\}$$

denote the ball and sphere, respectively, centered at x with radius r. The abbreviations $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $B^n = B^n(1)$, $S^{n-1} = S^{n-1}(1)$ will be used frequently. For $t \in \mathbb{R}$ and $a \in \mathbb{R}^n \backslash \{0\}$ we denote

$$\mathcal{P}(a,t) = \{x \in \mathbb{R} : x.a = t\} \cup \{\infty\}.$$

Then $P(a,t)$ is a hyperplane in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ perpendicular to the vector a, at distance $\frac{t}{|a|}$ from the origin.

Definition 8. (homeomorphism map). A map $f : \Omega \to \Omega'$ is called a homeomorphism if:
1- f is bijection, and
2- f and its inverse mapping $f^{-1} : \Omega' \to \Omega$ are both continuous.

Theorem 3. *Every one-to-one and continuous mapping of an open set of the plane onto a plane set is a homeomorphism.*

Definition 9. (Sense-preserving homeomorphisms.). Let Ω and Ω' be domains in $\overline{\mathbb{R}}^n$. We call a C^1-homeomorphism $f : \Omega \to \Omega'$ sense-preserving (orientation-preserving) if $J_f(x) > 0$ for all $x \in \Omega \backslash \{\infty, f^{-1}(\infty)\}$. If $J_f(x) < 0$ for all $x \in \Omega \backslash \{\infty, f^{-1}(\infty)\}$ then we call f sense-reversing (orientation-reversing).

Theorem 4. *For any simple closed curve in the plane, there is a homeomorphism of the plane which takes that curve into the standard circle.*

Definition 10. Let Ω, Ω' be domains in $\mathbb{R}^n$. A homeomorphism $f : \Omega \to \Omega'$ is called conformal if f is in $C^1(\Omega)$, $J_f(x) \neq 0$ for all $x \in \Omega$, and $|f'(x)h| = |f'(x)||h|$ for all $x \in \Omega$ and $h \in \mathbb{R}^n$. If Ω, Ω' are domains in $\mathbb{R}^n$, a homeomorphism $f : \Omega \to \Omega'$ is conformal if its restriction to $\Omega \backslash \{\infty, f^1(\infty)\}$ is conformal.

Example 1. Some basic examples of conformal mappings are the following elementary transformations.
(1) A reflection in $\mathcal{P}(a, t)$:

$$f_1(x) = x - 2(x.a - t)\frac{a}{|a|^2}, \quad f_1(\infty) = \infty.$$

(2) An inversion (reflection) in $S^{n-1}(a, r)$:

$$f_2(x) = a + r^2 \frac{(x - a)}{|x - a|^2}, \quad f_2(a) = \infty, \quad f_2(\infty) = a.$$

(3) A translation $f_3(x) = x + a, \quad a \in \mathbb{R}^n, \quad f_3(\infty) = \infty.$
(4) A stretching by a factor $k > 0 : f_4(x) = kx, \quad f_4(\infty) = \infty.$
(5) An orthogonal mapping, *i.e.* a linear map f_5 with

$$|f_5(x)| = |x|, \quad f_5(\infty) = \infty.$$

Definition 11. (Möbius transformation)
A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a Möbius transformation if $f = g_1 \circ g_2 \circ ... \circ g_p$, where each g_j is one of the elementary transformations in example (1 (1)-(5)) and p is a positive integer. Equivalently f is a Möbius transformation if $f = h_1 \circ h_2 \circ ... \circ h_m$ where each h_j is a reflection in a sphere or in a hyperplane and m is a positive integer.

It follows from the inverse function theorem and the chain rule that the set of all conformal mappings of $\mathbb{R}^n$ is a group.

Theorem 5. *Each Möbius transformation* $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ *is a homeomorphism.*

1.1.3 Admissible metric

We need to introduce the notion of admissible metric. Every $\gamma \in \Gamma$ shall be a locally rectifiable.

Definition 12. (Admissible metric): Let Γ be a family of curves in $\overline{\mathbb{R}}^n$. A metric ρ is called an admissible metric if it satisfies the following conditions:
i. $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a Borel measurable function,
ii. $\rho \geq 0$ and
iii. $\int_\gamma \rho ds \geq 1$

for each locally rectifiable curve $\gamma \in \Gamma$. By $F(\Gamma)$ we will mean the family of admissible functions.

Example 2. Let Γ be a curve family, suppose that γ contained in a Borel set $\Omega \subset R^n$ for all $\gamma \in \Gamma$, and $l(\gamma) \geq r > 0$ for all $\gamma \in \Gamma$, where γ is locally rectifiable. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\rho^*(x) = \begin{cases} \frac{1}{r} & \text{if } x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

Then $\rho^*(x)$ is admissible. Since

$$\int_\gamma \rho ds = \frac{1}{r} l(\gamma) \geq \frac{1}{r}.r = 1$$

1.1.4 Definition of the Modulus of family of curves

Definition 13. Let Γ be a family of curves in $\mathbb{R}^n$. Denote m the n-dimensional Lebesgue measure in $\mathbb{R}^n$. The modulus (or conformal modulus) of Γ defined by

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho^n dm.$$

where the infimum is taken over all metrics ρ in $F(\Gamma)$.

Remark 1. *Note in particular that the modulus of the collection of all non-locally rectifiable curves is zero.*

If $F(\Gamma) = \emptyset$, since $\rho(x) = \infty$ belongs to $F(\Gamma)$. Then $M(\Gamma) = \infty$. Clearly $0 \leq M(\Gamma) \leq \infty$.

Remark 2. *Observe that, if* $\Gamma_1 \subset \Gamma_2$ *then* $M(\Gamma_1) \leq M(\Gamma_2)$.

1.1.5 Properties of the Modulus

The basic properties of the modulus we take it from Väissälä's book([65]Ch 1).

Theorem 6. $M(\Gamma)$ *is an outer measure in the space of all curve families in* $\mathbb{R}^n$. *That is,*
1. $M(\emptyset) = 0$
2. If $\Gamma_1 \subset \Gamma_2$ *in* $\mathbb{R}^n$, *then* $M(\Gamma_1) \leq M(\Gamma_2)$, *and*
3.If $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, *then*

$$M(\Gamma) \leq \sum_{n=1}^{\infty} M(\Gamma_n) \tag{1.1}$$

proof. 1. Since the zero function belongs to $F(\emptyset)$, $M(\emptyset) = 0$.
2. If $\Gamma_1 \subset \Gamma_2$ Then $F(\Gamma_2) \subset F(\Gamma_1)$, if the infimum is taken on both sides. Thus $M(\Gamma_1) \leq M(\Gamma_2)$.
3. We note that (2.9) always holds if the right hand side is infinite. If it is finite, then given $\epsilon > 0$, we can for every n choose a function ρ_n admissible for Γ_n such that

$$\int \rho_n^n dm \leq M(\Gamma_n) + 2^{-n}\epsilon.$$

The function $\rho = \left(\sum \rho_n^P\right)^{1/P}$ is admissible for Γ. Consequently, we have

$$M(\Gamma) \leq \int \rho^n dm = \sum_{n=1}^{\infty} \int \rho_n^P dm \leq \sum_{n=1}^{\infty} M(\Gamma_n) + \epsilon,$$

and hence (2.9) □

Theorem 7. *$F(\Gamma) = \emptyset$ if and only if Γ contains a constant curve.*

Lemma 1. *$M(\Gamma) = 0$ if and only if there exist a Borel function $0 \leq \rho_0 \in L^2(\Omega)$, such that*

$$\int_\gamma \rho_0 ds = \infty \quad \textit{and} \quad \int_{\mathbb{R}^n} \rho_0^n dm < \infty$$

for every locally rectifiable $\gamma \in \Gamma$.

proof. If there exists a function ρ with the above properties then all the functions $\rho_0/n, n = 1, 2, ...,$ admissible for Γ and we have

$$M(\Gamma) \leq \lim_{n\to\infty} \frac{1}{n^2} \int_\gamma \rho_0^n dm = 0.$$

Conversely, if $M(\Gamma) = 0$ then there exists a sequence $\rho_l, \rho_2, ...$ of functions which are admissible for Γ, and satisfy

$$\int_{\mathbb{R}^n} \rho_n^p dm < 4^{-n}, \quad n = 1, 2, ...$$

Then $\rho = \sum \rho_n$ is a Borel function, and

$$\begin{aligned} \int_\gamma \rho^p dm &= \int_\gamma \Big(\sum_{n=1}^{\infty} 2^{-n/2} 2^{n/2} \rho_n\Big)^p dm \leq \\ &\leq \int_\gamma \sum_{n=1}^{\infty} 2^{-n/2} \sum_{n=1}^{\infty} 2^{n/2} \rho_n^p dm = \\ &= \sum_{n=1}^{\infty} 2^{n/2} \int_\gamma \rho_n^p dm < \sum_{n=1}^{\infty} 2^{-n/2} = 1 \end{aligned}$$

Since each ρ_n is admissible for Γ, we have further

$$\int_\gamma \rho dm = \sum_{n=1}^{\infty} \int_\gamma \rho_n dm = \infty$$

for every locally rectifiable $\gamma \in \Gamma$, and the lemma is proved. □

Theorem 8. *Let $c > 0$ and define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = cx$. Denote the image of a curve family $\Gamma \in \mathbb{R}^n$ under f by $c\Gamma$. Then*

$$M(c\Gamma) = c^{n-p} M(\Gamma)$$

Proof. $f(x) = cx$ implies that $|f'(x)| = c$. Then, $|J_f(x)| = c^n$ for all $x \in \mathbb{R}^n$. It suffices to show

$$c^{n-p}M_p(\Gamma) \leq M(c\Gamma) \quad \text{or} \quad M(\Gamma) \leq c^{p-n}M(c\Gamma).$$

Let $\hat{\rho} \in F(c\Gamma)$ and define $\rho(x) = c\hat{\rho}(f(x))$. Hence,

$$1 \leq \int_{f\circ\gamma} \hat{\rho} ds \leq \int_{\gamma} (\hat{\rho} \circ f).|f'(x)|ds = \int_{\gamma} (\hat{\rho} \circ f).cds = \int_{\gamma} \rho ds.$$

This implies that $\rho \in F(\Gamma)$, and so

$$\begin{aligned} M(\Gamma) \leq \int_{\mathbb{R}^n} \rho^p dm &= \int_{\mathbb{R}^n} c^p (\hat{\rho} \circ f)^p dm \\ &= c^{p-n} \int_{\mathbb{R}^n} c^n (\hat{\rho} \circ f)^p dm \\ &= c^{p-n} \int_{\mathbb{R}^n} (\hat{\rho} \circ f)^p |J_f(x)| dm = c^{p-n} \int_{\mathbb{R}^n} \hat{\rho}^p dm. \end{aligned}$$

Now by taking the infimum over all $\rho \in F(\Gamma)$, we obtain:

$$M(c\Gamma) = c^{n-p}M(\Gamma)$$

□

1.1.6 Modulus in conformal mappings

Let $\Omega \subset \overline{\mathbb{R}^n}$, and $f : \Omega \to \overline{\mathbb{R}^n}$ be a continuous function. Suppose that Γ is a family of curves in Ω Then $\Gamma' = \{f \circ \gamma : \gamma \in \Gamma\}$ is a family of curves in $f(\Omega)$. Γ' is called the image of Γ under f.

Theorem 9. *Let Ω, Ω' are domains in $\overline{\mathbb{R}}^n$, and if $f : \Omega \to \Omega'$ is conformal, then $M(\Gamma) = M(\Gamma')$, for all $\Gamma \subset \Omega$.*

Proof. Let $\rho' \in F(\Gamma')$ and define $\rho'(f(x)) \cdot |f'(x)|$. We have $\rho \in F(\Gamma)$, since for all locally rectifiable $\gamma \in \Gamma$

$$\begin{aligned} \int_{\gamma} \rho ds &= \int_{\gamma} \rho'(f(x)) \cdot |f'(x)| ds \\ &= \int_{f\circ\gamma} \rho'(x) ds \geq 1. \end{aligned}$$

Now since $\rho \in F(\Gamma)$, this leads us to

$$M(\Gamma) \leq \int_{\Omega} \rho dm$$

$$= \int_{\Omega} (\rho'(f(x)) \cdot |f'(x)|)^n dm$$
$$= \int_{\Omega} \rho'(f(x))^n \cdot |f'(x)|^n dm \quad \text{since } f \text{ is conformal}$$
$$= \int_{\Omega} (\rho'(f(x)) \cdot J_f(x) dm$$
$$= \int_{\Omega'} (\rho'(f(x)) \leq \int_{\mathbb{R}^n} \rho'^n dm.$$

Now taking the infimum, we obtain $M(\Gamma) \leq M(\Gamma')$.
We recall that f^{-1} is conformal if f is conformal. Hence $M(\Gamma') \leq M(\Gamma)$ and so,

$$M(\Gamma) = M(\Gamma')$$

□

1.2 Geometric Definition Of Quasiconformal space Mappings.

1.2.1 The dilatations

1.2.1.1 The dilatation of homeomorphism

Let Ω, Ω' are domains in $\overline{\mathbb{R}}^n$, let Γ be a curve family in Ω, and $\Gamma' = \{f \circ \gamma : \gamma \in \Gamma\}$ the image of family Γ under f. We define

$$K_I(f) = \sup \frac{M(\Gamma')}{M(\Gamma)} \quad \text{and} \quad K_O(f) = \inf \frac{M(\Gamma)}{M(\Gamma')}.$$

Where the suprema are taken over all $\gamma \subset \Gamma$ such that $M(\Gamma)$ and $M(\Gamma')$ are not both 0 or ∞. We say $K_I(f)$ is the inner dilatation of f, $K_O(f)$ is the outer dilatation of f, and $K(f) = \max\{K_O(f), K_I(f)\}$ is the maximal dilatation of f.
It follows from the definitions that the dilatations are positive numbers, possibly infinite, and we note that $K_I \geq 1$ or $K_O \geq 1$, hence $K \geq 1$.

Theorem 10. *([[58]][Theorem 3.1.2])*
Let $f : \Omega \to \Omega'$ be a homeomorphism. The following properties hold for all $x \in \Omega$:
1. $K_I(f^{-1}) = K_O(f)$. 2. $K_O(f^{-1}) = K_I(f)$. 3. $K(f^{-1}) = K(f)$.
4. $K_I(f \circ g) \leq K_I(f)K_I(g)$. 5. $K_O(f \circ g) \leq K_O(f)K_O(g)$. 6. $K(f \circ g) \leq K(f)K(g)$.

proof. Let Γ be a family of curves in Ω. By the definition of Γ', the results are clear if any of the dilatations are infinite. Hence, we will assume K_O and K_I to be finite. Recall $f : \Omega \to \Omega', \Gamma \subset \Omega$, and $\Gamma' = f\Omega \subset \Omega'$. To show Relation (1) we note

$$\frac{M(\Gamma')}{M(\Gamma)} = \frac{M(\Gamma')}{M(f^{-1}\Gamma')} \leq K_O(f^{-1})$$

and by taking the supremum over all Γ we obtain $K_I(f) \leq K_O(f^{-1})$.
To show Relation (2) we note

$$\frac{M(\Gamma)}{M(\Gamma')} = \frac{M(f^{-1}\Gamma')}{M(\Gamma')} \leq K_I(f^{-1})$$

and by taking the supremum over all Γ we obtain $K_O(f) \leq K_I(f^{-1})$.
Relation (3) follows from Relations (1) and (2). Now we want to prove relation (4) we see that:

$$\frac{M(\Gamma')}{M(\Gamma)} = \frac{M((f \circ g)\Gamma)}{M(\Gamma)} = \frac{M((f \circ g)\Gamma)}{M(g\Gamma)} . \frac{M(g\Gamma)}{M(\Gamma)} \leq K_I(f).K_I(g).$$

Taking the supremum over all Γ gives us Relation (4). Relation (5) follows in a similar fashion. Relations (4) and (5) together gives us

$$K_I(f \circ g) \leq K_I(f) \max\{K_I(g), K_O(g)\}.$$

hence, $K_I(f \circ g) \leq K_I(f)K(g)$. Similarly, $K_O(f \circ g) \leq K_O(f)K(g)$. Therefore

$$K_I(f \circ g) \leq K(f)K(g)$$
$$K_O(f \circ g) \leq K(f)K(g)$$

and so, $K(f \circ g) \leq K(f)K(g)$. □

1.2.1.2 The dilatation of Linear Mapping

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection, we define the following quantities:

$$H_I(A) = \frac{|detA|}{\ell(A)^n}, \quad H_O(A) = \frac{|A|^n}{|detA|}, \quad H(A) = \frac{|A|}{\ell(A)}$$

where $\ell(A) = \min_{||x||=1} |Ax|$, We say the quantities H_I, H_O, and H are the inner, outer, and linear dilatations of A, respectively. Obviously, all three dilatations are > 1.

Let $f : \Omega \to \Omega'$ be a homeomorphism, we say that f is a diffeomorphism, if f and f^{-1} are both belong to C^1. Equivalently, a diffeomorphism is a C^1-homeomorphism whose jacobian $J_f(x) \neq 0$. If f is a diffeomorphism, then

$$H_I(f'(x)) = \frac{|J_f(x)|}{\ell(f'(x))^n}, \quad H_O(f'(x)) = \frac{|f'(x)|^n}{|J_f(x)|}$$

In geometric sense, $H(A)$ measures the eccentricity of the ellipsoid $E(A)$ while $H_I(A)$ and $H_O(A)$ relate the volume of $E(B_n)$ to the volumes of the inscribed and circumscribed balls centered about $E(A)$. H(A) is the ratio of the greatest and the smallest semiaxis of $E(A)$.

Theorem 11. *([[58]][Theorem 3.1.5]) If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection, then*

$$1. H_O(A) \leq H_I(A)^{n-1}, \qquad 2. H_I(A) \leq H_O(A)^{n-1}, \qquad 3. H(A)^n = H_I(A) \cdot H_O(A),$$
$$4. H(A) \leq \min\{H_I(A), H_O(A)\} \leq H(A)^{n/2} \leq \max\{H_I(A), H_O(A)\} \leq H(A)^{n-1}.$$

proof. Let $a_1 \geq a_2 \geq ... \geq a_n$ be the semi-axis of $E(A)$. We want to prove 1, we know $a_n^{n-1} \leq a_2...a_n$. From this we get:

$$\begin{aligned}
a_n^{n-2} &\leq a_2...a_{n-1} \\
(a_n^{n-2})^n &\leq (a_2...a_{n-1})^n \\
a_n(a_n^{n^2-2n})a_1^{n-1} &\leq a_n(a_2...a_{n-1})^n a_1^{n-1} \\
a_n^{n^2-2n+1} \cdot a_1^{n-1} &\leq (a_1...a_{n-1})^{n-1}(a_2...a_n) \\
(a_n^{n-1})^{n-1} \cdot a_1^{n-1} &\leq (a_1...a_{n-1})^{n-1}(a_2...a_n) \\
\frac{a_1^{n-1}}{a_2...a_n} &\leq \frac{(a_1...a_{n-1})^{n-1}}{(a_n^{n-1})^{n-1}} \\
H_O(A) &\leq H_I(A)^{n-1}.
\end{aligned}$$

Now, we need prove 2 We have $a_1...a_{n-1} \leq a_1^{n-1}$. From this we obtain:

$$\begin{aligned}
a_1^{n-1} &\geq a_1...a_{n-1} \\
a_1^{n-2} &\geq a_2...a_{n-1} \\
(a_1^{n-2})^n = a_1^{n^2-2n} &\geq (a_2...a_{n-1})^n \\
a_n^{n-1} \cdot a_1^{n^2-2n} \cdot a_1 &\geq a_n^{n-1}(a_2...a_{n-1})^n a_1 \\
a_1^{n^2-2n+1} a_n^{n-1} = a_n^{n-1} \cdot (a_1^{n-1})^{n-1} &\geq a_n^{n-1}(a_2...a_{n-1})^n a_1 \\
(a_1^{n-1})^{n-1} \cdot a_n^{n-1} &\geq (a_1...a_{n-1})(a_2...a_n)^{n-1} \\
(\frac{a_1^{n-1}}{a_2...a_n})^{n-1} &\geq \frac{a_1...a_{n-1}}{a_n^{n-1}} \\
H_I(A) &\leq H_O(A)^{n-1}.
\end{aligned}$$

We prove 3

$$\begin{aligned}
H(A)^n = (\frac{a_1}{a_n})^n &= \frac{a_1^n}{a_n^n} \\
&= \frac{a_1^n}{a_n^n} \cdot \frac{a_2...a_{n-1}}{a_2...a_{n-1}} \\
&= \frac{a_1^{n-1}}{a_n^{n-1}} \cdot \frac{a_1...a_{n-1}}{a_2...a_n} \\
&= \frac{a_2...a_n}{a_n^{n-1}} \cdot \frac{a_1^{n-1}}{a_2...a_n} = H_I(A)H_O(A)
\end{aligned}$$

To prove 4 we will introduce in parts:
- $H(A) \leq H_O(A)$.

We have

$$\begin{aligned}
a_1...a_{n-1} &\leq a_1^{n-1} \\
a_1...a_{n-1} \cdot a_n &\leq a_1^{n-1} \cdot a_n \\
\frac{a_1}{a_n} \cdot a_n &\leq \frac{a_1^{n-1}}{a_2...a_{n-1}} \\
H(A) &\leq H_O(A)
\end{aligned}$$

• $H(A) \leq H_I(A)$.
We have

$$\begin{aligned} a_n^{n-1} &\leq a_2 ... a_n \\ a_1 \cdot a_n^{n-1} &\leq a_1 ... a_n = (a_1 ... a_{n-1}) \cdot a_n \\ \frac{a_1}{a_n} &\leq \frac{a_1 ... a_{n-1}}{a_n^{n-1}} \\ H(A) &\leq H_I(A) \end{aligned}$$

And so, we conclude that $H(A) \leq \min\{H_O(A), H_I(A)\}$.

Next, we want to prove that $\max\{H_O(A), H_I(A)\} \leq H(A)^{n-1}$.
• $H_O(A) \leq H(A)^{n-1}$.

$$\begin{aligned} a_n^{n-1} &\leq a_2 ... a_n \\ a_1^{n-1} a_n^{n-1} &\leq a_1^{n-1} a_2 ... a_n \\ \frac{a_1^{n-1}}{a_2 ... a_n} &\leq \frac{a_1^{n-1}}{a_n^{n-1}} \\ H_O(A) &\leq H(A)^{n-1} \end{aligned}$$

• $H_I(A) \leq H(A)^{n-1}$.

$$\begin{aligned} a_1 ... a_{n-1} &\leq a_1^{n-1} \\ \frac{a_1 ... a_{n-1}}{a_n^{n-1}} &\leq \frac{a_1^{n-1}}{a_n^{n-1}} \\ H_I(A) &\leq H(A)^{n-1} \end{aligned}$$

We conclude $\max\{H_O(A), H_I(A)\} \leq H(A)^{n-1}$. From part 3, we can get

$$\min\{H_O(A), H_I(A)\}^2 \leq H(A)^n \leq \max\{H_O(A), H_I(A)\}^2.$$

□

1.2.1.3 Geometric Definition Of Quasiconformal space Mappings.

Definition 14. If $K(f) = K < \infty$, we say that f is K-quasiconformal. The map f is K-quasiconformal in geometric sense if and only if

$$\frac{1}{K} M(\Gamma) \leq M(\Gamma') \leq K M(\Gamma)$$

for every curve family Γ in Ω. f is quasiconformal if $K(f) < \infty$.

From theorem (10) we get the following corollary

Corollary 1. *If f is K_1-quasiconformal and g is K_2-quasiconformal,, then f^{-1} is K_1-quasiconformal, and $h = f \circ g$ is K_1K_2-quasiconformal.*

1.2.1.4 Examples.

1. Let $a \neq 0$ be a real number, and let $f(x) = |x|^{a-1}x$. We can extend f to a homeomorphism $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ by defining $f(0) = 0, f(\infty) = \infty$ for $a > 0$ and $f(0) = \infty, f(\infty) = 0$

for $a < 0$. Then f is quasiconformal with

$$K_I(f) = |a|, \quad K_O(f) = |a|^{n-1} \quad \text{if} \quad |a| \geq 1,$$
$$K_I(f) = |a|^{1-n}, \quad K_O(f) = |a|^{-1} \quad \text{if} \quad |a| \leq 1.$$

2. Let (r, ϕ, z) be the cylindrical coordinates of a point $x \in \mathbb{R}^n$, i.e. $r \geq 0$, $0 \leq \phi \leq 2\pi, z \in \mathbb{R}^{n-2}$, and

$$\begin{cases} x_1 = r\cos\phi, \\ x_2 = r\sin\phi, \\ x_j = z_{j-2} \qquad \text{for } 3 \leq j \leq n \end{cases}$$

The domain Ω_α, defined by $0 < \phi < \alpha$, is called a wedge of angle α, $\alpha \in (0, 2\pi)$. Let $0 < \alpha \leq \beta < 2\pi$. The folding $f : \Omega_\alpha \to \Omega_\beta$, defined by

$$f(r, \phi, z) = (r, \beta\phi/\alpha, z),$$

is quasiconformal with $K_I(f) = \beta/\alpha, K_O(f) = (\beta/\alpha)^{n-1}$.

1.3 Analytic Definition of Quasiconformal Space Mappings

1.3.1 Partial derivatives

Suppose that A is an open set in $\mathbb{R}^n$ and that $f : A \to \mathbb{R}^n$ is a mapping. If the *ith* partial derivative of f exists at a point $x \in A$, we denote it by $\partial_i f(x)$. That is

$$\partial_i f(x) = \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t}$$

If f is differentiable at x, then all partial derivatives exist, and $\partial_i f(x) = f'(x)e_i$.

1.3.2 ACL-Property

Definition 15. (Absolutely continuous functions on an interval.) The function $f : (a, b) \to \mathbb{R}^m$ is absolutely continuous on the interval (a, b) if for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\sum_{i=1}^{n} ||f(b_i) - f(a_i)|| < \epsilon,$$

then for every finite sequence of non-intersecting intervals $a_i \leq x \leq b_i, i = 1, ..., n$ contained in (a, b), such that

$$\sum_{i=1}^{n} ||b_i - a_i|| < \delta$$

Remark 3. *([Equivalent definition]). The following conditions for a real-valued function f on a compact interval $[a, b]$ are equivalent*
(1) f is absolutely continuous;
(2) f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)dt$$

for all x on $[a, b]$;
(3) there exists a Lebesgue integrable function g on $[a, b]$ such that

$$f(x) = f(a) + \int_a^x g(t)dt$$

for all x on $[a, b]$.

Let $Q = \{x \in \mathbb{R}^n : x_i \in [a_i, b_i]\}$ be a closed n-interval, if $f : Q \to \mathbb{R}^m$ is continuous, for almost any $(c_1, c_2, ..., c_n) \in Q$, and $f_i : [a_i, b_i] \to \mathbb{R}^m$, where $f_i(x) = f(c_1, c_2, ..., c_{i-1}, x, c_{i+1}, ..., c_n)$, $i = 1, 2, ..., n$, are absolutely continuous.

Now, we define absolutely continuously on lines.

Definition 16. (ACL on an interval): A function $f|_Q$ is called ACL (absolutely continuous on lines) if $f|_Q$ is continuous and $f|_Q$ is absolutely continuous on almost every line segment in Q parallel to the coordinate axes.

Definition 17. (ACL): Let Ω be a domain, $Q \subset \Omega$. A function $f : \Omega \to \mathbb{R}^m$, is ACL if for every n-interval $Q \subset \Omega$, the function $f|_Q$ is ACL.

Remark 4. *A complex valued function f is ACL in Ω if its real and imaginary parts are ACL in Ω.*

Definition 18. Let Ω, Ω' are a domains in $\mathbb{R}^n$, and $f : \Omega \to \Omega'$ be a homeomorphism, we say that f is ACL if $f|_{\Omega\setminus\{\infty, f^{-1}(\infty)\}}$ is ACL.

Definition 19. (ACL^p):Let $f : \Omega \to \mathbb{R}^m$, is ACL for every closed n-interval Q, we say that f is in ACL^p or in $ACL^p(\Omega)$, If $p \geq 1$, and such a function has partial derivatives $D_i f(x)$ *a.e.* in Ω, these partial derivatives of f are locally L^p-integrable, and they are Borel functions.

1.3.3 Analytic definition of quasiconformal space mapping

Definition 20. :
A homeomorphism $f : \Omega \to \mathbb{R}^n, n \geq 2$, of a domain Ω in $\mathbb{R}^n$ is called quasiconformal in analytic sense if f is satisfying the following conditions
1. f is ACL^n, and
2. f is differentiable *a.e.*, and
3. there exists a constant $K, 1 \leq K < \infty$ such that

$$|f'(x)|^n \leq K|J_f(x)|, \quad |f'(x)| = \max_{|h|=1} |f'(x)h|$$

a.e. in Ω, where $f'(x)$ is the formal derivative.

The smallest $K \geq 1$ for which this inequality is true is called the outer dilatation of f and denoted by $K_O(f)$. If f is quasiconformal, then the smallest $K \geq 1$ for which the inequality

$$|J_f(x)| \leq Kl(f'(x))^n, \quad l(f'(x)) = \min_{|h|=1} |f'(x)h|$$

holds *a.e.* in Ω is called the inner dilatation of f and denoted by $K_I(f)$. The maximal dilatation of f is the number $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K$, f is said to be K-quasiconformal. It is well-known that

$$K_O(f) \leq K_I^{n-1}(f), \quad K_I(f) \leq K_O^{n-1}(f) \tag{1.2}$$

1.4 Equivalence of the Definitions

Now we are ready to formulate the theorem satisfying that Geometric definition of quasiconformal mappings is equivalent to analytic definition of it, and prove that result.

Theorem 12. *Let $f : \Omega \subset \overline{\mathbb{R}}^n \to \Omega'$ be a homeomorphism. For all $\Gamma \subset \Omega$ the following are equivalent:*
1. $\frac{1}{K} M(\Gamma) \leq M(\Gamma') \leq KM(\Gamma)$
2. f is ACL^n, almost every where differentiable, and

$$\frac{1}{K}|f'(x)|^n \leq |J_f(x)| \leq K.l(f'(x))^n. \tag{1.3}$$

1.5 The Beltrami Equation

There is another approach to the theory of planar quasiconformal mappings. Directly from the analytic definition we see that there is a measurable function $\mu : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$, where $\mathbb{C}$ be the complex plane, such that

$$f_{\overline{z}} = \mu(z) f_z \tag{1.4}$$

where

$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y), \quad \text{and} \quad f_z = \frac{1}{2}(f_x - if_y).$$

are formal derivatives of f with respect to $\overline{z}$ and z, $z = x + iy$, while f_x and f_y are partial derivatives of f with respect to x and y, respectively.
In the real variables x, y, u, and v, (1.4) can be written in the form of the system

$$\begin{cases} v_y = \alpha u_x + \beta u_y \\ -v_x = \beta u_x + \gamma u_y \end{cases}$$

where α, β, and γ are given measurable functions in x and y. The complex dilatation μ of f is in the unit ball of L^∞ when f is quasiconformal. Indeed,

$$||\mu||_\infty = \frac{K-1}{K+1} < 1.$$

Equation (1.4) is called the complex Beltrami equation, and the function $\mu_f = f_z / f_{\overline{z}}$ is called the Beltrami coefficients of f or the complex dilatation of f.

Example 3. Suppose $f(z) = az + b\overline{z}$ is an orientation preserving linear map of $\mathbb{C}$, with $|a| > |b|$. Then a little geometric calculation will show that $\mu_f(z) = b/a$, and f maps the unite circle to an ellipse and the ratio of the major and minor axes of this ellipse is

$$K = \frac{|a| + |b|}{|a| - |b|} = \frac{1 + |\mu_f|}{1 - |\mu_f|}$$

Example 4. Let $f(z) = -z \log |z|^2$, where $|z| \leq r = e^{-2}$. Then $f : \mathbb{D}(0, r) \to \mathbb{D}(0, 4r)$ is a homeomorphism and

$$f_{\overline{z}} = -\frac{z}{\overline{z}}, \quad f_z = -1 - \log |z|^2, \quad \mu_f(z) = \frac{z}{\overline{z}(1 + \log |z|^2)}.$$

Thus f is quasiconformal with a continuous Beltrami coefficient, and yet f is not C^1.

1.5.1 Chain rule for Beltrami coefficients

Lemma 2. *([[53]][Lemma 9.4]) If μ_f is Beltrami coefficients for $f_{\overline{z}} = \mu(z) f_z$ and μ_h is Beltrami coefficients for $h_{\overline{z}} = \mu(z) h_z$, then,*

$$\mu_{h \circ f}(z) = \frac{\mu_f(z) + \mu_h \circ f(z).\frac{\overline{\partial f(z)}}{\partial f(z)}}{1 + \mu_h \circ f(z).\mu_f(z).\frac{\overline{\overline{\partial} f(z)}}{\partial f(z)}}$$

$$\mu_{h \circ f^{-1}}(z) = \frac{\mu_h(z) - \mu_f(z)}{1 - \mu_h(z).\overline{\mu_f}(z)}.\frac{\partial f(z)}{\overline{\partial f(z)}}$$

Corollary 2. *Let $f : \Omega_1 \to \Omega_2$, $g : \Omega_1 \to \Omega_3$, and $h : g : \Omega_2 \to \Omega_4$ be quasiconformal maps. Then,*
1- We have that $\mu_f = \mu_g$ almost everywhere if and only if $f \circ g^{-1}$ is conformal;
2- If h is conformal, then $\mu_{h \circ f} = \mu_f$ almost everywhere;
3- If f is conformal, then $\mu_{h \circ f} = (\mu_h \circ f)\frac{\overline{f'}}{f'}$ almost everywhere.

1.5.2 Classification of Beltrami equations

We say that μ is bounded in Ω if $||\mu||_\infty < 1$, and that μ is locally bounded in Ω if $\mu\,|_A$ is bounded whenever A is a relatively compact subdomain of Ω. The Beltrami equation is divided into three cases according to the nature of $\mu(z)$ in Ω:
1. The classical case if $||\mu||_\infty < 1$,
2. The degenerated case if $|\mu| < 1$ almost everywhere and $||\mu||_\infty = 1$.
3. The alternating case if $|\mu| < 1$ almost everywhere in a part of Ω and $1/|\mu| < 1$ almost everywhere in the remaining part of Ω.

1.5.3 Solution the Beltrami equation

By writing $h : \Omega \to \mathbb{C}$, we assume that Ω is a domain in $\mathbb{C}$, which is an open and connected set and that h is continuous. A function $h : \Omega \to \mathbb{C}$ is a solution of (1.4), if h is ACL in Ω, and its ordinary partial derivatives, which exist $a.e.$ in Ω, satisfy (1) $a.e.$ in Ω.
A solution $h : \Omega \to \mathbb{C}$ of (1.4) which is a homeomorphism of Ω into $\mathbb{C}$ is called μ-homeomorphism or μ-conformal mapping. In the above cases (1) and (2), a solution $h : \Omega \to \mathbb{C}$ of (1.4) will be called elementary if h is open and discreet, meaning that h maps every open set onto an open set and that the preimage of every point in Ω consists of isolated points. Let $h : \Omega \to \mathbb{C}$ be an elementary solution. The complex dilatation of

h is defined by

$$\mu_h(z) = \mu(z) = h_{\overline{z}}(z)/h_z(z),$$

if $h_z(z) \neq 0$ and by $\mu(z) = 0$ if $h_{\overline{z}}(z) = 0$. For such a mapping, the dilatation is

$$K_h(z) = K_\mu(z) = \frac{1+|\mu|}{1-|\mu|}.$$

Note that $K_h < \infty$ *a.e.* if and only if $|\mu| < 1$ *a.e.* If $h \in ACL$, then h has partial derivatives h_x and h_y *a.e.* and, thus, by the well-known Gehring-Lehto theorem every ACL homeomorphism $h : \Omega \to \mathbb{C}$ is totally differentiable *a.e.*. For a sense-preserving ACL homeomorphism $h : \Omega \to \mathbb{C}$, the Jacobian $J_h(z) = |h_z|^2 - |h_{\overline{z}}|^2$ is nonnegative *a.e.*, and since $||\mu||_\infty < 1$, h is quasiconformal.

1.5.3.1 Principal Solutions

Definition 21. A quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ is said to be normalized at 0, 1, ∞ if

$$f(0) = 0, f(1) = 1, \quad \text{and} \quad f(\infty) = \infty$$

With μ as above, we will call the solutions to the Beltrami equation (1.4) normalized by the condition $f(z) = z + o(1)$ near ∞ the principal solutions.

Theorem 13. *(([7], Theorem [5.3.2]) Let $|\mu(z)| \leq k < 1$ for almost every $z \in \mathbb{C}$. Then there is a solution $f : \overline{\mathbb{C}} \to \mathbb{C}$ to the Beltrami equation*

$$f_{\overline{z}}(z) = \mu(z) f_z(z) \quad \text{for almost every} \quad z \in \mathbb{C} \tag{1.5}$$

which is a K-quasiconformal homeomorphism normalized by the three conditions

$$f(0) = 0, \quad f(1) = 1, \quad \text{and} \quad f(\infty) = \infty$$

Furthermore, the normalized solution of f is unique.

Lemma 3. *([7], lemma [5.3.5.]): Suppose we have a sequence of Beltrami coefficients $\{\mu_n\}_{n\in\mathbb{N}}$ such that*

$$||\mu_n||_{L^\infty(\mathbb{C})} \leq k < 1, \quad \text{for all} \quad n \in \mathbb{N}$$

and such that the pointwise limit

$$\mu(z) := \lim_{n\to\infty} \mu_n(z)$$

exists almost everywhere. Let f_n be the normalized solutions to

$$f_{\overline{z}} = \mu_n(z) f_z, \quad n \in \mathbb{N}$$

Then the limit $f(z) = \lim_{n\to\infty} f_n(z)$ exists, the convergence is uniform on compact subsets of $\mathbb{C}$ and f solves the limiting Beltrami equation,

$$f_{\overline{z}} = \mu(z) f_z,$$

almost everywhere.

Theorem 14. *(Stoilow Factorization): Let* $f : \Omega \to \Omega'$ *be a homeomorphic solution to the Beltrami equation*

$$f_{\overline{z}} = \mu(z) f_z \tag{1.6}$$

with $|\mu(z)| \leq k < 1$ *almost everywhere in* Ω*. Suppose* g *is any other solution to (1.6) on* Ω*. Then there exists a holomorphic function* $\Phi : \Omega' \to \mathbb{C}$ *such that*

$$g(z) = \Phi(f(z)), \quad z \in \Omega$$

Corollary 3. *If* μ *is measurable with* $|\mu(z)| \leq k < 1$ *at almost every* $z \in \mathbb{C}$*, then the normalized solution to the Beltrami equation* $f_z = \mu(z) f_z$ *is unique.*

proof. If f_1 and f_2 are normalized solutions to the same Beltrami equation, then Stoilow factorization implies $f_1 = \phi \circ f_2$, where ϕ is holomorphic in $\mathbb{C}$. Since f_1 and f_2 are homeomorphisms, $\phi : \mathbb{C} \to \mathbb{C}$ is conformal and by similarities of Conformal mappings in the plane. Since ϕ fixes 0, 1 and ∞, ϕ is the identity. □

Theorem 15. *([48], Existence Theorem p. 194) If* Ω *is an arbitrary domain and* μ *an arbitrary measurable function in* Ω *with*

$$\sup_{z \in \Omega} |\mu(z)| < 1,$$

then there exists a quasiconformal map f *of* Ω *with* $\mu_f = \mu$ *almost everywhere in* Ω*.*

Chapter 2

Lipschitz Spaces

2.1 Lipschitz Spaces

2.1.1 Definition of Lipschitz Spaces

Definition 22. Let $\Omega \subset \mathbb{R}^n$, and $0 < \alpha \leq 1$. A function $f : \Omega \to \mathbb{C}^n$ or $(\mathbb{R}^n)$. is said to belong to the Lipschitz space Lip_α if there is a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L(||x - y||^\alpha) \tag{2.1}$$

for all $x; y \in \Omega$. L, and α are called respectively Lipschitz constant and exponent (of f on Ω).

For $0 < \alpha \leq 1$, let $Lip_\alpha(\Omega)$ denote the Lip_α functions defined on Ω. Note that $||x|| = \sqrt{x_1^2 + ... + x_n^2}$.

Example 5. Set $\Omega = [a, b]$, and $f(x) = x$. Then

$$|f(x_1) - f(x_2)| = ||x_1 - x_2|| \tag{2.2}$$

That implies that $f \in Lip_1([a, b])$.

A function is Lipschitz if it is in Lip_1.

2.1.2 Some Properties of Lipschitz Spaces

2.1.2.1 Linearity

Theorem 16. *$Lip_\alpha(\Omega)$ is a linear space.*

Proof. If $f(x) \in Lip_\alpha(\Omega)$, and $\lambda \in \mathbb{R}$ or $\mathbb{C}$ is constant, then, there is a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L||x - y||^\alpha,$$

hence

$$\begin{aligned} |\lambda f(x_1) - \lambda f(x_2)| =& |\lambda(f(x_1) - f(x_2))| \\ &= |\lambda||f(x_1) - f(x_2)| \\ &\leq |\lambda|L||x - y||^\alpha \end{aligned}$$

and therefore $\lambda f(x) \in Lip_\alpha(\Omega)$.
If $f, g \in Lip_\alpha(\Omega)$, then

$$|f(x) - f(y)| \leq L_f||x - y||^\alpha,$$

and

$$|g(x) - g(y)| = L_g||x - y||^\alpha$$

Then $(f + g) \in Lip_\alpha(\Omega)$ becomes of,

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| =& |f(x) + g(x) - f(y) - g(y)| \\ &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq L_f||x - y||^\alpha + L_g||x - y||^\alpha \\ &= (L_f + L_g)||x - y||^\alpha \end{aligned}$$

□

2.1.2.2 Differentiability

Let $\Omega \subset \mathbb{R}^n$ is open, we say that $f : \Omega \to \mathbb{R}^m$ is differentiable at $a \in \Omega$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to a} \frac{|f(x) - f(a) - L(x - a)|}{|x - a|} = 0.$$

If such a linear map L exists, it is unique, called the derivative of f at a, and denoted by $Df(a)$. We also note that $f = (f_1, ..., f_m)$ is differentiable at a if and only if each of the coordinate functions f_i are differentiable at a.

Theorem 17. *(Lebesgue).*
Let $f : (a, b) \to \mathbb{R}$ be Lipschitz. Then f is differentiable at almost every point in (a, b).

Theorem 18. *(Rademacher's theorem).*
Let $\Omega \subset \mathbb{R}^n$ be open, and let $f : \Omega \to \mathbb{R}^n$ be Lipschitz. Then f is differentiable at almost every point in Ω.

proof. We may assume for simplicity and without loss of generality that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz. The proof splits into three parts.
Step 1. The one-dimensional result is used to conclude that the partial derivatives $(\frac{\partial f}{\partial x_i})$ of f exists almost everywhere. This gives us a candidate for the total derivative, namely the (almost everywhere defined) formal gradient

$$\nabla f(x) := (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}). \tag{2.3}$$

Next, it is shown that all directional derivatives exist almost everywhere, and can be given in terms of the gradient. Finally, by using the fact that there are only countably many directions in $\mathbb{R}^n$, the total derivative is shown to exist; it is only in this last step that the Lipschitz condition is seriously used.
We will now carry out these steps. The first claim is a direct consequence of Lebesgue

Theorem. Indeed, for every $x, v \in \mathbb{R}^n$, the function

$$f_{x,v}(t) := f(x+tv), \quad t \in \mathbb{R}$$

is Lipschitz as a function of one real variable, and hence differentiable at almost every $t \in \mathbb{R}$. Keeping now $v \in \mathbb{R}^n$ fixed, we conclude from Fubini theorem and the preceding remark that

$$D_v f(x) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} \tag{2.4}$$

exists for almost every $x \in \mathbb{R}^n$. (To be precise here, in order to use Fubini theorem, one has to first show that the set of those points x for which the limit in (2.4) exists is measurable.) In particular, as

$$D_{e_i} f(x) := \frac{\partial f}{\partial x_i}$$

for each $i = 1, ..., n$, where e_i is the ith standard basis vector in $\mathbb{R}^n$, the formal gradient $\nabla f(x)$ as given above in (2.3) exists at almost every $x \in \mathbb{R}^n$.
Step 2., we show that for every $v \in \mathbb{R}^n$

$$D_v f(x) := v.\nabla f(x), \tag{2.5}$$

for almost every $x \in \mathbb{R}^n$. To do so, fix $v = (v_1, ..., v_n) \in \mathbb{R}^n$. Then fix a test function $\eta \in C_0^\infty(\mathbb{R}^n)$. We have that

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \eta(x) D_v f(x) dx &= \int\limits_{\mathbb{R}^n} \eta(x) \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} \\
&= \lim_{t \to 0} \int\limits_{\mathbb{R}^n} \eta(x) \frac{f(x+tv) - f(x)}{t} \\
&= \lim_{t \to 0} \int\limits_{\mathbb{R}^n} -f(x) \frac{\eta(x) - \eta(x-tv)}{t} dx \\
&= -\int\limits_{\mathbb{R}^n} f(x) \lim_{t \to 0} \frac{\eta(x) - \eta(x-tv)}{t} dx \\
&= -\int\limits_{\mathbb{R}^n} f(x) D_v \eta(x) dx \\
&= -\sum_{i=1}^{n} v_i \int\limits_{\mathbb{R}^n} f(x) \frac{\partial \eta}{\partial x_i}(x) dx \\
&= \sum_{i=1}^{n} v_i \int\limits_{\mathbb{R}^n} \eta(x) \frac{\partial f}{\partial x_i}(x) dx \\
&= \int\limits_{\mathbb{R}^n} v \cdot \nabla f(x) \eta(x) dx
\end{aligned}$$

Because η was arbitrary, equality (2.5) holds for almost every $x \in \mathbb{R}^n$.
Step 3., We want to prove the original claim. To this end, fix a countable dense set of directions in $\mathbb{R}^n$; that is, fix a countable dense set of vectors (v_i) in $\partial \mathbb{B}^n$. By the first

two steps, we infer that there is a set $A \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \backslash A| = 0$ and that

$$D_{v_i} f(a) = v_i . \nabla f(a) \tag{2.6}$$

for every v_i and for every $a \in A$, where we also understand that both sides of (2.6) exist (the gradient $\nabla f(a)$ is still understood formally as in (2.8)). Now fix $a \in A$. For $v \in \partial \mathbb{B}^n$ and $t \in \mathbb{R}, t \neq 0$, set

$$D(v,t) := \frac{f(a+tv) - f(a)}{t} - v.\nabla f(x)$$

To prove the differentiability of f at a, we need to show that $D(v,t) \to 0$ as $t \to 0$ independently of v. To do this, fix $\epsilon > 0$. Then choose an ϵ-dense set of vectors $v_1, ..., v_N$ from the chosen dense collection of directions; i.e., for each $v \in \partial \mathbb{B}^n$ we have that $|v - v_i| < \epsilon$ for some $i \in \{1, ..., N\}$. We then find that

$$\begin{aligned} |D(v,t) - D(v_i,t)| \leq & \left| \frac{f(a+tv) - f(a+tv_i)}{t} \right| + |(v - v_i).\nabla f(a)| \\ & \leq C.||(v - v_i)|| \leq C.\epsilon \end{aligned}$$

where C is a constant independent of v, this constant depends only on $||\nabla f(a)||$ and the Lipschitz constant, by the Lipschitz assumption. Because $\lim_{t \to 0} D(v_i, t) = 0$ for each v_i, we can choose $\delta > 0$ such that

$$D(v_i, t) \leq \epsilon$$

for $|t| < \delta$, for each $i = 1, ..., N$. By combining the preceding inequalities, we obtain that

$$|D(v,t)| \leq C.\epsilon$$

whenever $|t| < \delta$, where C is independent of v, as required. This completes the proof of Rademachers theorem. □

2.1.2.3 Majorants

Definition 23. Let $\omega : [0, \infty) \to [0, \infty)$ be a continuous function, we say that ω is a majorant function if

$$\omega(0) = 0, \quad \omega \quad \text{is increasing, and} \quad \omega(t)/t \quad \text{is decreasing.} \tag{2.7}$$

For a majorant function ω we define Lipschitz space functions analogously to Definition (22) as follows

Definition 24. Let ω be a majorant, and given a subset Ω of $\mathbb{C}^n$ or $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}^m$ is said to belong to Lipschitz space $\Lambda_\omega(\Omega)$ if there is a constant $L = L(f) = L(f; \Omega)$ such that

$$|f(x) - f(y)| \leq L\omega(|x - y|), \tag{2.8}$$

for all $x, y \in \Omega$.
The set of functions satisfying (2.8) is denoted by $\Lambda_\omega(\Omega, L)$, hence $\Lambda_\omega(\Omega) = \bigcup_{L \geq 0} \Lambda_\omega(\Omega, L)$

Lemma 4. *If ω is convex and there are $C, \delta > 0$ such that*

$$|f(x) - f(y)| \leq C.\omega(|x - y|), \tag{2.9}$$

for all $x, y \in \Omega$ with $|x - y| < \delta$, then $f \in \Lambda_\omega(\Omega)$

Proof. Let k be an integer such that $\overline{\Omega}$ is covered by k balls of radius $\delta/3$ centered at points in Ω. Let $x_0; y_0 \in \Omega$ such that $|x_0 - y_0| \geq \delta$. Then, there is $x_1, ..., x_k \in \Omega$ such that

$$|x_j - x_{j+1}| < \delta, \quad j = 0, 1, ..., k$$

where $x_{k+1} = y_0$. So,

$$\begin{aligned} |f(x_0) - f(y_0)| &\leq \sum_{j=0}^{k} |f(x_j) - f(x_{j+1})| \leq (k+1)C.\omega(\delta) \\ &\leq (k+1)C.\omega(|x_0 - y_0|). \end{aligned}$$

□

Definition 25. A majorant function ω is called regular if there is $C > 0$, for all $\delta > 0$ sufficiently small,

$$\int_0^\delta \frac{\omega(t)}{t} dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt \leq C.\omega(\delta). \tag{2.10}$$

Example 6.

$$\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ -t^\alpha \ln t & \text{if } 0 < t \leq \frac{1}{e} \\ e^{-\alpha} & \text{if } t > \frac{1}{e} \end{cases}$$

If $0 < \alpha < 1$, then $\omega(t)$ is a majorant function. And a regular majorant because

$$\int_0^\delta \frac{\omega(t)}{t} dt = \int_0^\delta -t^{\alpha-1} \ln t dt = -\frac{\delta^\alpha \ln \delta}{\alpha} + \frac{\delta^\alpha}{\alpha^2} \leq C(-\delta^\alpha \ln \delta).$$

$$\begin{aligned} \delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt =& \delta \int_\delta^{1/e} -t^{\alpha-2} \ln t dt + \delta \int_{1/e}^\infty \frac{e^{-\alpha}}{t^2} dt \\ &= \frac{-\delta^\alpha \ln \delta}{1-\alpha} + \delta\Big(\frac{e^{1-\alpha}}{(1-\alpha)^2} - \frac{e^{1-\alpha}}{1-\alpha} + e^{1-\alpha}\Big) - \frac{\delta^\alpha}{(1-\alpha)^2} \\ &\leq C(-\delta^\alpha \ln \delta). \end{aligned}$$

Hence,

$$\int_0^\delta \frac{\omega(t)}{t} dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt \leq C.(-\delta^\alpha \ln \delta).$$

Definition 26. A majorant ω is called fast if there is $C > 0$, such that

$$\int_0^\delta \frac{\omega(t)}{t} dt \le C.\omega(\delta), \tag{2.11}$$

and it is called slow if there is $C > 0$ such that

$$\delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt \le C.\omega(\delta). \tag{2.12}$$

Remark 5. *Notes that a majorant ω is called regular if and only if it is both fast and slow.*

Definition 27. We say Ω is a Λ_ω-extension domain if each pair of points $x, y \in \Omega$ can be joined by a rectifiable curve $\gamma \subset \Omega$ satisfying

$$\int_\gamma \frac{\omega(d(z,\partial\Omega))}{d(z,\partial\Omega)} \le C\omega(|x-y|), \tag{2.13}$$

where $d(z,\partial\Omega) = \inf\{||z-w|| : w \in \partial\Omega\}$.
Yet another piece of notation will be needed. Given two sets Ω_1 and Ω_2 (in $\mathbb{C}^n$ or $\mathbb{R}^n$), we write$\Lambda_\omega(\Omega_1, \Omega_2)$ for the class of those continuous functions f on Ω_1, Ω_2 which satisfy (2.9), with some C; whenever $x \in \Omega_1$ and $y \in \Omega_2$.

Theorem 19. *([16], Theorem [1]): Let ω be a fast majorant, and let Ω be a Λ_ω-extension domain in $\mathbb{C}^n$: For a holomorphic function f on Ω, the following are equivalent:*
1. $f \in \Lambda_\omega(\Omega)$,
2. $|f| \in \Lambda_\omega(\Omega)$,
3. $|f| \in \Lambda_\omega(\Omega, \partial\Omega)$.
Where $\Lambda_\omega(\Omega)$ is the Lipschitz space associated with a majorant ω, and $\Lambda_\omega(\Omega, \partial\Omega) = \{f : \overline{\Omega} \to \mathbb{C}\ (\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n) : |f(x) - f(y)| \le C\omega(|x-y|)$ for $x \in \Omega$, and $y \in \partial\Omega\}$

proof. We want to prove that (3) implies (1), fix a point $z \in \Omega$ and consider the function h, defined on the unit ball by

$$h(w) = f(z + d(z)w)/M_z, \quad w \in \mathbb{B}^n,$$

where

$$d(z) := d(z,\partial\Omega) \quad \text{and} \quad M_z := \sup\{|f(\xi)| : ||\xi - z|| < d(z)\}.$$

Hence

$$h(0) = f(z)/M_z, \quad \nabla h(0) = \frac{d(z)}{M_z}\nabla f(z).$$

Since h is holomorphic in $\mathbb{B}^n$ and takes values in $\mathbb{D}$, we have

$$|\nabla h(0)| \le 1 - |h(0)|^2 \le 2(1 - |h(0)|). \tag{2.14}$$

we deduce from 2.14 that

$$d(z)|\nabla f(z)| \le 2(M_z - |f(z)|). \tag{2.15}$$

Now if $\xi \in \partial\Omega$ is a point with $|\xi - z| = d(z)$, then

$$\begin{aligned} |f(w)| - |f(z)| \leq & ||f(w)| - |f(\xi)|| + ||f(\xi)| - |f(z)|| \\ & \leq C\omega(2d(z)) + C\omega(d(z)) \leq 3C\omega(d(z)), \end{aligned} \tag{2.16}$$

using condition 3, and the fact that $\omega(2t) \leq 2\omega(t)$. Taking the supremum over $w \in \mathbf{B}(z, dz))$, we get

$$M_z - |f(z)| \leq \text{const.}\omega(d(z)), \tag{2.17}$$

and substituting into 2.15 gives

$$|\nabla f(z)| \leq \text{const.}\frac{\omega(d(z))}{d(z)}, \quad z \in \Omega. \tag{2.18}$$

Finally, given two points $x, y \in \Omega$, let us join them by a curve $\gamma \subset \Omega$ satisfying (2.13). Integrating (2.18) along h, we obtain

$$|f(x) - f(y)| \leq \int_\gamma |\nabla f(z)| ds(z) \leq \text{const.} \int_\gamma \frac{\omega(d(z))}{d(z)} ds(z). \tag{2.19}$$

A combination of (2.19) and (2.13) yields

$$|f(x) - f(y)| \leq \text{const.}\omega(|x - y|),$$

and we arrive at (1). $\square$

We fix some notation. At the risk of abusing terminology, we say increasing to mean non-decreasing and $a \lesssim b$ to mean $a \leq Cb$ for some constant $C > 0$.

Lemma 5. *Let u be a harmonic function on a smoothly bounded domain Ω in $\mathbb{R}^n$. If $u \in \Lambda_\omega(\Omega)$, then*

$$|\nabla u(x)| \lesssim \frac{\omega(\delta(x))}{\delta(x)}, \quad x \in \Omega.$$

where $\delta(x)$ is the Euclidean distance of x to $b\Omega$.

Proof. Fix $x_0 \in \Omega$. Let $\epsilon > 0$ such that $B(x_0, \epsilon) \subset\subset \Omega$. Now, by the Poisson integral formula, for $x \in B(x_0, \epsilon)$

$$\begin{aligned} \nabla u(x) = & \frac{1}{\omega_{n-1}\epsilon} \int_{|\xi|=\epsilon} u(x_0 + \xi) \nabla_x \Big(\frac{\epsilon^2 - |x - x_0|^2}{|x - x_0 - \xi|^n} \Big) d\sigma(\xi) \\ = & \frac{1}{\omega_{n-1}\epsilon} \int_{|\xi|=\epsilon} (u(x_0 + \xi) - u(x_0)) \nabla_x \Big(\frac{\epsilon^2 - |x - x_0|^2}{|x - x_0 - \xi|^n} \Big) d\sigma(\xi). \end{aligned}$$

Calculating ∇_x inside the integral, setting $x = x_0$, and estimating we get

$$|\nabla u(x)| \leq \frac{n}{\epsilon} \sup_{|\xi|=\epsilon} |u(x_0 + \xi) - u(x_0)| \lesssim \frac{\omega(\epsilon)}{\epsilon}.$$

Now, let $\epsilon = \delta(x_0)/2$ to obtain

$$|\nabla u(x)| \lesssim \frac{\omega(\frac{\delta(x_0)}{2})}{\delta(x_0)} \leq \frac{\omega(\delta(x_0))}{\delta(x_0)} \quad \text{(since } \omega \text{ is increasing).}$$

□

2.2 Moduli of continuity

Measuring the smoothness of a function by differentiability is too crude for many purposes in approximation. More subtle measurements are provided by the moduli of continuity and the moduli of smoothness. The main idea of modulus of continuity (present already in the notion of derivative) is to measure the difference between the function and its translate. Since there are many ways to measure the size of a function, we can have many different moduli of continuity. In this chapter we will only consider the following modulus.

Definition 28. Let $\Omega \subset \mathbb{R}^n$ be any compact set, and let $f(x)$, be any continuous function on Ω, $x = (x_1, x_2, ..., x_n) \in \Omega$. The function

$$\omega_f(\delta) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in\Omega}} |f(x) - f(y)|, \qquad 0 \leq \delta \leq daim \quad \Omega. \tag{2.20}$$

is called the modulus of continuity of $f(x)$.

Clearly, $\omega_f(\delta)$ is non-decreasing, and a constant for $\delta \geq \quad diam\Omega$, if Ω is bounded. The function ω is continuous at $\delta = 0$ if and only if f is uniformly continuous on Ω.

2.2.1 Some properties of modulus Of continuity

In this section we formulate and prove some lemmas and theorems that characterize the main properties of such modulus, and we get this part from IM Kolodiy, F. Hildebrand paper [39].

Lemma 6. *If* $0 < \delta_1 \leq \delta_2$, *then* $\omega(\delta_1, f) \leq \omega(\delta_2, f)$

Lemma 7. $f(x)$ *is uniformly continuous on* Ω *if and only if*

$$\lim_{\delta\to 0} \omega(\delta, f) = \omega(0) = 0$$

2.2.1.1 The continuity of the functions $\omega_f(\delta)$

Lemma 8. *For any continuous function* $f(x)$ *on a compact set* Ω*, the function* $\omega_f(\delta)$ *is continuous from the right.*

Proof. The function $\phi(x, y) \equiv |f(x)-f(y)|$ is continuous with respect to the combination of the variables (x, y) on the compact set $\{(x, y) \in \Omega \times \Omega, |x - y| \leq \delta\}$, and so there are points x_δ, and y_δ such that $|x_\delta - y_\delta| \leq \delta$ and

$$\omega(\delta, f) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in\overline{\Omega}}} \phi(x, y) = \phi(x_\delta, y_\delta) = |f(x_\delta) - f(y_\delta)|,$$

Consider the sequence $\{\delta_n\}$, $\delta_n > \delta$, converging to δ. For each δ_n there exists a pair of points $(x_{\delta_n}, y_{\delta_n})$ such that

$$|x_{\delta_n} - y_{\delta_n}| \leq \delta_n, \quad \omega(\delta_n, f) = |f(x_{\delta_n}) - f(y_{\delta_n})|.$$

Since $(x_{\delta_n}, y_{\delta_n}) \in \Omega \times \Omega$, there is a subsequence $(x_{\delta_{n'}}, y_{\delta_{n'}})$, such that $x_{\delta_{n'}} \to \overline{x}$ and $y_{\delta_{n'}} \to \overline{y}$. Thus $|\overline{x} - \overline{y}| \leq \delta$. But the function

$$\phi(x, y) \equiv |f(x) - f(y)|$$

is continuous on $\Omega \times \Omega$, and so

$$\lim_{\substack{\delta_{n'} \to \delta \\ \delta_{n'} > \delta}} |f(x_{\delta_{n'}}) - f(y_{\delta_{n'}})| = |f(\overline{x}) - f(\overline{y})|.$$

Hence

$$\omega(\delta, f) \geq |f(\overline{x}) - f(\overline{y})| = \lim_{\substack{\delta_{n'} \to \delta \\ \delta_{n'} > \delta}} |f(x_{\delta_{n'}}) - f(y_{\delta_{n'}})| = \lim_{\substack{\delta_{n'} \to \delta \\ \delta_{n'} > \delta}} \omega(\delta_{n'}, f) = \omega(\delta + 0, f),$$

and so

$$\omega(\delta, f) \geq \omega(\delta + 0, f).$$

On the other hand the monotonicity of $\omega(\delta, f)$, implies that

$$\omega(\delta + 0, f) \geq \omega(\delta, f).$$

Combining the last two inequalities, we obtain

$$\omega(\delta + 0, f) = \omega(\delta, f).$$

□

Theorem 20. *For any function $f(x)$ continuous on a compact and convex set Ω, the function $\omega(\delta, f)$ is continuous from the left if and only if Ω satisfies the following condition A:*
For any $\delta \geq 0, x, y \in \Omega, x \neq y$, there are points $x', y' \in \Omega$ such that

$$|x' - x| \leq \delta, \quad |y' - y| \leq \delta,$$

and

$$|x' - y'| \leq |x - y|.$$

proof. **Necessity.**
Let Ω be a compact in $\mathbb{R}^n$ that for any continuous $f(x), x \in \Omega$, the function $\omega(\delta, f)$ is continuous from the left. We show that Ω satisfies condition A. Assume the contrary, ie, that the condition A, not satisfied. Then we construct a continuous function $f_0(x), x \in \Omega$, for which $\omega(\delta, f_0)$ is discontinuous from the left at some point δ_0, that is a contradiction.
Indeed, the failure of a condition means that: There is $\delta_0 > 0$ and there exist $x_0, y_0 \in$

$\Omega, x_0 \neq y_0$, that for any $x', y' \in \Omega$ such that $|x' - x_0| < \delta_0, |y' - y_0| < \delta_0$ we have

$$|x' - y'| \geq |x_0 - y_0|$$

Consider the function

$$f_0(x) = \begin{cases} 1 - \frac{|x-x_0|}{\delta_0}, & \text{if } x \in K(x_0) = \{x : x \in \Omega, |x - x_0| < \delta_0\} \\ -(1 - \frac{|x-y_0|}{\delta_0}) & \text{if } x \in K(y_0) = \{x : x \in \Omega, |x - y_0| < \delta_0\} \\ 0 & \text{if } x \in \Omega \setminus (K(x_0) \cup K(y_0)) \end{cases}$$

For this function

$$\omega(\delta, f_0) = \begin{cases} \leq \frac{\delta}{\delta_0}, & \text{if } 0 \leq \delta < \delta_0, \\ \leq 1 & \text{if } \delta_0 \leq \delta < |x_0 - y_0| \\ = 2 & \text{if } |x_0 - y_0| \leq \delta < diam\Omega \end{cases}$$

Clearly, with $\omega(\delta, f_0)$ is discontinuous on the left at the point $\delta_0 = |x_0 - y_0|$.
Sufficiency.
Let the compact Ω satisfies Condition A and let $f(x)$ any continuous function on Ω. We will prove that $\omega(\delta, f)$ is continuous on the left. Since $f(x)$ is uniformly continuous on the compact Ω, then for any $\epsilon > 0$, there exists $\delta > 0$ such that for $|x - x'| \leq \delta, \quad |y - y'| \leq \delta$

$$|f(x) - f(x')| \leq \frac{\epsilon}{2}, \quad |f(y) - f(y')| \leq \frac{\epsilon}{2}.$$

By assumption A, for $\delta > 0$ and for any $x, y \in \Omega, x \neq y, |x - y| \leq \delta$, (where δ - any number in the interval $(0, diam\Omega]$, there are points $x', y' \in \Omega$ such that $|x - x'| \leq \delta, |y - y'| \leq \delta$ and and the inequality

$$|x' - y'| \leq |x - y| \leq \delta.$$

then

$$\begin{aligned} |f(x) - f(y)| \leq |f(x) - f(x')| + |f(x') - f(y')| + \\ + |f(x') - f(y)| \leq \epsilon + \sup_{\substack{|x'-y'| \leq \delta \\ x', y' \in \Omega}} |f(x') - f(y')| = \\ = \epsilon + \sup_{t<\delta} \sup_{\substack{|x'-y'| \leq t \\ x', y' \in \Omega}} |f(x') - f(y')| = \epsilon + \sup_{t<\delta} \omega(t, f) = \\ = \epsilon + \lim_{\substack{t \to \delta \\ t < \delta}} \omega(t, f) = \epsilon + \omega(\delta - 0, f). \end{aligned}$$

Hence,

$$\omega(\delta - 0, f) \leq \omega(\delta, f) \leq \epsilon + \omega(\delta - 0, f)$$

and, by the arbitrariness of $\epsilon > 0$ About

$$\omega(\delta - 0, f) = \omega(\delta, f).$$

Theorem (20) is proved. □

Corollary 4. *For any continuous function $f(x)$ on a compact set Ω, $\omega(\delta, f)$ is continuous if and only if Ω satisfies condition A.*

2.2.1.2 Subadditivity of function $\omega(\delta, f)$

Theorem 21. *A compact set Ω is convex if only if for any continuous function $f(x)$ on Ω the function $\omega(\delta, f)$ is semiaddiitive, i.e.,*

$$\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2), \quad \delta_1, \delta_2 \in \Omega$$

proof. **Sufficiency.**
Let Ω be convex and compact. We prove that for any continuous function $f(x)$, $x \in \Omega$, the function $\omega(\delta, f)$ is subadditive. Let x and y be any two points in Ω such that $|x - y| \leq \delta_1 + \delta_2$ where δ_1 and δ_2 - any non-negative number. On the segment joining x and y, we take a point $z \in \Omega$ such that

$$|x - z| \leq \delta_1, \quad |y - z| \leq \delta_2. \quad \text{since } \Omega \text{ is convex, } z \in \Omega$$

Obviously,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(z)| + |f(z) - f(y)| \leq \\ &\leq \sup_{\substack{|x-z|\leq\delta_1 \\ x,z\in\Omega}} |f(x) - f(z)| + \sup_{\substack{|y-z|\leq\delta_2 \\ y,z\in\Omega}} |f(z) - f(y)| = \\ &= \omega(\delta_1, f) + \omega(\delta_2, f), \end{aligned}$$

therefore,

$$\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2), \quad \delta_1, \delta_2 \in \Omega.$$

Necessity.
Suppose that for any continuous function $f(x)$ on the compact set Ω the function $\omega(\delta, f)$ is subadditive. We need to prove that the to Ω convex. Assume the contrary, i.e. that Ω is not convex. Then we construct a continuous function $f_0(x), x \in \Omega$, for which the function of $\omega(\delta, f_0)$ is not a semi-additive. Thus, suppose that Ω is a compact convex. Then there is $x_0, y_0 \in \Omega$ such that the line segment joining these points belongs entirely to Ω. Consequently, there are non-negative numbers δ_1 and δ_2 such that $|x_0 - y_0| = \delta_1 + \delta_2$, and the point $z = (\delta_2 x_0 + \delta_1 y_0)(\delta_1 + \delta_2)$ does not belongs to Ω. Obviously, there is a $\epsilon > 0$, such that $K_1 \cup K_2 = \emptyset$, where

$$\begin{aligned} K_1 &= \{x : x \in \Omega, |x - {}_0| < \delta_1 + \epsilon\} \\ K_2 &= \{x : x \in \Omega, |x - y_0| < \delta_2 + \epsilon\}. \end{aligned}$$

Consider the function

$$f_0(x) = \begin{cases} h_1 \cdot (1 - \frac{|x-{}_0|}{\delta_1+\epsilon} & \text{if} \quad x \in K_1, \\ -h_2 \cdot (1 - \frac{|x-{}_0|}{\delta_2+\epsilon} & \text{if} \quad x \in K_2, \\ 0 & \text{if} \quad x \in \Omega \backslash K_1 \cap K_2 \end{cases}$$

where for a given δ_1 and δ_2 are positive numbers h_1 and h_2 are related by

$$h_1(\delta_1 + \epsilon)^{-1} = h_2(\delta_2 + \epsilon)^{-1}$$

It is clear that

$$\omega(\delta_1+\delta_2, f_0) = h_1 + h_2,$$
$$\omega(\delta_1, f_0) \le \delta_1 \frac{h_1}{\delta_1+\epsilon},$$
$$\omega(\delta_2, f_0) \le \delta_2 \frac{h_2}{\delta_2+\epsilon}$$

Consequently, the

$$\begin{aligned} \omega(\delta_1, f_0) + omega(\delta_2, f_0) \le & \delta_1 \frac{h_1}{\delta_1+\epsilon} + \delta_2 \frac{h_2}{\delta_2+\epsilon} \\ & < (\delta_1+\epsilon)\frac{h_1}{\delta_1+\epsilon} + (\delta_2+\epsilon)\frac{h_2}{\delta_2+\epsilon} \\ & = h_1 + h_2 = \omega(\delta_1+\delta_2, f_0). \end{aligned}$$

So we have constructed a continuous function $f_0(x)$ and pointed such numbers δ_1 and δ_2 that

$$\omega(\delta_1+\delta_2, f_0) > \omega(\delta_1, f_0) + \omega(\delta_2, f_0)$$

Theorem (21) is proved □

Lemma 9. *If* $\lambda > 0$*, then*

$$\omega(\lambda\delta, f) \le (1+\lambda)\omega(\delta, f).$$

proof. Let n be integer such that $n \le \lambda < n+1$, then $\omega(\lambda\delta, f) \le \omega((n+1)\delta, f)$. Suppose $|x_1 - x_2| < (n+1)\delta$ and $x_1 < x_2$. We divide $[x_1, x_2]$ into $n+1$ equal parts each of length $(x_1 - x_2)/(n+1)$ by means of the points

$$z_i = x_1 + i(x_2 - x_1)/(n+1), \quad i = 0, 1, ..., n+1.$$

Then

$$|f(x_1) - f(x_2)| = |\sum_{i=0}^{n}[f(z_{i-1} - f(z_i)]| \le \sum_{i=0}^{n} |f(z_{i-1} - f(z_i)| \le (n+1)\omega(\delta, f).$$

Thus, $\omega((n+1)\delta, f) \le (n+1)\omega(\delta, f)$.
But $n+1 \le \lambda + 1$, and the lemma is proved. □

2.3 Harmonic mapping

The subject of harmonic maps is vast and has found many applications, and it would require a very long book to cover all aspects, even superficially. We first consider relevant aspects of harmonic functions on Euclidean space

A real valued function f on an open set $\Omega \subseteq \mathbb{R}^n$ is called harmonic on Ω if $f \in C^2$ on Ω (that is, all first and second partial derivatives of f exist and are continuous on Ω), and

$$\Delta f :\equiv \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial x_1^2} + ... + \frac{\partial^2 f}{\partial x_n^2} = 0, \quad (x_1, ..., x_n) \in \Omega$$

The operator Δ is called the Laplace operator or Laplacian. We say that a function f defined on a set (not necessarily open) $A \subset \mathbb{R}^n$ is harmonic on A if f can be extended to a function harmonic on an open set containing A. A one-to-one mapping $f = (f_1, ..., f_n)$ is a harmonic mapping if the all coordinate functions are harmonic. Thus areal-valued harmonic function is a harmonic mapping of a domain $\Omega \subseteq \mathbb{R}^n$ if and only if it is univalent (or one-to-one) in Ω.

We denote the Euclidean open ball in $\mathbb{R}^n$ of center $a \in \mathbb{R}^n$ and radius $r > 0$ by $B(a, r) := \{x \in \mathbb{R}^n : |x - a| < r\}$ (which we will sometimes write $B^n(a, r)$ to emphasize that its dimension is n), the corresponding sphere by $S(a, r) \equiv \partial B(a, r)$, the unit ball $B(0, 1)$ by B, and its boundary (unit sphere) by $\partial B \equiv S$.

2.3.1 Mean value property

Let ω_n denotes volume of the unit ball in $\mathbb{R}^n$ which define by

$$\omega_n = \begin{cases} \frac{\pi^{n/2}}{(n/2)!} & \text{if } n \text{ is even} \\ \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{1.3.5...n} & \text{if } n \text{ is odd.} \end{cases}$$

And let ω^*_{n-1} denotes the (unnormalized) surface area of the unite sphere in $\mathbb{R}^n$ define by $\omega^*_{n-1} = n\omega_n$. Then the volume measure of the ball $B(a, r)$ in $\mathbb{R}^n$ is $V(B^n(a, r)) = r^n\omega_n$, and the surface area of the sphere $S(a, r)$ in $\mathbb{R}^n$ is $Area(S^{n-1}(a, r)) = r^{n-1}n\omega_n$.

Definition 29. (Mean values). Let f be a Borel function on $\overline{B}(a, r)$ which is bounded above or below, the mean value of f over the sphere is :

$$\frac{1}{Area(S(a, r))} \int\limits_{S(a,r)} f(\xi)ds(\xi),$$

and over the ball is

$$\frac{1}{V(B(a, r))} \int\limits_{B(a,r)} f(x)dV(x).$$

where ds denotes surface-area measure, $dV = dV_n = dx_1...dx_n$ denotes Lebesgue volume measure on $\mathbb{R}^n$.

The first expression gives f as an average over the boundary of the ball, and the second as an average over the ball.

Now we may write the mean value properties in the following equivalent ways:
A continuous real valued function f in a domain $\Omega \subset \mathbb{R}^n$ has mean value property over spheres, if

$$f(a) = \frac{1}{n\omega_n} \int\limits_S f(a + r\xi)ds(\xi) := \int\limits_S f(a + r\xi)d\sigma(\xi),$$

for every ball $B(a, r) \subset \Omega$, where σ denotes the normalized surface-area measure on S (so that $\sigma(S) = 1$).

And f has the mean value property for balls, if

$$f(a) = \frac{1}{\omega_n} \int\limits_{B} f(a + rx) dV(x), \tag{2.21}$$

for every ball $\overline{B}(a, r) \subset \Omega$. In particular (when n=2):

$$f(a) = \frac{1}{2\pi} \int\limits_{0}^{2\pi} f(a + re^{i\theta}) d\theta.$$

for every disk $\mathbb{D}(a, r) \subset \Omega \subset \mathbb{R}^2$.
It is an important fact that the Mean value properties are equivalent to harmonicity of real harmonic functions

Definition 30. (Harmonic function). Let Ω be an open subset of $\mathbb{R}^n$ and $f \in C(\Omega, \mathbb{R})$. f is harmonic on Ω if and only if f satisfies the mean value equality

$$f(a) = \frac{1}{V(B(a, r))} \int\limits_{B(a,r)} f(x) dV(x); \tag{2.22}$$

for all $\overline{B}(a, r) \subset \Omega$.
Or equivalently

$$f(a) = \frac{1}{Area(S(a, r))} \int\limits_{S(a,r)} f(x) ds(x); \tag{2.23}$$

for all $\overline{B}(a, r) \subset \Omega$. In fact, if f is harmonic on Ω and $\overline{B}(a, r) \subset \Omega$, then

$$f(a) = \frac{1}{V(B(a, r))} \int\limits_{B(a,r)} f(x) dV(x) =$$

$$= \frac{n}{r^n.Area(S(B(a, r))} \int\limits_{0}^{r} \int\limits_{S(a,\rho)} f(\xi) \rho^{n-1} ds(\xi) d\rho.$$

Which implies

$$r^n f(a) = n \int\limits_{0}^{r} \frac{1}{Area(S(B(a, r))} \int\limits_{S(a,\rho)} \rho^{n-1} f(\xi) ds(\xi) d\rho.$$

Taking derivatives with respect to r on both sides it follows that

$$nr^{n-1} f(a) = \frac{nr^{n-1}}{Area(S(B(a, r))} \int\limits_{S(a,r)} f(\xi) ds(\xi).$$

Hence

$$f(a) = \frac{1}{Area(S(a, r))} \int\limits_{S(a,r)} f(\xi) ds(\xi).$$

This means $f(a)$ equals the average of f over the sphere $S(a,r)$.

Theorem 22. *([66] Theorem7) If $f = u + iv$ is analytic in a domain $\Omega \subset \mathbb{C}$, then each of the functions u and v is harmonic in Ω*

In this case the imaginary part of a analytic function f is called a harmonic conjugate of the real part of f.

Theorem 23. *([4] Theorem 1.28) If u is harmonic on a domain $\Omega \subseteq \mathbb{R}^n$, then u is real analytic in Ω.*

Suppose that Ω is simple connected domain and let u be harmonic on Ω. Then there is an analytic function f on Ω with $Ref = u$. This means that for such a function u there exists a harmonic function v defined on Ω such that $f = u + iv$ is analytic on Ω. Now we can prove next theorem.

Theorem 24. *([10] Theorem 4.31). If $f = u + iv$ is harmonic in a simply-connected domain Ω, then $f = g + \overline{h}$, where g and h are analytic.*

proof. Since u and v are real harmonic functions on a simply-connected domain, then the discussion before the statement of this theorem shows that there exists analytic functions f_1 and f_2 such that $u = Ref_1$ and $v = Imf_2$. Hence,

$$f = u + iv = Ref_1 + iImf_2 = \frac{f_1 + \overline{f_1}}{2} + i\frac{f_2 - \overline{f_2}}{2i} = \frac{f_1 + f_2}{2} + \frac{\overline{f_1 - f_2}}{2} = g + \overline{h}$$

□

2.3.2 Subharmonic Function

subharmonic functions are related to harmonic function as follows. If the values of a subharmonic function are no larger than the values of a harmonic function on the boundary of a ball, then the values of the subharmonic function are no larger than the values of the harmonic function also inside the ball. Subharmonic functions are of a particular importance in complex analysis, where they are intimately connected to holomorphic functions.

Definition 31. **(Upper semi-continuous function):** Let $\Omega \subset \mathbb{R}^n$, a function $u : \Omega \to [-\infty, +\infty)$ is said to be upper semicontinuous at a point $a \in \Omega$ if for any number $C > u(a)$ there exists a number $\delta = \delta(a, C)$ such that $u(x) < C$ whenever $|x - a| < \delta$ and $x \in \Omega$. A function u is said to be semicontinuous on the set Ω if it is upper semicontinuous at each point of Ω

An equivalent definition for u to be upper semicontinuous on Ω is to require the sets $\{x \in \Omega : u(x) < C\}$ be open in Ω for every $C \in \mathbb{R}$. Another equivalent definition for upper semicontinuous $u(a) \geq \limsup_{x\to a} u(x)$ for all $a \in \Omega$.

Remark 6. *: Note that upper semi-continuous functions are allowed to take value $-\infty$.*

Definition 32. Let Ω be an open subset of $\mathbb{R}^n$, and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function. We say that u is subharmonic function on Ω if u satisfy the following mean value inequality:

$$u(a) \leq \frac{1}{Area(S(a,r))} \int_{S(a,r)} u(x) d\sigma(x), \tag{2.24}$$

for all $\overline{B}(a,r) \subset \Omega$.

An equivalent definition is obtained using property:

$$u(a) \leq \frac{1}{Vol(B(a,r))} \int\limits_{B(a,r)} u(x)dV(x), \tag{2.25}$$

for all $\overline{B}(a,r) \subset \Omega$.

Remark 7. *Note that from the definition the subharmonic functions are allowed to take value $-\infty$, for an important example the function $\log|z-a|$.*

Example 7. Every harmonic function is subharmonic

2.3.3 Simple properties of subharmonic function

The subharmonic functions are a much more flexible tool than holomorphic, or even harmonic functions. An immediate consequence of the sub-mean value property is the maximum principle for subharmonic functions. There is no minimum principle for subharmonic functions, in other words subharmonic functions do not satisfy the minimum principle, for example $u(x) = |x|^2$ is subharmonic function on $\mathbb{R}^n$, but it is not harmonic .

1-If u is subharmonic on Ω, then Cu is subharmonic in Ω for any constant $C \geq 0$
2- If the functions $u_1(x), ..., u_m(x)$ are subharmonic in a domain $\Omega \subset \mathbb{R}^n$, then the functions $\sum_{i=1}^{m} u_i$, and $\max_{1\leq i\leq m} u_i(x)$ are also subharmonic in Ω
3-the limit of a uniformly convergent sequence of subharmonic functions is subharmonic function
4- the limit of a monotone decreasing sequence of subharmonic function is subharmonic function

Proposition 1. *Assume Ω is domain and $u \in C^2(\Omega)$. Then u is subharmonic in Ω if and only if $\Delta u(z) \geq 0, z \in \Omega$*

2.3.4 Poisson integral formula

: The Poisson integral formula shows that if $f(x)$ is harmonic in a ball $B(a,r)$ and continuous in the closed ball $\overline{B}(a,r)$, then its value at any interior point is completely determined by its values on the boundary $\partial B(a,r)$. Thus the Poisson integral is meaningful for every (bounded piecewise) continuous function $U(e^{i\theta})$ on the circle (or even for Lebesgue integrable functions).

Definition 33. (Poisson integral): Let f be integrable in the sphere $S(a,r) = \partial B(a,r)$ with respect to the surface measure dS and define

$$P[f](x) = \int\limits_{S(a,r)} P(x,y)v(y)dS(y),$$

for all $x \in B(a,r)$.
This is called the Poisson integral formula of f on $S(a,r)$.

The Poisson integral formula of the real continuous function f on the boundary $\partial\mathbb{D} := \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ of the unit disk is

$$P[f](z) = \frac{1}{2\pi}\int_0^{2\pi} P(\rho,\theta - t)f(t)dt \qquad (0 \leq \rho < 1, 0 \leq \theta < \pi).$$

where

$$P(\rho,\theta - t) = \frac{1-\rho^2}{1-2\rho\cos(\theta - t)+\rho^2} \qquad (z = \rho e^{i\theta}, 0 \leq \rho < 1).$$

Lemma 10. *: If $f(z)$ is harmonic in the unit disc $\mathbb{D}$ and continuous in the closed unit disc $\mathbb{D}$, then $f(z)$ satisfies the Poisson integral formula:*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}).P_\rho(t-\theta)d\theta := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(t-\theta)}).P_\rho(\theta)d\theta \qquad (z = \rho e^{i\theta}, 0 \leq \rho < 1).$$

2.4 Subharmonicity of $|f|^q$ of harmonic quasiregular mapping in space

We denote the Euclidean norm in $\mathbb{R}^n$ by $||.||$, and let $\Omega \subset \mathbb{R}^n$ be a region. Let $f(x) = (f_1(x), ..., f_n(x)) : \Omega \to \mathbb{R}^n$ with formal differential matrix

$$Df(x) = \Big\{\frac{\partial f_j(x)}{\partial x_i}\Big\}_{i,j=1}^n$$

is K-quasiregular, and we set

$$||Du|| = \left(\sum_{i,j=1}^n \left|\frac{\partial f_j(x)}{\partial x_i}\right|\right)^{1/2}$$

If $f = (f_1, ..., f_n)$ is a harmonic mapping defined in Ω, then the function $|f|^p$ for $p \geq 1$ is subharmonic in Ω and therefore has the sub-mean-value property over balls. If $p < 1$, then $|f|^p$ need not be subharmonic but, by a result of Hardy-Littlewood and others, there exists a constant $K = K(n;p) < 1$ such that

$$|f(x)|^p \leq Kf^{-n}\int_{B_r(x)} |f|^p dm$$

whenever $B_r(x) \subset \Omega$.

Proposition 2. *([6, Ch. VII 3, p.217]). Let $u = (u_1, ..., u_n) : \Omega \to \mathbb{R}^n$, be harmonic, let $\Omega_0 = \Omega \backslash u^{-1}(0)$, let $q \in \mathbb{R}$. Then for $x \in \Omega_0$*

$$\Delta|u|^q = q\Big[|u|^{q-2}\sum_{k=1}^n |\nabla u_k|^2 + (q-2)|u|^{q-4}\sum_{k=1}^n \Big(u \cdot \frac{\partial u}{\partial x_k}\Big)\Big]$$

Proof. Write $v := |u|^q = (u_1^2 + ... + u_n^2)^p$, for $p := q/2$. A direct computation gives

$$\begin{aligned} v_{x_1} =& p(u_1^2 + ... + u_n^2)^{p-1}(2u_1u_{1x_1} + ... + 2u_nu_{nx_1}) \\ =& q(u_1^2 + ... + u_n^2)^{p-1}(u_1u_{1x_1} + ... + u_nu_{nx_1}) \end{aligned}$$

and further

$$\begin{aligned} v_{x_1x_1} =& q\{2(p-1)(u_1^2 + ... + u_n^2)^{p-2}(u_1u_{1x_1} + ... + u_nu_{nx_1})^2 + \\ &+ (u_1^2 + ... + u_n^2)^{p-1}.[u_1u_{1x_1x_1} + (u_1u_{1x_1})^2 + ... + u_nu_{nx_1x_1} + (u_nu_{nx_1})^2]\}. \end{aligned}$$

Therefore

$$\begin{aligned} \Delta v =& v_{x_1x_1} + ... + v_{x_nx_n} \\ =& q\{|u|^{q-2}\Big[(u_1\Delta u_1 + ... + u_n\Delta u_n) + \Big(\sum_{k=1}^n u_{1x_k}^2 + ... \\ &+ \sum_{k=1}^n u_{nx_k}^2\Big)\Big] + (q-2)|u|^{q-4}\sum_{k=1}^n\Big(\sum_{j=1}^n u_ju_{jx_k}^2\Big)\} \\ =& q\{|u|^{q-2}\Big(\sum_{k=1}^n u_{1x_k}^2 + ... + \sum_{k=1}^n u_{nx_k}^2\Big) + (q-2)|u|^{q-4}\sum_{k=1}^n\Big(\sum_{j=1}^n u_j.\frac{\partial u_j}{\partial x_k}\Big)^2\} \\ =& q|u|^{q-4}\{|u|^2\sum_{j=1}^n\Big(\sum_{k=1}^n u_ju_{jx_k}^2\Big) + (q-2)\sum_{k=1}^n\Big(\sum_{j=1}^n u_j.\frac{\partial u_j}{\partial x_k}\Big)^2\} \\ & q|u|^{q-4}\{|u|^2\sum_{j=1}^n|\nabla u_j|^2 + (q-2)\sum_{k=1}^n\Big(\sum_{j=1}^n u_j\cdot\frac{\partial u_j}{\partial x_k}\Big)^2\} \end{aligned}$$

□

Theorem 25. *([6][Theorem 2.1])*
If $f : \Omega \to \mathbb{R}^n$, be a K-quasiregular harmonic mapping in $\Omega \subset \mathbb{R}^n$, then $|f|^q$ is subharmonic for some $0 < q = q(K, n) < 1$.

proof. Fix such a map $f : \Omega \to \mathbb{R}^n$, and set $\Omega_0 = \Omega - f^{-1}(0)$. $|f|^q$ is subharmonic at each point $x \in f^{-1}(0)$ for any $q > 0$. Hence we have to prove that $\Delta(|f|^q) \geq 0$ on Ω_0 for some $q = q(n, K) < 1$. Since f is quasiregular, the set $Z = \{x \in \Omega_0 : \det Du(x) = 0\}$ has measure zero (see [**?**]), it is also closed since f is smooth. In particular $\Omega_1 = \Omega_0 - Z$ is dense in Ω_0 and thus it suffices to prove that $\Delta|u|^q \geq 0$ on Ω_1. From Proposition 3, we obtain

$$\Delta|f|^q = q\left[|f|^{q-2}||Df||^2 + (q-2)|f|^{q-4}\left|\sum_{j=1}^n f_j\nabla f_j\right|^2\right]$$

So we need $0 < q = q(K, n) < 1$. such that

$$|f|^2 \cdot ||Df||^2 + (q-2)\left|\sum_{j=1}^n f_j\nabla f_j\right|^2 \geq 0$$

on Ω_1 or, equivalently, we need a constant $C = C(K, n) < 1$ such that

$$\left| \sum_{j=1}^{n} f_j(x) \nabla f_j(x) \right|^2 \leq C|f(x)|^2 ||Df(x)||^2$$

for all $x \in \Omega_1$. After normalization, we see that it suffices to find a constants $C = C(K, n) < 1$ such that

$$\sup_{|z|=1} \left| \sum_{j=1}^{n} z_j \nabla f_j(x) \right| \leq C||Df(x)||$$

At each point $x \in \Omega_1$ the matrix $D_u(x)$ is K-quasiconformal and its transpose $D_u(x)^t$ is also K-quasiconformal. Here we identify matrices with corresponding linear maps. To finish the proof we need the following lemma.

Lemma 11. *Let a matrix $A = (a_{ij}) \in GL(\mathbb{R}^n)$ be K-quasiconformal, for each $K \geq 1$ and $n \geq 2$ there is a constant $C = C(K, n) < 1$ such that*

$$\sup_{|x|=1} \left| \sum_{j=1}^{n} x_j A e_j \right| \leq C \Big(\sum_{i,j=1}^{n} a_{ij}^2 \Big)^{1/2}$$

proof. It is suffices to prove lemma for normalized matrices : $\sum\limits_{i,j=1}^{n} a_{ij}^2 = 1$. So, we set

$$GL_K(n) = \left\{ A = \Big(a_{ij}\Big)_{i,j=1}^{n} : A \quad \text{is } K\text{-quasiconformal}, \sum_{i,j=1}^{n} a_{ij}^2 = 1 \right\}$$

This set of matrices is compact and the function

$$\phi(A) = \sup_{|x|=1} \left| \sum_{j=1}^{n} x_j A e_j \right|$$

is continuous on the space of all $n \times n$ matrices, hence it attains its maximum at $A_0 \in GL_K(n)$ on the compact $GL_K(n)$. Set $\phi(A_0) = C$. Assume $C = 1$. Clearly $C \leq 1$. Then $\sup\limits_{|x|=1} \left| \sum\limits_{j=1}^{n} x_j A_0 e_j \right| = 1$, and this supremum is attained at $z \in \mathbb{S}^{n-1}$.

$$1 = \left| \sum_{j=1}^{n} z_j A_0 e_j \right| \leq \sum_{j=1}^{n} \left| z_j A_0 e_j \right| \leq \left(\sum_{j=1}^{n} z_j^2 \right)^{1/2} \cdot \left(\sum_{j=1}^{n} \left| A_0 e_j \right|^2 \right)^{1/2} = 1.$$

Hence, all the above inequalities are in fact equalities and therefor the vectors $z_j A_0 e_j, 1 \leq j \leq n$ are collinear. But this leads to a contradiction with invertibility of A_0. Thus, $C < 1$ and our lemma is proved. This ends the proof of theorem (25). □

2.5 Estimation of the Poisson Kernel

I get this part from Krantz papers [41], and [42].

We want to estimate the size of the Poisson kernel $P_\Omega(x,t) = P(x,t)$ of Ω. Let $\Omega \subset \mathbb{R}^n$ be a connected open set, $x \in \Omega$ and $t \in \partial\Omega$. It is often possible to calculate P_Ω explicitly. For example,
⋆ The Poisson kernel of the disc $\mathbb{D} \subseteq \mathbb{R}^2$ is

$$P_{\mathbb{D}}(x,t) = \frac{1}{2\pi}.\frac{1-|x|^2}{|x-t|^2}.$$

⋆ The Poisson kernel for the upper halfplane $\mathbb{H} = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\}$ is given by

$$P_{\mathbb{H}}(x,t) = \frac{1}{\pi}.\frac{x_2}{(x_1-t)^2 + x_2^2}.$$

⋆ The Poisson kernel for the unit ball $\mathbb{B} \subseteq \mathbb{R}^n$ is given by

$$P_{\mathbb{B}}(x,t) = \frac{\Gamma(n/2)}{2\pi^{n/2}}.\frac{1-|x|^2}{|x-t|^n}.$$

Where Γ is the classical gamma function
⋆ The Poisson kernel for the upper halfspace $\mathbb{H}^{n+1} \equiv \{x = (x_1, ..., x^{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ (with $x = (x_1, ..., x_{n+1}) = (x', x_{n+1})$) is given by

$$P_{\mathbb{H}}^{n+1}(x,t) = c_n \frac{x_{n+1}}{(|x'-t|^2 + x_{n+1}^2)^{[n+1]/2}}.$$

Where

$$c_n = \frac{\Gamma([n+1]/2)}{\pi^{[n+1]/2}}.$$

Here we need to have size estimates for the Poisson kernel on a fairly general domain (say a bounded domain with C^2 boundary). The standard asymptotic is

$$P_\Omega(x,t) \asymp \frac{\delta(x)}{|x-y|^n}. \tag{2.26}$$

Where $\delta(x)\delta_\Omega(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$.
Now, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. This means that there is a C^2, real-valued function ρ such that

$$\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}$$

and $\nabla\rho = 0$ on $\partial\Omega$. Thus $\partial\Omega$ is a regularly imbedded C^2 hypersurface in $\mathbb{R}^n$.

Theorem 26. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. Let $P_\Omega : \Omega \times \partial\Omega \to \mathbb{R}^+$ be the Poisson kernel for Ω. Then there are constants $C_1, C_2 > 0$ such that*

$$C_1 \frac{\delta(x)}{|x-y|^n} \leq P_\Omega(x,y) \leq C_2 \frac{\delta(x)}{|x-y|^n}. \tag{2.27}$$

proof. For convenience, we write

$$P(x,t) \asymp \frac{\delta(x)}{|x-y|^n}.$$

instead of (2.27). If $K \subset \Omega$ is a compact set, then the estimate we desire is trivial for $x \in K$ and $y \in \partial\Omega$. For then $|x-y| \geq c > 0$, $\delta(x)$ is bounded above, and we get a universal bound above and below on $\delta(x)/|x-y|^n$. A similar comment applies if x is near the boundary and y is far from x. So we may concentrate our attention on x near the boundary and y near x.

Now fix a point $P \in \partial\Omega$ and a point $P_0 \in \Omega$ such that the segment $\overline{P_0 P}$ is normal to the boundary at P. We shall dilate coordinates with center P_0. We assume that P_0 is close to $\partial\Omega$ - within a tubular neighborhood of the boundary-and we set $\epsilon = dist(P_0, P)$. We assume that coordinates have been rotated and centered so that

1. The point P is the origin $(0, 0, ..., 0)$,
2. The normal direction $\overrightarrow{PP_0}$ is the positive x_n-direction.

We write $x = (x_1, ..., x_n)$, $x \in \mathbb{R}^n$. We set $P_0 = (P_0^1, ..., P_0^n)$. With the normalization of coordinates, in fact $P_0 = (P_0^1, P_0^2, ..., P_0^n) = (0, ..., 0, +\epsilon)$. Now define

$$\Phi_\epsilon(x) = (\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}, ..., \frac{x_n}{\epsilon})$$

Clearly, that the mapping Φ_ϵ sends the point P_0 to $(1, 0, ..., 0)$. The first thing to prove

$$\lim_{\epsilon \to 0^+} \Phi_\epsilon(\Omega) = \mathbb{H}^n$$

To see this, we first check that if the defining function ρ, expanded about the point P, is given by

$$\rho(x) = \sum_{i=1}^{n} a_i^1 x_i + \sum_{i,k=1}^{n} a_{ik}^2 x_i x_k + ... = x_n + \sum_{i,k=1}^{n} a_{ik}^2 x_i x_k.$$

Then

$$\begin{aligned} \rho_\epsilon(s) \equiv & \frac{1}{\epsilon} \cdot \Big[\rho \circ \Phi_\epsilon^{-1}(s) \Big] \\ &= \frac{1}{\epsilon} \cdot \Big[-\epsilon s_n + \sum_{i,k=1}^{n} a_{ik}^2 x_i x_k + ... \Big] \\ &= -s_n + \epsilon \cdot \Big[\sum_{i,k=1}^{n} a_{ik}^2 x_i x_k + ... \Big] \end{aligned}$$

Clearly, as $\epsilon \to 0$, the transferred defining function ρ_ϵ tends to the linear defining function $\rho_0(s) \equiv -s_n$. In other words, the domains $\Phi_\epsilon(\Omega) \equiv \Omega_\epsilon$ converge (in an appropriate sense) to the standard halfspace. This last information is useful because we know the Poisson kernel for a halfspace.

Now we may take advantage of the facts accrued by setting $\Omega_\epsilon = \Phi_\epsilon(\Omega)$, letting $d\sigma$ be $(n-1)$-dimensional area measure on $\partial\Omega$, $d\sigma_\epsilon$ to be $(n-1)$-dimensional area measure on $\partial\Omega_\epsilon$, and taking f to be a function that is continuous on $\overline{\Omega}_\epsilon$ and harmonic on Ω_ϵ.

Further, we let $x \in \Omega$ and set $s = \Phi_\epsilon(x)$. Then we calculate that

$$f(s) = f(\Phi_\epsilon(x)) = \int\limits_{\partial\Omega_\epsilon} P_{\Omega_\epsilon}(\Phi_\epsilon(x), t) f(t) d\sigma_\epsilon(t)$$

$$= \int\limits_{\partial\Omega} P_{\Omega_\epsilon}(\Phi_\epsilon(x), \Phi_\epsilon(\tau)) f(\Phi_\epsilon(\tau)) det J_{\Phi_\epsilon(\tau)} d\sigma(\tau).$$

It is crucial to note here that the integral is over an $(n-1)$-dimensional hypersurface, and hence the Jacobian determinant is that of an $(n-1) \times (n-1)$ matrix. Now let us write

$$f \circ \Phi_\epsilon(x) = \int\limits_{\partial\Omega} P_\Omega(x, \tau)[f \circ \Phi_\epsilon(\tau)] d\sigma(\tau) = \int\limits_{\partial\Omega} K_\epsilon(x, \tau)[f \circ \Phi_\epsilon(\tau)] d\sigma(\tau)$$

Since this identity holds true for any choice of continuous f on the boundary of Ω_ϵ (with unique harmonic extension to Ω_ϵ), we may conclude that

$$P_\Omega(x, \tau) = K_\epsilon(x, \tau).$$

The identity (2.27) is the key to our result, for we know asymptotically what K_ϵ looks like. In particular, we know (see [[40]], Section [1.3]) on any smoothly bounded domain $\mathbb{H}$ that the Poisson kernel is a normal derivative of the Greens function:

$$P_{\mathbb{H}}(x, y) = \frac{\partial}{\partial \nu} G_{\mathbb{H}}(x, y).$$

Now with P, P_0 fixed as before, let W be a small, smoothly bounded, topologically trivial domain with these properties:
i. $W \subseteq \Omega$,
ii. $P_0 \in W$, $P \in \partial W$,
iii. $\partial W \cap \partial\Omega$ is a relative neighborhood of P in $\partial\Omega$.
Now the key observation at this point is that, when $\epsilon > 0$ is small, then the Poisson kernel for $\Phi_\epsilon(W)$ at interior points of the line segment $\Phi_\epsilon(\overline{PP_0})$ is very near to the Poisson kernel of the upper half space $\mathbb{H}^n$ at those same points.
As a result, we may calculate the Poisson kernel on Ω by instead calculating the kernel on W. In turn, it then suffices to calculate the kernel on $\mathbb{H}^n$. Thus we see that, for x on the interior of the line segment $\overline{PP_0}$

$$\begin{aligned} K_\epsilon(x, \tau) =& \epsilon^{-(n-1)} . P_{\Omega_\epsilon}(\Phi_\epsilon(x), \Phi_\epsilon(\tau)) \\ &\approx \epsilon^{-(n-1)} \frac{\Phi_\epsilon(x)_n}{(|\Phi'_\epsilon(x) - \Phi_\epsilon(\tau)|^2 + [\Phi_\epsilon(x)_n/\epsilon]^2)^{n/2}} \\ &= \epsilon^{-(n-1)} . \frac{x_n/\epsilon}{(|x'/\epsilon - \tau/\epsilon|^2 + [x_n/\epsilon]^2)^{n/2}} \\ &= \frac{x_n}{(|x' - \tau|^2 + [x_n]^2)^{n/2}} \end{aligned}$$

Unraveling the notation, we find that we have proved the approximation (2.27). □

Chapter 3

Moduli of continuity of harmonic quasiregular mappings on bounded domains

Let $\Omega \subset \mathbb{R}^n$ be a domain (connected, non-empty, open set). Harmonic quasiregular (briefly, *hqr*) mappings in the plane were studied first by O. Martio in [56], for a review of this subject and further results see [49] and references cited there. Moduli of continuity of harmonic quasiregular mappings in $\mathbb{B}^n$ were studied by several authors, see [45], [38], [8]. In this paper, our main goal is to extended one of the main results from [6] to more general domains in $\mathbb{R}^n$. In fact the following theorem was proved in [6].

Theorem 27. *[[6], Theorem A] If $u : \overline{\mathbb{B}}^n \to \mathbb{R}^n$ is a continuous map which is K-quasiregular map and harmonic in $\mathbb{B}^n$, then $\omega_u(\delta) \leq C\omega_f(\delta)$ for $\delta > 0$, where $f = u\,|_{S^{n-1}}$ and C is a constant depending only on K, ω_f and n.*

We use two methods to extend this result. The first method is to use the following theorem from [6].

Theorem 28. *[[6], Theorem B] There is a constant $q = q(K, n) \in (0, 1)$ such that $|u|^q$ is subharmonic in $\Omega \subset \mathbb{R}^n$ whenever $u : \Omega \to \mathbb{R}^n$ is a K- quasiregular harmonic map.*

The above theorem combined with Poisson integral representation gives Theorem 27. The main point is that a similar argument can be carried out without using explicit formula for the Poisson kernel. In fact suitable estimates are sufficient, and these rely on pointwise estimates for the Poisson kernel which are available in the case of bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary, see [41],[42]. We prove a version of Theorem 27 for domains Ω with C^2 boundary, see Theorem 29 below.

The second method is essentially based on a capacity estimate of O. Martio and R. Näkki [50]. Let us introduce needed terminology and notation.

Throughout this paper $\Omega \subset \mathbb{R}^n$ is bounded domain, $\delta(x) = \operatorname{dist}(x, \Omega^c)$ and $B_x = B(x, \delta(x)/2)$ for $x \in \Omega$. If Ω has C^2 boundary, then P_Ω denotes the Poisson kernel for Ω.

Given a subset E of $\mathbb{C}^n$ or $\mathbb{R}^n$; a function $f : E \to \mathbb{C}$ (or, more generally, a mapping f from E into $\mathbb{C}^m$ or $\mathbb{R}^m$) is said to belong to the Lipschitz space $\Lambda_\omega(E)$ if there is a constant $L = L(f) = L(f; E)$ such that

$$|f(x) - f(y)| \leq L\omega(|x - y|) \tag{3.1}$$

for all $x, y \in E$, or equivalently, $\omega_f(|x - y|) \leq L\omega(|x - y|)$ for $x, y \in E$. Here $\omega : [0, +\infty) \to [0, +\infty)$ is a majorant in the sense of Hinkkanen, see [28], which means ω

is non-decreasing and $\omega(2t) \le 2\omega(t)$. In that case we also say that f is ω-Lipschitz function. We remark that ω need not be continuous, that we may have $\omega(0) > 0$ and that $\omega(At) \le A\omega(t)$ for all $t \ge 0$ and $A \ge 1$. The most important special case is $\omega_\alpha(t) = t^\alpha$, $0 < \alpha \le 1$, when we get classical concept of Lipschitz or Hölder continuity.
There has been much work on Lipschitz-type properties of quasiconformal mappings. This topic was treated, among many other papers, in [19].
Following [22] and [47], we say that a function f belongs to the local Lipschitz space $\mathrm{loc}\,\Lambda_\omega(\Omega, L)$ if (3.1) holds, with a fixed $L \ge 0$, whenever $x \in \Omega$ and $y \in B_x$. We set $\mathrm{loc}\,\Lambda_\omega(\Omega) = \cup_{L\ge 0}\mathrm{loc}\,\Lambda_\omega(\Omega, L)$. If $\omega(t) = t^\alpha$, $0 < \alpha \le 1$, we use notation $\Lambda_\alpha(\Omega)$, $\mathrm{loc}\,\Lambda_\omega(\Omega)$ and $\mathrm{loc}\,\Lambda_\alpha(\Omega, L)$.
A domain Ω is a Λ_ω-extension domain if $\Lambda_\omega(\Omega) = \mathrm{loc}\,\Lambda_\omega(\Omega)$.
A compact set E in $\mathbb{R}^n$ is called c-uniformly perfect, $0 < c < 1$, if E contains at least two points and if for each $x \in E$ and $0 < r < \mathrm{diam}(E)$, the spherical ring $B(x; r) \setminus \overline{B}(x; cr)$ meets E.
If V is a subset of $\mathbb{R}^n$ and $u : V \to \mathbb{R}^m$, we define

$$\mathrm{osc}_V u = \sup\{|u(x) - u(y)| \, : x, y \in V\}\,.$$

For $\Omega \subset \mathbb{R}^n$ let $OC^1(\Omega)$ denote the class of all $f \in C^1(\Omega, \mathbb{R}^n)$ such that

$$\delta(x)|f'(x)| \le C\,\mathrm{osc}_{B_x} f, \qquad x \in \Omega \tag{3.2}$$

We denote by $OC^2(\Omega)$ the class of all $f \in C^2(\Omega, \mathbb{R}^n)$ such that for some constant C we have

$$\sup_{B_x} \delta^2(x)|\Delta f(x)| \le C\ \mathrm{osc}_{B_x} f, \qquad x \in \Omega. \tag{3.3}$$

It was observed in [54] that $OC^2(\Omega) \subset OC^1(\Omega)$. Note that every harmonic mapping $f : \Omega \to \mathbb{R}^n$ is in $OC^2(\Omega)$.
We also show that under some conditions a function $f \in OC^2(\Omega)$ is ω-Lipschitz function on Ω if and only if it satisfies Hardy-Littlewood (C, ω)- property:

$$\delta(x)|f'(x)| \le C\omega(\delta(x)), \qquad x \in \Omega.$$

Relying on this characterization and a result from [9] (Lemma (13) below) we also prove a version of Theorem 27 for Λ_ω-extension domain with c-uniformly perfect boundary and quasiconformal mappings, where ω is a majorant.
Finally, we give a simple proof of K. M. Dyakonov's result on relation between moduli of continuity of $|f|$ and f, see [19].
We follow the usual convention, letter C denotes a constant that can change its value from one occurrence to the next.

3.1 Auxiliary results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. Clearly, an explicit formula for the Poisson kernel is available only in special cases, like the ball. The following technical lemma is used in the next section when we consider smoothly bounded domains.

Lemma 12. *Assume Ω has C^1 boundary. Then there is a constant C depending only on Ω, such that*

$$\mathrm{area}\big(\partial\Omega \cap \mathbb{B}(z_0, r)\big) \le Cr^{n-1} \tag{3.4}$$

for all $r > 0$ and all $z_0 \in \partial\Omega$.

Proof. We have a local parametrization of $\partial\Omega$:

$$\begin{aligned} x_1 &= x_1(u_1, u_2, ..., u_{n-1}) \\ x_2 &= x_2(u_1, u_2, ..., u_{n-1}) \\ &\vdots \\ x_n &= x_n(u_1, u_2, ..., u_{n-1}) \end{aligned}$$

i.e. $x = x(u), x = (x_1, x_2, ..., x_n)$, where $x_j \in C^1(U)$, $U \subset \mathbb{R}^{n-1}$.

Since $\partial\Omega$ is compact, it suffices to prove the estimate (3.4) for $z_0 \in x(K)$, where $K \subset U$ is compact.
Now fix a compact $K \subset U$. We have

$$\text{area}\big(x(S)\big) = \int_S \sqrt{g}du$$

where $g = det(g_{ij})_{i,j=1}^{n-1}$, $g_{ij} = \sum_{m=1}^{n} \frac{\partial x_m}{\partial u_i}\frac{\partial x_m}{\partial u_j}$.

Note that $g_{ij} \in C(U)$, so $\sqrt{g}$ is a strictly positive continuous function on U. Therefore $0 < c \le \sqrt{g} \le C < +\infty$ on K. Let $u_1, u_2 \in K$ and $z_1 = x(u_1), z_2 = x(u_2)$. Since $x = x(u)$ is a parametrization, we have

$$|z_1 - z_2| \asymp |u_1 - u_2| \,. \tag{3.5}$$

Setting $z_0 = x(u_0)$ and using (3.5), we see that there is a constant M such that

$$\begin{aligned} \text{area}\big(\partial\Omega \cap \mathbb{B}(z_0, r)\big) &\le \text{area}\Big(x\big(\mathbb{B}(u_0, Mr)\big)\Big) = \int_{\mathbb{B}(u_0,Mr)} \sqrt{g}du \\ &\le C\int_{\mathbb{B}(u_0,Mr)} du \le C\,\text{Vol}_{n-1}\big(\mathbb{B}(u_0, Mr)\big) \\ &= C(Mr)^{n-1} \\ &= Cr^{n-1}. \quad \square \end{aligned}$$

3.2 The case of C^2 boundary

In this section we follow the first method described in the introduction and obtain the following generalization of Theorem 27.

Theorem 29. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary, and assume $u : \overline{\Omega} \to \mathbb{R}^n$ is a continuous map which is K- quasiregular and harmonic in Ω, then $\omega_u(\delta) \le C\omega_f(\delta)$ for $\delta > 0$, where $f = u\,|_{\partial\Omega}$ and C is a constant depending only on K, ω_f and n.*

Proof. The proof is similar to the proof in [6], but with additional technical difficulties due to the lack of an explicit formula for P_Ω. Instead we rely on Lemma 12 and crucial estimates (2.27), using a dyadic decomposition of $\partial\Omega$.
Let us recall some properties of ω_f:

$$\omega_f(\delta_1 + \delta_2) \le C\big(\omega_f(\delta_1) + \omega_f(\delta_2)\big), \qquad \omega_f(\lambda\delta) \le C\lambda\omega_f(\delta)$$

valid for $\delta, \delta_1, \delta_2 > 0$ and $\lambda \ge 1$. First, fix an exponent $q = q(K, n) < 1$ from Theorem 28. Fix $w \in \partial\Omega$ and $z \in \Omega$. Then $\varphi(\xi) = |u(w) - u(\xi)|^q$ is subharmonic in Ω and therefore we have

$$\varphi(z) \leq \int_{\partial\Omega} P_\Omega(z,\xi)\varphi(\xi)d\sigma(\xi).$$

But, for $\xi \in \partial\Omega$ we have

$$\begin{aligned}\varphi(\xi) = |u(w) - u(\xi)|^q &\leq \omega_f(|w-\xi|)^q \\ &\leq \omega_f(|w-z| + |z-\xi|)^q \\ &\leq C[\omega_f(|w-z|)^q + \omega_f(|z-\xi|)^q],\end{aligned}$$

and integration against Poisson kernel gives

$$\varphi(z) \leq C\Big[\omega_f(|w-z|)^q + \int_{\partial\Omega} P_\Omega(z,\xi)\omega_f(|z-\xi|)^q d\sigma(\xi)\Big].$$

Let $z_0 \in \partial\Omega$ be the closest point on the boundary to $z \in \Omega$. Then

$$|z-\xi| \asymp \delta(z) + |z_0 - \xi|$$

for $\xi \in \partial\Omega$. Therefore

$$\omega_f(|z-\xi|) \leq C\omega_f\big(\delta(z) + |z_0-\xi|\,\big) \leq C\frac{\delta(z)+|z_0-\xi|}{\delta(z)}\omega_f\big(\delta(z)\big).$$

By Theorem 26 we get

$$\varphi(z) \leq C\omega_f(|w-z|)^q + C\int_{\partial\Omega}(\delta(z))^{1-q}\frac{\big(\delta(z)+|z_0-\xi|\,\big)^q}{|z-\xi|^n}d\sigma(\xi).\omega_f\big(\delta(z)\big)^q.$$

Next we prove that the integral appearing above is bounded as a function of $z \in \Omega$. Set $\delta(z) = \delta$. Since $|z-\xi| \geq C\big(\delta(z) + |z_0-\xi|\,\big)$ we get

$$\int_{\partial\Omega}(\delta(z))^{1-q}\frac{\big(\delta(z)+|z_0-\xi|\,\big)^q}{|z-\xi|^n}d\sigma(\xi) \leq C\delta^{1-q}\int_{\partial\Omega}\big(\delta+|z_0-\xi|\,\big)^{q-n}d\sigma(\xi).$$

Now, we use the following decomposition of $\partial\Omega$: $\partial\Omega = \bigcup_{k=0}^{\infty} M_k$ where

$$M_k = \{\xi \in \partial\Omega : 2^{k-1}\delta \leq d(\xi, z_0) < 2^k\delta\}, \quad k \geq 1,$$

and

$$M_0 = \{\xi \in \partial\Omega : d(\xi, z_0) < \delta\}.$$

Using Lemma 12 we obtain:

$$\begin{aligned}\int_{\partial\Omega}(\delta(z))^{1-q}\frac{\big(\delta(z)+|z_0-\xi|\,\big)^q}{|z-\xi|^n}d\sigma(\xi) &\leq C\delta^{1-q}\sum_{k=0}^{\infty}\int_{M_k}\big(\delta+|z_0-\xi|\,\big)^{q-n}d\sigma(\xi) \\ &\leq C\delta^{1-q}\sum_{k=0}^{\infty}\int_{M_k}(2^k\delta)^{q-n}d\sigma(\xi)\end{aligned}$$

$$\leq C \sum_{k=0}^{\infty} 2^{k(q-n)} \delta^{1-n} \text{area}\,(M_k)$$
$$\leq C \sum_{k=0}^{\infty} 2^{k(q-n)} \delta^{1-n} (2^k \delta)^{n-1}$$
$$\leq C \sum_{k=0}^{\infty} 2^{k(q-1)} < +\infty.$$

Note that here we used $q \in (0,1)$. Hence we get

$$\varphi(z) \leq C[\omega_f(|w-z|)^q + \omega_f(\delta(z))^q] \leq C\omega_f(|w-z|)^q,$$

and therefore we proved

$$|u(w) - u(z)| \leq C\omega_f(|w-z|) \qquad \text{for} \quad w \in \partial\Omega, z \in \Omega.$$

In view of Lemma A.1. from [13] this estimate suffices to complete the proof. □.
If we assume that f is quasiconformal, then we can significantly relax the C^2-assumption on the boundary, see Theorem 30 below.

3.3 The case of uniformly perfect boundary

In this section we work with much more general domains, but here we consider only quasiconformal harmonic (or more general OC^2) mappings.

Proposition 3. *Let $f \in C^1(\Omega, \mathbb{R}^n)$ and let ω be a continuous majorant such that $\omega_*(t) = \omega(t)/t$ is non-increasing for $t > 0$. Assume f satisfies the following property:*

$$\delta(x)|f'(x)| \leq C\omega(\delta(x)), \quad x \in \Omega, \qquad (HL(\omega, C))$$

which we call Hardy-Littlewood (C, ω)-property. Then

$$f \in \operatorname{loc}\Lambda_\omega(L;\Omega), \qquad (\operatorname{loc}\Lambda_\omega).$$

If in addition f is harmonic in Ω or, more generally, $f \in OC^2(\Omega)$, then $(HL(\omega, C))$ is equivalent with $(\operatorname{loc}\Lambda_\omega)$.

Proof. Let us prove that $(HL(\omega, C))$ implies $(\operatorname{loc}\Lambda_\omega)$. If $y \in B_x$, then

$$|f(y) - f(x)| \leq \int_{[x,y]} |f'(z)| ds(z) \leq |y-x| \max_{z \in [x,y]} |f'(z)|$$
$$\leq C|y-x| \max_{z\in[x,y]} \frac{\omega(\delta(z))}{\delta(z)}.$$

Now, for every $z \in [x,y] \subset B_x$ we have $|x-y| \leq \delta(x)/2 \leq \delta(z)$, and since $\omega_\star$ is non-increasing we get $\omega(\delta(z))/\delta(z) \leq \omega(|x-y|)/|x-y|$. This, combined with the above estimate, gives $|f(y) - f(x)| \leq C\omega(|y-x|)$. Next we assume $f : \Omega \to \mathbb{R}^n$ is a harmonic mapping in $\operatorname{loc}\Lambda_\omega(\Omega, L)$. We set $M_x(r) = \max\{|f(y)| : |y-x| = r\}$ for $x \in \Omega$, $0 \leq r < \delta(x)$. Since f is harmonic we have $r|f'(x)| \leq C_n M_x(r)$ and $f \in \operatorname{loc}\Lambda_\omega(\Omega, L)$ implies $M_x(r) \leq L\omega(r)$. Therefore $r|f'(x)| \leq C_n L\omega(r)$ for $0 < r < \delta(x)$. Letting $r \to \delta(x)$ we deduce $(HL(\omega, C))$ with $C = C_n L$. Now we give a proof for the more general case of $f \in OC^2(\Omega)$. We assume $f \in \operatorname{loc}\Lambda_\omega(L, \Omega)$. Let us choose $x \in \Omega$. Since $\operatorname{diam} B_x = \delta(x)$, we have

$$\sup_{B_x} \delta^2(y)|\Delta f(y)| \le \operatorname{osc}_{B_x} f \le L\omega(\delta(x)).$$

Since $\delta(y) \asymp \delta(x)$ for $y \in B_x$ the above estimate gives

$$|\Delta f(y)| \le CL\frac{\omega(\delta(x))}{\delta^2(x)}, \qquad y \in B_x.$$

Next we use gradient estimates for Poisson equation in the ball B_x, see Theorem 3.9 from [**?**] and obtain

$$|f'(x)| \le C\left[\frac{1}{\delta(x)}\sup_{\partial B_x}|f| + \delta(x)\sup_{B_x}|\Delta f|\right].$$

Since both f' and Δf do not change if we replace f with $f - f(x)$ we see that

$$\begin{aligned} |f'(x)| &\le C\left[\frac{1}{\delta(x)}\operatorname{osc}_{\partial B_x}|f| + \delta(x)\sup_{B_x}|\Delta f|\right] \\ &\le CL\frac{\omega(\delta(x))}{\delta(x)} \end{aligned}$$

Note that a similar argument appeared in [**?**], it was used to prove inclusion $OC^2(\Omega) \subset OC^1(\Omega)$. □

An immediate consequence of the above proposition is the following corollary.

Corollary 5. *Let ω be a continuous majorant such that $\omega_\star(t) = \omega(t)/t$ is non-increasing for $t > 0$ and let $\Omega \subset \mathbb{R}^n$ be a domain which has Λ_ω-extension property. Then an OC^2 mapping (in particular a harmonic mapping) $f : \Omega \to \mathbb{R}^n$ belongs to $\Lambda_\omega(\Omega)$ if and only if it has Hardy-Littlewood (C, ω) property.*

Remark 8. *If the mapping f in the Proposition 3 and Corollary 5 extends continuously to $\overline{\Omega}$, then the assumption of continuity of ω can be omitted.*

Theorem 4 can be restated: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. Assume $u : \overline{\Omega} \to \mathbb{R}^n$ is a continuous map which is K- quasiregular map and harmonic in Ω, and $f \in \Lambda_\omega(\partial\Omega; L)$ where $f = u\,|_{\partial\Omega}$ Then

$$u \in \Lambda_\omega(\Omega; CL), \qquad C = C(K, \omega_f, n, \Omega). \tag{3.6}$$

If Ω is a Λ_ω-extension domain, then (3.6) is equivalent to Hardy-Littlewood (ω, C_1)-property: $\delta(x)|f'(x)| \le C_1\omega(\delta(x))$ for all $x \in \Omega$. Special cases of this result, for the disk and unit ball and holomorphic functions are well know as Privalov theorem. Hardy-Littlewood theorem is concerned by characterization of Lipschitz spaces in terms of growth of derivative.
The following result is contained in Theorem 3.2 from [9].

Lemma 13. *Let the boundary of a bounded domain Ω in $\mathbb{R}^n$ be c-uniformly perfect. If f is a continuous mapping of $\overline{\Omega}$ into $\mathbb{R}^n$ which is quasiconformal in Ω and if*

$$|f(x) - f(y)| \le \omega(|x - y|) \tag{3.7}$$

for all $x, y \in \partial\Omega$ and for some majorant ω, then

$$|f(x) - f(y)| \leq C\omega(|x - y|) \tag{3.8}$$

for all $y \in \partial\Omega$ and $x \in \Omega$, where C depends only on $c, n, K(f)$ and $\operatorname{diam}(\Omega)$.

Using Lemma 13 we prove the following generalization of Theorem 27.

Theorem 30. *Let the boundary of a bounded domain Ω in $\mathbb{R}^n$ be c-uniformly perfect. Assume f is a continuous mapping of $\overline{\Omega}$ into $\mathbb{R}^n$ which is quasiconformal in Ω and*

$$|f(x) - f(y)| \leq \omega(|x - y|), \qquad x, y \in \partial\Omega \tag{3.9}$$

for some majorant ω. Assume one of the following two conditions is satisfied.
a) f is harmonic in Ω.
b) $f \in OC^2(\Omega)$, $\omega(t)/t$ is non increasing for $t > 0$ and Ω is an Λ_ω-extension domain.
Then the following estimate holds:

$$|f(x) - f(y)| \leq C\omega(|x - y|), \qquad x, y \in \Omega. \tag{3.10}$$

Proof. Let us assume f is harmonic. By Lemma 2, estimate (3.10) holds for all $x \in \partial\Omega$ and all $y \in \Omega$. Using Lemma A. 1. from [13] we deduce that the same estimate is valid for all $x, y \in \Omega$. Now we consider condition b). Fix a point $x \in \Omega$. Choose a point $\xi \in \partial\Omega$ such that $|x - \xi| = \delta(x)$ and set $f_0(z) = f(z) - f(\xi)$, $z \in \overline{\Omega}$. We employ again gradient estimates for the Poisson equation, as in the proof of Proposition 3. Since $f' = f_0'$ and $\Delta f = \Delta f_0$ we obtain

$$|f'(x)| \leq C_n \left[\frac{1}{\delta(x)} \sup_{\partial B_x} |f_0| + \delta(x) \sup_{B_x} |\Delta f| \right]. \tag{3.11}$$

However, since $B_x \subset B(\xi, 3\delta(x)/2)$, Lemma 2 gives

$$\sup_{\partial B_x} |f_0(z)| \leq \sup_{\partial B_x} |f(z) - f(\xi)| \leq C\omega(3\delta(x)/2) \leq C\omega(\delta(x)).$$

Also, OC^2 condition gives $\sup_{B_x} |\Delta f| \leq C\delta^{-2}(x)$. These estimates, combined with (3.11) give $|f'(x)| \leq C\omega(\delta(x))/\delta(x)$. Hence we proved that f has Hardy-Littlewood (C, ω) property. Now the result follows from Corollary 5. □

3.3.1 Dyakonov's result

Now we give a simple proof of a Dyakonov's result from [19] which relates moduli of continuity of f and $|f|$ in the special case of quasiconformal f. Our proof is based on distortion property of quasiconformal mappings (see [21], p.383, [65], p.63):

$$\overline{B}(f(x), c_*\delta_*(x)) \subset f(B_x) \subset B(f(x), C_*\delta_*(x)), \qquad x \in \Omega \tag{3.12}$$

for a K-quasiconformal mapping $f : \Omega \to f(\Omega) = \Omega'$, where $\delta_*(x) = \operatorname{dist}(f(x), \partial\Omega')$.

Theorem 31. *Suppose $f : \Omega \to f(\Omega) = \Omega'$ is quasiconformal in domain $\Omega \subset \mathbb{R}^n$. Let $0 < \alpha \leq 1$. If $|f| \in \operatorname{loc}\Lambda_\alpha(\Omega, L)$, then $f \in \operatorname{loc}\Lambda_\alpha(\Omega, CL)$.*
If, in addition, Ω is a Λ_α-extension domain, then $f \in \Lambda_\alpha(\Omega)$.

Proof. Let us choose $x \in \Omega$ and set $R(x) = c_*\delta_*(x)$. We first prove the following:

$$\exists\, x_1, x_2 \in B_x : \quad |f(x_1)| - |f(x_2)| \geq R(x). \tag{3.13}$$

Let l be the line passing through 0 and $f(x)$, it intersects the sphere $\partial B(f(x), R(x))$ at points y_1 and y_2. By the first inclusion in (3.12) these two points lie in $f(B_x)$, hence $x_k = f^{-1}(y_k) \in B_x$, $k = 1, 2$. We consider two cases:
a) If $0 \notin B(f(x), R(x))$ and $|y_2| \geq |y_1|$, then $|y_2| - |y_1| = 2R(x)$.
b) If $0 \in B(f(x), R(x))$, then for example $0 \in [y_1, f(x)]$ and if we choose $x_1 = x$, we find $|y_2| - |f(x)| = R(x)$ and this yields (3.13). Now we obtain, using (3.13), that

$$c_*\delta_*(x) = R(x) \leq |f(x_1)| - |f(x_2)| \leq L|x_1 - x_2|^\alpha \leq L\delta(x)^\alpha.$$

Using the second inclusion in (3.12) we obtain

$$|f(z_1) - f(z_2)| \leq 2C_*\delta_*(x) \leq 2\frac{C_*}{c_*}L\delta(x)^\alpha,$$

and this completes the proof. □

Hence, as an immediate corollary we get K.M. Dyakonov results for quasiconformal mappings:
Theorem Dy Suppose Ω is a Λ_α-extension domain, $0 < \alpha \leq 1$, and f is a K-quasiconformal mapping of Ω onto $f(\Omega) \subset \mathbb{R}^n$. The following two conditions are equivalent:
a) $f \in \Lambda_\alpha(\Omega)$,
b) $|f| \in \Lambda_\alpha(\Omega)$.
If, in addition, Ω is a uniform domain and if $\alpha \leq K^{1/(1-n)}$, then these conditions are equivalent to
c) $|f| \in \text{loc}\,\Lambda_\alpha(\Omega)$.

3.4 Harmonic quasiconformal maps of quadrant

In this section we relate some results regarding HQC self maps of the quadrant $Q = \{z : z = x + iy, x, y > 0\}$. These results are from [2].
Let $\Pi^+ = \{z : z = x + iy, x > 0\}$. The following theorem can be proven considering norm of the directional derivative of f:

Theorem 32. *Suppose $f : \Omega \to f(\Omega) = \Omega'$ is quasiconformal harmonic in domain $\Omega \subset \mathbb{C}$. Suppose that $f(z) = Re\, G(z) + iIm\, H(z)$ where G and H are holomorphic, and $g(z) = G'(z)$, $h(z) = H'(z)$.*
Then $g(z) = \varphi(z)h(z)$ where φ is holomorphic and $\varphi(\Omega)$ is relatively compact subset of Π^+.

This result can be used to characterize HQC self maps of Q. Namely, the following result holds true:

Theorem 33. *The necessary and sufficient condition that $f : Q \to Q$ is quasiconformal harmonic homeomorphism of $\bar{Q}$ to $\bar{Q}$ is given as follows.*
Suppose that $f(z) = Re\, G(z) + iIm\, H(z)$ where G and H are holomorphic, and $g(z) = G'(z)$, $h(z) = H'(z)$.
Then the conditons are:
1) $g(z) = \varphi(z)h(z)$ where φ is holomorphic and $\varphi(\Omega)$ is relatively compact subset of Π^+.
2) g and h map Q into $\bar{\Pi}^+$,
3) g is real on imaginary axis, while h is real on real axis.

Bibliography

[1] A. Ababaoub, M. Arsenović, M. Mateljević, and A. Shkheam, *Moduli of continuity of harmonic quasiregular mappings on bounded domains*, Annales Academic Scientiarum Fennicae Mathematica, volume 38, 2013, 1-9

[2] A. Ababaoub, A. Shkheam, V. Božin, *Harmonic quasiconformal maps of the quadrant*, draft paper

[3] L. V. Ahlfors, *Lectures On Quasiconformal Mappings*, D. Van Nostrand Company (Canada), LTD, 1966.

[4] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, Mathematics Subject Classification (2000): 31-01, 31B05, 31C05, Springer, Editorial Board, S. Axler, F.W. Gehring, and K.A. Ribet, New York, 2000.

[5] Astala, K. Gehring, F.W., *Quasiconformal analogues of theorems of Koebe and Hardy-Littlewood*, Mich.Math.J. 32 (1985) 99-107.

[6] M. Arsenović, V. Božin, V. Manojlović, *Moduli of continuity of Harmonic Quasireglur Mappings in $\mathbb{B}^n$*, Potential Anal. 81, (2011), 34:283-291.

[7] K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, p. cm. (Princeton mathematical series ; 48), 2009

[8] M. Arsenović, V. Kojić, M. Mateljević, *On Lipschitz continuity of harmonic quasiregular maps on the unit ball in $\mathbb{R}^n$*, Ann. Acad. Sci. Fenn. 33, 315-318 (2008).

[9] M. Arsenovic, V. Manojlovic and R. Näkki, *Boundary Modulus Of Continuity And Quasiconformal Mappings*, Annales Academi, Scientiarum Fennic, Mathematica Volumen 37, 2012, 107118

[10] M.Brilleslyper, M. Dorff, J. McDougall, J. Rolf, B. Schaubroeck, R. Stankewitz, and K. Stephenson, *Exploring Complex Analysis*, NSF Grants No. DUE-0632969, No. DUE-0632976, No.DUE-0633125, and No. DUE-0633266, 2011.

[11] Z. M. Balogh, K. Fässler, and I. D. Platis, *Modulus Of Curve Families And Extremality Of Spiral-Stretch Maps*, 2000 Mathematics Subject Classification. 30C62,30C75,2009.

[12] J. Cooper,*Working Analysis*, Elsevier B.V. All rights reserved, 2005.

[13] Caffarelli, L.A., Kinderlehrer, D.: *Potential methods in variational inequalities*, J. Anal. Math. 37, (1980) 285-295.

[14] K. M. Dyakonov, *Holomorphic functions and quasiconformal mappings with smooth moduli*, Elsevier, Advances in Mathematics 187 (2004) 146172.

[15] K.K. Dube, *Fundamentals Of Complex Analysis: Theory And Applications*, 1.K. international publishing House Pvt. Ltd. 2009.

[16] B.A .Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern Geometry- Methods and Applications* 2^{nd}*(Graduate texts in mathematics,93-)*, 1984, 1992,Springer-Verlag New York,Inc.

[17] R. DeVore, G. Lorentz, *Constructive Approximation*, Springer-Verlag, .

[18] K. M. Dyakonov, *Equivalent norms on Lipschitz-type spaces of holomorphic functions*, Acta Math. 178 (1997), no. 2, 143167.

[19] K.M. Dyakonov, *Holomorphic functions and quasiconformal mappings with smooth moduli*, Advances in Mathematics, 187 (2004) 146-172

[20] F. Gehring, *Quasiconformal Mappings in Euclidean Spaces*, Ann Arbor, MI 48109-1109, USA, 2005.

[21] F. Gehring, *Rings an qc mappings in space*, TAMS **103** , 1962, 353-393

[22] F. W. Gehring and O. Martio, *Lipschitz-classes and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math., 10(1985), 203219.

[23] F. W. Gehring, K. Hag, *The Ubiquitous Quasidisk*, Mathematics subject Classification. Primiry 30C62, Secoundry 30C20, 30C45, 30C65. 2010.

[24] V. Gutlyanskii, V. Ryazanov, U. Srebro, and E. Yakubov, *The Beltrami Equation: A Geometric Approach*, Springer-Verlag,, 2010

[25] F. W. Gehring, J. Väisälä, *On the geometric definition for quasiconformal mappings*, Commentarii Mathematici Helvetici, 1962, Volume 36, Issue 1, pp 19-32.

[26] I. Holopainen, *Quasiregular mappings and the p-Laplace operator*, 1991 Mathematics Subject Classification. Primary 58J60, 30C65; Secondary 53C20, 31C12,35J60.

[27] J. Heinonen, *Lectures On Lipschitz Analysis*, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

[28] A. Hinkkanen, *Modulus of continuity of harmonic functions*, J. Anal. Math. 50, 1988, 129.

[29] V. P. Havin, *A generalization of the Privalov-Zygmund theorem on the modulus of continuity of the conjugate function*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. 6 (1971), no. 2-3, 252258, ibid. 6 (1971), no. 4, 265287.

[30] J. W. Iverson, *Möbius Transformations* , Accessed Mar, 18, 2007.

[31] T. Iwaniec, and G. Martin, *Geometric Function Theory and Non linear Analysis*, Clarendon Press Oxford, 2001.

[32] L. V. Kovalev, *Hölder spaces of quasiconformal mappings*, 2000 Mathematics Subject Classification:Primary 30C62, Secoundary 26B35. 75(89) (2004), 87-94.

[33] R. Kühnau, *Geometric Function Theory*, Handbook Of Complex Analysis, V I and II, Elsever, 2005.

[34] D. Kalaj, *Lipschitz spaces and harmonic mappings*, Ann. Acad. Sci. Fenn. 34, 475-485, (2009).

[35] D. Kalaj, *A priori estimate of gradient of a solution to certain differential inequality and quasiconformal mappings* ,Journal d'Analyse Mathmatique (ISSN:0021-7670), Volume 118, Issue 5, Feb. 2012, Page(s) 23 pages [11406]

[36] Kiwon Kim, *Hardy-Littlewood property with the inner length metric*, Commun. Korean Math. Soc. 19 (2004), No. 1, pp. 53-62

[37] R. Kühnau *Handbook Of Complex Analysis Geometric Function Theory*, Volume 2, Elsevier B.V. All rights reserved, 2005.

[38] D. Kalaj, *Lipschitz spaces and harmonic mappings*, Ann. Acad. Sci. Fenn. 34, 475-485, (2009).

[39] I.M. Kolodiy, F. Hildebrand, *Some properties of the module of coninuity.* In Russian language, Mathematical Notes, UDC 517.5, v. 9, No 5 (1971), 495-500

[40] S. G. Krantz, *Function theory of several Complex Variables,*1^{st} *ed*,John Wiley and Sons,New York,1982.

[41] S. G. Krantz, *Calculation and estimation of the Poisson kernel*, J. Math. Anal. Appl. 302(2005)143-148.

[42] S. G. Krantz, *Some New Thought on Maximal Functions and Poisson Integral*,J. Math. Anal. Appl. 302(2010)143-148.

[43] I. M. Kolodiy, F. Khiĺdebrand, *Some properties of the modulus of continuity*, Mathematical notes of the Academy of Sciences of the USSR May 1971, Volume 9, Issue 5, pp 285-288

[44] D. Kalaj, Vesna Manojlovic, *Subharmonicity of the Modulus of Quasiregular Harmonic*, Journal of Mathematical Analysis and Applications (ISSN:0022-247X), Volume 381, Issue 2, May 2011, Page(s) 6 pages [11078]

[45] V. Kojić, M. Pavlović, *Subharmonicity of* $|f|^p$ *for quasiregular harmonic functions with applications*, J. Math. Anal. Appl. 342, 742-746 (2008).

[46] H. Lin, *Classical Complex Analysis A Geometric Approch* , vol 1,world Scientific Publishing Co. Pte. Ltd, 2011

[47] V. Lappalainen, Lip_h*-extension domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. Disserta- tiones 56, 1985.

[48] O. Lehto, K. Virtanen, *Quasiconformal Mappings in the Plane*, Translated from the German by K. W. Lucas, Second Edition, Springer-Verlag, 1973

[49] M. Mateljević, *Quasiconformality of harmonic mappings between Jordan domains*, Filomat 26:3 (2012), 479- 510.

[50] O. Martio, R. Näkki, *Boundary Höllder continuity and quasiconformal mappings*, JLMS (2) 44 (1991) 339-350

[51] R. W. Mathews, J. H. Mathews, *Complex Analysis for Mathematics And Engineering*, Jones and Bartlett publishers Inc, fifth Edition, 2006.

[52] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, *Moduli in Modern Mapping Theory With 12 Illustrations*, Springer Science+Business Media, LLC, 2009

[53] N. Mercer, and R. Stankewitz, *Introduction to Quasiconformal Mappings in the plane with an application to quasiconformal surgery*,

[54] M. Mateljević and M. Vuorinen, *On uniform continuity of harmonic maps*, to appear in Journal of Inequalities and Applications, Volume 2010, Article ID 178732, 19 pages doi:10.1155/2010/1787; arXiv: 0709.4546v1

[55] Craig A Nolder, *A Quasiregular Analogue of a Theorem of Hardy and Littlewood*,Trans. Amer. Math. Soc. 331 (1992), 215-226

[56] O. Martio, *On harmonic quasiconformal mappings*, Ann. Acad. Sci. Fenn., A 1, 425, 3-10 (1968).

[57] J. Olsen, *The Geometry of Möbius Transformations* , Spring, 2010

[58] A. Purcell, *Analysis of quasiconformal maps in* $\mathbb{R}^n$, Graduate School Theses and Dissertations, University of South Florida, 2006

[59] N. Papamichael,N. Stylianopoulos, *Numerical Conformal Mapping: Domain Decomposition and the Mapping of Quadrilateral*, World Scientific Publishing Co.Ltd., 2010.

[60] A. Rasila, *Introduction to quasiconformal mappings in n-space*, Proceedings of the International Workshop on Quasiconformal, 19, 2006.

[61] J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parbolic PDEs and the Theory of Global Attractors*, Cambredage University Press

[62] T. J. Rivlin, *An Introduction to the Approximation of Functions*, Dover Pblication, Inc., Mineola, N. Y. 11501, 1969. Springer-Verlag, Berlin, 1993.

[63] A. I. Stepanets, *Uniform Approximations by Trigonometric Polynomials*, Ridderprent bv. Netherland.2001.2001.

[64] E. M. Sten, *Boundary behavior of Holomorphic Functions of Several Complex Variables*,Princeton University Press, Princeton, 1972.

[65] J. Väisälä , *Lectures on n-Dimensional Quasiconformal Mappings*, Spring-Verlsg, Heidelberg' New Yark, AMS Subject Classifications (1970): 30A 60

[66] E.B.Saff,A.D.Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, Third Edition.

[67] A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings*, Spring-Verlsg, Bogotá-Valpraiso, 2011, 2001.

[68] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture notes in Math, vol 1319, Springer-Verla, Berlin 1988

[69] A. Zygmund, *Trigonometric series.* 2nd ed. Vols. I, II, Cambridge University Press, New York, 1959.

Prilog 1.

Izjava o autorstvu

Potpisani Ali Elhrari Ababoub
broj indeksa 2046/2009

Izjavljujem

da je doktorska disertacija pod naslovom

"Lipschitz space and quasiconformal mappings"

- rezultat sopstvenog istraživačkog rada,
- da predložena disertacija u celini ni u delovima nije bila predložena za dobijanje bilo koje diplome prema studijskim programima drugih visokoškolskih ustanova,
- da su rezultati korektno navedeni i
- da nisam kršio autorska prava i koristio intelektualnu svojinu drugih lica.

Potpis doktoranda

U Beogradu, ________________

____

Prilog 2.

Izjava o istovetnosti štampane i elektronske verzije doktorskog rada

Ime i prezime autora: *Ali Elhrari Ababoub*
Broj indeksa: 2045/2009
Studijski program: ______________________
Naslov rada: *"Lipschitz space and quasiconformal mappings"*
Mentor: *redovni Prof. dr. Miodrag Mateljević*

Potpisani *Ali Elhrari Ababoub*

Izjavljujem da je štampana verzija mog doktorskog rada istovetna elektronskoj verziji koju sam predao za objavljivanje na portalu **Digitalnog repozitorijuma Univerziteta u Beogradu**.

Dozvoljavam da se objave moji lični podaci vezani za dobijanje akademskog zvanja doktora nauka, kao što su ime i prezime, godina i mesto rodjenja i datum odbrane rada.

Ovi lični podaci mogu se objaviti na mrežnim stranicama digitalne biblioteke, u elektronskom katalogu i u publikacijama Univerziteta u Beogradu.

Potpis doktoranda

U Beogradu, ______________

____

Prilog 3.

Izjava o korišćenju

Ovlašćujem Univerzitetsku biblioteku "Svetozar Marković" da u Digitalni repozitorijum Univerziteta u Beogradu unese moju doktorsku disertaciju pod naslovom:

"Lipschitz space and quasiconformal mappings"

koja je moje autorsko delo.

Disertaciju sa svim prilozima predao sam u elektronskom formatu pogodnom za trajno arhiviranje.

Moju doktorsku disertaciju pohranjenu u Digitalni repozitorijum Univerziteta u Beogradu mogu da koriste svi koji poštuju odredbe sadržane u odabranom tipu licence Kreativne zajednice (Creative Commons) za koju sam se odlučio.

1. Autorstvo
2. Autorstvo - nekomercijalno
3. Autorstvo - nekomercijalno - bez prerade
4. Autorstvo - nekomercijalno - deliti pod istim uslovima
5. Autorstvo - bez prerade
6. Autorstvo - deliti pod istim uslovima

(Molimo da zaokružite samo jednu od šest ponudjenih licenci, kratak opis licenci dat je na poledjini lista).

Potpis doktoranda

U Beogradu, ________________

____

1. Autorstvo - Dozvoljavate umnožavanje, distribuciju i javno saopštavanje dela, i prerade, ako se navede ime autora na način odredjen od strane autora ili davaoca licence, čak i u komercijalne svrhe. Ovo je najslobodnija od svih licenci.

2. Autorstvo - nekomercijalno. Dozvoljavate umnožavanje, distribuciju i javno saopštavanje dela, i prerade, ako se navede ime autora na način odredjen od strane autora ili davaoca licence. Ova licenca ne dozvoljava komercijalnu upotrebu dela.

3. Autorstvo - nekomercijalno - bez prerade. Dozvoljavate umnožavanje, distribuciju i javno saopštavanje dela, bez promena, preoblikovanja ili upotrebe dela u svom delu, ako se navede ime autora na način odredjen od strane autora ili davaoca licence. Ova licenca ne dozvoljava komercijalnu upotrebu dela. U odnosu na sve ostale licence, ovom licencom se ograničava najveći obim prava korišćenja dela.

4. Autorstvo - nekomercijalno - deliti pod istim uslovima. Dozvoljavate umnožavanje, distribuciju i javno saopštavanje dela, i prerade, ako se navede ime autora na način odredjen od strane autora ili davaoca licence i ako se prerada distribuira pod istom ili sličnom licencom. Ova licenca ne dozvoljava komercijalnu upotrebu dela i prerada.

5. Autorstvo - bez prerade. Dozvoljavate umnožavanje, distribuciju i javno saopštavanje dela, bez promena, preoblikovanja ili upotrebe dela u svom delu, ako se navede ime autora na način odredjen od strane autora ili davaoca licence. Ova licenca dozvoljava komercijalnu upotrebu dela.

6. Autorstvo - deliti pod istim uslovima. Dozvoljavate umnožavanje, distribuciju i javno saopštavanje dela, i prerade, ako se navede ime autora na način odredjen od strane autora ili davaoca licence i ako se prerada distribuira pod istom ili sličnom licencom. Ova licenca dozvoljava komercijalnu upotrebu dela i prerada. Slična je softverskim licencama, odnosno licencama otvorenog koda.