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Serija II T. 19. Zagreb 1964. Broj 3–4 POSEBNI OTISAK

GLASNIK MATEMATIČKO - FIZIČKI I ASTRONOMSKI
PERIODICUM MATHEMATICO - PHYSICUM ET ASTRONOMICUM

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Monotone mappings between some kinds of ordered sets

Monotona preslikavanja među nekim vrstama uređenih skupova

Z a g r e b 1 9 6 4

Štamparski zavod „Ognjen Prica" Zagreb, Savska cesta 31, 1964

GLASNIK MAT. - FIZ. I ASTR.
Tom 19. — No. 3—4 — 1964.

MONOTONE MAPPINGS BETWEEN SOME KINDS OF ORDERED SETS

Đuro Kurepa, Zagreb

Dedicated to Professor A. D. Wallace for his 60th birthday

0. *Introduction.*

$(O, <)$ or simply O, like $(O_1, <_1)$, $(O_2, <_2)$ will denote sets O, O_1, O_2, ordered (totally or partially) by means of $<, <_1, <_2, \ldots$ respectively.

0.1. If $(O_1, <_1)$ and $(O_2, <_2)$ are (partially or totally) ordered sets, a mapping f from O_1 to O_2 is said to be increasing (isotone, order preserving) or a member of

$$\uparrow_= ((O_1, <_1), (O_2, <_2)) = \{f; x \in O_1 \Longrightarrow f x_1 \in O_2\},$$

provided

$$\{x, y\} \subseteq O_1 \wedge x \leq_1 y \Longrightarrow f x \leq_2 f y_2;$$

if moreover $x <_1 y \Rightarrow f x <_2 f y$, f is said to be strongly or strictly increasing.

0.2. The set of all increasing functions from $(O_1, <_1)$ to $(O_2, <_2)$ is denoted by

$$\uparrow_= ((O_1, <_1), (O_2, <_2)) \tag{1}$$

or shorter by $\uparrow_= (O_1, O_2)$.

0.3. The set of all strongly increasing functions from $(O_1, <_1)$ to $(O_2, <_2)$ is denoted by

$$\uparrow ((O_1, <_1), (O_2, <_2)) \quad \text{or} \quad \uparrow (O_1, O_2). \tag{1}$$

An important problem is to determine the last set, for given O_1, O_2.

0.4. Varying the sets $(O_1, <_1)$, $(O_2, <_2)$, one is varying considerably the sets **0.2** (1) and **0.3** (1). In particular, the problem arises to determine the existence of some member of the set (1) having a certain given property. Among the ordered sets some are quite characteristic, like $(PS, \subset)$, η_α, lattices, etc.

Ovaj rad je financirao Savezni fond za naučni rad i Republički fond za naučni rad SRH.

0.5. Partitive sets $PS, P'S, P_a S$ etc.

For any set S one defines $PS = \{X; X \subseteq S\}$, $P'S = \{X; O \subset \subset X \subseteq S\}$, $P_a S = \{X; X \subseteq S \wedge kX < a\}$; a is a given cardinal number.

0.6. Left ideals of $(O, <)$. Operator IO.

IO or $I(O, <)$ consists of all the initial section or left ideals of $(O, <)$; in other words

$$X \in I(O, <) \Leftrightarrow X \subseteq O \wedge (X = \emptyset \vee x \in X \rightarrow O(\cdot, x] \subseteq X),$$

where

$$O(\cdot, x] = \{y; y \in O \wedge y \leq x\}.$$

Then one has the graphs or diagrams $(I(O, <), \subset)$ and $(I(O, <), \supset)$.

0.7. The set $w(O, <)$ (resp. $w'(O, <)$ or $\omega(O, <)$) consists of all the well-ordered subsets of $(O, <)$, the empty set $\emptyset$ being included (being not included).

0.8. $w'_0 = \sigma(O, <) = \{x; x \in w(O, <) \wedge x \neq \emptyset, x$ is bounded in $(O, <)\}$.

0.9. $w_0(O, <) = \{x; x$ is a well-ordered bounded subset of $(O, <)\}$.

0.10. The operators L, L_0, L', L'_0.

$L(O, <)$ consists of all the chains of $(O, <)$,

L' consists of all the non empty members of $L(O, <)$,

L_b consists of all the bounded members of $L(O, <)$,

L'_b consists of all the non empty bounded members of $L(O, <)$.

0.11. Operators ^-L, ^-L_b, $^-L'$, $^-L'_b$ (anti L, anti L', etc.).

The definition is obtained from Section **0.10** by replacing chains by antichains.

0.12. Relations $-\!|$, $|\!-$. Relations $=\!|$, $|\!=$.

For sequences or ordered sets A, B the relation $A -\!| B$ or $B |\!- A$ means that A is a proper initial section of B; we set

$$A =\!| B \Leftrightarrow A = B \wedge A -\!| B.$$

Thus, for a given $(O, <)$, we have the ordered sets $(X(O, <), -\!|)$, for the operators $X \in \{w, w_0, w', w'_0, L, L_0, L', L'_0\}$.

Instead of w'_0, we wrote previously σ. All these sets are trees.

0.13. Left ideal closure. Right ideal closure.

For $X \subseteq (O, <)$, let

$$0X = \bigcup_x O(\cdot, x], \ (x \in X); \ 0\emptyset = \emptyset,$$

$$1X = \bigcup_x O[x, \cdot), \ (x \in X); \ 1\emptyset = \emptyset.$$

Consequently, $0X$ (resp. $1X$) is the minimal initial (terminal) section of $(O, <)$ containing the set X.

0.14. For any family F of sets $\subseteq (O, <)$ we put

$$0F = \{0X;\ X \in F\},$$
$$1F = \{1X;\ X \in F\}.$$

In particular we have the diagrams

$$(0\,w\,(O, <), \subset), (0\,L\,(O, <), \subset), (0\,\sigma\,(O, <), \subset), (0\,A\,(O, <), \subset), \text{ etc.}$$

0.15. Stellarity number $s\,(O, <) = \inf\{kF; F$ being composed of chains $\subseteq (O, <)$ and $\bigcup_{X \in F} X = O\}$.

0.16. Antistellarity number $\neg s\,(O, <) = a\,(O, <) = \inf\{kF; F \subseteq A\,(O, <) \wedge \bigcup F = O\}$.

0.17. Number $\Gamma(O, <)$.

The first ordinal number, which is not imbeddable in $(O, <)$, is denoted by $\Gamma(O, <)$ or ΓO.

0.18. The consideration of the sets $(w\,O, -|)$, $\uparrow((w\,O, \rightarrow)\,(O, <))$ was initiated by the present author who proved in particular that $\uparrow(\sigma X, X) = \emptyset$, for $R \in \{Ra, Re\}$; the topic was then studied by S. Ginsburg [1] (Theorem 10) who proved in particular that $\uparrow i\,(O, \rightarrow), (O, <) = \emptyset$, for every infinite totally ordered group $(O, <)$ and for $i = \omega$; if moreover $(O, <)$ is a totally ordered field, then one could write also $i = \sigma$.

An ordered set $(O, <)$ is called by Ginsburg a k-set or a k'-set according as to whether the set

$$\uparrow((w\,O, -|, (O, <)) \tag{1}$$

is empty or non-empty; every member of the set (1) is called by Ginsberg a k-function on O[1].

1. *Some theorems on strictly increasing functions.*

1.1. Theorem. $\uparrow((w\,(O, <), -|, (O, <)) = \emptyset$, *for every ordered set* (cf. **0.1, 0.5.2, 0.5.6**). *There exists no strictly increasing mapping of the set* $(w\,(O, <), -|)$ *into the set* $(O, <)$; *in other words, for any non-empty ordered set* $(O, <)$ *and increasing mapping* f *from* $(w\,O, -|)$ *into* $(O, <)$ *there exists a well-ordered subset* W *of* $(O, <)$

[1] »Question. Do there exist two k'-sets E and F such that $E \times F$ is a k-set?« ([1], p. 588.) (if E, F are both simply ordered k-sets, so is also $E \times F$, this set being ordered by the method of the last differences ([1], p. 587, Theorem 8).

on which the function f is constant and $kW > 1$. *Symbolically,* $\uparrow (wO, O) = \emptyset$, *for every* $(O, <)$. (Đ. Kurepa [11], théorème 1)[2].

Proof. Suppose, on the contrary, that there exists a mapping $f : (wO, —|) \to (O, <)$ such that $x —| x'$ in $wO \Longrightarrow fx < fx'$. Since the void set $\emptyset$ is a member of wO satisfying $\emptyset —| x$, for every non empty set $x \leqslant wE$, one should have

$$f\emptyset < fx \, (x \leqslant wE, \; x \neq \emptyset).$$

Let

$$x_1 = f\{fe_0\}; \text{ then } x_0 < x_1; \text{ if } x_2 = f\{fe_1\},$$

$$e_0 = \emptyset, \; e_1 = e_0 \cup \{fe_0\} = \{fe_0\}, \; e_2 = e_1 \cup \{fe_1\} = \{fe_0, fe_1\};$$

assume that $0 < \alpha < \gamma E$ and that for every $\alpha_0 < \alpha$ one has defined the well-ordered sets e_{α_0} and that these sets form an α-chain in the tree $(wO, —|)$, i. e. that $\xi < \eta < \alpha_0 \Longrightarrow e_\xi —| e_\eta$; let us define e_α as $e_{\alpha-1} \cup \{fe_{\alpha-1}\}$ or as the union of all the sets e_{α_0}, where $(\alpha_0 < < \alpha)$, according as to whether α is of the first kind or of the second kind. The set e_α should be a member of wE; therefore fe_α would be a member of $(O, <)$. Consequently, for every $\alpha < \Gamma O$ (for ΓO see. **0.10**) one would have the well-ordered set e_α such that

$$\alpha < \beta < \gamma O \Longrightarrow e_\alpha —| e_\beta;$$

therefore, by hypothesis,

$$\alpha < \beta < \gamma O \Longrightarrow fe < fe_\beta$$

and the elements

$$fe_\alpha \quad (\alpha < \gamma O)$$

would constitute a well-ordered subset of $(O, <)$ of order type γO, contradicting the definition of ΓO as the first ordinal not imbeddable into $(O, <)$.

1.2. Theorem. $\uparrow ((PO, \subset), (O, <)) = \emptyset$, (1)

$\uparrow (PO, \supset), (O, <)) = \emptyset$ (cf. **0.1, 0.5**). (2)

Proof. As a matter of fact, if f were a member of the set (1), then the restriction f_0 of f in the set wO would yield (contrary to **1.1.**) a member of $\uparrow w((O, <), (O, <))$, because for sets $X, Y \leqslant PO$, the relation $X —| Y$ implies $X \subset Y$.

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[2] In Mathematical Reviews **17** (1956), 1065, reviewing this paper, S. Ginsburg writes: »Three results are stated, the first being incorrect [counterexample: Let E be the negative integers. For each subset S of E let $f(S) = \max \{x \mid x \text{ in } E\} - 1$]«. This example is inadequate for the situation, because the void set which is a member of ωE is omitted; this example shows that $\uparrow((\omega E, —|), (E <)) \neq \emptyset$; here $(E, <)$ may stay for any inversely ordered set (S. Ginsburg [1], p. 586, Corollary).

The second part of the theorem is a consequence of the first part of the theorem and of the fact that every partitive set $(PS, \subset)$ is isomorphic to its dual $(PS, \supset)$.

1.3. Theorem. *If $f \in \uparrow (iO, O)$ (where $i = \omega$ or σ), then $f\{x\} > x$, for no $x \in O$; in particular, if $(O, <)$ is a chain, then $f\{x\} \leq x$, for every $x \in O$.*

Proof. Suppose on the contrary that f be a strictly increasing mapping of $(\omega O, \dashv)$ into $(O, <)$ and that for some $x_0 \in O$ one has $x_0 < f\{x_0\}$. Let $x_1 = f x_0$ and $f\{x_0, x_1\} = x_2, f\{x_0, x_1, x_2\} = x_3$, etc. As in **1.1.** one would define, for $0 < \alpha < \Gamma(O, <)$, the point $x_\alpha = f\{x_0, x_1, \ldots, x_{\alpha_0} \ldots\}$ $\alpha_0 < \alpha$, which yields the strictly increasing γ-sequence

$$x_0, x_1, \ldots, x_\alpha, \ldots \qquad (\alpha < \gamma O),$$

contradicting the definition of γO.

1.4. Analogously, one proves the following

Theorem. *If $i \in \{w, \sigma\}$ and $f \in \uparrow (iO, O)$, then*

$$fY > X^{3} \text{ for no } X \in iO.$$

The proof is analogous to the proof of Theorem **1.1.** (replacing $\emptyset$ by any $X \in iO$ satisfying $fX > X$.

1.5. Theorem. *If $f \in \uparrow (\omega O)$ and if $(O, <)$ is a left complete chain and the function $f_0 x = f\{x\}$ is increasing in $(O, <)$, then the fixpoints of the function f_0 constitute a non empty left complete ordered set.*

The theorem is a consequence of Theorem **1.3.** and of the theorem 1 in Đ. Kurepa [12].

1.6. Theorem. *If $\Gamma O > \omega$ (cf. **0.18**), then $\uparrow((P_{\aleph 0} O, \subset), (O, <)) \neq \emptyset$.*

Proof. It is sufficient to consider any infinite well-ordered subset $\{c_0 < c_1 < \ldots\}$ and, for any $X \subseteq O$ satisfying

$$kX < \aleph_0, \text{ to put } fX = c_{kX}.$$

1.7. Lemma. *If $kX < kO$, for every $X \in wO$, then $\uparrow((P_{kO} O, \subset)(O, <)) = \emptyset$.*

As a matter of fact, in this case $w(O, \leq) \subseteq P_{kO} O$; consequently, if f were a strictly increasing mapping of $(P_{kO} O, \subset)$ into $(O, <)$, then the restriction of the same mapping on $w(O, <)$ would yield a member of $\uparrow (w(O, <), (O, <))$ contrary to Theorem **1.1.**

1.8. Theorem.

$$\uparrow((P(O, <), \subset), (O, <)) \cup \uparrow((P(O, <), \supset), (O, <))) = \emptyset. \qquad (1)$$

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[3] For ordered sets A, B one defines

$$A < B \Longleftrightarrow \dot{A} \leq \dot{B}$$

$$A \leq B \Longleftrightarrow \dot{A} \leq \dot{B}.$$

The first summand in (1) is empty because of Theorem 1.1. and of the inclusion $w(O, <) \subseteq P(O, <)$. The second summand in (1) is empty because the ordered sets $(PO, \subset)$, $(PO, \supset)$ are isomorphic and moreover one has the following:

1.8.1. Lemma. *If i is an isomorphism from $(O, <)$ onto $(O_1, <_1)$, then*

$$\uparrow((O, <), (O_2, <_2)) \neq \emptyset \Leftrightarrow \uparrow((O, <), (O_2, <_2)) \neq \emptyset .$$

In particular, $f \in \uparrow((O, <), (O_2, <_2)) \Rightarrow f i^{-1} \in \uparrow(O_1, O_2)$.

1.9. Theorem. *Let O be any subset of the set R of real numbers such that $\Gamma O = \omega_1$ and let O be conditionally complete* (i. e. contains sup X of every bounded subset X of O); *then* $\uparrow(\sigma O, O) = \emptyset$. *In particular,* $\uparrow((\sigma R, -|), (R, <)) = \emptyset$.

The proof is based on the fact that the tree $(\sigma Ra, -|)$ is not a union of $\leq \aleph_0$ of its antichains (cf. Kurepa [10a] [9] p. 37, Theorem **2.1**) the last proposition is implied by the equality $\uparrow(\sigma Ra, Ra) = \emptyset$; the last formula was proved in Kurepa [10] p. 89, Theorem **3.1.** and [10a] p. 40, Theorem **3.1**; another proof was given by S. Ginsburg ([1], p. 588) .

1.9.1. Now, let us suppose that there exists a strictly increasing mapping f of σO into O

$$f \in \uparrow(\sigma O, O). \qquad (1)$$

1.9.2. For any $X \in \sigma O$, let $\overline{X}$ be the closure of the set X in the ordered space $(O, <)$. Then $x \in 0\, O \Rightarrow \overline{x} \in \sigma O$ and $\{x, y\} \subset \sigma O$ and $X -| Y$ yield $\overline{X} =| \overline{Y}$, the equality $\overline{x} = \overline{y}$ holding if and only if the point sup X is the last point in the well-ordered set Y; then, γX is of the second kind.

Firstly, since X is a non empty well-ordered bounded set, so is also $\overline{X}$; therefore, $\overline{x} \subset \sigma O$, the set O being, by assumption, conditionally right complete. Furthermore, if $x -| y$, then sup $x \in \overline{x} \subseteq \overline{y}$; if $x \cup \{\sup x\} = y$, then $\overline{x} = \overline{x \cup \{\sup x\}} = \overline{y}$, i. e. $\overline{x} = \overline{y}$.

Conversely, $\overline{x} = \overline{y}$ and $x -| y$ imply that the set y contains no point $>$ sup x; and since $y \setminus x \neq \emptyset$ (because of $x -| y$), one has necessarily sup $x \in y$ and sup x is the last point in y.

1.9.3. Function g.

For $x \in \sigma O$, let

$g(x) = f\overline{x}$, provided Γx is of the first kind, and let

$g(x) = f\overline{x} \setminus \{\sup x\}$, provided Γx is of the second kind (cf. **0.17**; for well-ordered sets W the number ΓW coincides with the order type of W).

One should have $g \in \uparrow(\sigma O, O)$, i. e. $x, y \in w O \wedge x -| y \Rightarrow$
$\Rightarrow fx < fy$ and $\{fx, fy\} \subset O$.

Case $\bar{x} \dashv \bar{y}$. If $\Gamma x, \Gamma y \in I$, then $g(x) = f(\bar{x}) < f(\bar{y}) = g(y)$. If ΓX is of the first kind and Γy of the second kind, then

$$g x = f \bar{x} < f(\bar{y} \setminus \{\sup y\}) = g y ;$$

here occurs the sign because $\bar{x} \dashv \bar{y} \setminus \{\sup y\}$ (a consequence of the relations $x \dashv y, -\Gamma x + \Gamma y \geq \omega_0$). The remaining two cases: $\Gamma x \in II \wedge \Gamma y \in I$, $\Gamma x \in II \wedge \Gamma y \in II$ are discussed in an analogous way.

Case $\bar{x} = \bar{y}$. Since $x \dashv y$, one has $y = x \cup \{\sup x\}$, $\Gamma x \in II$; hence, $g x = f(\bar{x} \setminus \{\sup x\}) = f(\bar{y} \setminus \{\sup x\}) < f(\bar{y}) = g(y)$.

Consequently, $f x < g y$.

1.9.4. If $x \in \sigma O$ and Γx is of the second kind, then every immediate successor x^+ of x is of the form $x \cup \{b\}$ with $b \in O\,[\sup x, \cdot)$ and $g(x^+) \geq g(x \cup \{\sup x\}) > g(x)$.

Firstly, $x^+ = x \cup \{b\}$; secondly,

$$g x^+ = g(x \cup \{b\}) = f\overline{x \cup \{b\}}) = f(\bar{x} \cup \{b\} > f \bar{x} = g x .$$

1.9.5. Now let us conclude and show that there would be a strictly increasing function from $\sigma R a$ to $R a$, contrarily to Theorem **3.1** in Đ. Kurepa [10].

For $x \in R_0\, \sigma O = \{y ; y \in \sigma O \text{ with } \sigma O(\cdot, y) = \emptyset\}$, let $r_0(x)$ be such that $r_0(x) \in R a$ and $r_0(x) < g(x)$. For every $y \in \sigma O$ with $\Gamma y \in I$ denote by y^- the immediate predecessor of y; i. e. for every such y the point y^- is the last one in the set $\sigma O(\cdot, y)$; let $0 < a < \omega_1$ and suppose that on the set $\sigma O(\cdot, a) = \bigcup R_\xi\, \sigma O\ (\xi < a)$ a strictly increasing function r_a be defined such that it takes values in the set $(O, <)$ and that

$$r_\xi \mid \sigma O(\cdot, \xi) \qquad (\xi < a) \tag{1}$$

be an a-sequence of the more and more extending functions; let us define also functions $r_a(\sigma O(\cdot, a]$, extending the functions (1), by setting, for every $x \in R_a(\sigma O)$, any member of $(O, <)$ such that

$$r_{a-1}(x^-) < r_a(x) < g(x), \text{ provided } a \in (I),$$
$$g(x) < \uparrow r_a(x) < g(x \cup \{\sup x\}), \text{ provided } a \in II.$$

The definition should be possible for every $a < \omega_1 (= \gamma O)$; putting then $r(x) = r_a(x)$, for every $x \in R_a\, \sigma O$ and every $a < \gamma\, \sigma O$, one should have $r \in \uparrow(\sigma(O, <) \dashv), (R a, <))$.

1.9.6. Let $E = r\, \sigma O$; then the set σO would be the union of the $\dashv$-antichains $^{-}r x$ $(X \in O)$; since $O \subseteq R a$, this would mean that the set $(\sigma O, \dashv)$ is a union of $\leq \aleph_0$ antichains, thus, the antistellarity of $(\sigma O, \dashv)$ would be $\leq \aleph_0$,

$$^{-}s(\sigma O, \dashv) \leq \aleph_0 . \tag{1}$$

And this very relation is impossible, the implied relation (1) contradicting Theorem **1.10** which follows. This completes the proof of Theorem **1.9.**

1.10. Theorem. *Let O be any subset of $\bar{\eta}_0 (= \bar{R}a = Re)$ such that $\Gamma O = \omega_1$. Then the antistellarity number of the tree $(\sigma O, —|)$ is $\aleph_1$,*

$$\neg s(\sigma O, —|) = \aleph_1 \text{ (cf. } \mathbf{0.16}, \mathbf{0.12}) .$$

Proof. **1.10.1.** At first, the ordered set of rationals is imbeddable into $(O, <)$ i. e.

$$\text{order type } \eta \leq \text{order type } (O, <) . \tag{1}$$

We have two cases:

1.10.1.1. First case: $kO = \aleph_0$. Then (1) was proved in Đ. Kurepa [8] (p. 146, Theorem 1).

1.10.1.2. *Second case:* $kO > \aleph_0$. In this case

$$\neg s\, \sigma O = \aleph_1 \text{ (cf. Theorem } \mathbf{1.10}) .$$

As a matter of fact, the set Ra (or its type η) is similar to a subset of $(O, <)$, i. e. η is imbeddable into O. In other words, let O_0 be the set of all the points x_0 of $(O, <)$ which are not points of bilateral accumulation of $(O, <)$; the set O_0 is countable because every x_0 is an extremal point of an interval $I(x_0)$ of the set η $(= Re)$ and such that $I(x_0) \cap O = \{x_0\}$; consequently, the sets Int $I(x_0)$ $(x_0 \in O_0)$ are pairwise disjoint; therefore, $kO_0 \leq \aleph_0$; this and $kO > > \aleph_0$ imply that the set $O_1 = O \setminus O_0$ is of a cardinality $\geq \aleph_1$ and has no consecutive points. According to a well-known theorem of Cantor, this implies that η_0 is imbeddable in O_1 and a fortiori in O. Thus, formula (1) is completely proved.

1.10.2. Any isomorphic imbedding of η_0 into O implies an isomorphic imbedding of the tree $(\sigma\eta, —|)$ into the tree $(\sigma O, —|)$; therefore, one has

$$\text{order type } (\sigma\eta, —|) \leq \text{order type } (\sigma O, —|)$$

and hence, for the antistellarity numbers one has.

1.10.3. Lemma. $\neg s(\sigma\eta, —|) \leq \neg s(\sigma O, —|)$.

1.10.4. Now, in Đ. Kurepa [10], p. 87, Theorem **2.1** and [11] p. 37, Theorem **2.1** it was proved that $\neg s(\sigma\eta, —|) = \aleph_1$. This formula and Lemma **1.10.3.** yield.

Lemma. $\neg s(\sigma O —|) \geq \aleph_1$.

1.10.5. The antistellartiy number of the tree $(\sigma O, —|)$ is $\leq \aleph_1$. For abreviation, put $T = (\sigma O, —|)$. Then, denoting by $R_0 X$ the set of all the initial elements of X, one has the following disjoint partition of T into antichains $R_\xi T$:

$$T = R_0 T \cup R_1 T \cup \ldots = \bigcup_\xi R_\xi T,$$

where $R_\alpha T = R_0 (T \setminus \bigcup_{\xi<\alpha} R_\xi T)$, for every ordinal $O < \alpha$.

Now, certainly $R_{\omega_1} = \emptyset$; otherwise, if $a \in R_{\omega_1} T$, then the union of the well-ordered subsets $X \in T(\cdot, a)$ would yield a non-countable well-ordered set $\subset \overline{\eta}$, contradicting a well-known theorem of Cantor.

The two Lemmas **1.10.4, 1.10.5** yield the requested Theorem **1.10.**

1.11. On the existence of strictly increasing functions and ewerywhere dense subsets.

If there exists a strictly increasing function from $(O, <)$ to $(O_1, <_1)$ and if X_1 is an everywhere dense subset of $(O_1, <_1)$, does there exist also a strictly increasing function from $(O, <)$ to $(X_1, <_1)$? Not, necessarily!

T h e o r e m. *There exist ordered sets* $(O, <)$, $(O_1, <_1)$ *such that* $\uparrow((O, <), (O_1, <_1)) \neq \emptyset$ *and that for some everywhere dense part* X_1 *of* $(O_1, <_1)$ *one has* $\uparrow((O, <), (X_1, <_1)) = \emptyset$; *in particular,*

$$\uparrow((w\, R\, a, -|), R\, e) \neq \emptyset \quad \textit{and} \quad \uparrow((w\,(R\, a, -|), (R\, a, <)) = \emptyset\,. \qquad (1)$$

The last equality being a special case of Theorem **1.1**, let us prove (1); even a stronger result holds:

1.12. $\uparrow((P\, R\, a, \subset), (R\, e, <)) \neq \emptyset$ (Sierpiński [13], p. 240).

In fact, let $r_1, r_2, \ldots$ be a normal well-ordering of the set $R\, a$; for $x_k \in R\, a$, put $f\, x_k = \Sigma r_n^{-2}$, n satisfying $r_n < k$; put also $f\, \emptyset = 0$; then f is a member of the set in **1.12.**

2. *Intervention of the antistellarity number* $\neg s\, O$ (cf. **0.16**).

The question of the existence of a strictly increasing mapping on any $(O, <)$ into η_0 or in general into η_σ is the subject of the following theorem.

2.1. T h e o r e m. *For any regular number* $\aleph_\sigma$ *one has*

$$\neg s\,(O, <) \leq \aleph_\sigma \Rightarrow \uparrow((S, <); \eta_\sigma);$$

in other words, if an ordered set $(O, <)$ *is the union of* $\leq \aleph_\sigma$ *antichains, then there exists a stricly increasing mapping of* $(O, <)$ *to* $(\eta_\sigma, <)$ *(the case* $\sigma = O$ was proved in Đ. Kurepa [7], p. 837, Theorem 1).

2.2. *Proof.*

2.2.1. Let

$$A_\xi \; (\xi < \alpha \leq \omega_\sigma) \qquad (1)$$

be a sequence of pairwise disjoint antichains exhausting the set $(O, <)$. The case $a < \omega_\sigma$ offering no difficulty, let us consider that in (1) we have $a = \omega_\sigma$. Set, for every $0 < \nu < \omega_\sigma$,

$$F_\nu = \cup A_{\nu'}, (\nu' < \nu); \tag{2}$$

we shall define a sequence of one-valued functions f_ν on $F_\nu (\nu < a)$ such that f_ν be an extension of $f_{\nu'}$, for every $\nu' < \nu$.

2.2.2. Let $W\eta_\sigma$ be any normal well-ordering of η_σ; consequently, the order type $\gamma W\eta_\sigma$ is an initial ordinal $\geq \omega_\sigma$. To start with, let $f_1 F_1 = R_0 W\eta_\sigma$ (the set formed by the first member of $W\eta_\sigma$). Let $1 < \nu < a$ and suppose that, for $1 < \nu < \omega_\sigma$ and every $\nu' < \nu$, the following condition $K(\nu')$ holds:

2.2.3. Condition $K(\nu') : \Gamma f F_{\nu'}, \Gamma f F^*_{\nu'} < \omega_\sigma$, where $\Gamma(X, <)$ is the first ordinal which is not imbeddable into $(X, <)$). Let us define f_ν on F_ν. If ν is of the second kind, we put, for every $a \in F_\nu$, $f_\nu(a) = f_{\nu'}(a)$, where $\nu' < \nu$ such that $f_\nu(a)$ be defined. The number ω_σ being regular, one is aware that the condition $K(\nu)$ holds.

2.2.4. If the number ν is of the first kind, the function f_ν shall extend the function $f_{\nu-1}$ and coincide in $F_{\nu-1}$ with $f_{\nu-1}$; for $a \in F_\nu \setminus F_{\nu-1}$, let us consider the sets

$$f_{\nu-1} F_{\nu-1}(\cdot, a), f_{\nu-1} F_{\nu-1}(a, \cdot) F_{\nu-1}. \tag{1}$$

The condition $K(\nu-1)$ implies that the first set in (1) is empty or cofinal to an ordinal number $< \omega_\sigma$, and that the second set in (2) is empty or coinitial to the inverse of an ordinal number $< \omega_\sigma$; by the property of η_σ one concludes that there exists some member of η_σ located between the two sets (1); the first such point occurring in the well-order $W\eta_\sigma$ shall be denoted by $f_\nu(a)$. This means that the function $f_\nu | F_\nu$ is defined.

2.2.5. Let us prove that the condition $K(\nu)$ holds. But this is implied by the fact that every non-empty open interval of the ordered set $f_\nu F_\nu$ contains a point of $f_{\nu-1} F_{\nu-1}$, this resulting from the definition of $f_\nu(a)$ as the first element in the well-ordering $W\eta_\sigma$ located between the two sets (1).

2.2.6. By transfinite induction, $f_\nu | F_\nu$ is defined for every $\nu < \omega_1$; putting

$$f = \sup f_\nu | F_\nu \quad (\nu < \omega_\sigma), \tag{1}$$

one obtains a requested member of the set $\uparrow (O; \eta_\sigma)$. Of course, the formula (1) means that $\mathrm{Dom}\, f = \cup \mathrm{Dom}\, f_\nu (\nu < \omega_\sigma)$ and that, for every $x \in \mathrm{Dom}\, f$, one has $fx = f_\nu x$, for every ν satisfying $x \in \mathrm{Dom}\, f$.

2.2.7. Now, we have the relation

$$k\,\eta_\sigma = \aleph_\sigma \langle \quad \rangle 2^{\aleph\sigma - 1} = \aleph_\sigma,$$

for σ of the first kind, and

$$2^{\aleph_\xi} \leq \aleph_\sigma (\xi < \sigma),$$

for σ of the second kind (cf. F. Hausdorff [2], p. 180).

Consequently, one has the following.

2.3.8. Theorem.

$$[\neg s (O, <) \leq \aleph_\sigma \Longleftrightarrow \uparrow ((O, <), \eta_\sigma) \neq \emptyset] \Longleftrightarrow \xi < \sigma \Longleftrightarrow 2^{\aleph_\xi} \leq \aleph_\sigma,$$

for every regular $\aleph_\sigma$.

2.4. The converse of Theorem **2.1.** One might wonder whether the converse of Theorem **2.1** holds. It is so, provided $k\, \eta_\sigma = \aleph_\sigma$. As a matter of fact, it f is a strictly increasing function of $(O, <)$ to η_σ, one has

$$O = \bigcup_x f^{-1} x \qquad (x \leqslant \eta\, \sigma)\, .$$

Everyone of these summands being an antichain, the formula yields that the antistellarity of $(O, <)$ is $\leq \aleph_\alpha$ (it is to be noted that for every chain L one has $\uparrow (O, L) \neq \emptyset \Rightarrow s\, O \leq k\, L$).

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MONOTONA PRESLIKAVANJA MEĐU NEKIM VRSTAMA UREĐENIH SKUPOVA

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Sadržaj

0.2. Skup svih uzlaznih (odnosno strogo uzlaznih) preslikavanja uređena skupa $(O_1, <_1)$ u uređen skup $(O_2, <_2)$ označuje se sa (1).

0.3. Analogno vrijedi i za strogo uzlazna preslikavanja.

0.6. IO označuje skup svih početnih komada uređena skupa $(O, <)$.

0.7. wO (odnosno $w'O$ ili ωO) označuje skup svih dobro uređenih podskupova od $(O, <)$ pri čemu prazni skup uključujemo (isključujemo). $w_0 O$ (odnosno $w'_0 O$) dobije se promatrajući samo omeđene članove.

0.13. Definicija od $0X, 1X$ je dana odgovarajućim formulama u **0.13.**

0.14. Definicija od $0F, 1F$, za svaku obitelj F skupova, razabire se iz formula u **0.14.**

1.1. Teorem. *Ne postoji čisto uzlazno preslikavanje od* $(w(O, <), -|)$ *u* $(O, <)$.

1.9. Teorem. *Neka je O proizvoljan skup realnih brojeva sa svojstvom* $\Gamma O = \omega_1$ *i koji sadrži* sup X, *za svako omeđeno* $X \subseteq O$; *tada je* $\uparrow(\sigma O, O) = \emptyset$.

1.10. Teorem. *Ako je O podskup skupa R realnih brojeva sa svojstvom* $\Gamma O = \omega_1$ *tada vrijedi* (1).

2.1. Teorem. *Svaki regularni broj* $\aleph_\sigma$ *zadovoljava* (1); drugim *riječima, ako je uređen skup* $(O, <)$ *unija od* $\leq \aleph_\sigma$ *antilanaca, tada postoji čisto uzlazna funkcija od* $(O, <)$ *ka* $(\eta_\sigma, <)$.

(Primljeno 1. II 1964.)