METODA ZA PRORAČUN NESTACIONARNIH RAVANSKIH LAMINARNIH GRANIČNIH SLOJEVA

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UVODNA RAZMATRANJA

Nestacionarni granični slojevi i pored toga što su vrlo interesantni i važni nisu dovoljno proučeni. Broj metoda za rešavanje problema ovih graničnih slojeva, za razliku od stacionarnih, je vrlo mali. Naime, postoje samo približne metode, dok opšte metode iz klase tačnih rešenja do sada nije bilo. Najopštiji slučaj do sada, stepeni t^{α} i eksponencijalni $e^{\beta t}$ zakon promene brzine spoljnog potencijalnog strujanja sa vremenom, koji je obuhvatio dobar deo do tada već rešenih slučajeva: nagli trzaj $\alpha = 0$, postupno ubrzanje $\alpha = 1$, Gertler-ovo rešenje za $\alpha = 2, 3,$ i 4, rešio je Vatson [6]. Pored toga, bilo je rešeno i dosta specijalnih problema kao: oscilovanje ploče, oscilovanje cilindra itd. Treba odmah napomenuti da je 1960 god i H.A.Hasan [2] sa Virdžinia politehničkog instituta dao jednu metodu pretpostavljajući

$$U(x,t) = \frac{\nu}{l}\left(\frac{2\nu t}{l^2}\right)^{-\frac{1}{2}(1+\lambda)} h(s)$$

$$\psi(x,y,t) = \nu\left(\frac{2\nu t}{l^2}\right)^{-\frac{1}{2}\lambda} \chi(s,\sigma)$$

gde su:

l - karakteristična dužina

λ - realan ili kompleksan broj

$$s = \left(\frac{x}{l}\right)\left(\frac{2\nu t}{l^2}\right)^{-\frac{1}{2}(1-\lambda)}$$

$$\sigma = \frac{y}{\sqrt{2\nu t}}$$

$$h(s) = s^{\alpha}\sum_{n=0}^{\infty} a_n s^n$$

α - ceo broj; $a_0 = 0$

Da bi rešio diferencijalne jednačine Hasan uvodi nove promenljive u obliku

$$\xi = s \qquad \eta = 6 \cdot f(s)$$

i pretpostavlja da je

$$\chi = \frac{h}{f} \phi(\xi, \eta)$$

te dobija sisteme diferencijalnih jednačina za odredjivanje funkcije $\phi(\xi, \eta)$.

Metoda Hasan-a poseduje niz slabosti:Pre svega,oblik pretpostavljene funkcije U(x,t) znatno ograničava primenu metode,pošto će se retko javiti u praktičnim problemima.Rešavanjem spoljnog potencijalnog strujanja,nekom od poznatih metoda,ili eksperimentalnim ispitivanjem oblika funkcije brzine spoljnog potencijalnog strujanja,videće se da,gornji oblik će se vrlo retko javiti.Možda bi trebalo izdvojiti slučaj $U(x,t) = x^n t^m$,koji je praktično moguć,a može se rešavati ovom metodom Najinteresantniji i praktično najviše moguć slučaj $U(x,t) = U(x)\Omega(t)$ ostaje van domašaja ove metode.

Sem gore navedenog mogao bi se staviti prigovor na komplikovanost metode za praktična rešavanja,što je posledica višeznačnosti koordinata,i na nepogodnost tabulisanja univerzalnih funkcija,pošto se u diferencijalnoj jednačini za odredjivanje prvog koeficijenta-funkcije ϕ_0 javljaju dva parametra α i a_0 koji se ne mogu eliminisati.

Navedeni razlozi govore da ova metoda ustvari ima više matematički značaj, a manji,za rešavanje problema teorije nestacionarnih laminarnih graničnih slojeva.

Znači,i pored svega,opšte i praktično primenljive metode,do sada nije bilo.

Zahvaljujući uvodjenju novih nezavisnih specijalnih promenljivih τ i η,umesto starih promenljivih t i y,ta metoda je data.Metoda pretpostavlja jedno ograničenje za oblik funkcije rasporeda brzina spoljnog potencijalnog strujanja.Naime, traži se da ta funkcija bude data u obliku da razdvaja promenljive tj.$U(x,t)=U(x)\Omega(t)$.Medjutim,ovo ne pretstavlja nikakvo ograničenje,jer će se kao što smo maločas videli uglavnom ta funkcija javljati baš u takvom obliku,bilo da je dobivena teoriskim razmatranjem potencijalnog strujanja bilo eksperimentalno,ili će joj se moći sa dovoljnom približnošću dati takav oblik.To znači da,naše ograničenje za raspored brzina spoljnog potencijalnog strujanja je ustvari i praktično uslovljeno.

Sada se može preci na izlaganje suštine nove metode. Praktično, kod proračuna nas interesuje kako se na nekom odredjenom mestu duž konture opticanog tela, menjaju veličine, koje karakterišu granični sloj, sa vremenom. U tom cilju kod nove metode se koordinata x ostavlja nepromenjena, jer figuriše kao parametar, a umesto promenljivih t i y uvode se nove specijalne promenljive u obliku

$$\tau = \frac{1}{\nu}\int_0^t \Omega^2(t)\,dt \qquad \eta = \frac{\Omega(t)\cdot y}{\nu\sqrt{3\tau}}$$

gde su:

$\Omega(t)$– funkcija koja pokazuje promenu brzine spoljnjeg potencijalnog strujanja sa vremenom.

ν – kinematska viskoznost.

Rešenje jednačina nestacionarnih laminarnih graničnih slojeva biće dato u obliku specijalnog reda po stacionarnoj brzini U(x) i njenim izvodima U', U'',....., pri čemu se postavlja uslov da funkcija U(x) i njeni izvodi $\frac{dU}{dx}$, ... do reda "n" budu neprekidni, sa koeficijentima koji su funkcije promenljivih τ i η . Na ovaj način od parcijalne jednačine u kojoj su figurisale tri promenljive dolazimo do sistema parcijalnih jednačina, ali u kojima figurišu sada samo dve promenljive τ i η . Dati sistem je rekurzivan. Rešenja ovog sistema parcijalnih jednačina daće koeficijente-funkcije specijalnog reda. Ta rešenja su data u obliku stepenih redova po τ sa koeficijentima, koji su funkcije od η . Ovi koeficijenti-funkcije promenljive η biće dati u obliku linearnih kombinacija jedared za svagda tabuliranih univerzalnih funkcija.

Specijalni red je tako izabran da već prvi član u njemu zadovoljava tačno granične uslove, a ostali članovi da koriguju prvi član reda samo u unutrašnjosti graničnog sloja duž konture. Isto tako stepeni redovi po τ su izabrani da vodeći član u stepenom redu za prvi koeficijenat-funkciju specijalnog reda zadovoljava tačno granične uslove, a ostali članovi u redu da vrše korekciju ovog člana unutar sloja po vremenu t (odnosno τ).

Pošto se rešenje diferencijalne jednačine graničnog sloja razvija u dva pravca: duž konture,odnosno u pravcu x - ose,specijalni red i po vremenu t (odnosno τ) - stepeni redovi,to treba i pokazati konvergenciju u ta dva pravca.Pošto vodeći član naznačenih redova (po x i τ) zadovoljava tačno granične uslove,to treba očekivati da će on dosta dugo i po x i po t da aproksimira rešenje.Korekture koje donose ostali članovi trebalo bi da dodju što kasnije do izražaja i po x i po t,pa bi nada za dobru konvergenciju bila ostvarena.Za dobrotu konvergencije stepenih redova značajan je jedan momenat.Po vremenu t (odnosno τ) u intervalu od 0 do ∞ nema singulariteta,kao što je to tačka odvajanja po koordinati x,koja je u znatnoj meri kvarila konvergenciju redova [1] upotrebljenih za rešavanje jednačina stacionarnih graničnih slojeva.Medjutim,pošto se ovde taj efekat neće javiti,to je i nada za bolju konvergenciju ovih redova opravdanija,nego što je to bio slučaj kod redova za rešavanje stacionarnih graničnih slojeva [1].

U radu [3] ukazano je da nas zaustavljanje na već trećem članu specijalnog reda dovodi sasvim u blizinu tačnih rezultata,što govori da je konvergencija specijalnog reda verovatno sasvim dobra tj on verovatno sasvim brzo konvergira tačnim vrednostima.

Ovde su samo iznešene indicije u vezi sa konvergencijom,koju je skoro nemoguće dokazati,već je moguće samo praktično potvrditi.Medjutim,iz prednjeg izlaganja se vidi da su nade za dobru konvergenciju opravdane potpuno.

Koordinate τ i η moraju biti izabrane tako da nisu funkcije i od promenljive x tj.da su jednoznačne,jer će se samo tada metoda odlikovati jednostavnošću.Prema ovome proračun bi se jednostavno proveo po vremenu t (odnosno τ) tj.našli bi se koeficijenti-funkcije specijalnog reda,a zatim bi se išlo duž konture (duž x-ose) i tražile veličine graničnog sloja na pojedinim mestima na konturi.Znači,specijalni red bi nas vodio duž konture,a njegovi koeficijenti bi pokazivali promenu veličina graničnog sloja sa vremenom t (odnosno τ).

§ 1 TRANSFORMACIJA JEDNAČINA GRANIČNOG SLOJA

Diferencijalne jednačine nestacionarnih laminarnih i ravnih graničnih slojeva imaju sledeći oblik

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad 1.$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 2.$$

sa graničnim uslovima

$$u = v = 0 \qquad y = 0$$
$$u = U(x,t) \qquad y = \infty \qquad 2'.$$

gde su:

- x – odstojanje duž zida konture koju optiče tečnost
- y – upravno odstojanje od zida
- t – vreme
- $u(x,y,t)$ – komponenta brzine u pravcu x ose
- $v(x,y,t)$ – komponenta brzine u pravcu y ose
- ν – kinematska viskoznost
- $U(x,t)$ – unapred zadata brzina spoljašnjeg potencijalnog strujanja na granici graničnog sloja

Ako sa indeksima x,y,t označimo parcijalne izvode po odnosnim koordinatama i uvedemo funkciju strujanja $\Psi(x,y,t)$ izrazima

$$u = \Psi_y \qquad v = -\Psi_x \qquad \text{sa} \quad \Psi(x,0,t) = 0 \qquad 3.$$

biće jednačina kontinuiteta identički zadovoljena, a jednačina 1 svodi se na oblik

$$\Psi_{yt} + \Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = U_t + UU_x + \nu\Psi_{yyy} \qquad 4.$$

sa graničnim uslovima

$$\Psi_x(x,y,t) = \Psi_y(x,y,t) = 0 \qquad y = 0$$
$$\Psi_y(x,y,t) \to U(x,t) \qquad y \to \infty \qquad 4'.$$

Uvedimo nove bezdimenzione specijalne promenljive

$$\tau = \frac{1}{\nu}\int_0^t \Omega^2(t)\,dt \qquad \eta = \frac{\Omega(t)\,Y}{\nu\sqrt{3\tau}}$$

i funkciju strujanja pretstavimo u obliku

$$\Psi(x,y,t) = \nu\sqrt{3\tau}\;U(x)\,F(x,\eta,\tau) \tag{6.}$$

Iz izraza 5.vidi se,da usled $\Omega(t) > 0$ (ili $\Omega(t) < 0$) za $t > 0$ promenljiva τ monotono raste sa porastom vremena t.Jednoznačnost promenljivih 5. je zagarantovana svuda i jedino za $t=0$ sme da bude $\Omega(t)=0$.

Potražimo odgovarajuće parcijalne izvode funkcije strujanja

$$\Psi_y = U\cdot\Omega\,F_\eta$$
$$\Psi_{yy} = U\frac{\Omega^2}{\nu\sqrt{3\tau}}\,F_{\eta\eta}$$
$$\Psi_{yyy} = U\frac{\Omega^3}{\nu^2\,3\tau}\,F_{\eta\eta\eta}$$
$$\Psi_{yt} = U\Omega' F_\eta + U\frac{\Omega^3}{\nu}F_{\eta\tau} + U\Omega'\eta F_{\eta\eta} - U\frac{3}{2}\frac{\Omega^3}{\nu}\frac{\eta}{3\tau}F_{\eta\eta}$$
$$\Psi_x = \nu\sqrt{3\tau}\,U'F + \nu\sqrt{3\tau}\,U F_x$$
$$\Psi_{xy} = U'\Omega F_\eta + U\Omega F_{x\eta}$$

Smenom u diferencijalnu jednačinu 4. ova se svodi na novi oblik

$$U\Omega'F_\eta + U\frac{\Omega^3}{\nu}F_{\eta\tau} + U\Omega'\eta F_{\eta\eta} - U\frac{3}{2}\frac{\Omega^3}{\nu}\frac{\eta}{3\tau}F_{\eta\eta} + U\Omega F_\eta\left(U'\Omega F_\eta + U\Omega F_{x\eta}\right)$$
$$- U\frac{\Omega^2}{\nu\sqrt{3\tau}}F_{\eta\eta}\left(\nu\sqrt{3\tau}\,U'F + \nu\sqrt{3\tau}\,UF_x\right) = U\Omega' + \Omega^2 UU' + U\frac{\Omega^3}{\nu 3\tau}F_{\eta\eta\eta}$$

Ako se leva i desna strana jednačine pomnoži izrazom $\frac{\nu 3\tau}{U\Omega^3}$ i sredi dobiće se konačan oblik gornje jednačine

$$F_{\eta\eta\eta} + \alpha(\tau)\left(1 - F_\eta - \eta F_{\eta\eta}\right) + \frac{3}{2}\eta F_{\eta\eta} - 3\tau F_{\eta\tau} + \beta(\tau)\Big[U'(1 -$$
$$- F_\eta^2 + F F_{\eta\eta}) + U\left(F_x F_{\eta\eta} - F_\eta F_{x\eta}\right)\Big] = 0 \tag{7.}$$

gde su uvedene oznake za izraze

$$\alpha(\tau) = \frac{\Omega'\nu 3\tau}{\Omega^3} \qquad \beta(\tau) = \frac{\nu 3\tau}{\Omega}$$

Granični uslovi 4' transformišu se na novi oblik

$$F(x,0,\tau)=F_\eta(x,0,\tau)=0$$
$$\lim_{\eta\to\infty} F_\eta(x,\eta,\tau)=1 \qquad 4.''$$

Jednačina 7. je nelinearna parcijalna jednačina trećeg reda, sa tri promenljive. Ako pretpostavimo da je funkcija rasporeda spoljne potencijalne brzine U(x) i njeni izvodi $\frac{dU}{dx}, \frac{d^2U}{dx^2}, \ldots$ do reda n neprekidni, tada rešenje parcijalne jednačine 7. možemo tražiti u obliku sledećeg reda

$$F(x,\eta,\tau)=F_0(\eta,\tau)+U'F_1(\eta,\tau)+U'^2F_2(\eta,\tau)+UU''F_{2a}(\eta,\tau)+U'^3F_3(\eta,\tau)$$
$$+UU'U''F_{3a}(\eta,\tau)+U^2U'''F_{3b}(\eta,\tau)+\cdots \qquad 8.$$

Ako pretpostavljeno rešenje u gornjem obliku unesemo u jednačinu 7.

$$F_{0\eta\eta\eta}+U'F_{1\eta\eta\eta}+U'^2F_{2\eta\eta\eta}+UU''F_{2a\eta\eta\eta}+\cdots+\alpha(\tau)\Big[1-$$
$$-(F_{0\eta}+U'F_{1\eta}+U'^2F_{2\eta}+UU''F_{2a\eta}+\cdots)-\eta(F_{0\eta\eta}+U'F_{1\eta\eta}+$$
$$+U'^2F_{2\eta\eta}+UU''F_{2a\eta\eta}+\cdots)\Big]+\frac{3}{2}\eta(F_{0\eta\eta}+U'F_{1\eta\eta}+U'^2F_{2\eta\eta}+$$
$$+UU''F_{2a\eta\eta}+\cdots)-3\tau(F_{0\eta\tau}+U'F_{1\eta\tau}+U'^2F_{2\eta\tau}+\cdots)+$$
$$+\beta(\tau)\Big\{U'\Big[1-(F_{0\eta}+U'F_{1\eta}+U'^2F_{2\eta}+UU''F_{2a\eta}+\cdots)^2+$$
$$+(F_0+U'F_1+U'^2F_2+UU''F_{2a}+\cdots)(F_{0\eta\eta}+U'F_{1\eta\eta}+U'^2F_{2\eta\eta}+$$
$$+UU''F_{2a\eta\eta}+\cdots)\Big]+U\Big[(U''F_1+2U'U''F_2+\cdots)(F_{0\eta\eta}+$$
$$+U'F_{1\eta\eta}+U'^2F_{2\eta\eta}+UU''F_{2a\eta\eta}+\cdots)-(F_{0\eta}+U'F_{1\eta}+U'^2F_{2\eta}+$$
$$+\cdots)(U''F_{1\eta}+2U'U''F_{2\eta}+\cdots)\Big]\Big\}=0$$

dobijamo posle uporedjivanja sledeći rekurzivni sistem parcijalnih jednačina za odredjivanje nepoznatih koeficijenata-funkcija $F_0(\eta,\tau), F_1(\eta,\tau), \dots\dots\dots$ specijalnog reda 8:

a. $F_{0\eta\eta\eta} + \alpha(\tau)(1 - F_{0\eta} - \eta F_{0\eta\eta}) + \frac{3}{2}\eta F_{0\eta\eta} - 3\tau F_{0\eta\tau} = 0$ 9.

b. $F_{1\eta\eta\eta} - \alpha(\tau)(F_{1\eta} + \eta F_{1\eta\eta}) + \frac{3}{2}\eta F_{1\eta\eta} - 3\tau F_{1\eta\tau} + \beta(\tau)(1 - F_{0\eta}^2 + F_0 F_{0\eta\eta}) = 0$

c. $F_{2\eta\eta\eta} - \alpha(\tau)(F_{2\eta} + \eta F_{2\eta\eta}) + \frac{3}{2}\eta F_{2\eta\eta} - 3\tau F_{2\eta\tau} + \beta(\tau)(-2F_{0\eta}F_{1\eta} +$

$+ F_0 F_{1\eta\eta} + F_1 F_{0\eta\eta}) = 0$

c. $F_{2a\eta\eta\eta} - \alpha(\tau)(F_{2a\eta} + \eta F_{2a\eta\eta}) + \frac{3}{2}\eta F_{2a\eta\eta} - 3\tau F_{2a\eta\tau} + \beta(\tau)(F_1 F_{0\eta\eta} - F_{0\eta}F_{1\eta}) = 0$

d. $F_{3\eta\eta\eta} - \alpha(\tau)(F_{3\eta} + \eta F_{3\eta\eta}) + \frac{3}{2}\eta F_{3\eta\eta} - 3\tau F_{3\eta\tau} + \beta(\tau)(-2F_{0\eta}F_{2\eta} -$

$- F_{1\eta}^2 + F_2 F_{0\eta\eta} + F_1 F_{1\eta\eta} + F_0 F_{2\eta\eta}) = 0$

d. $F_{3a\eta\eta\eta} - \alpha(\tau)(F_{3a\eta} + \eta F_{3a\eta\eta}) + \frac{3}{2}\eta F_{3a\eta\eta} - 3\tau F_{3a\eta\tau} + \beta(\tau)(-3F_{0\eta}F_{2a\eta}$

$+ 2F_{2a}F_{0\eta\eta} + 2F_2 F_{0\eta\eta} - 2F_{0\eta}F_{2\eta} + F_0 F_{2a\eta\eta} + F_1 F_{1\eta\eta} - F_{1\eta}^2) = 0$

d. $F_{3b\eta\eta\eta} - \alpha(\tau)(F_{3b\eta} + \eta F_{3b\eta\eta}) + \frac{3}{2}\eta F_{3b\eta\eta} - 3\tau F_{3b\eta\tau} + \beta(\tau)(F_{2a}F_{0\eta\eta} -$

$- F_{0\eta}F_{2a\eta}) = 0$

. . . .

U gornjim jednačinama centralnu ulogu igraju funkcije

$$\alpha(\tau) = \frac{\Omega' \nu 3\tau}{\Omega^3} \quad i \quad \beta(\tau) = \frac{\nu 3\tau}{\Omega} \qquad 10.$$

jer specijalni podaci svakog posebnog problema još preko njih dolaze eksplicitno do izražaja.

Funkciju $\alpha(\tau)$ možemo izraziti i na sledeći način

$$\alpha(\tau) = 3\tau \frac{d}{d\tau} \ln \frac{\Omega}{\Omega_0} \qquad 11.$$

jer je $d\tau = \frac{\Omega^2}{\nu}dt$. Vreme t je jednoznačno prekoprve od jednačina 5. izraženo u funkciji od τ. Konstantna brzina Ω_0 u gornjoj jednačini uvedena je samo zbog dimenzione korektnosti. $\alpha(\tau)$ je sa τ pri zadanoj $\Omega(t)$ jednoznačna funkcija t .

$$\alpha(\tau) = 3\nu\tau \frac{\Omega'(t)}{\Omega^3(t)} = \frac{3}{2} \frac{\tau(t)\tau''(t)}{\tau'^2(t)}$$

Zbog centralnog značaja ove funkcije ubuduće ćemo je zvati "GLAVNA FUNKCIJA".

Glavna funkcija može se protumačiti kao "lokalni parametar oblika" $\frac{\delta^2 U_t}{\nu U}$ ili $\frac{\delta^2 \Omega'}{\nu \Omega}$ [3] gde je $\delta(x,t)$ neka od mera za debljinu graničnog sloja. Kao mera za debljinu graničnog sloja kod nestacionarnih slučajeva pogodno je uzeti debljinu istiskivanja δ^* [3]. Ako debljinu istiskivanja i debljinu pada impulsa transformišemo na nove promenljive dobija se:

Pošto je $\eta = \dfrac{\Omega Y}{\nu\sqrt{3\tau}}$ odnosno $\dfrac{d\eta}{dY} = \dfrac{\Omega}{\nu\sqrt{3\tau}}$

to se izraz za debljinu istiskivanja

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dY$$

svodi na sledeći oblik

$$\delta^* = \frac{\left\{3\nu\int_0^t \Omega^2 dt\right\}^{1/2}}{\Omega}\eta_\infty - \int_0^\infty F_\eta \frac{\left\{3\nu\int_0^t \Omega^2 dt\right\}^{1/2}}{\Omega} d\eta = \frac{\left\{3\nu\int_0^t \Omega^2 dt\right\}^{1/2}}{\Omega}\left\{\eta_\infty - \int_0^\infty F_\eta d\eta\right\}$$

$$= \frac{\left\{3\nu\int_0^t \Omega^2 dt\right\}^{1/2}}{\Omega}\left\{\eta_\infty - F_\infty\right\} = \frac{\left\{3\nu\int_0^t \Omega^2 dt\right\}^{1/2}}{\Omega}\eta_0 \qquad 12.$$

gde je $\eta_0 = \lim_{\eta\to\infty}(\eta - F)$

$$\vartheta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dY = \frac{\left\{3\nu\int_0^t \Omega^2 dt\right\}^{1/2}}{\Omega}\int_0^\infty F_\eta (1 - F_\eta) d\eta \qquad 13.$$

Iz izraza za debljinu istiskivanja sleduje:

$$\delta^{*2} = \frac{3\nu\int_0^t \Omega^2 dt}{\Omega^2}\eta_0^2$$

pa smenom u izraz za glavnu funkciju dobivamo

$$\alpha(\tau) = \frac{1}{\eta_0^2(t)}\,\frac{\delta^{*2}\Omega'}{\nu\Omega} \qquad 14.$$

ili $\dfrac{\delta^{*2}\Omega'}{\nu\Omega} = -\dfrac{\left.\frac{\partial p}{\partial x}\right)_L}{\frac{\mu\Omega}{\delta^*}/\delta^*}$ — i pretstavlja odnos sila pritisaka i sila trenja

Druga funkcija što ulazi u jednačine 9 $\beta(\tau)$ pomnožena sa U' pretstavljala bi "usputni parametar oblika" [3]

$$U'\beta(\tau) = \frac{U'\nu 3\tau}{\Omega} = \frac{U' 3\int_0^t \Omega^2 dt}{\Omega} = \frac{U_x\, 3\int_0^t U^2 dt}{U^2} \qquad 15.$$

Ako u proizvod $U'\beta(\tau)$ unesemo izraz za debljinu istiskivanja dobija se :

$$U'\beta(\tau) = \frac{\delta^{*2}U_x}{\nu} = \frac{\delta^{*2}U'}{\nu}\Omega \qquad 15'.$$

odakle se vidi da zaista pretstavlja usputni parametar oblika. Sa druge strane vidi se da je

$$\frac{\delta^{*2}U_x}{\nu} = -\frac{\left.\partial p/\partial x\right)_u}{\frac{\mu\Omega U}{\delta^*}/\delta^*}$$

Zbir ova dva parametra lokalnog i usputnog dao bi ukupan parametar oblika [3]

$\alpha(\tau) + U'\beta(\tau) = \gamma(x,\tau)$-ukupan parametar oblika

Samim postupkom rešavanja jednačina 7 mi smo učinili da centralnu ulogu igra "lokalni parametar oblika" $\alpha(\tau)$ dok smo sa "usputnim parametrom oblika" $U'\beta(\tau)$ učinili da nam figuriše kao parametar. Kasnije ćemo se vratiti i u većoj meri osvetliti značaj ovih funkcija.

Na isti način kao i jednačinu 4 možemo transformisati i granične uslove 4' na nove promenljive

$$F(x,o,\tau) = F_\eta(x,o,\tau) = 0$$

$$\lim_{\eta \to \infty} F_\eta(x,\eta,\tau) = 1 \qquad 7'.$$

a isto tako i granične uslove za svaku aproksimaciju

$$F_o(o,\tau) = F_{o\eta}(o,\tau) = 0$$

$$F_{o\eta}(\eta,\tau) \to 1 \qquad za \quad \eta \to \infty$$

$$9'.$$

$$F_1(o,\tau) = F_{1\eta}(o,\tau) = 0$$

$$F_{1\eta}(\eta,\tau) \to 0 \qquad za \quad \eta \to \infty$$

. . . .

. . . .

§ 2 NALAŽENJE VEZA IZMEDJU VELICINA GRANIČNOG SLOJA I NOVO UVEDENIH PROMENLJIVIH

Za debljinu istiskivanja δ^* i debljinu pada impulsa ϑ već smo našli izraze u funkciji novih promenljivih. Ostaje nam da nadjemo izraz za tangencijalni napon u funkciji novih promenljivih

$$N_\tau = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0}$$

Pošto je $\frac{u}{U} = F_\eta$ to je $\frac{\partial}{\partial y}\left[\frac{u}{U}\right] = \frac{\partial F_\eta}{\partial \eta}\frac{\partial \eta}{\partial y}$ ili

$$\frac{u_y}{U} = F_{\eta\eta}\frac{\Omega}{\{3\nu\int_0^t \Omega^2 dt\}^{1/2}}$$

Sada je
$$\frac{N_\tau(x,\tau)}{\varrho\,\Omega_\infty^2} = U(x,t)\frac{\Omega(t)}{\Omega_\infty^2}\frac{1}{\sqrt{3\tau}}F_{\eta\eta}(x,o,\tau) \qquad 16.$$

Nadjimo veze $\Omega = \Omega(\tau)$ i $t = t(\tau)$:

Iz prve od funkcija 5 imamo

$$d\tau = \frac{1}{\nu}\Omega^2 dt \qquad 17$$

da odavde i iz izraza za glavnu funkciju $\alpha(\tau)$ dobijamo

$$\frac{1}{3}\,\frac{\alpha(\tau)}{\tau}\,d\tau = d\ln\frac{\Omega}{\Omega_o}$$

ako se integrali i leva i desna strana ove jednačine

$$\frac{1}{3}\int\frac{\alpha(\tau)}{\tau}\,d\tau = \int d\ln\frac{\Omega}{\Omega_o}$$

ja se:

$$\Omega(t) = \Omega_o\, e^{\frac{1}{3}\int_{\tau_o}^{\tau}\frac{\alpha(\tau)}{\tau}d\tau}$$

Uvodjenjem oznake $g(\tau) = e^{\frac{1}{3}\int_{\tau_o}^{\tau}\frac{\alpha(\tau)}{\tau}d\tau}$ dobija se

je
$$\Omega(t) = \Omega_o\, g(\tau)$$

Iz jednačine 17 vodeći računa o 19 dobija se

$$\nu\frac{d\tau}{dt} = \Omega_o^2\, g^2(\tau)$$

a odavde:

$$t - t_o = \frac{\nu}{\Omega_o^2}\int_{\tau_o}^{\tau}\frac{d\tau}{g^2(\tau)}$$

Inverzna funkcija $\tau(t)$ može se odavde u svakom pojedinom slučaju izračunati. Za dalji rad, koji proizilazi od unapred zadate glavne funkcije $\alpha(\tau)$, dobili smo sledeće izraze

$$u(x,y,t) = U(x)\,\Omega_o\, g(\tau)\, F_\eta(x,\eta,\tau)$$

$$t - t_o = \frac{\nu}{\Omega_o^2}\int_{\tau_o}^{\tau}\frac{d\tau}{g^2(\tau)} \qquad y = \frac{\nu}{\Omega_o}\,\frac{\sqrt{3\tau}}{g(\tau)}\,\eta$$

U praktičnim problemima obično će biti zadata funkcija $\Omega(t)$. Tada treba odrediti glavnu funkciju $\alpha(\tau)$. Ona će biti data u funkciji od vremena t tj. $\alpha(t)$, jer je i $\Omega = \Omega(t)$, ali iz izraza $\tau = \frac{1}{\nu}\int^{t}\Omega^2 dt$ možemo inverzijom naći $t = t(\tau)$ pa prema tome i $\alpha = \alpha(\tau)$.

U daljem radu uzimaćemo da je funkcija $\Omega(t)$ data u obliku beskonačnog reda ili polinoma po vremenu t.

§ 3 RAZVIJANJE U RED FUNKCIJE $\Omega(t)$ I GLAVNE FUNKCIJE $\alpha(\tau)$ I NJIHOVA VEZA.

Pretpostavimo da je funkcija $\Omega(t)$ analitička funkcija u intervalu $0 \le t < \infty$ i da se može predstaviti u vidu stepenog reda

$$\Omega(t) = \sum_{K=0}^{\infty} \Omega_K t^K$$

Posmatraćemo sledeće slučajeve:

$$\text{A} \qquad \Omega(t) = \Omega_0 + \Omega_1 t + \Omega_2 t^2 + \cdots \qquad \Omega_0 \neq 0$$

$$\text{B} \qquad \Omega(t) = \Omega_1 t + \Omega_2 t^2 + \Omega_3 t^3 + \cdots \qquad \Omega_1 \neq 0 \qquad 22$$

Veliki broj praktičnih problema je obuhvaćen sa ova dva slučaja. Kasnije ćemo problem uopštiti i obuhvatiti daleko veći broj problema. Sada ćemo za ova dva slučaja dati oblik glavne funkcije

Slučaj A: $\Omega_0 \neq 0$

Ako smenimo izraz za $\Omega(t)$ u obliku reda u izraz za τ dobija se:

$$\tau = \frac{1}{\nu}\int_0^t \Omega^2 dt = \frac{1}{\nu}\int_0^t \Big(\sum_0^{\infty} \Omega_K t^K\Big)\Big(\sum_0^{\infty} \Omega_i t^i\Big) dt = \frac{1}{\nu}\int_0^t \Big(\sum_0^{\infty}\sum_0^{\infty} \Omega_K \Omega_i t^{K+i}\Big) dt$$

$$= \frac{1}{\nu}\Big(\sum_0^{\infty}\sum_0^{\infty} \frac{1}{K+i+1}\,\Omega_K \Omega_i t^{K+i+1}\Big) \qquad 23$$

ili:

$$\tau = \frac{1}{\nu}\left(\Omega_0^2 t + \Omega_0\Omega_1 t^2 + \cdots\cdots\right) \qquad 23'$$

Iz stava o inverziji sleduje

$$t(\tau) = C_1\tau + C_2\tau^2 + C_3\tau^3 + \cdots\cdots \qquad 24$$

Smenom u prvu od jednačina 10. dobija se

$$\alpha(\tau) = \frac{3\Omega' \int^t \Omega^2 dt}{\Omega^3} = 3\,\frac{(\Omega_1 + 2\Omega_2 t + \cdots)(\Omega_0^2 t + \Omega_0\Omega_1 t^2 + \cdots)}{(\Omega_0^3 + 3\Omega_0^2\Omega_1 t + \cdots)} =$$

$$= 3\left\{\frac{\Omega_1}{\Omega_0} t + \cdots\cdots\right\}$$

S obzirom na 24. sleduje

$$\alpha(\tau) = \alpha_1\tau + \alpha_2\tau^2 + \alpha_3\tau^3 + \cdots\cdots = \sum_{i=0}^{\infty} \alpha_i \tau^i \qquad 25$$

Izračunavanje koeficijenata α_i:

Iz izraza za glavnu funkciju dobija se:

$$3\Omega' \int_0^t \Omega^2 dt = \Omega^3 \sum_{K=0}^{\infty} \alpha_K \left\{\frac{1}{\nu}\int_0^t \Omega^2 dt\right\}^K$$

ili:

$$3\Big\{(\Omega_1 + 2\Omega_2 t + 3\Omega_3 t^2 + \cdots\cdots)\Big[\Omega_0^2 t + \Omega_0\Omega_1 t^2 + \tfrac{1}{3}(2\Omega_0\Omega_2 + \Omega_1^2)t^3 +$$

$$+ \tfrac{1}{2}(\Omega_0\Omega_3 + \Omega_1\Omega_2)t^4 + \cdots\cdots\Big]\Big\} = (\Omega_0 + \Omega_1 t + \Omega_2 t^2 + \cdots\cdots)^3 \Big\{$$

$$\Big\{\alpha_0 + \alpha_1\Big[\tfrac{1}{\nu}(\Omega_0^2 t + \Omega_0\Omega_1 t^2 + \cdots\cdots)\Big] + \alpha_2\Big[\tfrac{1}{\nu}(\Omega_0^2 t + \Omega_0\Omega_1 t^2 + \cdots)\Big]^2$$

$$+ \alpha_3\Big[\tfrac{1}{\nu}(\Omega_0^2 t + \cdots\cdots)\Big]^3 + \cdots\cdots\cdots\cdots\Big\}$$

Izjednačujući koeficijente uz iste stepene po t

ija se:

$$= 0$$

$$3a_1$$

$$9a_1^2 + 6a_2 \qquad (26)$$

$$27a_1^3 - 33a_1a_2 + 9a_3$$

$$-81a_4 + 142a_1^2a_2 - 22a_2^2 - 54a_1a_3 + 12a_4$$

$$_5 = 243a_1^5 + \frac{2415}{10}a_1^2a_3 + \frac{988}{5}a_1a_2^2 - \frac{1093}{2}a_1^3a_2 - 60a_2a_3 - 81a_1a_4 + 15a_5$$

e je

$$a_i = \frac{\Omega_i \nu^i}{\Omega_o^{2i+1}}$$

Za inverzni problem pri unapred zadanoj glavnoj funkciji $\alpha(\tau) = \sum_0^\infty \alpha_\kappa \tau^\kappa$ sa $\alpha_o = 0$ treba izračunati brzinu odnosno funkciju $\Omega(t) = \sum_{i=0}^\infty \Omega_i t^i$ sa $\Omega_o \neq 0$ .Odgovarajući koeficijenti dobijaju se jednoznačno iz definicione formule $a_i = \frac{\Omega_i \nu^i}{\Omega_o^{2i+1}}$

Koeficijente a_i možemo dobiti rešavanjem jednačina:

$$a_1 = \frac{1}{3}\alpha_1$$

$$a_2 = \frac{1}{6}(\alpha_1^2 + \alpha_2) \qquad (27)$$

$$a_3 = \frac{1}{54}(5\alpha_1^3 + 11\alpha_1\alpha_2 + 6\alpha_3)$$

$$a_4 = \frac{1}{648}(54\alpha_4 + 35\alpha_1^4 + 122\alpha_1^2\alpha_2 + 108\alpha_1\alpha_3 + 33\alpha_2^2)$$

$$a_5 = \frac{1}{2624400}(77598\alpha_1^5 + 401436\alpha_1^3\alpha_2 + 460080\alpha_1^2\alpha_3 + 276858\alpha_1\alpha_2^2 + 194400\alpha_2\alpha_3 + 393660\alpha_1\alpha_4 + 174960\alpha_5)$$

Sada će definicija pomoćnih koeficijenata a_i dati koeficijente Ω_i funkcije $\Omega(t)$

$$\Omega_o \neq 0 \qquad \Omega_i = \frac{a_i \Omega_o^{2i+1}}{\nu^i}$$

Uvedimo bezdimenzionalne veličine

$$\bar{t} = \frac{V}{L}t \qquad \bar{\Omega}_\kappa = \frac{\Omega_\kappa}{\Omega_o}\frac{L^\kappa}{V^\kappa} \qquad (28)$$

dalje je

$$\bar{\tau} = \frac{\tau}{Re} \qquad \bar{\alpha}_\kappa = \alpha_\kappa Re^\kappa \qquad (29)$$

Tada postoje sledeće relacije:

$$\Omega(t) = \sum_{\kappa=0}^\infty \Omega_\kappa t^\kappa \longleftrightarrow \bar{\Omega}(\bar{t}) = \sum_{\kappa=0}^\infty \bar{\Omega}_\kappa \bar{t}^\kappa$$

$$\alpha(\tau) = \sum_{\kappa=0}^\infty \alpha_\kappa \tau^\kappa \longleftrightarrow \bar{\alpha}(\bar{\tau}) = \sum_{\kappa=0}^\infty \bar{\alpha}_\kappa \bar{\tau}^\kappa \qquad (30)$$

što znači da sistem obrazaca ostaje formalno isti. Međutim jednostavnije je raditi sa veličinama sa crtom.

Slučaj B: $\Omega_0 = 0 \quad \Omega_1 \neq 0$

U ovom slučaju je

$$\tau = \frac{1}{\nu}\int_0^t \Omega^2 dt = \frac{1}{\nu}\int_0^t \Big(\sum_0^\infty \Omega_i t^i\Big)\Big(\sum_0^\infty \Omega_k t^k\Big)dt = \frac{1}{\nu}\sum_{k=1}^\infty \sum_{i=1}^\infty \frac{1}{k+i+1}\,\Omega_k \Omega_i\, t^{k+i+1}$$

ili

$$\tau = \frac{1}{\nu}\left(\frac{1}{3}\Omega_1^2 t^3 + \cdots\cdots\cdots\right) \qquad 31$$

Inverzijom se dobija

$$t = c_{1/3}\tau^{1/3} + c_{2/3}\tau^{2/3} + \cdots\cdots \qquad 31'$$

Izraz za glavnu funkciju u ovom slučaju svodi se na oblik

$$\alpha(\tau) = \frac{3\Omega' \int_0^t \Omega^2 dt}{\Omega^3} = 3\frac{(\Omega_1 + \cdots)(\frac{1}{3}\Omega_1^2 t^3 + \cdots)}{\Omega_1^3 t^3 + \cdots} = 1 + d_1 t + \cdots$$

odnosno ako se unese inverzna funkcija 31' dobiva se da je

$$\alpha(\tau) = \sum_{k=0}^\infty \alpha_{k/3}\tau^{k/3} \qquad \text{sa } \alpha_0 = 1 \qquad 32.$$

Izračunavanje koeficijenata: Ako se u izrazu

$$3\left[(\Omega_1 + 2\Omega_2 t + \cdots)(1/3\,\Omega_1^2 t^3 + \cdots)\right] = (\Omega_1^3 t^3 + 3\Omega_1^2\Omega_2 t^4 + \cdots)$$
$$\left\{\alpha_0 + \alpha_{1/3}\left[\frac{1}{\nu}\left(\frac{1}{3}\Omega_1^2 t^3 + \cdots\right)\right]^{1/3} + \cdots\cdots\right\}$$

izjednače koeficijenti uz iste stepene t dobiće se

$$\alpha_0 = 1$$

$$\alpha_{1/3} = \frac{1}{2}a_{1/3} \qquad 33.$$

$$\alpha_{2/3} = -\frac{23}{20}a_{1/3}^2 + \frac{6}{5}a_{2/3}$$

$$\alpha_1 = \frac{203}{120}a_{1/3}^3 - \frac{296}{30}a_{1/3}a_{2/3} - \frac{4}{3}a_1$$

$$\alpha_{4/3} = -\frac{919}{600}a_{1/3}^4 + \frac{5226}{150}a_{1/3}^2 a_{2/3} + \frac{9}{7}a_{1/3}a_1 - \frac{1088}{175}a_{2/3}^2 - \frac{22}{21}a_{4/3}$$

$$\alpha_{5/3} = -\frac{1839}{7200}a_{1/3}^5 - \frac{2931}{30}a_{1/3}^3 a_{2/3} + \frac{2004}{735}a_{1/3}^2 a_1 + \frac{9576}{175}a_{1/3}a_{2/3} + \frac{539}{588}a_{1/3}a_{4/3} - \frac{39}{20}a_{2/3}a_1 - \frac{3}{4}a_{5/3}$$

gde je:

$$a_{i/3} = \frac{\Omega_{i+1}(3\nu)^{i/3}}{\Omega_1^{2/3\,i+1}}$$

I ovde u cilju lakšeg rada uvedimo bezdimenzione veličine

$$\bar{t} = \frac{V}{L}t \qquad \bar{\tau} = \frac{\tau}{Re} \qquad \bar{\alpha}_k = \alpha_k Re^k \qquad 34 \text{ i } 35.$$

tada važe iste one jednačine gore napisane.

Slučaj B': specijalni slučaj slučaja B

Funkcija $\Omega(t)$ data je u obliku sledećeg reda

$$\Omega(t) = \sum_{k=0}^\infty \Omega_{3k+1}\, t^{3k+1} \qquad (k = 0, 1, \cdots) \qquad 36$$

vaj slučaj glavna funkcija ima sledeći oblik

$$(\tau) = \frac{3\Omega' \int^{t} \Omega^2 dt}{\Omega^3} = \frac{3(\Omega_1 + 4\Omega_4 t^3 + \cdots)(\frac{1}{3}\Omega_1^2 t^3 + \cdots\cdots)}{\Omega_1^3 t^3 + 3\Omega_1^2 t^6 + \cdots\cdots} =$$

$$= 1 + d_3 t^3 + d_6 t^6 + \cdots\cdots$$

$$\nu\tau = \frac{1}{3}\Omega_1^2 t^3 + \frac{1}{3}\Omega_1 \Omega_4 t^6 + \cdots\cdots$$

erzijom se dobija

$$t = \tau^{1/3}(C_{1/3} + C_1 \tau + \cdots\cdots) \qquad 36'$$

smenom gore u izraz za glavnu funkciju dobija se

$$\alpha(\tau) = 1 + \alpha_1 \tau + \alpha_2 \tau^2 + \cdots\cdots \qquad 37.$$

edjivanje koeficijenata glavne funkcije:

$$3\left\{(\Omega_1 + 4\Omega_4 t^3 + \cdots)(\tfrac{1}{3}\Omega_1^2 t^3 + \cdots)\right\} = (\Omega_1^3 t^3 + 3\Omega_1^2 \Omega_4 t^6 + \cdots)$$

$$\left\{\alpha_0 + \alpha_1\left[\tfrac{1}{\nu}(\tfrac{1}{3}\Omega_1^2 t^3 + \cdots)\right] + \alpha_2\left[\tfrac{1}{\nu}(\tfrac{1}{3}\Omega_1^2 t^3 + \cdots)\right]^2 + \cdots\right\}$$

zjednačujući koeficijente uz iste stepene t dobija se:

$$a_i = \frac{\nu^i \Omega_{3i+1}}{\Omega_1^{2i+1}}$$

$$\alpha_0 = 1$$

$$\alpha_1 = 6a_1 \qquad 38.$$

$$\alpha_2 = 42a_2 - 69a_1^2$$

$$\alpha_3 = \frac{567}{2}a_3 - \frac{1431}{2}a_1 a_2 + 630a_1^3$$

$$\alpha_4 = \frac{4212}{5}a_4 - \frac{25758}{5}a_1 a_3 - \frac{7353}{2}a_2^2 + 9072a_1^2 a_2 - 4815a_1^4$$

$$\alpha_5 = 3240a_5 - 16200a_2 a_3 - \frac{87966}{5}a_1 a_4 + \frac{318087}{5}a_1^2 a_3 + \frac{235386}{5}\cdot a_2^2 a_1 - 109377a_1^3 a_2 + 27702a_1^5$$

I ovde je korisno veličine strujne ravni učiniti bezdimenzionalnim. Po analogiji sa istim postupkom kod slučaja A i ovde ćemo imati

$$\bar{t} = \frac{V}{L}t \qquad \bar{\Omega}_{3K+1} = \frac{\Omega_{3K+1}}{\Omega_1}\frac{V^{3K+1}}{L^{3K+1}} \qquad 39.$$

$$\bar{\tau} = \frac{\tau}{Re} \qquad \bar{\alpha}_K = \alpha_K Re^K \qquad 40.$$

Tada će gornja jednačina 38. ostati formalno ista ako se istovremeno mesto α_K stavi $\bar{\alpha}_K$ a mesto Ω_{3K+1} se stavi $\bar{\Omega}_{3K+1}$.

Vrlo lako se može rešiti inverzan problem ako je unapred zadano $\alpha(\tau)$

§ 4 UOPŠTAVANJE POSTAVLJENOG PROBLEMA

Postavlja se sada pitanje kojim spoljnim strujanjima (po vremenu) pripada glavna funkcija data sledećim oblicima

$$\alpha(\tau) = \alpha_0 + \alpha_1 \tau + \alpha_2 \tau^2 + \cdots \qquad 41.$$

ili mnogo opštiji slučaj

II $$\alpha(\tau) = \alpha_0 + \alpha_{1/3} \tau^{1/3} + \alpha_{2/3} \tau^{2/3} + \cdots \qquad 42.$$

Da bi odgovorili na ovo pitanje vratićemo se na jednačinu 19′ u koju ćemo zameniti glavnu funkciju $\alpha(\tau)$ sa gornjim izrazima I i II te ćemo dobiti

I $$\Omega(t) = \Omega_0 \left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\alpha_0} (q_0 + q_1 \tau + q_2 \tau^2 + \cdots)$$

II $$\Omega(t) = \Omega_0 \left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\alpha_0} (q_0 + q_{1/3} \tau^{1/3} + q_{2/3} \tau^{2/3} + \cdots) \qquad 43.$$

Na isti način ćemo preko 20. prevesti τ u t

$$t - t_0 = \frac{\nu}{\Omega_0^2} \int_{\tau_0}^{\tau} \frac{d\tau}{g^2(\tau)}$$

Za naša dva slučaja sleduje:

I $$t(\tau) = \tau^{1-\frac{2}{3}\alpha_0} (r_0 + r_1 \tau + r_2 \tau^2 + \cdots)$$

II $$t(\tau) = \tau^{1-\frac{2}{3}\alpha_0} (r_0 + r_{1/3} \tau^{1/3} + r_{2/3} \tau^{2/3} + \cdots) \qquad 44.$$

Ako sada nadjene vrednosti 44. smenimo u 43. biće

I $$\Omega(t) = t^{\frac{\alpha_0}{3-2\alpha_0}} (S_0 + S_1 t^{2m+1} + S_2 t^{2(2m+1)} + \cdots) \qquad 45.$$

II $$\Omega(t) = t^{\frac{\alpha_0}{3-2\alpha_0}} (S_0 + S_{1/3} t^{\frac{1}{3}(2m+1)} + \cdots) \qquad 46.$$

gde je $m = \frac{\alpha_0}{3 - 2\alpha_0}$ odavde je $\alpha_0 = \frac{3m}{2m+1}$ $\quad m \neq -\frac{1}{2}$

Posmatrajmo raspored brzina po vremenu dat sa 45.

Za $\alpha_0 = 0$ $(m=0)$ dobija se naš slučaj A

Za $\alpha_0 = 1$ $(m=1)$ dobija se naš specijalan slučaj B′

Posmatrajmo sada raspored brzina po vremenu dat sa 46.

Za $\alpha_0 = 0$ $(m=0)$ dobija se opšti slučaj B″ tj. dobija se raspored brzina $\Omega(t) = \Omega_0 + \Omega_{1/3} t^{1/3} + \cdots$

Za $\alpha_0 = 1$ $(m=1)$ dobija se naš opšti slučaj B

Smatrajmo još neke slučajeve glavne funkcije.

Specijalan slučaj glavne funkcije $\alpha(\tau)=\alpha_0$:

Za ovaj slučaj glavne funkcije iz 19' dobijamo

$$\Omega(t)=\Omega_0 e^{\frac{1}{3}\int_{\tau_0}^{\tau}\frac{\alpha_0}{\tau}d\tau}=\Omega_0\left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\alpha_0} \qquad 47.$$

iz 20. dobijamo

$$t-t_0=\frac{\nu}{\Omega_0^2}\int_{\tau_0}^{\tau}\frac{d\tau}{\left(\frac{\tau}{\tau_0}\right)^{2/3\alpha_0}}=\frac{\nu}{\Omega_0^2}\int_{\tau_0}^{\tau}\left(\frac{\tau_0}{\tau}\right)^{\frac{2}{3}\alpha_0}d\tau=$$

$$=\begin{cases}\dfrac{3\nu\tau_0}{\Omega_0^2(3-2\alpha_0)}\left[\left(\dfrac{\tau_0}{\tau}\right)^{\frac{2}{3}\alpha_0-1}-1\right] & za\ \alpha_0\neq\frac{3}{2}\\[2ex] \dfrac{\nu\tau_0}{\Omega_0^2}\ln\left(\dfrac{\tau}{\tau_0}\right) & za\ \alpha_0=\frac{3}{2}\end{cases}$$

Pogodnim izborom raspoložive vrednosti t_0 $\left(t_0=\frac{3\nu\tau_0}{\Omega_0^2(3-2\alpha_0)}\right)$ može se ovaj rezultat napisati u sledećem obliku

$$t=\begin{cases}t_0\left(\dfrac{\tau_0}{\tau}\right)^{\frac{2}{3}\alpha_0-1} & za\ \alpha_0\neq\frac{3}{2}\\[2ex] t_0\left[1+\ln\left(\dfrac{\tau}{\tau_0}\right)\right] & za\ \alpha_0=\frac{3}{2}\end{cases} \qquad 48.$$

Za inverznu funkciju τ imamo

$$\frac{\tau}{\tau_0}=\begin{cases}\left(\dfrac{t}{t_0}\right)^{\frac{3}{3-2\alpha_0}} & za\ \alpha_0\neq\frac{3}{2}\\[2ex] e^{\frac{t-t_0}{t_0}} & za\ \alpha_0=\frac{3}{2}\end{cases} \qquad 49.$$

Ili smenom u 47 dobija se

$$\Omega(t)=\begin{cases}\Omega_0\left(\dfrac{t}{t_0}\right)^{m} & za\ \alpha_0\neq\frac{3}{2}\\[2ex] \Omega_0 e^{\frac{1}{3}\frac{t-t_0}{t_0}} & za\ \alpha_0=\frac{3}{2}\end{cases} \qquad 50.$$

gde je $\quad m=\dfrac{\alpha_0}{3-2\alpha_0}$

Ispitivanje funkcije $\beta(\tau)$:

Samim postupkom rešavanja jednačina nestacionarnog graničnog sloja mi smo učinili da je značaj glavne funkcije $\alpha(\tau)$ kao lokalnog parametra oblika primaran dok smo usputni parametar oblika učinili da u jednačinama figuriše kao paraar te je automatski njegov značaj postao sekundaran.Zbog ga će lokalni parametar oblika $\alpha(\tau)$ i dati raspored brzina o vremenu),koji se može rešavati ovom metodom.Da vidimo sada ako će se ponašati funkcija $\beta(\tau)$ tj.kakav će oblik imati za do sada proučene oblike funkcije $\Omega(t)$.

Slučaj A: $\Omega = \Omega_0 + \Omega_1 t + \Omega_2 t^2 + \cdots$

Za ovaj slučaj funkcija $\beta(\tau)$ imaće sledeći oblik

$$\beta(\tau) = \frac{\vartheta 3\tau}{\Omega}$$

Iz 23' i gornjeg izraza za funkciju $\Omega(t)$ dobija se

$$\beta(\tau) = \frac{3(\Omega_0^2 t + \Omega_0\Omega_1 t^2 + \cdots)}{\Omega_0 + \Omega_1 t + \cdots} = a_1 t + a_2 t^2 + \cdots$$

odnosno,ako se vodi računa o izrazu 24.dobiće se da je

$$\beta(\tau) = \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 + \cdots \qquad 51.$$

Nalaženje koeficijenata $\beta_1, \beta_2, \ldots\ldots$:

Ako u jednakosti

$$3\{\Omega_0^2 t + \Omega_0\Omega_1 t^2 + \cdots\} = (\Omega_0 + \Omega_1 t + \cdots)\{\beta_0 + \beta_1[\tfrac{1}{\vartheta}(\Omega_0^2 t + \cdots)] + \cdots\}$$

izjednačimo izraze uz iste stepene t dobija se:

$$\begin{aligned}
\beta_0 &= 0\\
\beta_1 &= 3b_1\\
\beta_2 &= -3b_2\\
\beta_3 &= -3b_3 + 6b_1a_1^2\\
\beta_4 &= -3b_4 + 14b_2a_2 - 14b_1a_1^3\\
\beta_5 &= -3b_5 + \tfrac{33}{2}b_4a_1 - \tfrac{105}{2}b_3a_1^2 + 7b_1a_2^2 + 35b_1a_1^4
\end{aligned} \qquad 52.$$

gde su:

$$b_i = \frac{\vartheta^i\Omega_{i-1}}{\Omega_0^{2i}} \qquad a_i = \frac{\Omega_i\vartheta^i}{\Omega_0^{2i+1}}$$

Slučaj B: $\Omega = \Omega_1 t + \Omega_2 t^2 + \Omega_3 t^3 + \cdots$

Iz jednakosti 31 i datog izraza za funkciju $\Omega(t)$ dobija se

$$\beta(\tau) = \frac{\frac{1}{3}\Omega_1^2 t^3 + \cdots}{\Omega_1 t + \cdots} = q_1 t^2 + q_2 t^3 + \cdots$$

odnosno zbog 31 gornji izraz dobija konačan oblik

$$\beta(\tau) = \beta_{2/3}\tau^{2/3} + \beta_1\tau + \beta_{4/3}\tau^{4/3} + \cdots \qquad 53.$$

[N]alaženje koeficijenata $\beta_{1/3}, \beta_{2/3}, \ldots\ldots\ldots$:

[A]ko u jednakosti

$$3\left\{\frac{1}{3}\Omega_1^2 t^3 + \cdots\cdots\right\} = (\Omega_1 t + \cdots\cdots)\left\{\beta_0 + \beta_{1/3}\left[\frac{1}{\nu}\left(\frac{1}{3}\Omega_1^2 t^3 + \cdots\right)\right]^{1/3} + \cdots\right\}$$

izjednačimo izraze uz iste stepene t dobiće se traženi koeficijenti

$$\begin{aligned}
&\beta_0 = 0\\
&\beta_{1/3} = 0\\
&\beta_{2/3} = b_{2/3}\\
&\beta_1 = -\frac{1}{2} b_1\\
&\beta_{4/3} = \frac{1}{2} b_{2/3} a_{1/3}^2 - \frac{3}{5} b_{4/3}\\
&\beta_{5/3} = -3 b_5 + \frac{33}{2} b_4 a_1 - \frac{105}{2} b_3 a_1^2 + 7 b_1 a_2^2 + 35 b_1 a_1^4
\end{aligned} \qquad 54.$$

gde su:

$$a_{i/3} = \frac{\Omega_{i+1}(3\nu)^{i/3}}{\Omega_1^{2i/3+1}} \qquad b_{i/3} = \frac{\Omega_{i-1}(3\nu)^{i/3}}{\Omega_1^{2i/3}}$$

Na isti način kao i ranije i ovde se mogu uvesti bezdimenzionalne, pošto je rad sa njima mnogo jednostavniji.

Specijalan slučaj $\alpha(\tau) = \alpha_0$:

U paragrafu 4 videli smo da gornjem obliku glavne funkcije odgovara sledeći oblik rasporeda brzina (po vremenu) spoljnjeg potencijalnog strujanja

$$\Omega(t) = \Omega_0 t^m$$

Ovaj oblik spoljne brzine dozvoljava slična rešenja, koja su iscrpno proučena u radu [4]. Videli smo da se glavna funkcija za ovaj slučaj svela na α_0, medjutim, funkcija $\beta(\tau)$ će se svesti na

$$\beta(\tau) = \beta_K \tau^K \qquad 55.$$

što ćemo i pokazati:

Funkcija $\beta(\tau)$ za gornji slučaj rasporeda brzina dobija sledeći oblik

$$\beta(\tau) = \frac{\nu 3\tau}{\Omega} = \frac{3\int_0^t \Omega^2 dt}{\Omega} = \frac{3\Omega_0^2 \int_0^t t^{2m} dt}{\Omega_0 t^m}$$

odnosno ako se izvrši integraljenje

$$\beta(\tau) = 3\frac{\Omega_0}{2m+1} t^{m+1} \qquad 56.$$

Sa druge strane iz izraza

$$\tau = \frac{\Omega_0^2}{\nu}\int_0^t t^{2m} dt = \frac{\Omega_0^2}{\nu}\frac{1}{2m+1} t^{2m+1} \qquad 57.$$

inverzijom se lako dobija

$$t = \left[\frac{(2m+1)\nu}{\Omega_o^2} \right]^{\frac{1}{2m+1}} \tau^{\frac{1}{2m+1}} \qquad 59$$

te je sada:

$$\beta(\tau) = 3\frac{\Omega_o}{2m+1} \left[\frac{(2m+1)}{\Omega_o^2}\nu\right]^{\frac{m+1}{2m+1}} \tau^{\frac{m+1}{2m+1}} = \beta_{\frac{m+1}{2m+1}} \tau^{\frac{m+1}{2m+1}} \qquad 59'$$

ili ako uvedemo oznaku $K = \frac{m+1}{2m+1}$

dobija se

$$\beta(\tau) = \beta_K \tau^K$$

§5 FIZIČKO OPRAVDANJE KOORDINATA

Ovde ćemo formalno pokazati opravdanost uvodjenja novih promenljivih τ i η u datom obliku i njihovu fizikalnost. Prethodno ćemo se kratko zadržati na samom putu kojim se išlo pri uvodjenju novih promenljivih za rešavanje stacionarnih graničnih slojeva i na Gertler-ovim promenljivim ξ i η [1]. Oduvek se javljala težnja da se potencijalne i strujne linije potencijalnog strujanja oko konture iskoriste kao koordinatne linije za proračun viskoznog strujanja oko iste konture. Pri opticanju neke konture na zidu se stvara granični sloj u kome je potencijalno strujanje zamenjeno viskoznim. Može se zamisliti da je brzina na samoj konturi zamenjena sa $U(x)$, jer je sloj u kome se stvara viskozno strujanje relativno tanak. U tom potencijalnom strujanju u blizini zida kojim je zamenio stvarno viskozno strujanje Gertler je aproksimativno zamenio potencijalnu funkciju sa $\int_0^x U(x)dx$, a funkciju strujanja sa $U(x)\cdot y$ i sa njima formirao svoje promenljive $\bar{\xi}$ i $\bar{\eta}$ a zatim i konačne promenljive ξ i η [1].

U slučaju stacionarnog strujanja potencijalna funkcija se menja samo sa promenljivom x, dok će u slučaju nestacionarnog strujanja ona biti funkcija i od x i od vremena t, a strujna funkcija će se menjati još i sa odstojanjem y od tela. U slučaju nestacionarnih strujanja proračun se provodi tako da na odredjenom mestu duž konture posmatramo promenu veličina graničnog sloja sa vremenom, zato ćemo mi mesto promenljivih t i y formirati nove promenljive $\bar{\tau}$ i $\bar{\eta}$, koje će fizikalno predstavljati funkcionalnu zavisnost promene potencijala brzine sa vremenom $\varphi_1(t)$ i strujne funkcije $\psi_1(t,y)$. Naime, iz Koši-Lagranžeove jednačine obzirom na pretpostavljeni oblik funkcije brzine spoljnog potencijalnog strujanja možemo formalno dobiti

$$\varphi_1(t) = \int_0^t \Omega^2(t)\,dt = \bar{\tau}$$

a za promenu strujne funkcije

$$\psi_1(y,t) = \Omega(t)\cdot y = \bar{\eta}$$

Na ovaj način smo strujnu ravan y,t preveli u strujnu ravan φ_1, ψ_1. Od ovih promenljivih $\varphi_1 = \bar{\tau}$ i $\psi_1 = \bar{\eta}$ možemo formirati i konačne promenljive τ i η odnosno konačno strujnu ravan τ, η.

Na ovaj način smo problem preveli u ravan η, τ a u isto vreme i učinili ga nezavisnim od promenljive x. A zatim će rešenje, za svako x duž konture, biti u izvesnoj razmeri preko funkcija U(x) i njenih izvoda prevedeno u stvarno rešenje tj. da bude funkcija i promenljive x

Struktura promenljive η bila je diktirana-ukazana oblikom specijalnih jedinih promenljivih "sličnih graničnih slojeva" [4] . Promenljiva sličnih graničnih slojeva predstavljala je bezdimenzionalno rastojanje od zida, a imala je oblik

$$\eta = K\frac{y}{\sqrt{\nu t}}$$

što znači da bi za slučaj stepenog porasta brzine spoljnog potencijalnog strujanja sa vremenom, naša promenljiva η trebala da predje u promenljivu sličnih rešenja η. Ovo se može lako pokazati da je zaista tako ako se vodi računa da je

$$\eta = \frac{\Omega\cdot y}{\nu\sqrt{3\tau}} = \frac{\Omega_0 t^m y}{\nu\sqrt{\frac{3}{\nu}\int_0^t \Omega_0^2 t^{2m} dt}} = \sqrt{\frac{2m+1}{3}}\,\frac{y}{\sqrt{\nu t}}$$

Medjutim da bi u opštim problemima dobili redukovano odstojanje od zida morala se ova promenljiva η uopštiti

Može se još pokazati da promenljive $\bar{\eta}$ i τ predstavljaju lokalne Re-brojeve. Opšte, Re-broj može se napisati kao

$$Re = \frac{V\cdot L}{\nu} = \frac{V^2 T}{\nu}$$

Ako izvršimo formalno uporedjenje ovog izraza sa našom novom promenljivom τ videćemo da imaju isti oblik tj.

$$d\tau = \frac{\Omega^2 dt}{\nu}$$

I za drugu promenljivu $\bar{\eta}$ može se pokazati da predstavlja lokalni Re-broj. Pošto je

$$\bar{\eta} = \frac{\Omega \cdot Y}{\nu}$$

odnosno formalno

$$\bar{\eta} = \frac{V \cdot Y}{\nu} = \frac{V^2 T}{\nu}$$

Na ovaj način smo pokazali fizikalnost i opravdanost uvodjenja novih promenljivih u obliku u kojem su uvedene.

§ 6 REŠAVANJE DOBIJENOG SISTEMA PARCIJALNIH JEDNAČINA POMOĆU REDOVA

Izborom reda za modulisanu strujnu funkciju $F(x,\eta,\tau)$ u obliku 8. mi smo od parcijalne jednačine 7. dobili sistem parcijalnih jednačina 9. Ako uvedemo sledeće oznake za pojedine članove specijalnog reda-aproksimacije

$$F_0 = F \qquad F_1 = \Phi \qquad F_2 = H \qquad F_3 = L$$

gde su F, Φ, H i L funkcije od η i τ, rekurzivni sistem parcijalnih jednačina 9. dobija sledeći oblik:

a. $F_{\eta\eta\eta} + \alpha(\tau)(1 - F_\eta - \eta F_{\eta\eta}) + \frac{3}{2}\eta F_{\eta\eta} - 3\tau F_{\eta\tau} = 0$ 60.

b. $\Phi_{\eta\eta\eta} - \alpha(\tau)(\Phi_\eta + \eta\Phi_{\eta\eta}) + \frac{3}{2}\eta\Phi_{\eta\eta} - 3\tau\Phi_{\eta\tau} + \beta(\tau)(1 - F_\eta^2 + FF_{\eta\eta}) = 0$

c. $H_{\eta\eta\eta} - \alpha(\tau)(H_\eta + \eta H_{\eta\eta}) + \frac{3}{2}\eta H_{\eta\eta} - 3\tau H_{\eta\tau} + \beta(\tau)(-2F_\eta\Phi_\eta + F\Phi_{\eta\eta} + \Phi F_{\eta\eta}) = 0$

d. $Ha_{\eta\eta\eta} - \alpha(\tau)(Ha_\eta + \eta Ha_{\eta\eta}) + \frac{3}{2}\eta Ha_{\eta\eta} - 3\tau Ha_{\eta\tau} + \beta(\tau)(\Phi F_{\eta\eta} - F_\eta\Phi_\eta) = 0$

e. $L_{\eta\eta\eta} - \alpha(\tau)(L_\eta + \eta L_{\eta\eta}) + \frac{3}{2}\eta L_{\eta\eta} - 3\tau L_{\eta\tau} + \beta(\tau)(-2F_\eta H_\eta - \Phi_\eta^2 + HF_{\eta\eta} +$
$+ \Phi\Phi_{\eta\eta} + FH_{\eta\eta}) = 0$

f. $La_{\eta\eta\eta} - \alpha(\tau)(La_\eta + \eta La_{\eta\eta}) + \frac{3}{2}\eta La_{\eta\eta} - 3\tau La_{\eta\tau} + \beta(\tau)(-3F_\eta Ha_\eta +$
$+ 2Ha F_{\eta\eta} + 2HF_{\eta\eta} - 2F_\eta H_\eta + FHa_{\eta\eta} + \Phi\Phi_{\eta\eta} - \Phi_\eta^2) = 0$

g. $Lb_{\eta\eta\eta} - \alpha(\tau)(Lb_\eta + \eta Lb_{\eta\eta}) + \frac{3}{2}\eta Lb_{\eta\eta} - 3\tau Lb_{\eta\tau} + \beta(\tau)(Ha F_{\eta\eta} - F_\eta Ha_\eta) = 0$

sa graničnim uslovima

a. $F(0,\tau) = F_\eta(0,\tau) = 0$
$F_\eta(\eta,\tau) \to 1 \quad$ za $\eta \to \infty$ 61.

b. $\Phi(0,\tau) = \Phi_\eta(0,\tau) = 0$
$\Phi_\eta(\eta,\tau) \to 0 \quad$ za $\eta \to \infty$

granični uslovi za ostale aproksimacije biće isti kao za b.

Do sada smo iscrpno proučili vezu izmedju spoljne brzine po vremenu $\Omega(t)$ i glavne funkcije $\alpha(\tau)$ i funkcije $\beta(\tau)$ te smo potpuno završili pripreme za rešavanje problema graničnog sloja.

Proučićemo najpre široku klasu I svih glavnih funkcija koje se u posmatranom intervalu $0 \leq \tau < \infty$ mogu razviti u stepeni red

$$\alpha(\tau) = \alpha_0 + \alpha_1 \tau + \alpha_2 \tau^2 + \cdots\cdots$$

sa proizvoljnim (racionalnim) α_0. Videli smo u § 4 da njima u jednoznačnoj korespodenciji (u oblasti Rejnoldsove sličnosti) pripadaju spoljne brzine oblika

$$\Omega(t) = t^m (S_0 + S_1 t^{2m+1} + \cdots\cdots)$$

U primeni nas interesuju vrednosti parametra $0 \leq m < \infty$ $(0 \leq \alpha_0 < \frac{3}{2})$. Medjutim od većeg praktičnog interesa su specijalni slučajevi $\alpha_0 = 0$ i $\alpha_0 = 1$. Slučaj $\alpha_0 = 0$ mi smo proučili u § 4. i označili smo ga sa slučaj A.

Slučaj A: $\alpha_0 = 0$ $(m = 0)$

$$\Omega(t) = \Omega_0 + \Omega_1 t + \Omega_2 t^2 + \cdots\cdots$$

sa $\Omega_0 \neq 0$ (inače proizvoljno), Ω_i - proizvoljni (i=1, 2, ...). Izračunavanje koeficijenata α_i iz koeficijenata Ω_i dato je jednačinama 27. a koeficijenata β_K sa jednačinama 52.

Ako uvedemo $\alpha(\tau)$- stepeni red I i $\beta(\tau)$ - u obliku 51. u sistem parcijalnih jednačina 60. onda će tražena rešenja $F(\eta,\tau)$, $\Phi(\eta,\tau)$, ... biti pristupačna za računanje kao stepeni redovi po τ sa koeficijentima, koji su funkcije reduciranog rastojanja od zida η

$$F(\eta,\tau) = \sum_{K=0}^{\infty} F_K(\eta)\tau^K \qquad \Phi(\eta,\tau) = \sum_{K=0}^{\infty} \Phi_K(\eta)\tau^K$$

$$H(\eta,\tau) = \sum_{K=0}^{\infty} H_K(\eta)\tau^K \qquad L(\eta,\tau) = \sum_{K=0}^{\infty} L_K(\eta)\tau^K$$

Ako ovo zamenimo u 60. svaka od parcijalne jednačina 60. će se rastaviti na rekurzivni sistem običnih diferencijalnih jednačina za odredjivanje koeficijenata-funkcija svakog od gornjih stepenih redova.

a. $F_o''' + \tau F_1''' + \cdots + (\alpha_o + \tau\alpha_1 + \cdots)\left[1 - (F_o' + \tau F_1' + \cdots) - \eta(F_o'' + \tau F_1'' + \cdots)\right]$
$+ \frac{3}{2}\eta(F_o'' + \tau F_1'' + \cdots) - 3\tau(F_1' + 2\tau F_2' + \cdots) = 0$

Izjednačujuci izraze uz iste stepene τ dobija se rekurzivni sistem običnih diferencijalnih jednačina

$$F_o''' + \alpha_o(1 - F_o' - \eta F_o'') + \frac{3}{2}\eta F_o'' = 0$$
$$F_1''' - \alpha_o(F_1' + \eta F_1'') + \alpha_1(1 - F_o' - \eta F_o'') + \frac{3}{2}\eta F_1'' - 3F_1' = 0$$
$$F_2''' - \alpha_o(F_2' + \eta F_2'') - \alpha_1(F_1' + \eta F_1'') + \alpha_2(1 - F_o' - \eta F_o'') + \frac{3}{2}\eta F_2'' - 6F_2' = 0$$

ili da bi dobili rekurentnu formulu u 60a. ćemo smeniti $F = \sum\limits_{o}^{\infty} F_K \tau^K$

$$\sum_{K=0}^{\infty} F_K''\tau^K + \sum_{i=0}^{\infty}\alpha_i\tau^i\left[1 - \sum_{K=0}^{\infty} F_K'\tau^K - \eta\sum_{K=0}^{\infty} F_K''\tau^K\right] + \frac{3}{2}\eta\sum_{K=0}^{\infty} F_K''\tau^K - 3\sum K F_K'\tau^K = 0$$

Ako svedemo sve $\sum$ da budu pod njima stepeni τ^K napr.:

$$\sum_{i=0}^{\infty}\alpha_i\tau^i \cdot \sum_{K=0}^{\infty} F_K'\tau^K = \sum_{K=0}^{\infty}\sum_{i=0}^{\infty}\alpha_i F_K'\tau^{K+i} =$$

mesto K+i stavimo n tj. K+i=n dobiće se da je

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\alpha_i F_{n-i}'\tau^n$$

Sada možemo mesto „n" staviti K (izmena slova) i dobiti konačno rekurentnu formulu za sistem a:

$$F_K''' + \alpha_K(1 - F_o' - \eta F_o'') + \frac{3}{2}\eta F_K'' - 3KF_K' = \sum_{i=0}^{\infty}\alpha_i(F_{K-i}' + \eta F_{K-i}'') \qquad K = 1, 2, \ldots$$

b. $\Phi_o''' + \tau\Phi_1''' + \cdots - (\alpha_o + \alpha_1\tau + \cdots)\left[(\Phi_o' + \tau\Phi_1' + \cdots) + \eta(\Phi_o'' + \tau\Phi_1'' + \cdots)\right] +$
$+ \frac{3}{2}\eta(\Phi_o'' + \cdots) - 3\tau(\Phi_1' + 2\Phi_2'\tau + \cdots) + (\beta_o + \beta_1\tau + \cdots)\left[1 - (F_o' + \cdots) + (F_o + \cdots)(F_o'' + \cdots)\right] = 0$

Izjednačujuci izraze uz iste stepene τ dobija se sistem običnih diferencijalnih jednačina

$$\Phi_o''' - \alpha_o(\Phi_o' + \eta\Phi_o'') + \frac{3}{2}\eta\Phi_o'' + \beta_o(1 - F_o'^2 + F_o F_o'') = 0$$
$$\Phi_1''' - \alpha_o(\Phi_1' + \eta\Phi_1'') - \alpha_1(\Phi_o' + \eta\Phi_o'') + \frac{3}{2}\eta\Phi_1'' - 3\Phi_1' +$$
$$+ \beta_o(-2F_o'F_1' + F_o F_1'' + F_o'' F_1) + \beta_1(1 - F_o'^2 + F_o F_o'') = 0$$

Da bi dobili rekurentnu formulu unećemo u 60b. izraze u obliku redova

$$\Phi = \sum_{K=0}^{\infty}\Phi_K\tau^K \qquad F = \sum_{K=0}^{\infty} F_K\tau^K \qquad \alpha = \sum_{i=0}^{\infty}\alpha_i\tau^i \qquad \beta = \sum_{i=0}^{\infty}\beta_i\tau^i$$

Vodeći računa da je $F\eta^2 = \sum_{K=0}^{\infty}\sum_{j=0}^{\infty} F_K' F_j' \tau^{K+j} = \sum_{j=0}^{\infty}\sum_{K=0}^{\infty} F_j' F_{K-j}' \tau^K$
i da je $\beta F\eta^2 = \sum \beta_i \tau^i \sum\sum F_j' F_{K-j}' \tau^K = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{K=0}^{\infty} \beta_i F_j' F_{K-j-i}' \tau^K$ dobiće se
sledeća rekurentna formula

$$\Phi_K''' - \sum_{i=0}^{\infty}\alpha_i(\Phi_{K-i}' + \eta\Phi_{K-i}'') + \frac{3}{2}\eta\Phi_K'' - 3\cdot K\Phi_K' + \beta_K(1-F_0'^2+F_0F_0'') + \sum_{i=0}^{\infty} \beta_i\Big[-\sum_{j=0}^{K-i}F_j'F_{K-i-j}' + \sum_{j=0}^{K-i}F_j F_{K-i-j}''\Big] = 0 \qquad K=1,2,\dots$$

c. $$H_0''' + \tau H_1''' + \dots - (\alpha_0+\alpha_1\tau+\dots)\big[(H_0'+\tau H_1'+\dots)+\eta(H_0''+\tau H_1''+\dots)\big] + \frac{3}{2}\eta(H_0''+\tau H_1''+\dots) - 3\tau(H_1'+2H_2'\tau+\dots) + (\beta_0+\beta_1\tau+\dots)\big[-2(F_0'+\tau F_1'+\dots)(\Phi_0'+\tau\Phi_1''+\dots) + (\Phi_0+\dots)(F_0''+\tau F_1''+\dots) + (F_0+\dots)(\Phi_0''+\dots)\big] = 0$$

Izjednačujući izraze uz iste stepene τ dobija se rekurzivni sistem običnih diferencijalnih jednačina

$$H_0''' - \alpha_0(H_0'+\eta H_0'') + \frac{3}{2}\eta H_0'' + \beta_0(-2F_0'\Phi_0' + \Phi_0F_0'' + F_0\Phi_0'') = 0$$

$$H_1''' - \alpha_0(H_1'+\eta H_1'') + \frac{3}{2}\eta H_1'' - 3H_1' + \beta_0(-2F_0'\Phi_1' - 2F_1'\Phi_0' + \Phi_0F_1'' + \Phi_1F_0'' + F_0\Phi_1'' + F_1\Phi_0'') + \beta_1(-2F_0\Phi_0' + \Phi_0F_0'' + F_0\Phi_0'') = 0$$

Za dobijanje rekurentne formule u 60c. unećemo izraze u obliku redova te ćemo dobiti

$$H_K''' - \sum_{i=0}^{K}\alpha_i(H_{K-i}' + \eta H_{K-i}'') + \frac{3}{2}\eta H_K'' - 3KH_K' + \sum_{i=0}^{\infty}\beta_i\Big[-2\sum_{j=0}^{K-i}F_{K-i-j}\Phi_j' + \sum_{j=0}^{K-i}F_{K-i-j}\Phi_j'' + \sum_{j=0}^{K-i}F_{K-i-j}''\Phi_j\Big] = 0 \qquad K=1,2,\dots$$

d. $$H_{a0}''' + \tau H_{a0}''' + \dots - (\alpha_0+\alpha_1\tau+\dots)\big[(H_{a0}'+\tau H_{a1}'+\dots)+\eta(H_{a0}''+\dots)\big] + \frac{3}{2}\eta(H_{a0}''+\tau H_{a1}''+\dots) - 3\tau(H_{a1}'+2\tau H_{a2}'+\dots) + (\beta_0+\tau\beta_1+\dots)\big[(\Phi_0+\dots)\cdot(F_0''+\tau F_1''+\dots) - (F_0'+\tau F_1'+\dots)(\Phi_0'+\tau\Phi_1'+\dots)\big]$$

Izjednačujući izraze uz iste stepene τ dobićemo rekurzivni sistem običnih diferencijalnih jednačina

$$H_{a0}''' - \alpha_0(H_{a0}'+\eta H_{a0}'') + \frac{3}{2}\eta H_{a0}'' + \beta_0(F_0''\Phi_0 - F_0'\Phi_0') = 0$$

$$H_{a1}''' - \alpha_0 (H_{a1}' + \eta H_{a1}'') - \alpha_1 (H_{a0}' + \eta H_{a0}'') + \frac{3}{2}\eta H_{a1}'' - 3 H_{a1}' +$$
$$+ \beta_0 (\Phi_0 F_1'' + \Phi_1 F_0'' - F_0'\Phi_1' - F_1'\Phi_0') + \beta_1 (F_0''\Phi_0 - F_0'\Phi_0') = 0$$

. . . .

ili u obliku rekurentne formule

$$H_{aK}''' - \sum_{i=0}^{K} \alpha_i (H_{aK-i}' + \eta H_{aK-i}'') + \frac{3}{2}\eta H_{aK}'' - 3K H_{aK}' + \sum_{i=0}^{\infty} \Big[\sum_{j=0}^{K-i} F_j'' \Phi_{K-i-j} - \sum_{j=0}^{K-i} F_j' \Phi_{K-i-j}' \Big] = 0 \qquad K = 1, 2, \ldots$$

e. $$L_0''' + \tau L_1''' + \cdots - (\alpha_0 + \alpha_1 \tau + \cdots)\big[(L_0' + \tau L_1' + \cdots) + \eta (L_0'' + \tau L_1'' + \cdots)\big] + \frac{3}{2}\eta (L_0'' + \tau L_1'' + \cdots) - 3\tau (L_1' + 2\tau L_2' + \cdots) +$$
$$+ (\beta_0 + \beta_1 \tau + \cdots)\big[-2(F_0' + \cdots)(H_0' + \cdots) - (\Phi_0' + \tau \Phi_1' + \cdots)^2 + \cdots\big] = 0$$

Izjednačujući izraze uz iste stepene τ dobićemo rekurzivni sistem običnih diferencijalnih jednačina:

$$L_0''' - \alpha_0 (L_0' + \eta L_0'') + \frac{3}{2}\eta L_0'' + \beta_0 (-2 F_0' H_0' - \Phi_0'^2 + H_0 F_0'' + \Phi_0 \Phi_0'' + F_0 H_0'') = 0$$

. . . .

ili u obliku rekurentne formule:

$$L_K''' - \sum_{i=0}^{K} \alpha_i (L_{K-i}' + \eta L_{K-i}'') + \frac{3}{2}\eta L_K'' - 3K L_K' + \sum_{i=0}^{K} \beta_i \Big[-2 \sum_{j=0}^{K-i} F_j' H_{K-i-j}' - \sum_{j=0}^{K-i} \Phi_j' \Phi_{K-i-j}' + \sum_{j=0}^{\infty} H_j F_{K-i-j}'' + \sum_{j=0}^{K-i} \Phi_j \Phi_{K-i-j}'' + \sum_{j=0}^{K-i} F_j H_{K-i-j}'' \Big] = 0$$

k=1,2,3, ...

f. $$L_{a0}''' + \tau L_{a1}''' + \cdots - (\alpha_0 + \alpha_1 \tau + \cdots)\big[(L_{a0}' + \cdots) + \eta (L_{a0}'' + \cdots)\big] + \frac{3}{2}\eta \cdot (L_{a0}'' + \cdots) - 3\tau (L_{a1}' + \cdots) + (\beta_0 + \beta_1 \tau + \cdots)\big[-3 (F_0' + \cdots)(H_{a0}' + \cdots) + \cdots\big] = 0$$

ili u obliku rekurentne formule:

$$L_{aK}''' - \sum_{i=0}^{K} \alpha_i (L_{aK-i}' + \eta L_{aK-i}'') + \frac{3}{2}\eta L_{aK}'' - 3K L_{aK}' + \sum_{i=0}^{K} \beta_i \Big[-3 \sum_{j=0}^{K-i} F_j' H_{aK-i-j}' + 2 \sum_{j=0}^{K-i} H_{aj} F_{K-i-j}'' + 2 \sum_{j=0}^{K-i} H_j F_{K-i-j}'' - 2 \sum_{j=0}^{K-i} F_j' H_{K-i-j}' + \sum_{j=0}^{K-i} F_j H_{aK-i-j}'' + \sum_{j=0}^{K-i} \Phi_j \Phi_{K-i-j}'' - \sum_{j=0}^{K-i} \Phi_j' \Phi_{K-i-j}' \Big] = 0$$

$K = 1, 2, \ldots$

g. $L_{b0}''' - \alpha_0(L_{b0}' + \eta L_{b0}'') + \frac{3}{2}\eta L_{b0}'' + \beta_0(H_{a0}F_0'' - F_0'H_{a0}') = 0$

$L_{b1}''' - \alpha_0(L_{b1}' + \eta L_{b1}'') - \alpha_1(L_{b0}' + \eta L_{b0}'') + \frac{3}{2}\eta L_{b1}'' - 3L_{b1}' + \beta_0(H_{a0}F_1'' + H_{a1}F_0'' - F_0'H_{a1}' - F_1'H_{a0}') + \beta_1(H_{a0}F_0'' - F_0'H_{a0}') = 0$

. . . .

ili u obliku rekurentne formule:

$$L_{bk}''' - \sum_{i=0}^{k}\alpha_i(L_{bk-i}' + L_{bk-i}''\eta) + \frac{3}{2}\eta L_{bk}'' - 3k\cdot L_{bk}' + \sum_{i=0}^{k}\beta_i\left[\sum_{j=0}^{k-i}H_{aj}F_{k-i-j}'' - \sum_{j=0}^{k-i}F_j'H_{ak-i-j}'\right] = 0 \qquad k = 1,2,\ldots.$$

Granični uslovi:

$F_0'(0)=0$ $\quad$ $F_1'(0)=0\ldots\ldots\ldots F_n'(0)=0$

$F_0'(\infty)=1$ $\quad$ $F_1'(\infty)=0\ldots\ldots\ldots F_n'(\infty)=0$

(isto za $F_k(0)=0$, $k=0,1,\ldots.$)

$\phi_0'(0)=0$ $\quad$ $\phi_k'(0)=0$ $\quad$ $k=1,2,\ldots$ (isto za $\phi_k(0)=0$)

$\phi_0'(\infty)=0$ $\quad$ $\phi_k'(\infty)=0$ $\quad$ " "

$H_0'(0)=0$ $\quad$ $H_k'(0)=0$ $\quad$ $k=1,2,\ldots.$ (isto za $H_k(0)=0$)

$H_0'(\infty)=0$ $\quad$ $H_k'(\ )=0$ $\quad$ "

. . . .

granični uslovi za sve dalje aproksimacije su isti.

Pogledajmo sada izraz za komponentu brzine u pravcu konture tj u pravcu z-ose

$$u(x,y,t) = U(x)\Omega(t)\left[F_0'(\eta) + \tau F_1'(\eta) + \cdots + U'(\phi_0'(\eta) + \tau\phi_1'(\eta) + \cdots) + U'^2(H_0'(\eta) + \tau H_1'(\eta) + \cdots)\right]$$

videćemo da smo već sa prvim članom reda zadovoljili tačno spoljne granične uslove.Svi ostali članovi isčezavaju na spoljnom rubu graničnog sloja (tj za$\eta\to\infty$i svi izvodi prirodno takodje isčezavaju na samom zidu),oni samo donose sa porastom x i t (odnosno τ) korekture nultog približavanja uunutrašnjosti graničnog sloja. Sa ovim smo postigli cilj, uvodjenjem reda 8. § 1 i stepenih redova 62. mi smo postigli da već prvi član zadovoljava tačno spoljne granične uslove, a ostali članovi da vrše korekture u unutrašnjosti sloja pa će i nada za dobru konvergenciju specijalnog reda i redova 62. biti opravdana.

U svim linearnim diferencijalnim jednačinama za odredjivanje $F_k(\eta)$, $\Phi_k(\eta)$, . . . pojavljuju se parametri $\alpha_0, \alpha_1, \ldots$ zatim $\beta_0, \beta_1, \ldots$ Za slučaj sličnih rešenja imali smo da se glavna funkcija $\alpha(\tau)$ svodi na α_0, a sistem običnih diferencijalnih jednačina za prvu aproksimaciju svodi se samo na jednu diferencijalnu jednačinu za $F_0(\eta)$. U toj diferencijalnoj jednačini se sem promenljive η javlja i α_0 te je F_0 funkcija $F_0(\eta, \alpha_0)$ Kako je $F_0\ (\eta, \alpha_0)$ merodavno za izračunavanje svih $F_k(\eta)$, $\Phi_k(\eta)$ . . . k=1,2, . . . onda se ne može davanjem pogodnog oblika za $F_k(\eta)$, $\Phi_k(\eta)$ . . . . doći do funkcija koje su nezavisne od α_0 To se ne može očekivati ni sa fizičke tačke gledišta, jer su α_0 i $F_0\ (\eta, \alpha_0)$, koji regulišu početni profil odlučujući za dalje ponašanje pri strujanju. Medjutim uspeva se $F_k(\eta)$, $\Phi_k(\eta)$, . . . izraziti preko linearnih kombinacija funkcija koje ne zavise od koeficijenata $\alpha_1, \alpha_2, \ldots ; \beta_0, \beta_1, \ldots$. Mogu se tada te funkcije za svaku odredjenu vrednost α_0 tabulirati jedared za svagda.

Dovodjenje $F_k(\eta)$, . . . za usvojeno α_0 na univerzalne funkcije sprovodi se putem sledećih izraza:

$$F_0 = F_0$$
$$F_1 = \alpha_1 f_1$$
$$F_2 = \alpha_1^2 f_{11} + \alpha_2 f_2$$
$$F_3 = \alpha_1^3 f_{111} + \alpha_1 \alpha_2 f_{12} + \alpha_3 f_3$$
$$F_4 = \alpha_1^4 f_{1111} + \alpha_1^2 \alpha_2 f_{112} + \alpha_1 \alpha_3 f_{13} + \alpha_2^2 f_{22} + \alpha_4 f_4$$
$$F_5 = \alpha_1^5 f_{11111} + \alpha_1^3 \alpha_2 f_{1112} + \alpha_1^2 \alpha_3 f_{113} + \alpha_1 \alpha_2^2 f_{122} + \alpha_1 \alpha_4 f_{14} + \\ + \alpha_2 \alpha_3 f_{23} + \alpha_5 f_5$$

$$\Phi_0 = \beta_0 \varphi_0$$
$$\Phi_1 = \alpha_1 \beta_0 \varphi_{10} + \beta_1 \varphi_1$$
$$\Phi_2 = \alpha_1^2 \beta_0 \varphi_{110} + \alpha_1 \beta_1 \varphi_{11} + \alpha_2 \beta_0 \varphi_{20} + \beta_2 \varphi_2$$

. . . .
. . . .

$H_o = \beta_o^2 h_{oo}$

$H_1 = \alpha_1 \beta_o^2 h_{100} + \beta_o \beta_1 h_{o1}$

. . . .

$H_{ao} = \beta_o^2 h_{aoo}$

$H_{a1} = \alpha_1 \beta_o^2 h_{a100} + \beta_o \beta_1 h_{ao1}$

. . . .

$L_o = \beta_o^3 l_{ooo}$

. . . .

Ranije smo pokazali u § 4 pri ispitivanju funkcije $\beta(\tau)$ da je uvek $\beta_o = 0$. Sada izvršimo korekcije u svim jednačinama i ovim linearnim kombinacijama.

Pošto se u prvoj aproksimaciji ne pojavljuje funkcija $\beta(\tau)$ to se rekurzivni sistem običnih diferencijalnih jednačina neće izmeniti.

Medjutim u drugoj aproksimaciji ako stavimo $\beta_o = 0$ dobija se sledeći rekurzivni sistem:

$$\Phi_1''' - \alpha_o(\Phi_1' + \eta\Phi_1'') + \frac{3}{2}\eta\Phi_1'' - 3\Phi_1' = -\beta_1(1 - F_o'^2 + F_o F_o'')$$

$$\Phi_2''' - \alpha_o(\Phi_2' + \eta\Phi_2'') - \alpha_1(\Phi_1' + \eta\Phi_1'') + \frac{3}{2}\eta\Phi_2'' - 6\Phi_2' = -$$

$$-\beta_1(-2F_o'F_1' + F_o F_1'' + F_1 F_o'') - \beta_2(1 - F_o'^2 + F_o F_o'')$$

$$\Phi_3''' - \alpha_o(\Phi_3' + \eta\Phi_3'') - \alpha_1(\Phi_2' + \eta\Phi_2'') - \alpha_2(\Phi_1' + \eta\Phi_1'') + \frac{3}{2}\eta\Phi_3''$$

$$-9\Phi_3' = -\beta_1(-2F_o'F_2' - F_1'^2 + F_o F_2'' + F_1 F_1'' + F_2 F_o'') - \beta_2(-2F_o'F_1' +$$

$$+ F_o F_1'' + F_1 F_o'') - \beta_3(1 - F_o'^2 + F_o F_o'')$$

. . . .

Rekurentni obrazac svodi se na sledeći oblik:

$$\Phi_K''' - \sum_{i=0}^{K-1} \alpha_i (\Phi_{K-i}' + \eta\Phi_{K-i}'') + \frac{3}{2}\eta\Phi_K'' - 3K\Phi_K' +$$

$$+ \beta_K(1 - F_o'^2 + F_o F_o'') + \sum_{i=1}^{K-1} \beta_i \Big[- \sum_{j=0}^{K-i} F_j' F_{K-i-j}' +$$

$$+ \sum_{j=0}^{K-i} F_j F_{K-i-j}'' \Big] = 0 \qquad K = 2, 3, \ldots$$

c. Konačan oblik ovoga sistema dobićemo stavljajući $\beta_0=0$ i $\phi_0=0$ te su $H_0=0$ i $H_1=0$.

$$H_2'''-\alpha_0(H_2'+\eta H_2'')+\frac{3}{2}\eta H_2''-6H_2'+\beta_1(-2F_0'\phi_1'+\phi_1F_0''+F_0\phi_1'')=0$$

. . . .

Rekurentna formula svodi se na

$$H_K'''-\sum_{i=0}^{K-2}\alpha_i\left(H_{K-i}'+\eta H_{K-i}''\right)+\frac{3}{2}\eta H_K''-3KH_K'+\sum_{i=1}^{K-1}\beta_i\Big[-2\sum_{j=1}^{K-i}F_{K-i-j}'\phi_j'+$$
$$+\sum_{j=1}^{K-i}F_{K-i-j}\phi_j''+\sum_{j=1}^{K-i}F_{K-i-j}''\phi_j\Big]=0 \qquad K=2,3,\dots\dots$$

d. Za ovaj sistem se dobija

$$H_{a0}'''-\alpha_0(H_{a2}'+\eta H_{a2}'')+\frac{3}{2}\eta H_{a2}''-6H_{a2}'+\beta_1(\phi_1F_0''-F_0'\phi_1')=0$$

. . . .

i rekurentna formula

$$H_{aK}'''-\sum_{i=0}^{K-2}\alpha_i\left(H_{aK-i}'+\eta H_{aK-i}''\right)+\frac{3}{2}\eta H_{aK}''-3KH_{aK}'+\sum_{i=1}^{K-1}\beta_i\Big[\sum_{j=1}^{K-i}F_j''\phi_{K-i-j}-$$
$$-\sum_{j=1}^{K-i}F_j'\phi_{K-i-j}'\Big]=0 \qquad K=2,3,\dots\dots$$

e. Konačan oblik za ovaj sistem dobićemo stavljajući $\beta_0=0,\phi_0=0$ $H_0=0$ i $H_1=0$ te je i $L_0=0$ $L_{1,2}=0$ a ostaju

$$L_3'''-\alpha_0(L_3'+\eta L_3'')+\frac{3}{2}\eta L_3''-9L_3'+\beta_1(-2F_0'H_2'-\phi_1'^2+H_2F_0''+\phi_1\phi_1''+F_0H_2'')=0$$

. . . .

f. Za ovaj sistem dobija se

$$L_{a3}'''-\alpha_0(L_{a3}'+\eta L_{a3}'')+\frac{3}{2}\eta L_{a3}''-9L_{a3}'+\beta_1(-3F_0'H_{a2}'+2H_{a2}F_0''+2H_2F_0''-$$
$$-2F_0'H_2'+F_0H_{a2}''+\phi_1\phi_1''-\phi_1'^2)=0$$

. . . .

. . . .

g. Ovaj sistem dobija oblik

$$L_{b3}'''-\alpha_0(L_{b3}'+\eta L_{b3}'')+\frac{3}{2}\eta L_{b3}''-9L_{b3}'+\beta_1(H_{a2}F_0''-F_0'H_{a2}')=0$$

$$L_{b3}''' - \alpha_0 (L_{b3}' + \eta L_{b3}'') + \frac{3}{2}\eta L_{b3}'' - 9L_{b3}' + \beta_1 (H_{a2}F_0'' - F_0' H_{a2}') = 0$$

. . . .

i još rekurentne formule za ova tri zadnja sistema

e $$L_K''' - \sum_{i=0}^{K-3} \alpha_i (L_{K-i}' + \eta L_{K-i}'') + \frac{3}{2}\eta L_K'' - 3KL_K' + \sum_{i=1}^{K-2}\beta_i \Big[-2\sum_{j=2}^{K-i} H_j' F_{K-i-j}' - \sum_{j=1}^{K-i-1} \Phi_j' \Phi_{K-i-j}' + \sum_{j=2}^{K-i} H_j F_{K-i-j}'' + \sum_{j=1}^{K-i-1} \Phi_j \Phi_{K-i-j}'' + \sum_{j=2}^{K-i} H_j'' F_{K-i-j} \Big] = 0$$

f. $$L_{aK}''' - \sum \alpha_i (L_{aK-i}' + \eta L_{aK-i}'') + \frac{3}{2}\eta L_{aK}'' - 3KL_{aK}' + \sum_{i=1}^{K-2}\beta_i \Big[-3\sum_{j=2}^{K-i} H_{aj}' F_{K-i-j}' + 2\sum_{j=2}^{K-i} H_{aj} F_{K-i-j}'' + 2\sum_{j=2}^{K-i} H_j F_{K-i-j}'' - 2\sum_{j=2}^{K-i} H_j' F_{K-i-j}' + \sum_{j=2}^{K-i} H_{aj}'' F_{K-i-j} + \sum_{j=1}^{K-i-j} \Phi_j \Phi_{K-i-j}'' - \sum_{j=1}^{K-i-1} \Phi_j' \Phi_{K-i-j}' \Big] = 0$$

g. $$L_{bK}''' - \sum_{i=0}^{K-3} \alpha_i (L_{bK}' + \eta L_{bK}'') + \frac{3}{2}\eta L_{bK}'' - 3KL_{bK}' + \sum_{i=1}^{K-2}\beta_i \Big[\sum_{j=2}^{K-i} H_{aj} F_{K-i-j}'' - \sum_{j=2}^{K-i} H_{aj}' F_{K-i-j}' \Big] = 0 \qquad K = 3, 4, \ldots$$

Sada izvršimo ista sredjivanja i u linearnim kombinacijama za sve navedene sisteme tj. svuda umesto β_0 stavićemo nulu tj. $\beta_0 = 0$

Na ovaj način ćemo dobiti nove linearne kombinacije sa mnogo manje članova

$F_0 = F_0$

$F_1 = \alpha_1 f_1$

$F_2 = \alpha_1^2 f_{11} + \alpha_2 f_2$

$F_3 = \alpha_1^3 f_{111} + \alpha_1\alpha_2 f_{12} + \alpha_3 f_3$

$F_4 = \alpha_1^4 f_{1111} + \alpha_1^2\alpha_2 f_{112} + \alpha_1\alpha_3 f_{13} + \alpha_2^2 f_{22} + \alpha_4 f_4$

$F_5 = \alpha_1^5 f_{11111} + \alpha_1^3\alpha_2 f_{1112} + \alpha_1^2\alpha_3 f_{113} + \alpha_1\alpha_2^2 f_{122} + \alpha_1\alpha_4 f_{14} + \alpha_2\alpha_3 f_{23} + \alpha_5 f_5$

$\Phi_1 = \beta_1 \varphi_1$

$\Phi_2 = \alpha_1\beta_1 \varphi_{11} + \beta_2 \varphi_2$

$\Phi_3 = \alpha_1^2\beta_1 \varphi_{111} + \alpha_1\beta_2 \varphi_{12} + \alpha_2\beta_1 \varphi_{21} + \beta_3\varphi_3$

$\Phi_4 = \alpha_1^3\beta_1 \varphi_{1111} + \alpha_1\alpha_2\beta_1 \varphi_{121} + \alpha_1^2\beta_2 \varphi_{112} + \alpha_3\beta_1\varphi_{31} + \alpha_2\beta_2\varphi_{22} + \alpha_1\beta_3\varphi_{13} +$
$+ \beta_4\varphi_4$

$\Phi_5 = \alpha_1^4\beta_1 \varphi_{11111} + \alpha_1^2\alpha_2\beta_1 \varphi_{1121} + \alpha_1\alpha_3\beta_1\varphi_{131} + \alpha_2^2\beta_1\varphi_{221} + \alpha_4\beta_1\varphi_{41} +$

$$+\alpha_1^3\beta_2\varphi_{1112}+\alpha_1\alpha_2\beta_2\varphi_{122}+\alpha_3\beta_2\varphi_{32}+\alpha_1^2\beta_3\varphi_{113}+\alpha_2\beta_3\varphi_{23}+\alpha_1\beta_4\varphi_{14}+\beta_5\varphi_5$$

$$H_2=\beta_1^2h_{11}$$
$$H_3=\alpha_1\beta_1^2h_{111}+\beta_1\beta_2h_{12}$$
$$H_4=\alpha_1^2\beta_1^2h_{1111}+\alpha_2\beta_1^2h_{211}+\alpha_1\beta_1\beta_2h_{112}+\beta_1\beta_3h_{13}+\beta_2^2h_{22}$$
$$H_5=\alpha_1^3\beta_1^2h_{11111}+\alpha_1\alpha_2\beta_1^2h_{1211}+\alpha_3\beta_1^2h_{311}+\alpha_1^2\beta_1\beta_2h_{1112}+\alpha_1\beta_2^2h_{122}+$$
$$+\alpha_2\beta_1\beta_2h_{212}+\alpha_1\beta_1\beta_3h_{113}+\beta_2\beta_3h_{23}+\beta_1\beta_4h_{14}$$

$$H_{a2}=\beta_1^2h_{a11}$$
$$H_{a3}=\alpha_1\beta_1^2h_{a111}+\beta_1\beta_2h_{a12}$$
$$H_{a4}=\alpha_1^2\beta_1^2h_{a1111}+\alpha_2\beta_1^2h_{a211}+\alpha_1\beta_1\beta_2h_{a112}+\beta_1\beta_3h_{a13}+\beta_2^2h_{a22}$$
$$H_{a5}=\alpha_1^3\beta_1^2h_{a11111}+\alpha_1\alpha_2\beta_1^2h_{a1211}+\alpha_3\beta_1^2h_{a311}+\alpha_1^2\beta_1\beta_2h_{a1112}+\alpha_1\beta_2^2h_{a122}+$$
$$+\alpha_2\beta_1\beta_2h_{a212}+\alpha_1\beta_1\beta_3h_{a113}+\beta_2\beta_3h_{a23}+\beta_1\beta_4h_{a14}$$

$$L_3=\beta_1^3\ell_{111}$$
$$L_4=\alpha_1\beta_1^3\ell_{1111}+\beta_1^2\beta_2\ell_{112}$$
$$L_5=\alpha_1^2\beta_1^3\ell_{11111}+\alpha_2\beta_1^3\ell_{2111}+\alpha_1\beta_1^2\beta_2\ell_{1112}+\beta_1^2\beta_3\ell_{113}+\beta_1\beta_2^2\ell_{122}$$

Iste su kombinacije za La i Lb

. . . .

Sada ćemo na osnovu ovih linearnih kombinacija razdvojiti date sisteme a. b. c. . . . na nove sisteme u kojima ćemo imati veći broj diferencijalnih jednačina, ali u njima neće više figurisati konstante $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$

Novi sistemi diferencijalnih jednačina imaju sledeći oblik

$$F_0'''+\alpha_0(1-F_0'-\eta F_0'')+\frac{3}{2}\eta F_0''=0$$
$$f_1'''-\alpha_0(f_1'+\eta f_1'')+\frac{3}{2}\eta f_1''-3f_1'=-(1-F_0'-\eta F_0'')$$
$$f_{11}'''-\alpha_0(f_{11}'+\eta f_{11}'')+\frac{3}{2}\eta f_{11}''-6f_{11}'=(f_1'+\eta f_1'')$$
$$f_2'''-\alpha_0(f_2'+\eta f_2'')+\frac{3}{2}\eta f_2''-6f_2'=-(1-F_0'-\eta F_0'')$$
$$f_{111}'''-\alpha_0(f_{111}'+\eta f_{111}'')+\frac{3}{2}\eta f_{111}''-9f_{111}'=(f_{11}'+\eta f_{11}'')$$
$$f_{12}'''-\alpha_0(f_{12}'+\eta\cdot f_{12}'')+\frac{3}{2}\eta f_{12}''-9f_{12}'=(f_2'+\eta f_2'')+(f_1'+\eta f_1'')$$

$$f_3''' - \alpha_0 (f_3' + \eta f_3'') + \tfrac{3}{2} \eta f_3'' - 9 f_3' = - (1 - F_0' - \eta F_0'')$$

$$f_{1111}''' - \alpha_0 (f_{1111}' + \eta f_{1111}'') + \tfrac{3}{2} \eta f_{1111}'' - 12 f_{1111}' = (f_{111}' + \eta f_{111}'')$$

$$f_{112}''' - \alpha_0 (f_{112}' + \eta f_{112}'') + \tfrac{3}{2} \eta f_{112}'' - 12 f_{112}' = (f_{12}' + \eta f_{12}'') + (f_{11}' + \eta f_{11}'')$$

$$f_{13}''' - \alpha_0 (f_{13}' + \eta f_{13}'') + \tfrac{3}{2} \eta f_{13}'' - 12 f_{13}' = (f_3' + \eta f_3'') + (f_1' + \eta f_1'')$$

$$f_{22}''' - \alpha_0 (f_{22}' + \eta f_{22}'') + \tfrac{3}{2} \eta f_{22}'' - 12 f_{22}' = (f_2' + \eta f_2'')$$

$$f_4''' - \alpha_0 (f_4' + \eta f_4'') + \tfrac{3}{2} \eta f_4'' - 12 f_4' = - (1 - F_0' - \eta F_0'')$$

$$f_{11111}''' - \alpha_0 (f_{11111}' + \eta f_{11111}'') + \tfrac{3}{2} \eta f_{11111}'' - 15 f_{11111}' = (f_{1111}' + \eta f_{1111}'')$$

$$f_{1112}''' - \alpha_0 (f_{1112}' + \eta f_{1112}'') + \tfrac{3}{2} \eta f_{1112}'' - 15 f_{1112}' = (f_{112}' + \eta f_{112}'') + (f_{111}' + \eta f_{111}'')$$

$$f_{113}''' - \alpha_0 (f_{113}' + \eta f_{113}'') + \tfrac{3}{2} \eta f_{113}'' - 15 f_{113}' = (f_{13}' + \eta f_{13}'') + (f_{11}' + \eta f_{11}'')$$

$$f_{122}''' - \alpha_0 (f_{122}' + \eta f_{122}'') + \tfrac{3}{2} \eta f_{122}'' - 15 f_{122}' = (f_{22}' + \eta f_{22}'') + (f_{12}' + \eta f_{12}'')$$

$$f_{14}''' - \alpha_0 (f_{14}' + \eta f_{14}'') + \tfrac{3}{2} \eta f_{14}'' - 15 f_{14}' = (f_4' + \eta f_4'') + (f_1' + \eta f_1'')$$

$$f_{23}''' - \alpha_0 (f_{23}' + \eta f_{23}'') + \tfrac{3}{2} \eta f_{23}'' - 15 f_{23}' = (f_3' + \eta f_3'') + (f_2' + \eta f_2'')$$

$$f_5''' - \alpha_0 (f_5' + \eta f_5'') + \tfrac{3}{2} \eta f_5'' - 15 f_5' = - (1 - F_0' - \eta F_0'')$$

. . . .

b. $$\varphi_1''' - \alpha_0 (\varphi_1' + \eta \varphi_1'') + \tfrac{3}{2} \eta \varphi_1'' - 3 \varphi_1' = - (1 - F_0'^2 + F_0 F_0'')$$

$$\varphi_{11}''' - \alpha_0 (\varphi_{11}' + \eta \varphi_{11}'') + \tfrac{3}{2} \eta \varphi_{11}'' - 6 \varphi_{11}' = (\varphi_1' + \eta \varphi_1'') - (- 2 F_0' f_1' + F_0 f_1'' + f_1 F_0'')$$

$$\varphi_2''' - \alpha_0 (\varphi_2' + \eta \varphi_2'') + \tfrac{3}{2} \eta \varphi_2'' - 6 \varphi_2' = - (1 - F_0'^2 + F_0 F_0'')$$

$$\varphi_{111}''' - \alpha_0 (\varphi_{111}' + \eta \varphi_{111}'') + \tfrac{3}{2} \eta \varphi_{111}'' - 9 \varphi_{111}' = (\varphi_{11}' + \eta \varphi_{11}'') - (- 2 F_0' f_{11}' - f_1'^2 +$$
$$+ F_0 f_{11}'' + f_1 f_1'' + f_{11} F_0'')$$

$$\varphi_{12}''' - \alpha_0 (\varphi_{12}' + \eta \varphi_{12}'') + \tfrac{3}{2} \eta \varphi_{12}'' - 9 \varphi_{12}' = (\varphi_2' + \eta \varphi_2'') - (- 2 F_0' f_1' + F_0 f_1'' + f_1 F_0'')$$

$$\varphi_{21}''' - \alpha_0 (\varphi_{21}' + \eta \varphi_{21}'') + \tfrac{3}{2} \eta \varphi_{21}'' - 9 \varphi_{21}' = (\varphi_1' + \eta \varphi_1'') - (- 2 F_0' f_2' + F_0 f_2'' + f_2 F_0'')$$

$$\varphi_3''' - \alpha_0 (\varphi_3' + \eta \varphi_3'') + \tfrac{3}{2} \eta \varphi_3'' - 9 \varphi_3' = - (1 - F_0'^2 + F_0 F_0'')$$

$$\varphi_{1111}''' - \alpha_0 (\varphi_{1111}' + \eta \varphi_{1111}'') + \tfrac{3}{2} \eta \varphi_{1111}'' - 12 \varphi_{1111}' = (\varphi_{111}' + \eta \varphi_{111}'') - (- 2 F_0' f_{111}' -$$

$$-2f_1'f_{11}' + F_0 f_{111}'' + f_1 f_{11}'' + f_{111} F_0'' + f_{11} f_1'')$$

$$\varphi_{121}''' - \alpha_0(\varphi_{121}' + \eta\varphi_{121}'') + \frac{3}{2}\eta\varphi_{121}'' - 12\varphi_{121}' = (\varphi_{21}' + \eta\varphi_{21}'') + (\varphi_{11}' + \eta\varphi_{11}'') -$$
$$-(-2F_0'f_{12}' - 2f_1'f_2' + F_0 f_{12}'' + f_1'f_2' + f_2 f_1'' + f_{12}F_0'')$$

$$\varphi_{112}''' - \alpha_0(\varphi_{112}' + \eta\varphi_{112}'') + \frac{3}{2}\eta\varphi_{112}'' - 12\varphi_{112}' = (\varphi_{12}' + \eta\varphi_{12}'') - (-2F_0'f_{11}' - f_1'^2 +$$
$$+ F_0 f_{11}'' + f_1 f_1'' + f_{11}F_0'')$$

$$\varphi_{31}''' - \alpha_0(\varphi_{31}' + \eta\varphi_{31}'') + \frac{3}{2}\eta\varphi_{31}'' - 12\varphi_{31}' = (\varphi_1' + \eta\varphi_1'') - (-2F_0'f_3' + F_0 f_3'' +$$
$$+ f_3F_0'')$$

$$\varphi_{22}''' - \alpha_0(\varphi_{22}' + \eta\varphi_{22}'') + \frac{3}{2}\eta\varphi_{22}'' - 12\varphi_{22}' = (\varphi_2' + \eta\varphi_2'') - (-2F_0'f_2' + F_0 f_2'' + f_2F_0'')$$

$$\varphi_{13}''' - \alpha_0(\varphi_{13}' + \eta\varphi_{13}'') + \frac{3}{2}\eta\varphi_{13}'' - 12\varphi_{13}' = (\varphi_3' + \eta\varphi_3'') - (-2F_0'f_1' + F_0 f_1'' + f_1F_0'')$$

$$\varphi_4''' - \alpha_0(\varphi_4' + \eta\varphi_4'') + \frac{3}{2}\eta\varphi_4'' - 12\varphi_4' = -(1 - F_0'^2 + F_0F_0'')$$

$$\varphi_{11111}''' - \alpha_0(\varphi_{11111}' + \eta\varphi_{11111}'') + \frac{3}{2}\eta\varphi_{11111}'' - 15\varphi_{11111}' = (\varphi_{1111}' + \eta\varphi_{1111}'') - (-2F_0'f_{111}' -$$
$$-2f_1'f_{111}' - f_{11}'^2 + F_0 f_{1111}'' + f_1 f_{111}'' + f_{11}f_{11}'' + f_{111}f_1'' + f_{1111}F_0'')$$

$$\varphi_{1121}''' - \alpha_0(\varphi_{1121}' + \eta\varphi_{1121}'') + \frac{3}{2}\eta\varphi_{1121}'' - 15\varphi_{1121}' = (\varphi_{121}' + \eta\varphi_{121}'') + (\varphi_{111}' + \eta\varphi_{111}'') -$$
$$-(-2F_0'f_{112}' - 2f_1'f_{12}' - 2f_{11}'f_2' + F_0 f_{112}'' + f_1 f_{12}'' + f_2 f_{11}'' + f_2''f_{11} + f_1''f_{12} + f_{112}F_0'')$$

$$\varphi_{131}''' - \alpha_0(\varphi_{131}' + \eta\varphi_{131}'') + \frac{3}{2}\eta\varphi_{131}'' - 15\varphi_{131}' = (\varphi_{31}' + \eta\varphi_{31}'') + (\varphi_{11}' + \eta\varphi_{11}'') -$$
$$-(-2F_0'f_{13}' - 2f_1'f_3' + F_0 f_{13}'' + f_1 f_3'' + f_3 f_1'' + f_{13}F_0'')$$

$$\varphi_{221}''' - \alpha_0(\varphi_{221}' + \eta\varphi_{221}'') + \frac{3}{2}\eta\varphi_{221}'' - 15\varphi_{221}' = (\varphi_{21}' + \eta\varphi_{21}'') - (-2F_0'f_{22}' -$$
$$-f_2'^2 + F_0 f_{22}'' + f_2 f_2'' + f_{22}F_0'')$$

$$\varphi_{41}''' - \alpha_0(\varphi_{41}' + \eta\varphi_{41}'') + \frac{3}{2}\eta\varphi_{41}'' - 15\varphi_{41}' = (\varphi_1' + \eta\varphi_1'') - (-2F_0'f_4' + F_0 f_4'' + f_4F_0'')$$

$$\varphi_{1112}''' - \alpha_0(\varphi_{1112}' + \eta\varphi_{1112}'') + \frac{3}{2}\eta\varphi_{1112}'' - 15\varphi_{1112}' = (\varphi_{112}' + \eta\varphi_{112}'') - (-2F_0'f_{111}' -$$
$$-2f_1'f_{11}' + F_0 f_{111}'' + f_1 f_{11}'' + f_{11}f_1'' + f_{111}F_0'')$$

$$\varphi_{122}''' - \alpha_0(\varphi_{122}' + \eta\varphi_{122}'') + \frac{3}{2}\eta\varphi_{122}'' - 15\varphi_{122}' = (\varphi_{22}' + \eta\varphi_{22}'') + (\varphi_{12}' + \eta\varphi_{12}'') -$$
$$-(-2F_0'f_{12}' - 2f_1'f_2' + F_0 f_{12}'' + f_1 f_2'' + f_2 f_1'' + f_{12}F_0'')$$

$$\varphi_{32}''' - \alpha_0(\varphi_{32}' + \eta\varphi_{32}'') + \frac{3}{2}\eta\varphi_{32}'' - 15\varphi_{32}' = (\varphi_2' + \eta\varphi_2'') - (-2F_0'f_3' + F_0 f_3'' + f_3F_0'')$$

$$\varphi_{113}''' - \alpha_0(\varphi_{113}' + \eta\varphi_{113}'') + \frac{3}{2}\eta\varphi_{113}'' - 15\varphi_{113}' = (\varphi_{13}' + \eta\varphi_{13}'') - (-2F_0'f_{11}' - f_1'^2 + F_0 f_{11}'' + f_1 f_1'' + f_{11}F_0'')$$

$$\varphi_{23}''' - \alpha_0(\varphi_{23}' + \eta\varphi_{23}'') + \frac{3}{2}\eta\varphi_{23}'' - 15\varphi_{23}' = (\varphi_3' + \eta\varphi_3'') - (-2F_0'f_2' + F_0 f_2'' + f_2 F_0'')$$

$$\varphi_{14}''' - \alpha_0(\varphi_{14}' + \eta\varphi_{14}'') + \frac{3}{2}\eta\varphi_{14}'' - 15\varphi_{14}' = (\varphi_4' + \eta\varphi_4'') - (-2F_0'f_1' + F_0 f_1'' + f_1 F_0'')$$

$$\varphi_5''' - \alpha_0(\varphi_5' + \eta\varphi_5'') + \frac{3}{2}\eta\varphi_5'' - 15\varphi_5' = -(1 - F_0'^2 + F_0F_0'')$$

c. $$h_{11}''' - \alpha_0(h_{11}' + \eta h_{11}'') + \frac{3}{2}\eta h_{11}'' - 6h_{11}' = -(-2F_0'\varphi_1' + \varphi_1 F_0'' + F_0\varphi_1'')$$

$$h_{111}''' - \alpha_0(h_{111}' + \eta h_{111}'') + \frac{3}{2}\eta h_{111}'' - 9h_{111}' = (h_{11}' + \eta h_{11}'') - (-2F_0'\varphi_{11}' - 2f_1\varphi_1' + \varphi_1 f_1'' + \varphi_{11}F_0'' + F_0\varphi_{11}'' + f_1\varphi_1'')$$

$$h_{12}''' - \alpha_0(h_{12}' + \eta h_{12}'') + \frac{3}{2}\eta h_{12}'' - 9h_{12}' = -(-2F_0'\varphi_2' + \varphi_2F_0'' + F_0\varphi_2'') - (-2F_0'\varphi_1' + \varphi_1F_0'' + F_0\varphi_1'')$$

$$h_{1111}''' - \alpha_0(h_{1111}' + \eta h_{1111}'') + \frac{3}{2}\eta h_{1111}'' - 12h_{1111}' = (h_{111}' + \eta h_{111}'') - (-2F_0'\varphi_{111}' - 2f_1'\varphi_{11}' - 2f_{11}\varphi_1' + f_{11}''\varphi_1 + f_1''\varphi_{11} + F_0''\varphi_{111} + F_0\varphi_{111}'' + f_1\varphi_{11}'' + f_{11}\varphi_1'')$$

$$h_{211}''' - \alpha_0(h_{211}' + \eta h_{211}'') + \frac{3}{2}\eta h_{211}'' - 12h_{211}' = (h_{11}' + \eta h_{11}'') - (-2F_0'\varphi_{21}' - 2f_2'\varphi_1' + f_2''\varphi_1 + F_0''\varphi_{21} + F_0\varphi_{21}'' + f_2\varphi_1'')$$

$$h_{112}''' - \alpha_0(h_{112}' + \eta h_{112}'') + \frac{3}{2}\eta h_{112}'' - 12h_{112}' = (h_{12}' + \eta h_{12}'') - (-2F_0'\varphi_{12}' - 2f_1'\varphi_2' + f_1''\varphi_2 + F_0''\varphi_{12} + F_0\varphi_{12}'' + f_1\varphi_2'') - (-2F_0'\varphi_{11}' - 2f_1'\varphi_1' + f_1''\varphi_1 + F_0''\varphi_{11} + F_0\varphi_{11}'' + f_1\varphi_1'')$$

$$h_{13}''' - \alpha_0(h_{13}' + \eta h_{13}'') + \frac{3}{2}\eta h_{13}'' - 12h_{13}' = -(-2F_0'\varphi_3' + F_0''\varphi_3 + F_0\varphi_3'') - (-2F_0'\varphi_1' + F_0''\varphi_1 + F_0\varphi_1'')$$

$$h_{22}''' - \alpha_0(h_{22}' + \eta h_{22}'') + \frac{3}{2}\eta h_{22}'' - 12h_{22}' = -(-2F_0'\varphi_2' + F_0''\varphi_2 + F_0\varphi_2'')$$

$$h_{11111}''' - \alpha_0(h_{11111}' + \eta h_{11111}'') + \frac{3}{2}\eta h_{11111}'' - 15h_{11111}' = (h_{1111}' + \eta h_{1111}'') - (-2F_0'\varphi_{1111}' - 2f_1'\varphi_{111}' - 2f_{11}'\varphi_{11}' - 2f_{111}'\varphi_1' + \varphi_1 f_{111}'' + \varphi_{11}f_{11}'' + \varphi_{111}f_1'' + \varphi_{1111}F_0'' + F_0\varphi_{1111}'' + f_1\varphi_{111}'' + f_{11}\varphi_{11}'' + f_{111}\varphi_1'')$$

$$h_{1211}''' - \alpha_0(h_{1211}' + \eta h_{1211}'') + \frac{3}{2}\eta h_{1211}'' - 15h_{1211}' = (h_{211}' + \eta h_{211}'') + (h_{111}' + \eta h_{111}'') - (-2F_0''\varphi_{121}' - 2f_1'\varphi_{21}' - 2f_2'\varphi_{11}' - 2f_{12}'\varphi_1' + \varphi_1 f_{12}'' + \varphi_{11}f_2'' + \varphi_{21}f_1'' + \varphi_{121}F_0'' + F_0\varphi_{121}'' + f_1\varphi_{21}'' + f_2\varphi_{11}'' + f_{12}\varphi_1'')$$

$$h_{311}''' - \alpha_0(h_{311}' + \eta h_{311}'') + \frac{3}{2}\eta h_{311}'' - 15h_{311}' = (h_{11}' + \eta h_{11}'') - (-2F_0'\varphi_{31}' - 2f_3'\varphi_1' + \varphi_1 f_3'' +$$

$$+\varphi_{31}F_0''+F_0\varphi_{31}''+f_3\varphi_1'')$$

$$h_{1112}'''-\alpha_0(h_{1112}'+\eta h_{1112}'')+\frac{3}{2}\eta h_{1112}''-15h_{1112}'=(h_{112}'+\eta h_{112}'')-(-2F_0'\varphi_{112}'-2f_1'\varphi_{12}'-$$
$$-2f_{11}'\varphi_2'+f_{11}''\varphi_2+f_1''\varphi_{12}+F_0''\varphi_{112}+F_0\varphi_{112}''+f_1\varphi_{12}''+f_{11}\varphi_2'')-(-2F_0'\varphi_{111}'-$$
$$-2f_1'\varphi_{11}'-2f_{11}'\varphi_1'+\varphi_1f_{11}''+\varphi_{11}f_1''+\varphi_{111}F_0''+F_0\varphi_{111}''+f_1\varphi_{11}''+f_{11}\varphi_1'')$$

$$h_{122}'''-\alpha_0(h_{122}'+\eta h_{122}'')+\frac{3}{2}\eta h_{122}''-15h_{122}'=(h_{22}'+\eta h_{22}'')-(-2F_0'\varphi_{12}'-2f_1'\varphi_2'+$$
$$+\varphi_2f_1''+\varphi_{12}F_0''+F_0\varphi_{12}''+f_1\varphi_2'')$$

$$h_{212}'''-\alpha_0(h_{212}'+\eta h_{212}'')+\frac{3}{2}\eta h_{212}''-15h_{212}'=(h_{12}'+\eta h_{12}'')-(-2F_0'\varphi_{22}'-2f_2'\varphi_2'+$$
$$+f_2''\varphi_2+F_0''\varphi_{22}+F_0\varphi_{22}''+f_2\varphi_2'')-(-2F_0'\varphi_{21}'-2f_2'\varphi_1'+\varphi_1f_2''+\varphi_{21}F_0''+F_0\varphi_{21}''+f_2\varphi_1'')$$

$$h_{113}'''-\alpha_0(h_{113}'+\eta h_{113}'')+\frac{3}{2}\eta h_{113}''-15h_{113}'=(h_{13}'+\eta h_{13}'')-(-2F_0'\varphi_{13}'-2f_1'\varphi_3'+$$
$$+\varphi_3f_1''+\varphi_{13}F_0''+F_0\varphi_{13}''+f_1\varphi_3'')-(-2F_0'\varphi_{11}'-2f_1'\varphi_1'+\varphi_1f_1''+\varphi_{11}F_0''+F_0\varphi_{11}''+f_1\varphi_1'')$$

$$h_{23}'''-\alpha_0(h_{23}'+\eta h_{23}'')+\frac{3}{2}\eta h_{23}''-15h_{23}'=-(-2F_0'\varphi_3'+\varphi_3F_0''+F_0\varphi_3'')-(-2F_0'\varphi_2'+$$
$$+\varphi_2F_0''+F_0\varphi_2'')$$

$$h_{14}'''-\alpha_0(h_{14}'+\eta h_{14}'')+\frac{3}{2}\eta h_{14}''-15h_{14}''=-(-2F_0'\varphi_4'+\varphi_4F_0''+F_0\varphi_4'')-(-2F_0'\varphi_1'+\varphi_1F_0''+F_0\varphi_1'')$$

d. $h_{011}'''-\alpha_0(h_{011}'+\eta h_{011}'')+\frac{3}{2}\eta h_{011}''-6h_{011}'=-(\varphi_1F_0''-F_0'\varphi_1')$

$$h_{0111}'''-\alpha_0(h_{0111}'+\eta h_{0111}'')+\frac{3}{2}\eta h_{0111}''-9h_{0111}'=(h_{011}'+\eta h_{011}'')-(\varphi_1f_1''+\varphi_{11}F_0''-F_0'\varphi_{11}'-f_1'\varphi_1')$$

$$h_{012}'''-\alpha_0(h_{012}'+\eta h_{012}'')+\frac{3}{2}\eta h_{012}''-9h_{012}'=-(\varphi_2F_0''-F_0'\varphi_2')-(\varphi_1F_0''-F_0'\varphi_1')$$

$$h_{01111}'''-\alpha_0(h_{01111}'+\eta h_{01111}'')+\frac{3}{2}\eta h_{01111}''-12h_{01111}'=(h_{0111}'+\eta h_{0111}'')-(\varphi_1f_{11}''+\varphi_{11}f_1''+$$
$$+\varphi_{111}F_0''-F_0'\varphi_{111}'-f_1'\varphi_{11}'-f_{11}'\varphi_1')$$

$$h_{0112}'''-\alpha_0(h_{0112}'+\eta h_{0112}'')+\frac{3}{2}\eta h_{0112}''-12h_{0112}'=(h_{012}'+\eta h_{012}'')-(\varphi_2f_1''+\varphi_{12}F_0''-$$
$$-F_0'\varphi_{12}'-f_1'\varphi_2')-(\varphi_1f_1''+\varphi_{11}F_0''-F_0'\varphi_{11}'-f_1'\varphi_1')$$

$$h_{013}'''-\alpha_0(h_{013}'+\eta h_{013}'')+\frac{3}{2}\eta h_{013}''-12h_{013}'=-(\varphi_3F_0''-F_0'\varphi_3')-(\varphi_1F_0''-\varphi_1'F_0')$$

$$h_{022}'''-\alpha_0(h_{022}'+\eta h_{022}'')+\frac{3}{2}\eta h_{022}''-12h_{022}'=-(\varphi_2F_0''-\varphi_2'F_0')$$

$$h_{011111}'''-\alpha_0(h_{011111}'+\eta h_{011111}'')+\frac{3}{2}\eta h_{011111}''-15h_{011111}=(h_{01111}'+\eta h_{01111}'')-$$

$$-(\varphi_1 f_{111}'' + \varphi_{11} f_{11}'' + \varphi_{111} f_1'' + \varphi_{1111} F_0'' - F_0'\varphi_{1111}' - f_1'\varphi_{111}' - f_{11}'\varphi_{11}' - f_{111}'\varphi_1')$$

$$h_{a1211}''' - \alpha_0(h_{a1211}' + \eta h_{a1211}'') + \frac{3}{2}\eta h_{a1211}'' - 15 h_{a1211}' = (h_{a211}' + \eta h_{a211}'') + (h_{a111}' +$$
$$+ \eta h_{a111}'') - (\varphi_1 f_{12}'' + \varphi_{11} f_2'' + \varphi_{21} f_1'' + \varphi_{121} F_0'' - F_0'\varphi_{121}' - f_1'\varphi_{21}' - f_2'\varphi_{11}' - f_{12}'\varphi_1')$$

$$h_{a311}''' - \alpha_0(h_{a311}' + \eta h_{a311}'') + \frac{3}{2}\eta h_{a311}'' - 15 h_{a311}' = (h_{a11}' + \eta h_{a11}'') - (\varphi_1 F_3'' + \varphi_{31} F_0'' - \varphi_1' f_3' -$$
$$- \varphi_{31}' F_0')$$

$$h_{a1112}''' - \alpha_0(h_{a1112}' + \eta h_{a1112}'') + \frac{3}{2}\eta h_{a1112}'' - 15 h_{a1112}' = (h_{a112}' + \eta h_{a112}'') - (\varphi_{12} f_1'' + \varphi_{112} F_0'' -$$
$$- F_0'\varphi_{112}' - f_1'\varphi_{12}') - (\varphi_1 f_{11}'' + \varphi_{11} f_1'' + \varphi_{111} F_0'' - F_0'\varphi_{111}' - f_1'\varphi_{11}' - f_{11}'\varphi_1') - (\varphi_2 f_{11}'' - \varphi_2' f_{11}')$$

$$h_{a122}''' - \alpha_0(h_{a122}' + \eta h_{a122}'') + \frac{3}{2}\eta h_{a122}'' - 15 h_{a122}' = (h_{a22}' + \eta h_{a22}'') - (\varphi_2 f_1'' + \varphi_{12} F_0'' - F_0'\varphi_{12}' -$$
$$- \varphi_2' f_1')$$

$$h_{a212}''' - \alpha_0(h_{a212}' + \eta h_{a212}'') + \frac{3}{2}\eta h_{a212}'' - 15 h_{a212}' = (h_{a12}' + \eta h_{a12}'') - (\varphi_2 f_2'' + \varphi_{22} F_0'' -$$
$$- F_0'\varphi_{22}' - f_2'\varphi_2') - (\varphi_1 f_2'' + \varphi_{21} F_0'' - F_0'\varphi_{21}' - \varphi_1' f_2')$$

$$h_{a113}''' - \alpha_0(h_{a113}' + \eta h_{a113}'') + \frac{3}{2}\eta h_{a113}'' - 15 h_{a113}' = (h_{a13}' + \eta h_{a13}'') - (\varphi_3 f_1'' + \varphi_{13} F_0'' -$$
$$- F_0'\varphi_{13}' - f_1'\varphi_3') - (\varphi_1 f_1'' + \varphi_{11} F_0'' - F_0'\varphi_{11}' - f_1\varphi_1')$$

$$h_{a23}''' - \alpha_0(h_{a23}' + \eta h_{a23}'') + \frac{3}{2}\eta h_{a23}'' - 15 h_{a23}' = -(\varphi_3 F_0'' - F_0'\varphi_3') - (\varphi_2 F_0'' - F_0'\varphi_2')$$

$$h_{a14}''' - \alpha_0(h_{a14}' + \eta h_{a14}'') + \frac{3}{2}\eta h_{a14}'' - 15 h_{a14}' = -(\varphi_4 F_0'' - F_0'\varphi_4') - (\varphi_1 F_0'' - F_0'\varphi_1')$$

e. $$l_{11}''' - \alpha_0(l_{11}' + \eta l_{11}'') + \frac{3}{2}\eta l_{11}'' - 9 l_{11}' = -(-2F_0'h_{11}' - \varphi_1'^2 + h_{11}F_0'' + \varphi_1\varphi_1'' + F_0 h_{11}'')$$

$$l_{111}''' - \alpha_0(l_{111}' + \eta l_{111}'') + \frac{3}{2}\eta l_{111}'' - 12 l_{111}' = (l_{11}' + \eta l_{11}'') - (-2F_0'h_{111}' - 2f_1'h_{11}' -$$
$$- 2\varphi_1'\varphi_{11}' + h_{11}f_1'' + h_{111}F_0'' + \varphi_1\varphi_{11}'' + \varphi_{11}\varphi_1'' + F_0 h_{111}'' + f_1 h_{11}'')$$

$$l_{112}''' - \alpha_0(l_{112}' + \eta l_{112}'') + \frac{3}{2}\eta l_{112}'' - 12 l_{112}' = -(-2F_0'h_{12}' - 2\varphi_1'\varphi_2' + h_{12}F_0'' + \varphi_1\varphi_2'' +$$
$$+ \varphi_2\varphi_1'' + F_0 h_{12}'') - (-2F_0'h_{11}' - \varphi_1'^2 + h_{11}F_0'' + \varphi_1\varphi_1'' + F_0 h_{11}'')$$

$$l_{1111}''' - \alpha_0(l_{1111}' + \eta l_{1111}'') + \frac{3}{2}\eta l_{1111}'' - 15 l_{1111}' = (l_{111}' + \eta l_{111}'') - (-2F_0'h_{1111}' - 2f_1'h_{111}' -$$
$$- 2f_{11}'h_{11}' - 2\varphi_1'\varphi_{111}' - \varphi_{11}'^2 + f_{11}''h_{11} + f_1''h_{111} + F_0''h_{1111} + \varphi_1\varphi_{111}'' + \varphi_{11}\varphi_{11}'' + \varphi_{111}\varphi_1'' +$$
$$+ F_0 h_{1111}'' + f_1 h_{111}'' + f_{11} h_{11}'')$$

$$l_{211}''' - \alpha_0(l_{211}' + \eta l_{211}'') + \frac{3}{2}\eta l_{211}'' - 15 l_{211}' = (l_{11}' + \eta l_{11}'') - (-2F_0'h_{211}' - 2f_2'h_{11}' -$$
$$- 2\varphi_1'\varphi_{21}' + h_{11}f_2'' + h_{211}F_0'' + \varphi_1\varphi_{21}'' + \varphi_{21}\varphi_1'' + F_0 h_{211}'' + f_2 h_{11}'')$$

$$l_{1112}''' - \alpha_0(l_{1112}' + \eta l_{1112}'') + \frac{3}{2}\eta l_{1112}'' - 15 l_{1112}' = (l_{112}' + \eta l_{112}'') - (-2F_0'h_{112}' -$$

$$-2f_1'h_{12}'-2\varphi_1'\varphi_{12}'-2\varphi_{11}'\varphi_2'+h_{12}f_1''+h_{112}F_0''+\varphi_1\varphi_{12}''+\varphi_2\varphi_{11}''+\varphi_2''\varphi_{11}+\varphi_{12}\varphi_1''+$$
$$+F_0h_{112}''+f_1h_{12}''-(-2F_0'h_{111}'-2f_1'h_{11}'-2\varphi_1'\varphi_{11}'+h_{11}f_1''+h_{111}F_0''+\varphi_1\varphi_{11}''+\varphi_{11}\varphi_1''+$$
$$+F_0h_{111}''+f_1h_{11}'')$$

$$l_{113}'''-\alpha_0(l_{113}'+\eta l_{113}'')+\frac{3}{2}\eta l_{113}''-15l_{113}'=-(-2F_0'h_{13}'-2\varphi_1'\varphi_3'+h_{13}F_0''+$$
$$+\varphi_1\varphi_3''+\varphi_3\varphi_1''+F_0h_{13}'')-(-2F_0'h_{11}'-\varphi_1'^2+h_{11}F_0''+\varphi_1\varphi_1''+F_0h_{11}'')$$

$$l_{122}'''-\alpha_0(l_{122}'+\eta l_{122}'')+\frac{3}{2}\eta l_{122}''-15l_{122}'=-(-2F_0'h_{22}'-\varphi_2'^2+h_{22}F_0''+$$
$$+\varphi_2\varphi_2''+F_0h_{22}'')-(-2F_0'h_{12}'-2\varphi_1'\varphi_2'+h_{12}F_0''+\varphi_1\varphi_2''+\varphi_2\varphi_1''+F_0h_{12}'')$$

f. $$l_{011}'''-\alpha_0(l_{011}'+\eta l_{011}'')+\frac{3}{2}\eta l_{011}''-9l_{011}'=-(-3F_0'h_{011}'+2h_{011}F_0''+2h_{11}F_0''-$$
$$-2F_0'h_{11}'+F_0h_{011}''+\varphi_1\varphi_1''-\varphi_1'^2)$$

$$l_{0111}'''-\alpha_0(l_{0111}'+\eta l_{0111}'')+\frac{3}{2}\eta l_{0111}''-12l_{0111}'=(l_{011}'+\eta l_{011}'')-(-3F_0'h_{0111}'-$$
$$-3f_1'h_{011}'+2h_{011}f_1''+2h_{0111}F_0''+2h_{11}f_1''+2h_{111}F_0''-2F_0'h_{111}'-2f_1'h_{11}'+F_0h_{0111}''+$$
$$+f_1h_{011}''+\varphi_1\varphi_{11}''+\varphi_{11}\varphi_1''-2\varphi_1'\varphi_{11}')$$

$$l_{012}'''-\alpha_0(l_{012}'+\eta l_{012}'')+\frac{3}{2}\eta l_{012}''-12l_{012}'=-(-3F_0'h_{012}'+2h_{012}F_0''+2h_{12}F_0''-$$
$$-2F_0'h_{12}'+F_0h_{012}''+\varphi_1\varphi_2''+\varphi_2\varphi_1''-2\varphi_1'\varphi_2')-(-3F_0'h_{011}'+2h_{011}F_0''+2h_{11}F_0''-2F_0'h_{11}'+$$
$$+F_0h_{011}''+\varphi_1\varphi_1''-\varphi_1'^2)$$

$$l_{01111}'''-\alpha_0(l_{01111}'+\eta l_{01111}'')+\frac{3}{2}\eta l_{01111}''-15l_{01111}'=(l_{0111}'+\eta l_{0111}'')-(-3F_0'h_{01111}'-$$
$$-3f_1'h_{0111}'-3f_{11}'h_{011}'+2h_{011}f_{11}''+2h_{0111}f_1''+2h_{01111}F_0''+2h_{11}f_{11}''+2h_{111}f_1''+2h_{1111}F_0''$$
$$-2F_0'h_{1111}'-2f_1'h_{111}'-2f_{11}'h_{11}'+F_0h_{01111}''+f_1h_{0111}''+f_{11}h_{011}''+\varphi_1\varphi_{111}''+\varphi_{11}\varphi_{11}''+\varphi_{111}\varphi_1''$$
$$-2\varphi_1'\varphi_{111}'-\varphi_{11}'^2)$$

$$l_{0211}'''-\alpha_0(l_{0211}'+\eta l_{0211}'')+\frac{3}{2}\eta l_{0211}''-15l_{0211}'=(l_{011}'+\eta l_{011}'')-(-3F_0'h_{0211}'-$$
$$-3f_2'h_{011}'+2h_{011}f_2''+2h_{0211}F_0''+2h_{11}f_2''+2h_{211}F_0''-2F_0'h_{211}'-2f_2'h_{11}'+F_0h_{0211}''+$$
$$+f_2h_{011}''+\varphi_1\varphi_{21}''+\varphi_{21}\varphi_1''-2\varphi_1'\varphi_{21}')$$

$$l_{0112}'''-\alpha_0(l_{0112}'+\eta l_{0112}'')+\frac{3}{2}\eta l_{0112}''-15l_{0112}'=(l_{012}'+\eta l_{012}'')-(-3F_0'h_{0112}'-$$
$$-3f_1'h_{012}'+2h_{012}f_1''+2h_{0112}F_0''+2h_{12}f_1''+2h_{112}F_0''-2F_0'h_{112}'-2f_1'h_{12}'+F_0h_{0112}''+$$
$$+f_1h_{012}''+\varphi_1\varphi_{12}''+\varphi_2\varphi_{11}''+\varphi_{11}\varphi_2''+\varphi_{12}\varphi_1''-2\varphi_1'\varphi_{12}'-2\varphi_{11}'\varphi_2')-(-3F_0'h_{0111}'-3f_1'h_{011}'$$
$$+2h_{011}f_1''+2h_{0111}F_0''+2h_{11}f_1''+2h_{111}F_0''-2F_0'h_{111}'-2f_1'h_{11}'+F_0h_{0111}''+f_1h_{011}''+\varphi_1\varphi_{11}''+$$
$$+\varphi_{11}\varphi_1''-2\varphi_1'\varphi_{11}')$$

$$l_{013}'''-\alpha_0(l_{013}'+\eta l_{013}'')+\frac{3}{2}\eta l_{013}''-15l_{013}'=-(-3F_0'h_{013}'+2h_{013}F_0''+2h_{13}F_0''-$$
$$-2F_0'h_{13}'+F_0h_{013}''+\varphi_1\varphi_3''+\varphi_3\varphi_1''-2\varphi_1'\varphi_3')-(-3F_0'h_{011}'+2h_{011}F_0''+2h_{11}F_0''-2F_0'h_{11}'+$$
$$+F_0h_{011}''+\varphi_1\varphi_1''-\varphi_1'^2)$$

$$l_{0122}'''-\alpha_0(l_{0122}'+\eta l_{0122}'')+\frac{3}{2}\eta l_{0122}''-15l_{0122}'=-(-3F_0'h_{022}'+2h_{022}F_0''+$$

$$+2h_{22}F_o''-2F_o'h_{22}'+F_o h_{a22}''+\varphi\varphi_2''-\varphi_2'^2)-(-3F_o'h_{a12}'+2h_{a12}F_o''+2h_{12}F_o''-$$
$$-2F_o'h_{12}'+F_o h_{a12}''+\varphi_1\varphi_2''+\varphi_2\varphi_1''-2\varphi_1'\varphi_2')$$

g. $$f_{b11}'''-\alpha_o(f_{b11}'+\eta f_{b11}'')+\frac{3}{2}\eta f_{b11}''-9f_{b11}'=-(h_{a11}F_o''-F_o'h_{a11}')$$

$$f_{b111}'''-\alpha_o(f_{b111}'+\eta f_{b111}'')+\frac{3}{2}\eta f_{b111}''-12f_{b111}'=(f_{b11}'+\eta f_{b11}'')-(h_{a11}f_1''+$$
$$+h_{a111}F_o''-F_o'h_{a111}'-f_1'h_{a11}')$$

$$f_{b112}'''-\alpha_o(f_{b112}'+\eta f_{b112}'')+\frac{3}{2}\eta f_{b112}''-12f_{b112}'=-(h_{a12}F_o''-F_o'h_{a12}')-(h_{a11}F_o''-F_o'h_{a11}')$$

$$f_{b1111}'''-\alpha_o(f_{b1111}'+\eta f_{b1111}'')+\frac{3}{2}\eta f_{b1111}''-15f_{b1111}'=(f_{b111}'+\eta f_{b111}'')-(h_{a11}f_{11}''+$$
$$h_{a111}f_1''+h_{a1111}F_o''-F_o'h_{a1111}'-f_1'h_{a111}'-f_{11}'h_{a11}')$$

$$f_{b2111}'''-\alpha_o(f_{b2111}'+\eta f_{b2111}'')+\frac{3}{2}\eta f_{b2111}''-15f_{b2111}'=(f_{b111}'+\eta f_{b111}'')-(h_{a11}f_2''+$$
$$+h_{a211}F_o''-F_o'h_{a211}'-f_2'h_{a11}')$$

$$f_{b1112}'''-\alpha_o(f_{b1112}'+\eta f_{b1112}'')+\frac{3}{2}\eta f_{b1112}''-15f_{b1112}'=(f_{b112}'+\eta f_{b112}'')-(h_{a12}f_1''+$$
$$+h_{a112}F_o''-F_o'h_{a112}'-f_1'h_{a12}')-(h_{a11}f_1''+h_{a111}F_o''-F_o'h_{a111}'-f_1'h_{a11}')$$

$$f_{b113}'''-\alpha_o(f_{b113}'+\eta f_{b113}'')+\frac{3}{2}\eta f_{b113}''-15f_{b113}'=-(h_{a13}F_o''-F_o'h_{a13}')-(h_{a11}F_o''-$$
$$-F_o'h_{a11}')$$

$$f_{b122}'''-\alpha_o(f_{b122}'+\eta f_{b122}'')+\frac{3}{2}\eta f_{b122}''-15b_{b122}'=-(h_{a22}F_o''-h_{a22}'F_o')-(h_{a12}F_o''-$$
$$-h_{a12}'F_o')$$

. . . .

Na ovaj smo način učinili sve f ..., φ ..., h,........ za usvojeno α_o nezavisnim od ostalih koeficijenata α_k i β_k k=1,2,.......,te se stoga mogu sračunati i tabulisati. A onda se računanje pojedinih problema sastoji u elementarnim matematičkim operacijama.

Odgovarajući granični uslovi za sve f ..., φ ..., . . . glase sada:

$f\ldots(o) = f'\ldots(o) = f'\ldots(\infty) = o$

$\varphi\ldots(o) = \varphi'\ldots(o) = \varphi'\ldots(\infty) = o$

$h\ldots(o) = h'\ldots(o) = h'\ldots(\infty) = o$

. . . .

Sada bi produžna komponenta brzine imala sledeći oblik:

$$u(x,y,t)=U(x)\Omega(t)\Big\{F_o'+\alpha_1 f_1'\tau+(\alpha_1^2 f_{11}'+\alpha_2 f_2')\tau^2+\ldots$$
$$+U'[\beta_1\varphi_1'\tau+(\alpha_1\beta_1\varphi_{11}'+\beta_2\varphi_2')\tau^2+\ldots]+U'^2[$$
$$[\beta_1^2 h_{11}'\tau^2+(\alpha_1\beta_1^2 h_{111}'+\beta_1\beta_2 h_{12}')\tau^3+\ldots]+\ldots\Big\}$$

Proučimo još jedan slučaj tj naš slučaj B dobijen iz opšte klase II:

Slučaj B:

U ovom slučaju je

$\alpha_0 = 1$ (m=1)

a raspored brzina počinje sa linearnim članom

$$\Omega(t) = \Omega_1 t + \Omega_2 t^2 + \cdots\cdots$$

sa $\Omega_1 \neq 0$ (inače proizvoljno), Ω_i -proizvoljni (i = 2,3,..)

Izračunavanje koeficijenata α_k iz koeficijenata Ω_k dato je jednačinama 33, a koeficijenata β_k sa jednačinama 54.

U ovom slučaju glavna funkcija $\alpha(\tau)$ i funkcija $\beta(\tau)$ nisu više date kao stepeni redovi po τ, već kao stepeni redovi po $\tau^{1/3}$ tj.

$$\alpha(\tau) = \alpha_0 + \alpha_{1/3}\tau^{1/3} + \alpha_{2/3}\tau^{2/3} + \cdots\cdots$$

gde je: $\alpha_0 = 1$ i

$$\beta(\tau) = \beta_0 + \beta_{1/3}\tau^{1/3} + \beta_{2/3}\tau^{2/3} + \cdots\cdots$$

Ako unesemo $\alpha(\tau)$ i $\beta(\tau)$ u gornjem obliku, onda će tražena rešenja biti pristupačna za računanje kao stepeni redovi po $\tau^{1/3}$ sa koeficijentima koji su funkcije reduciranog rastojanja od zida.

$$F(\eta,\tau) = \sum_{k=0}^{\infty} F_{k/3}(\eta)\,\tau^{k/3} \qquad \Phi(\eta,\tau) = \sum_{k=0}^{\infty} \Phi_{k/3}(\eta)\,\tau^{k/3}$$

$$H(\eta,\tau) = \sum_{k=0}^{\infty} H_{k/3}(\eta)\,\tau^{k/3} \qquad L(\eta,\tau) = \sum_{k=0}^{\infty} L_{k/3}(\eta)\,\tau^{k/3}$$

Ako ove izraze smenimo u 60. svaka od parcijalnih jednačina tog sistema razdvojiće se na rekurzivni sistem običnih diferencijalnih jednačina za odredjivanje koeficijenata -funkcija svakog od gornjih stepenih redova.

a. Od prve jednačine sistema 60 dobićemo sledeći sistem običnih diferencijalnih jednačina:

$$F_0''' + \alpha_0(1 - F_0' - \eta F_0'') + \tfrac{3}{2}\eta F_0'' = 0$$

$$F_{1/3}''' - \alpha_0(F_{1/3}' + \eta F_{1/3}'') + \tfrac{3}{2}\eta F_{1/3}'' + \alpha_{1/3}(1 - F_0' - \eta F_0'') - F_{1/3}' = 0$$

$$\cdots\cdots\cdots\cdots$$

ili u obliku rekurentne formule

$$F'''_{k/3} + \alpha_{k/3}(1 - F_0' - \eta F_0'') + \frac{3}{2}\eta F''_{k/3} - K F''_{k/3} =$$
$$= \sum_{i=0}^{k-1} \alpha_{i/3}\left(F'_{\frac{k-i}{3}} + \eta F''_{\frac{k-i}{3}}\right)$$

za svako k=1,2,............

b. Parcijalna jednačina za sistem ϕ(60.) daje sledeći rekurzivni sistem običnih diferencijalnih jednačina

$$\phi_0''' - \alpha_0(\phi_0' + \eta\phi_0'') + \frac{3}{2}\eta\phi_0'' + \beta_0(1 - F_0'^2 + F_0F_0'') = 0$$
$$\phi'''_{1/3} - \alpha_0(\phi'_{1/3} + \eta\phi''_{1/3}) + \frac{3}{2}\eta\phi''_{1/3} - \phi'_{1/3} - \alpha_{1/3}(\phi_0' + \eta\phi_0'') + \beta_0(-$$
$$-2F_0'F'_{1/3} + F_0F''_{1/3} + F_0''F_{1/3}) + \beta_{1/3}(1 - F_0'^2 + F_0F_0'') = 0$$

. . . .

ili u obliku rekurentne formule:

$$\phi'''_{k/3} - \sum_{i=0}^{k}\alpha_{i/3}\left(\phi'_{\frac{k-i}{3}} + \eta\phi''_{\frac{k-i}{3}}\right) + \frac{3}{2}\eta\phi''_{k/3} - K\phi'_{k/3} + \beta_{k/3}(1 - F_0'^2 + F_0F_0'') +$$
$$+ \sum_{i=0}^{k-1}\beta_{i/3}\left[-\sum_{j=0}^{k-i}F'_{j/3}F'_{\frac{k-i-j}{3}} + \sum_{j=0}^{k-i}F_{j/3}F''_{\frac{k-i-j}{3}}\right] = 0 \qquad K = 1,2,\ldots$$

c. Treća parcijalna jednačina sistema 60. daje sledeći rekurzivni sistem običnih diferencijalnih jednačina

$$H_0''' - \alpha_0(H_0' + \eta H_0'') + \frac{3}{2}\eta H_0'' + \beta_0(-2F_0'\phi_0' + \phi_0F_0'' + F_0\phi_0'') = 0$$

. . . .

. . . .

ili u obliku rekurentne formule

$$H'''_{k/3} - \sum_{i=0}^{k}\alpha_{i/3}\left(H'_{\frac{k-i}{3}} + \eta H''_{\frac{k-i}{3}}\right) + \frac{3}{2}\eta H''_{k/3} - K H'_{k/3} + \sum_{i=0}^{k}\beta_{i/3}\left[-2\sum_{j=0}^{k-i}F'_{\frac{k-i-j}{3}}\phi'_{j/3}\right.$$
$$\left. + \sum_{j=0}^{k-i}F_{\frac{k-i-j}{3}}\phi''_{j/3} + \sum_{j=0}^{k-i}F''_{\frac{k-i-j}{3}}\phi_{j/3}\right] = 0 \qquad K = 1,2,\ldots$$

d. Ova jednačina iz sistema 60. daje sledeći rekurzivni sistem običnih diferencijalnih jednačina

$$H_{a0}''' - \alpha_0(H_{a0}' + \eta H_{a0}'') + \frac{3}{2}\eta H_{a0}'' + \beta_0(F_0''\phi_0 - F_0'\phi_0') = 0$$

. . . .

. . . .

ili u obliku rekuretne formule

$$H'''_{a\,k/3} - \sum_{i=0}^{k}\alpha_{i/3}\left(H'_{a\frac{k-i}{3}} + \eta H''_{a\frac{k-i}{3}}\right) + \frac{3}{2}\eta H''_{a\,k/3} - K H'_{a\,k/3} + \sum_{i=0}^{k}\beta_{i/3}\Big[$$
$$\Big[\sum_{j=0}^{k-i}F''_{j/3}\phi_{\frac{k-i-j}{3}} - \sum_{j=0}^{k-i}F'_{j/3}\phi'_{\frac{k-i-j}{3}}\Big] = 0 \qquad K = 1,2,\ldots\ldots$$

e. Ova jednačina daje sledeći sistem

$$L_o''' - \alpha_o(L_o' + \eta L_o'') + \frac{3}{2}\eta L_o'' + \beta_o(-2F_o'H_o' - \Phi_o'^2 + H_oF_o'' + \Phi_o\Phi_o'' + F_oH_o'') = 0$$

. . . .

. . . .

ili u obliku rekurentne formule

$$L'''_{k/3} - \sum_{i=0}^{k}\alpha_{i/3}\left(L'_{\frac{k-i}{3}} + \eta L''_{\frac{k-i}{3}}\right) + \frac{3}{2}\eta L''_{k/3} - kL'_{k/3} = -\sum_{i=0}^{k}\beta_{i/3}\Big[-2\sum_{j=0}^{k-i}F'_{j/3}H'_{\frac{k-i-j}{3}} - \sum_{j=0}^{k-i}\Phi'_{j/3}\Phi'_{\frac{k-i-j}{3}} + \sum_{j=0}^{k-i}H_{j/3}F''_{\frac{k-i-j}{3}} + \sum_{j=0}^{k-i}\Phi_{j/3}\Phi''_{\frac{k-i-j}{3}} + \sum_{j=0}^{k-i}F_{j/3}H''_{\frac{k-i-j}{3}}\Big]$$

za svako k=1,2, ...

f. Ova parcijalna jednačina daje sledeći rekurzivni sistem.

$$L'''_{ao} - \alpha_o(L'_{ao} + \eta L''_{ao}) + \frac{3}{2}\eta L''_{ao} + \beta_o(-3F_o'H'_{ao} + 2H_{ao}F_o'' + 2H_oF_o'' - 2F_o'H_o' + F_oH''_{ao} + \Phi_o\Phi_o'' - \Phi_o'^2) = 0$$

. . . .

. . . .

ili u obliku rekurentne formule

$$L'''_{ak/3} - \sum_{i=0}^{k}\alpha_{i/3}\left(L'_{a\frac{k-i}{3}} + \eta L''_{a\frac{k-i}{3}}\right) + \frac{3}{2}\eta L''_{ak/3} - kL'_{ak/3} + \sum_{i=0}^{k}\beta_{i/3}\Big[-3\sum_{j=0}^{k-i}F'_{j/3} \cdot H'_{a\frac{k-i-j}{3}} + 2\sum_{j=0}^{k-i}H_{aj/3}F''_{\frac{k-i-j}{3}} + 2\sum_{j=0}^{k-i}H_{j/3}F''_{\frac{k-i-j}{3}} - 2\sum_{j=0}^{k-i}F'_{j/3}H'_{\frac{k-i-j}{3}} + \sum_{j=0}^{k-i}F_{j/3}H''_{a\frac{k-i-j}{3}} + \sum_{j=0}^{k-i}\Phi_{j/3}\Phi''_{\frac{k-i-j}{3}} - \sum_{j=0}^{k-i}\Phi'_{j/3}\Phi'_{\frac{k-i-j}{3}}\Big] = 0 \qquad k=1,2,\ldots$$

g. Zadnja parcijalna jednačina daje sledeći sistem

$$L'''_{bo} - \alpha_o(L'_{bo} + \eta L''_{bo}) + \frac{3}{2}\eta L''_{bo} + \beta_o(H_{ao}F_o'' - F_o'H'_{ao}) = 0$$

$$L'''_{b1/3} - \alpha_o(L'_{b1/3} + \eta L''_{b1/3}) - \alpha_{1/3}(L'_{bo} + \eta L''_{bo}) + \frac{3}{2}\eta L''_{b1/3} - L'_{b1/3} + \beta_o(H_{ao}F''_{1/3} + H_{a1/3}F_o'' - F_o'H'_{a1/3} - F'_{1/3}H'_{ao}) + \beta_{1/3}(H_{ao}F_o'' - F_o'H'_{ao}) = 0$$

. . . .

ili u obliku rekurentne formule

$$L'''_{bk/3} - \sum_{i=0}^{k}\alpha_{i/3}\left(L'_{b\frac{k-i}{3}} + \eta L''_{b\frac{k-i}{3}}\right) + \frac{3}{2}\eta L''_{bk/3} - kL'_{bk/3} + \sum_{i=0}^{k}\beta_{i/3}\Big[\sum_{j=0}^{k-i}H_{aj/3} \cdot F''_{\frac{k-i-j}{3}} - \sum_{j=0}^{k-i}F'_{j/3}H'_{a\frac{k-i-j}{3}}\Big] = 0 \qquad k=1,2,\ldots$$

Granični uslovi:

$$F_0'(0)=0 \qquad F_k'(0)=0 \qquad k=1,2,\ldots$$

$$F_0'(\infty)=1 \qquad F_k'(\infty)=0 \qquad k=1,2,\ldots$$

$$\Phi_0'(0)=0 \qquad \Phi_k'(0)=0 \qquad k=1,2,\ldots$$

$$\Phi_0'(\infty)=0 \qquad \Phi_k'(\infty)=0 \qquad k=1,2,\ldots$$

isti su dalje granični uslovi i za H, Ha, L, La i L_b

Dovodjenje $F_{k/3}(\eta), \Phi_{k/3}(\eta), \ldots$ za usvojeno α_0 na univerzalne funkcije sprovodi se putem sledećih izraza

$$F_0=F_0$$

$$F_{1/3}=\alpha_{1/3} f_{1/3}$$

$$F_{2/3}=\alpha_{1/3}^2 f_{1/3\,1/3}+\alpha_{2/3} f_{2/3}$$

$$F_1=\alpha_{1/3}^3 f_{1/3\,1/3\,1/3}+\alpha_{1/3}\alpha_{2/3} f_{1/3\,2/3}+\alpha_1 f_1$$

$$F_{4/3}=\alpha_{1/3}^4 f_{1/3\,1/3\,1/3\,1/3}+\alpha_{1/3}^2\alpha_{2/3} f_{1/3\,1/3\,2/3}+\alpha_{1/3}\alpha_1 f_{1/3\,1}+\alpha_{2/3}^2 f_{2/3\,2/3}+\alpha_{4/3} f_{4/3}$$

$$F_{5/3}=\alpha_{1/3}^5 f_{1/3\,1/3\,1/3\,1/3\,1/3}+\alpha_{1/3}^3\alpha_{2/3} f_{1/3\,1/3\,1/3\,2/3}+\alpha_{1/3}^2\alpha_1 f_{1/3\,1/3\,1}+\alpha_{1/3}\alpha_{2/3}^2 f_{1/3\,2/3\,2/3}+\alpha_{1/3}\alpha_{4/3} f_{1/3\,4/3}$$
$$+\alpha_{2/3}\alpha_1 f_{2/3\,1}+\alpha_{5/3} f_{5/3}$$

$$\Phi_0=\beta_0\varphi_0$$

$$\Phi_{1/3}=\alpha_{1/3}\beta_0\varphi_{1/3\,0}+\beta_{1/3}\varphi_{1/3}$$

$$\Phi_{2/3}=\alpha_{1/3}^2\beta_0\varphi_{1/3\,1/3\,0}+\alpha_{1/3}\beta_{1/3}\varphi_{1/3\,1/3}+\alpha_{2/3}\beta_0\varphi_{2/3\,0}+\beta_{2/3}\varphi_{2/3}$$

. . . .

. . . .

$$H_0=\beta_0^2 h_{00}$$

$$H_{1/3}=\alpha_{1/3}\beta_0^2 h_{1/3\,00}+\beta_0\beta_{1/3} h_{0\,1/3}$$

$$H_{2/3}=\alpha_{1/3}^2\beta_0^2 h_{1/3\,1/3\,00}+\alpha_{2/3}\beta_0^2 h_{2/3\,00}+\alpha_{1/3}\beta_0\beta_{1/3} h_{1/3\,0\,1/3}+\beta_0\beta_{2/3} h_{0\,2/3}+$$
$$+\beta_{1/3}^2 h_{1/3\,1/3}$$

. . . .

. . . .

$$Ha_0=\beta_0^2 ha_{00}$$

$$Ha_{1/3}=\alpha_{1/3}\beta_0 ha_{1/3\,00}+\beta_0\beta_{1/3} ha_{0\,1/3}$$

$$Ha_{2/3}=\alpha_{1/3}^2\beta_0^2 ha_{1/3\,1/3\,00}+\alpha_{2/3}\beta_0^2 ha_{2/3\,00}+\alpha_{1/3}\beta_0\beta_{1/3} ha_{1/3\,0\,1/3}+\beta_0\beta_{2/3} ha_{0\,2/3}+$$
$$+\beta_{1/3}^2 ha_{1/3\,1/3}$$

. . . .

. . . .

$$L_0=\beta_0^3 l_{000}$$

. . . . . .

$L_{ao} = \beta_0^3 \ell_{a000}$

. . . .

. . . .

$L_{bo} = \beta_0^3 \ell_{b000}$

. . . .

. . . .

Formalno se može dobiti iz prethodnog slučaja A ako se svuda gde stoji k (1, 2, . . .) stavi k/3. Da nebi dalje odugovlačili sa pisanjem zadržaćemo se ovde. Pošto smo ranije u § 4 videli da su za ovaj slučaj B pri ispitivanju funkcije $\beta(\tau)$, $\beta_0 = 0$ i $\beta_{1/3} = 0$, to se onda dati sistemi običnih diferencijalnih jednačina i gornje linearne kombinacije univerzalnih funkcija prevode na ove oblike:

a. Ovaj sistem zadržava stari oblik

b.
$$\Phi_{2/3}''' - \alpha_0(\Phi_{2/3}' + \eta\Phi_{2/3}'') + \frac{3}{2}\eta\Phi_{2/3}'' - 2\Phi_{2/3}' + \beta_{2/3}(1 - F_0'^2 + F_0F_0'') = 0$$

$$\Phi_1''' - \alpha_0(\Phi_1' + \eta\Phi_1'') - \alpha_{1/3}(\Phi_{2/3}' + \eta\Phi_{2/3}'') + \frac{3}{2}\eta\Phi_1'' - 3\Phi_1' + \beta_{2/3}(-$$
$$-2F_0'F_{1/3}' + F_0F_{1/3}'' + F_{1/3}F_0'') + \beta_1(1 - F_0'^2 + F_0F_0'') = 0$$

. . . .

c.
$$H_{4/3}''' - \alpha_0(H_{4/3}' + \eta H_{4/3}'') + \frac{3}{2}\eta H_{4/3}'' - 4H_{4/3}' + \beta_{2/3}(-2F_0'\Phi_{2/3}' + \Phi_{2/3}F_0'' + F_0\Phi_{2/3}'') =$$
$$= 0$$

. . . .

d.
$$Ha_{4/3}''' - \alpha_0(Ha_{4/3}' + \eta Ha_{4/3}'') + \frac{3}{2}\eta Ha_{4/3}'' - 4Ha_{4/3}' + \beta_{2/3}(\Phi_{2/3}F_0'' + F_0'\Phi_{2/3}') = 0$$

. . . .

Za e, f i g nemamo jednačina, pošto smo se mi zadržali na stepenu 5/3 a u daljim sistemima e, f i g tek bi imali jednačine od stepena 6/3

Na isti način napišimo sada i linearne kombinacije, koje sada dobijaju sledeći oblik

$F_0 = F_0$

$F_{1/3} = \alpha_{1/3} f_{1/3}$

$F_{2/3} = \alpha_{1/3}^2 f_{1/3\,1/3} + \alpha_{2/3} f_{2/3}$

$F_1 = \alpha_{1/3}^3 f_{1/3\,1/3\,1/3} + \alpha_{1/3}\alpha_{2/3} f_{1/3\,2/3} + \alpha_1 f_1$

$$F_{4/3}=\alpha_{1/3}^{4} f_{1/3\,1/3\,1/3\,1/3}+\alpha_{1/3}^{2}\alpha_{2/3} f_{1/3\,1/3\,2/3}+\alpha_{1/3}\alpha_{1} f_{1/3\,1}+\alpha_{2/3}^{2} f_{2/3\,2/3}+\alpha_{4/3} f_{4/3}$$

$$F_{5/3}=\alpha_{1/3}^{5} f_{1/3\,1/3\,1/3\,1/3\,1/3}+\alpha_{1/3}^{3}\alpha_{2/3} f_{1/3\,1/3\,1/3\,2/3}+\alpha_{1/3}^{2}\alpha_{1} f_{1/3\,1/3\,1}+\alpha_{1/3}\alpha_{2/3}^{2} f_{1/3\,2/3\,2/3}+$$
$$+\alpha_{1/3}\alpha_{4/3} f_{1/3\,4/3}+\alpha_{2/3}\alpha_{1} f_{2/3\,1}+\alpha_{5/3} f_{5/3}$$

$$\Phi_{2/3}=\beta_{2/3}\varphi_{2/3}$$

$$\Phi_{1}=\alpha_{1/3}\beta_{2/3}\varphi_{1/3\,2/3}+\beta_{1}\varphi_{1}$$

$$\Phi_{4/3}=\alpha_{1/3}^{2}\beta_{2/3}\varphi_{1/3\,1/3\,2/3}+\alpha_{2/3}\beta_{2/3}\varphi_{2/3\,2/3}+\alpha_{1/3}\beta_{1}\varphi_{1/3\,1}+\beta_{4/3}\varphi_{4/3}$$

$$\Phi_{5/3}=\alpha_{1/3}^{3}\beta_{2/3}\varphi_{1/3\,1/3\,1/3\,2/3}+\alpha_{1/3}\alpha_{2/3}\beta_{2/3}\varphi_{1/3\,2/3\,2/3}+\alpha_{1}\beta_{2/3}\varphi_{1\,2/3}+\alpha_{1/3}^{2}\beta_{1}\varphi_{1/3\,1/3\,1}+$$
$$+\alpha_{2/3}\beta_{1}\varphi_{2/3\,1}+\alpha_{1/3}\beta_{4/3}\varphi_{1/3\,4/3}+\beta_{5/3}\varphi_{5/3}$$

$$H_{4/3}=\beta_{2/3}^{2} h_{2/3\,2/3}$$

$$H_{5/3}=\alpha_{1/3}\beta_{2/3}^{2} h_{1/3\,2/3\,2/3}+\beta_{2/3}\beta_{1} h_{2/3\,1}$$

$$Ha_{4/3}=\beta_{2/3}^{2} ha_{2/3\,2/3} \qquad Ha_{5/3}=\alpha_{1/3}\beta_{2/3}^{2} ha_{1/3\,2/3\,2/3}+\beta_{2/3}\beta_{1} ha_{2/3\,1}$$

Novi sistemi diferencijalnih jednačina za odredjivanje univerzalnih funkcija imaju sledeći oblik

a. $F_0'''+\alpha_0(1-F_0'-\eta F_0'')+\frac{3}{2}\eta F_0''=0$

$$f_{1/3}'''-\alpha_0(f_{1/3}'+\eta f_{1/3}'')+\frac{3}{2}\eta f_{1/3}''-f_{1/3}'=-(1-F_0'-\eta F_0'')$$

$$f_{1/3\,1/3}'''-\alpha_0(f_{1/3\,1/3}'+\eta f_{1/3\,1/3}'')+\frac{3}{2}\eta f_{1/3\,1/3}''-2f_{1/3\,1/3}'=(f_{1/3}'+\eta f_{1/3}'')$$

$$f_{2/3}'''-\alpha_0(f_{2/3}'+\eta f_{2/3}'')+\frac{3}{2}\eta f_{2/3}''-2f_{2/3}'=-(1-F_0'-\eta F_0'')$$

$$f_{1/3\,1/3\,1/3}'''-\alpha_0(f_{1/3\,1/3\,1/3}'+\eta f_{1/3\,1/3\,1/3}'')+\frac{3}{2}\eta f_{1/3\,1/3\,1/3}''-3f_{1/3\,1/3\,1/3}'=(f_{1/3\,1/3}'+\eta f_{1/3\,1/3}'')$$

$$f_{1/3\,2/3}'''-\alpha_0(f_{1/3\,2/3}'+\eta f_{1/3\,2/3}'')+\frac{3}{2}\eta f_{1/3\,2/3}''-3f_{1/3\,2/3}'=(f_{2/3}'+\eta f_{2/3}'')+(f_{1/3}'+\eta f_{1/3}'')$$

$$f_{1}'''-\alpha_0(f_{1}'+\eta f_{1}'')+\frac{3}{2}\eta f_{1}''-3f_{1}'=-(1-F_0'-\eta F_0'')$$

$$f_{1/3\,1/3\,1/3\,1/3}'''-\alpha_0(f_{1/3\,1/3\,1/3\,1/3}'+\eta f_{1/3\,1/3\,1/3\,1/3}'')+\frac{3}{2}\eta f_{1/3\,1/3\,1/3\,1/3}''-4f_{1/3\,1/3\,1/3\,1/3}'=(f_{1/3\,1/3\,1/3}'+$$
$$+\eta f_{1/3\,1/3\,1/3}'')$$

$$f_{1/3\,1/3\,2/3}'''-\alpha_0(f_{1/3\,1/3\,2/3}'+\eta f_{1/3\,1/3\,2/3}'')+\frac{3}{2}\eta f_{1/3\,1/3\,2/3}''-4f_{1/3\,1/3\,2/3}'=(f_{1/3\,2/3}'+\eta f_{1/3\,2/3}'')+$$
$$+(f_{1/3\,1/3}'+\eta f_{1/3\,1/3}'')$$

$$f_{1/3\,1}'''-\alpha_0(f_{1/3\,1}'+\eta f_{1/3\,1}'')+\frac{3}{2}\eta f_{1/3\,1}''-4f_{1/3\,1}'=(f_{1}'+\eta f_{1}'')+(f_{1/3}'+\eta f_{1/3}'')$$

$$f_{2/3\,2/3}'''-\alpha_0(f_{2/3\,2/3}'+\eta f_{2/3\,2/3}'')+\frac{3}{2}\eta f_{2/3\,2/3}''-4f_{2/3\,2/3}'=(f_{2/3}'+\eta f_{2/3}'')$$

$$f_{4/3}'''-\alpha_0(f_{4/3}'+\eta f_{4/3}'')+\frac{3}{2}\eta f_{4/3}''-4f_{4/3}'=-(1-F_0'-\eta F_0'')$$

$$f_{1/3\,1/3\,1/3\,1/3\,1/3}'''-\alpha_0(f_{1/3\,1/3\,1/3\,1/3\,1/3}'+\eta f_{1/3\,1/3\,1/3\,1/3\,1/3}'')+\frac{3}{2}\eta f_{1/3\,1/3\,1/3\,1/3\,1/3}''-5f_{1/3\,1/3\,1/3\,1/3\,1/3}'=$$
$$=(f_{1/3\,1/3\,1/3\,1/3}'+\eta f_{1/3\,1/3\,1/3\,1/3}'')$$

$$f'''_{1/3\,1/3\,1/3\,2/3} - \alpha_0\left(f'_{1/3\,1/3\,1/3\,2/3} + \eta f''_{1/3\,1/3\,1/3\,2/3}\right) + \frac{3}{2}\eta f''_{1/3\,1/3\,1/3\,2/3} - 5f'_{1/3\,1/3\,1/3\,2/3} =$$
$$= \left(f'_{1/3\,1/3\,1/3} + \eta f''_{1/3\,1/3\,1/3}\right) + \left(f'_{1/3\,1/3\,2/3} + \eta f''_{1/3\,1/3\,2/3}\right)$$

$$f'''_{1/3\,1/3\,1} - \alpha_0\left(f'_{1/3\,1/3\,1} + \eta f''_{1/3\,1/3\,1}\right) + \frac{3}{2}\eta f''_{1/3\,1/3\,1} - 5f'_{1/3\,1/3\,1} = \left(f'_{1/3\,1} + \eta f''_{1/3\,1}\right) +$$
$$+ \left(f'_{1/3\,1/3} + \eta f''_{1/3\,1/3}\right)$$

$$f'''_{1/3\,2/3\,2/3} - \alpha_0\left(f'_{1/3\,2/3\,2/3} + \eta f''_{1/3\,2/3\,2/3}\right) + \frac{3}{2}\eta f''_{1/3\,2/3\,2/3} - 5f'_{1/3\,2/3\,2/3} = \left(f'_{2/3\,2/3} +\right.$$
$$\left. + \eta f''_{2/3\,2/3}\right) + \left(f'_{1/3\,2/3} + \eta f''_{1/3\,2/3}\right)$$

$$f'''_{1/3\,4/3} - \alpha_0\left(f'_{1/3\,4/3} + \eta f''_{1/3\,4/3}\right) + \frac{3}{2}\eta f''_{1/3\,4/3} - 5f'_{1/3\,4/3} = \left(f'_{4/3} + \eta f''_{4/3}\right) + \left(f'_{1/3} + \eta f''_{1/3}\right)$$

$$f'''_{2/3\,1} - \alpha_0\left(f'_{2/3\,1} + \eta f''_{2/3\,1}\right) + \frac{3}{2}\eta f''_{2/3\,1} - 5f'_{2/3\,1} = \left(f_1' + \eta f_1''\right) + \left(f'_{2/3} + \eta f''_{2/3}\right)$$

$$f'''_{5/3} - \alpha_0\left(f'_{5/3} + \eta f''_{5/3}\right) + \frac{3}{2}\eta f''_{5/3} - 5f'_{5/3} = -\left(1 - F_0' - \eta F_0''\right)$$

b. $$\varphi'''_{2/3} - \alpha_0\left(\varphi'_{2/3} + \eta \varphi''_{2/3}\right) + \frac{3}{2}\eta \varphi'_{2/3} - 2\varphi'_{2/3} = -\left(1 - F_0'^2 + F_0 F_0''\right)$$

$$\varphi'''_{1/3\,2/3} - \alpha_0\left(\varphi'_{1/3\,2/3} + \eta \varphi''_{1/3\,2/3}\right) + \frac{3}{2}\eta \varphi''_{1/3\,2/3} - 3\varphi'_{1/3\,2/3} = \left(\varphi'_{2/3} + \eta \varphi''_{2/3}\right) -$$
$$- \left(-2F_0' f'_{1/3} + F_0 f''_{1/3} + f_{1/3} F_0''\right)$$

$$\varphi_1''' - \alpha_0\left(\varphi_1' + \eta \varphi_1''\right) + \frac{3}{2}\eta \varphi_1'' - 3\varphi_1' = -\left(1 - F_0'^2 + F_0 F_0''\right)$$

$$\varphi'''_{1/3\,1/3\,2/3} - \alpha_0\left(\varphi'_{1/3\,1/3\,2/3} + \eta \varphi''_{1/3\,1/3\,2/3}\right) + \frac{3}{2}\eta \varphi''_{1/3\,1/3\,2/3} - 4\varphi'_{1/3\,1/3\,2/3} = \left(\varphi'_{1/3\,2/3} + \eta \varphi''_{1/3\,2/3}\right)$$
$$- \left(-2F_0' f'_{1/3\,1/3} - f'^2_{1/3} + F_0 f''_{1/3\,1/3} + f_{1/3} f''_{1/3} + f_{1/3\,1/3} F_0''\right)$$

$$\varphi'''_{2/3\,2/3} - \alpha_0\left(\varphi'_{2/3\,2/3} + \eta \varphi''_{2/3\,2/3}\right) + \frac{3}{2}\eta \varphi''_{2/3\,2/3} - 4\varphi'_{2/3\,2/3} = \left(\varphi'_{2/3} + \eta \varphi''_{2/3}\right) - \left(-2F_0' f'_{2/3}\right.$$
$$\left. + F_0 f''_{2/3} + f_{2/3} F_0''\right)$$

$$\varphi'''_{1/3\,1} - \alpha_0\left(\varphi'_{1/3\,1} + \eta \varphi''_{1/3\,1}\right) + \frac{3}{2}\eta \varphi''_{1/3\,1} - 4\varphi'_{1/3\,1} = \left(\varphi_1' + \eta \varphi_1''\right) - \left(-2F_0' f'_{1/3} + F_0 f''_{1/3} + f_{1/3} F_0''\right)$$

$$\varphi'''_{4/3} - \alpha_0\left(\varphi'_{4/3} + \eta \varphi''_{4/3}\right) + \frac{3}{2}\eta \varphi''_{4/3} - 4\varphi'_{4/3} = -\left(1 - F_0'^2 + F_0 F_0''\right)$$

$$\varphi'''_{1/3\,1/3\,1/3\,2/3} - \alpha_0\left(\varphi'_{1/3\,1/3\,1/3\,2/3} + \eta \varphi''_{1/3\,1/3\,1/3\,2/3}\right) + \frac{3}{2}\eta \varphi''_{1/3\,1/3\,1/3\,2/3} - 5\varphi'_{1/3\,1/3\,1/3\,2/3} = \left(\varphi'_{1/3\,1/3\,2/3} +\right.$$
$$\left. + \eta \varphi''_{1/3\,1/3\,2/3}\right) - \left(-2F_0' f'_{1/3\,1/3\,1/3} - 2f'_{1/3} f'_{1/3\,1/3} + F_0 f''_{1/3\,1/3\,1/3} + f_{1/3} f''_{1/3\,1/3} + f_{1/3\,1/3} f''_{1/3} + f_{1/3\,1/3\,1/3} F_0''\right)$$

$$\varphi'''_{1/3\,2/3\,2/3} - \alpha_0\left(\varphi'_{1/3\,2/3\,2/3} + \eta \varphi''_{1/3\,2/3\,2/3}\right) + \frac{3}{2}\eta \varphi''_{1/3\,2/3\,2/3} - 5\varphi'_{1/3\,2/3\,2/3} = \left(\varphi'_{2/3\,2/3} + \eta \varphi''_{2/3\,2/3}\right)$$
$$+ \left(\varphi'_{1/3\,2/3} + \eta \varphi''_{1/3\,2/3}\right) - \left(-2F_0' f'_{1/3\,2/3} - 2f'_{1/3} f'_{2/3} + F_0 f''_{1/3\,2/3} + f_{1/3} f''_{2/3} + f_{2/3} f''_{1/3} + f_{1/3\,2/3} F_0''\right)$$

$$\varphi'''_{1\,2/3} - \alpha_0\left(\varphi'_{1\,2/3} + \eta \varphi''_{1\,2/3}\right) + \frac{3}{2}\eta \varphi''_{1\,2/3} - 5\varphi'_{1\,2/3} = \left(\varphi'_{2/3} + \eta \varphi''_{2/3}\right) - \left(-2F_0' f_1' + F_0 f_1'' + f_1 F_0''\right)$$

$$\varphi'''_{1/3\,1/3\,1} - \alpha_0(\varphi'_{1/3\,1/3\,1} + \eta\varphi''_{1/3\,1/3\,1}) + \frac{3}{2}\eta\varphi''_{1/3\,1/3\,1} - 5\varphi'_{1/3\,1/3\,1} = (\varphi'_{1/3\,1} + \eta\varphi''_{1/3\,1}) - (-2F_0' f'_{1/3\,1/3} - f'^2_{1/3} + F_0 f''_{1/3\,1/3} + f_{1/3} f''_{1/3} + f_{1/3\,1/3} F_0'')$$

$$\varphi'''_{2/3\,1} - \alpha_0(\varphi'_{2/3\,1} + \eta\varphi''_{2/3\,1}) + \frac{3}{2}\eta\varphi''_{2/3\,1} - 5\varphi'_{2/3\,1} = (\varphi_1' + \eta\varphi_1'') - (-2F_0' f'_{2/3} + F_0 f''_{2/3} + f_{2/3} F_0'')$$

$$\varphi'''_{1/3\,4/3} - \alpha_0(\varphi'_{1/3\,4/3} + \eta\varphi''_{1/3\,4/3}) + \frac{3}{2}\eta\varphi''_{1/3\,4/3} - 5\varphi'_{1/3\,4/3} = (\varphi'_{4/3} + \eta\varphi''_{4/3}) - (-2F_0' f'_{1/3} + F_0 f''_{1/3} + f_{1/3} F_0'')$$

$$\varphi'''_{5/3} - \alpha_0(\varphi'_{5/3} + \eta\varphi''_{5/3}) + \frac{3}{2}\eta\varphi''_{5/3} - 5\varphi'_{5/3} = -(1 - F_0'^2 + F_0 F_0'')$$

c. $$h'''_{2/3\,2/3} - \alpha_0(h'_{2/3\,2/3} + \eta h''_{2/3\,2/3}) + \frac{3}{2}\eta h''_{2/3\,2/3} - 4h'_{2/3\,2/3} = -(-2F_0'\varphi'_{2/3} + F_0''\varphi_{2/3} + F_0\varphi''_{2/3})$$

$$h'''_{1/3\,2/3\,2/3} - \alpha_0(h'_{1/3\,2/3\,2/3} + \eta h''_{1/3\,2/3\,2/3}) + \frac{3}{2}\eta h''_{1/3\,2/3\,2/3} - 5h'_{1/3\,2/3\,2/3} = (h'_{2/3\,2/3} + \eta h''_{2/3\,2/3}) -$$
$$-(-2F_0'\varphi'_{1/3\,2/3} - 2f'_{1/3}\varphi'_{2/3} + \varphi_{2/3} f''_{1/3} + \varphi_{1/3\,2/3} F_0'' + F_0\varphi''_{1/3\,2/3} + f_{1/3}\varphi''_{2/3})$$

$$h'''_{2/3\,1} - \alpha_0(h'_{2/3\,1} + \eta h''_{2/3\,1}) + \frac{3}{2}\eta h''_{2/3\,1} - 5h'_{2/3\,1} = -(-2F_0'\varphi_1' + \varphi_1 F_0'' + F_0\varphi_1'') -$$
$$-(-2F_0'\varphi'_{2/3} + \varphi_{2/3}F_0'' + F_0\varphi''_{2/3})$$

d. $$ha'''_{2/3\,2/3} - \alpha_0(ha'_{2/3\,2/3} + \eta ha''_{2/3\,2/3}) + \frac{3}{2}\eta ha''_{2/3\,2/3} - 4ha'_{2/3\,2/3} = -(\varphi_{2/3}F_0'' - \varphi'_{2/3}F_0')$$

$$ha'''_{1/3\,2/3\,2/3} - \alpha_0(ha'_{1/3\,2/3\,2/3} + \eta ha''_{1/3\,2/3\,2/3}) + \frac{3}{2}\eta ha''_{1/3\,2/3\,2/3} - 5ha'_{1/3\,2/3\,2/3} = (ha'_{2/3\,2/3} +$$
$$+ \eta ha''_{2/3\,2/3}) - (\varphi_{2/3} f''_{1/3} + \varphi_{1/3\,2/3}F_0'' - F_0'\varphi'_{1/3\,2/3} - \varphi'_{2/3} f'_{1/3})$$

$$ha'''_{2/3\,1} - \alpha_0(ha'_{2/3\,1} + \eta ha''_{2/3\,1}) + \frac{3}{2}\eta ha''_{2/3\,1} - 5ha'_{2/3\,1} = -(\varphi_1 F_0'' - F_0\varphi_1') -$$
$$-(\varphi_{2/3}F_0'' - F_0'\varphi'_{2/3})$$

. . . .

Kod e, f, g sistemi bi počeli tek od stepena 6/3 odnosno člana L_2, . . . , a mi smo se zadržali samo na članovima do reda 5/3 te ostale aproksimacije otpadaju.

Sada posmatrajmo još specijalan slučaj kada je:

$$\alpha(\tau) = \alpha_0$$

Specijalan slučaj:

$$\alpha(\tau)=\alpha_0$$

Ranije smo videli da ovoj vrednosti glavne funkcije odgovara sledeći oblik rasporeda brzina po vremenu

$$\Omega(t)=\Omega_0 t^m$$

gde je

$$m=\frac{\alpha_0}{3-2\alpha_0} \qquad za \qquad \alpha_0 \neq \frac{3}{2}$$

Rasporedu brzina datom u gornjem obliku odgovara sledeća vrednost funkcije $\beta(\tau)$

$$\beta(\tau)=\frac{\nu 3\tau}{\Omega}=\frac{3\int_0^t \Omega^2 dt}{\Omega}=3\frac{\int_0^t \Omega_0^2 t^{2m}dt}{\Omega_0 t^m}$$

ili kada se integrali dobija se:

$$\beta(\tau)=\frac{3}{2m+1}\Omega_0 t^{m+1}$$

Iz

$$\tau=\frac{1}{\nu}\int_0^t \Omega^2 dt=\frac{1}{\nu}\Omega_0^2\int_0^t t^{2m}dt=\frac{1}{\nu}\frac{\Omega_0^2}{2m+1}t^{2m+1}$$

sleduje

$$t=\left[\frac{\nu(2m+1)}{\Omega_0^2}\right]^{\frac{1}{2m+1}}\tau^{\frac{1}{2m+1}}$$

te je

$$\beta(\tau)=\frac{3\Omega_0}{2m+1}\left[\frac{\nu(2m+1)}{\Omega_0^2}\right]^{\frac{m+1}{2m+1}}\tau^{\frac{m+1}{2m+1}}$$

uvedimo oznaku

$$K=\frac{m+1}{2m+1}$$

dobićemo da je

$$\beta(\tau)=\beta_K \tau^K$$

Osvrnimo se sada ponovo na ulogu funkcija $\alpha(\tau)$ i $\beta(\tau)$ Sem klasifikacije problema, koji se mogu rešavati po ovoj metodi, a koju smo izvršili u §4 prema glavnoj funkciji $\alpha(\tau)$, ova nam funkcija daje i oblik rešenja za prvu jednačinu sistema 60. Analogno redu za $\alpha(\tau)$ dali smo rešenje za $F(\eta,\tau)$ u obliku istog reda tj. $\alpha(\tau)=\alpha_0+\alpha_1\tau+\cdots$ te je i rešenje za ovaj slučaj A dato u obliku reda $F(\eta,\tau)=F_0(\eta)+\tau F_1(\eta)+\cdots$

Za ostale parcijalne jednačine sistema 60. rešenja smo isto dali u obliku reda po τ, ali nam je oblik reda i početni član u tom redu bio diktiran oblikom reda funkcije $\beta(\tau)$ odnosno prvim (početnim) članom toga reda. Napr. u slučaju A imali smo da je $\beta(\tau)=\beta_1\tau+\cdots$ te je rešenje trebalo dati u obliku istog reda koji počinje sa linearnim članom tj $\Phi(\eta,\tau)=\Phi_1(\eta)\tau+\Phi_2(\eta)\tau^2+\cdots$ a u slučaju B imali smo $\beta(\tau)=\beta_{2/3}\tau^{2/3}+\cdots$ te je i rešenje trebalo dati u obliku $\Phi(\eta,\tau)=\Phi_{2/3}(\eta)\tau^{2/3}+\cdots$

U ovom specijalnom slučaju je $\beta(\tau)=\beta_K\tau^K$ te će i rešenje biti dato u obliku $\Phi(\eta,\tau)=\Phi_K(\eta)\tau^K$

Pošto se u trećoj parcijalnoj jednačini sistema 60. javlja proizvod funkcija $\beta(\tau)$ i $\Phi(\eta,\tau)$ to je očigledno da za slučaj A mora rešenje biti dato u obliku reda koji počinje sa kvadratnim članom $H(\eta,\tau)=H_2(\eta)\tau^2+\ldots$ U ovom specijalnom slučaju je $H(\eta,\tau)=H_{2K}(\eta)\tau^{2K}$

Dalje je očigledno da će za četvrtu jednačinu biti isto $H_a(\eta,\tau)=H_{a2k}(\eta)\tau^{2K}$ a za petu $L(\eta,\tau)=L_{3k}(\eta)\tau^{3k}$ i za šestu i sedmu $L_a(\eta,\tau)=L_{a3k}(\eta)\tau^{3K}$ i $L_b(\eta,\tau)=L_{b3K}(\eta)\tau^{3K}$

Sistem parcijalnih jednačina 9. za ovaj specijalan slučaj svodi se na sledeći oblik:

Oblik glavne funkcije $\alpha(\tau)=\alpha_0$ daje nam oblik rešenja koje ima sledeći izgled $F(\eta,\tau)=F_0(\eta)$, a diferencijalna jednačina za odredjivanje ove funkcije ima oblik

$$F_0'''+\alpha_0(1-F_0'-\eta F_0'')+\frac{3}{2}\eta F_0''=0$$

Druga jednačina sistema 9. ako se uvede $\beta(\tau)=\beta_K\tau^K$ i $\Phi=\Phi_K\tau^K$ dobija sledeći oblik

$$\tau^K\Phi_K'''-\alpha_0(\Phi_K'+\eta\Phi_K'')\tau^K+\frac{3}{2}\eta\Phi_K''\tau^K-3K\Phi_K'\tau^K+\beta_K\tau^K(1-F_0'^2+F_0F_0'')=0$$

ili

$$\Phi_K'''-\alpha_0(\Phi_K'+\eta\Phi_K'')+\frac{3}{2}\eta\Phi_K''-3K\Phi_K'+\beta_K(1-F_0'^2+F_0F_0'')=0$$

Treća jednačina se svodi na sledeći oblik ako se vodi računa da je $H_{2K}(\eta,\tau)=H_{2K}(\eta)\tau^{2K}$

$$H_{2K}'''-\alpha_0(H_{2K}'+\eta H_{2K}'')+\frac{3}{2}\eta H_{2K}''-6KH_{2K}'+\beta_K(-2F_0'\Phi_K'+F_0\Phi_K''+\Phi_KF_0'')=0$$

Četvrta jednačina se prevodi na sledeći oblik

$$H_{a2K}'''-\alpha_0(H_{a2K}'+\eta H_{a2K}'')+\frac{3}{2}\eta H_{a2K}''-6KH_{a2K}'+\beta_K(\Phi_KF_0''-F_0'\Phi_K')=0$$

Peta jednačina se svodi na

$$L_{3K}''' - \alpha_o(L_{3K}' + \eta L_{3K}'') + \frac{3}{2}\eta L_{3K}'' - 9K L_{3K}' + \beta_K(-2F_o'H_{2K}' - \phi_K'^2 + H_{2K}F_o'' + \phi_K\phi_K'' + F_oH_{2K}'') = 0$$

šesta jednačina dobija sledeći oblik

$$L_{a3K}''' - \alpha_o(L_{a3K}' + \eta L_{a3K}'') + \frac{3}{2}\eta L_{a3K}'' - 9K L_{a3K}' + \beta_K(-3F_o'H_{a2K}' + 2H_{a2K}F_o'' + 2H_{aK}F_o'' - 2F_o'H_{2K}' + F_oH_{a2K}'' + \phi_K\phi_K'' - \phi_K'^2) = 0$$

i sedma jednačina

$$L_{b3K}''' - \alpha_o(L_{b3K}' + \eta L_{b3K}'') + \frac{3}{2}\eta L_{b3K}'' - 9K L_{b3K}' + \beta_K(H_{a2K}F_o'' - F_o'H_{a2K}') = 0$$

Ili uvodeći

$$F_o = F_o$$
$$\phi_K = \beta_K \varphi_K$$
$$H_{2K} = \beta_K^2 h_{KK} \qquad H_{a2K} = \beta_K^2 h_{aKK}$$
$$L_{3K} = \beta_K^3 L_{KKK} \qquad L_{a3K} = \beta_K^3 L_{aKKK} \qquad L_{b3K} = \beta_K^3 l_{bKKK}$$

dobija se sledeći sistem diferencijalnih jednačina

$$F_o''' + \alpha_o(1 - F_o' - \eta F_o'') + \frac{3}{2}\eta F_o'' = 0$$

$$\varphi_K''' - \alpha_o(\varphi_K' + \eta\varphi_K'') + \frac{3}{2}\eta\varphi_K'' - 3K\varphi_K' = -(1 - F_o'^2 + F_oF_o'')$$

$$h_{KK}''' - \alpha_o(h_{KK}' + \eta h_{KK}'') + \frac{3}{2}\eta h_{KK}'' - 6K h_{KK}' = -(-2F_o'\varphi_K' + F_o\varphi_K'' + \varphi_K F_o'')$$

$$h_{oKK}''' - \alpha_o(h_{aKK}' + \eta h_{KK}'') + \frac{3}{2}\eta h_{aKK}'' - 6K h_{aKK}' = -(\varphi_K F_o'' - F_o'\varphi_K')$$

$$L_{KKK}''' - \alpha_o(L_{KKK}' + \eta l_{KKK}'') + \frac{3}{2}\eta L_{KKK}'' - 9K l_{KKK}' = -(-2F_o'h_{KK}' - \varphi_K'^2 + h_{KK}F_o'' + \varphi_K\varphi_K'' + F_oh_{KK}'')$$

$$L_{aKKK}''' - \alpha_o(L_{aKKK}' + \eta L_{aKKK}'') + \frac{3}{2}\eta L_{aKKK}'' - 9K L_{aKKK}' = -(-3F_o'h_{aKK}' + 2h_{aKK}F_o'' + 2h_{KK}F_o'' - 2F_o'h_{KK}' + F_oh_{aKK}'' + \varphi_K\varphi_K'' - \varphi_K'^2)$$

$$l_{bKKK}''' - \alpha_o(l_{bKKK}' + \eta l_{bKKK}'') + \frac{3}{2}\eta l_{bKKK}'' - 9K l_{bKKK}' = -(h_{aKK}F_o'' - F_o'h_{aKK}')$$

Gornjim jednačinama možemo dati i drugi oblik. Ranije smo videli da je

$$m = \frac{\alpha_o}{3 - 2\alpha_o} \quad ili\ odavde \quad \alpha_o = \frac{3m}{2m+1} \quad i\ da\ je \quad K = \frac{m+1}{2m+1}$$

Pošto je

$$\eta = \frac{\Omega Y}{\nu\sqrt{3\tau}} = \sqrt{\frac{4(2m+1)}{3}}\,\eta_1$$

gde je η_1 - promenljiva "sličnih rešenja", to gornje diferencijalne jednačine možemo izraziti u funkciji nove promenljive Napr. prevedimo prvu od gornjih jednačina, koju ćemo pisati u sledećem obliku:

$$F_o''' + \left(\frac{3}{2} - \alpha_o\right)\eta F_o'' - \alpha_o F_o' = -\alpha_o$$

Vodeći računa da je

$$\frac{dF_o'}{d\eta} = \sqrt{\frac{3}{4(2m+1)}}\, F_o'' \qquad \frac{d^2F_o'}{d\eta^2} = \frac{3}{4(2m+1)} F_o'''$$

Ako sada u datu jednačinu smenimo α_o i K u funkciji od m i vodimo računa o izvodima datim u gornjem obliku, jednačina se svodi na sledeći oblik

$$\frac{3}{4(2m+1)} F_o''' + \left(\frac{3}{2} - \frac{3m}{2m+1}\right)\eta_1\sqrt{\frac{4(2m+1)}{3}}\sqrt{\frac{3}{4(2m+1)}}F_o'' - \frac{3m}{2m+1}F_o' = -\frac{3m}{2m+1}$$

ili ako se sredi dobija se konačno

$$F_o''' + 2\eta_1 F_o'' - 4m\,(1-F_o') = 0$$

Na isti način transformišemo i ostale jednačine ali: prethodno rastavimo β_K na

$$\beta_K = \frac{3}{2m+1}\,\Omega_o\left[\frac{\nu(2m+1)}{\Omega_o^2}\right]^{\frac{m+1}{2m+1}} = \frac{3}{2m+1}\,\beta_K^*$$

i uradimo sledeću stvar; član $\frac{3}{2m+1}$ ostavimo u datim jednačinama, tako da je

$$\frac{3}{4(2m+1)}\varphi_K''' + \left(\frac{3}{2} - \frac{3m}{2m+1}\right)\eta_1\sqrt{\frac{4(2m+1)}{3}}\sqrt{\frac{3}{4(2m+1)}}\varphi_K'' - \left(3\frac{m+1}{2m+1} + \frac{3m}{2m+1}\right)\varphi_K' = -\frac{3}{2m+1}\left(1-F_o'^2+F_oF_o''\right)$$

dobiće se posle sredjivanja

$$\varphi_K''' + 2\eta_1\varphi_K'' - 4(2m+1)\varphi_K' = -4\left(1-F_o'^2+F_oF_o''\right)$$

Ostale jednačine na isti način se svode na sledeći oblik.

$$h_{KK}''' + 2\eta_1 h_{KK}'' - 4(3m+2)\,h_{KK}' = -4\left(-2F_o'\varphi_K' + F_o\varphi_K'' + \varphi_K F_o''\right)$$

$$h_{aKK}''' + 2\eta_1 h_{aKK}'' - 4(3m+2)\,h_{aKK}' = -4\left(\varphi_K F_o'' - F_o'\varphi_K'\right)$$

$$f_{KKK}''' + 2\eta_1 f_{KKK}'' - 4(4m+3)\,f_{KKK}' = -4\left(-2F_o'h_{KK}' - \varphi_K'^2 + h_{KK}F_o'' + \varphi_K\varphi_K'' + F_o h_{KK}''\right)$$

$$f_{aKKK}''' + 2\eta_1 f_{aKKK}'' - 4(4m+3)\,f_{aKKK}' = -4\left(-3F_o'h_{aKK}' + 2h_{aKK}F_o'' + 2h_{KK}F_o'' - 2F_o'h_{KK}' + F_o h_{aKK}'' + \varphi_K\varphi_K'' - \varphi_K'^2\right)$$

$$f_{bKKK}''' + 2\eta_1 f_{bKKK}'' - 4(4m+3) = -4\left(h_{aKK}F_o'' - F_o'h_{aKK}'\right)$$

gde su sada

$$F_0 = F_0 \qquad\qquad L_K = \beta_K^{*3} \ell_{KKK}$$
$$\Phi_K = \beta_K^{*} \varphi_K \qquad\qquad L_{aK} = \beta_K^{*3} \ell_{aKKK}$$
$$H_K = \beta_K^{*2} h_{KK} \qquad\qquad L_{bK} = \beta_K^{*3} \ell_{bKKK}$$

I strujna funkcija sada se transformiše na novi oblik

$$\Psi(x,y,t) = \nu\sqrt{3\tau}\, U F(x,\eta,\tau) = \nu\sqrt{3\tau}\, U(x) \Big\{ F_0(\eta) + U'(x)\tau^K \Phi_K(\eta) +$$
$$+ \left[U'^2 H_{2K}(\eta) + UU'' H_{a2K}(\eta)\right]\tau^{2K} + \left[U'^3 L_{3K}(\eta) + UU'U'' L_{a3K}(\eta) +\right.$$
$$\left. + U^2 U''' L_{b3K}(\eta)\right]\tau^{3K} + \dots\dots = \nu\sqrt{\frac{3}{2m+1}\,\frac{\Omega_0^2}{\nu}\, t^{2m+1}}\; U(x) \Big\{ F_0(\eta_1)\cdot$$
$$\sqrt{\frac{4(2m+1)}{3}} + U'\beta_K^{*}\varphi_K(\eta_1)\sqrt{\frac{4(2m+1)}{3}}\,\tau^K + \left[U'^2 h_{KK}(\eta_1) + UU'' h_{aKK}(\eta_1)\right]\cdot$$
$$\cdot \beta_K^{*2}\sqrt{\frac{4(2m+1)}{3}}\,\tau^{2K} + \dots\dots =$$
$$= 2\sqrt{\nu t}\; U(x)\,\Omega_0 t^m \Big\{ F_0(\eta_1) + U'\Omega_0 \left[\frac{\nu(2m+1)}{\Omega_0^2}\right]^K \left[\frac{\Omega_0^2}{\nu(2m+1)}\right]^K \cdot$$
$$(t^{2m+1})^K \varphi_K(\eta_1) + \left[U'^2 h_{KK}(\eta_1) + UU'' h_{aKK}(\eta_1)\right]\Omega_0^2 \left[\frac{\nu(2m+1)}{\Omega_0^2}\right]^{2K} \left[\frac{\Omega_0^2}{\nu(2m+1)}\right]^{2K} (t^{2m+1})^{2K} + \dots$$

ili ako se sredi dobija se konačno

$$\Psi(x,y,t) = 2\sqrt{\nu t}\; U(x)\,\Omega_0 t^m \Big\{ F_0(\eta_1) + \Omega_0 U' t^{m+1} \varphi_K(\eta_1) + \Omega_0^2 t^{2(m+1)} \cdot$$
$$\left[U'^2 h_{KK}(\eta_1) + UU'' h_{aKK}(\eta_1)\right] + \Omega_0^3 t^{3(m+1)} \left[U'^3 \ell_{KKK}(\eta_1) +\right.$$
$$\left. + UU'U'' \ell_{aKKK}(\eta_1) + U^2 U''' \ell_{bKKK}(\eta_1)\right] + \dots\dots$$

Vidi se da u slučaju ako je brzina spoljnog potencijalnog strujanja data u obliku $U(x,t)=U(x)\cdot t^m$, da se u tom slučaju i oblik strujne funkcije i jednačine za odredjivanje nepoznatih koeficijenata-funkcija iz naše metode svode na isti oblik koji je Vatson dobio pri rešavanju istog slučaja, što je siguran znak tačnosti ove metode.

§ 7 NALAZENJE VREMENA ODVAJANJA I PREDJENOG PUTA TELOM ZA TO VREME

Kod nestacionarnih graničnih slojeva interesantno je naći momenat početka odvajanja sloja sa površine tela tj. vreme koje prodje od početka kretanja do momenta kada prvi put počinje da se odvaja granični sloj. Od prvog trenutka odvajanja sloja pa nadalje tačka odvajanja će se pomerati duž konture sa vremenom sve do momenta dostizanja stacionarnog stanja. Zato je u toku proračuna interesantno tražiti vreme potrebno da tačka odvajanja dodje na neko mesto duž konture u intervalu od mesta prvog odvajanja do mesta gde bi se nalazila pri dostizanju stacionarnog stanja.

Tačku odvajanja ćemo naći kao i u slučaju stacionarnih graničnih slojeva iz uslova da je tangencijalni napon na zidu u toj tački ravan nuli. Samo što će ovde ta tačka biti funkcija koordinate **x** i vremena **t** (odnosno τ).

Za tangencijalni napon dobija se sledeći izraz

$$N_{tan} = \mu\left(\frac{\partial u}{\partial y}\right) = \mu\, U_{(x)} \frac{\Omega^2}{\nu\sqrt{3\tau}}\, F_{\eta\eta}(x,\eta,\tau)$$

ili na zidu: $y=0$; $\eta=0$

$$N_{tan} = \mu\, U \frac{\Omega^2}{\nu\sqrt{3\tau}}\, F_{\eta\eta}(x,0,\tau) = \mu\, U \frac{\Omega^2}{\nu\sqrt{3\tau}} \Big\{ F_0''(0) + F_1''(0)\tau + \cdots U'\big[\Phi_1''(0)\tau + \cdots\big] + \cdots \Big\}$$

a u tački odvajanja je $N_{tan} = 0 \;\therefore$

$$\left\{ \sum_{K=0}^{\infty} F_K''(0)\tau^K + U'\sum_{K=1}^{\infty}\Phi_K''(0)\tau^K + U'^2\sum_{K=2}^{\infty} H_K''(0)\tau^K + UU''\sum_{K=2}^{\infty} H_{aK}''(0)\tau^K + \cdots \right\} = 0$$

Te smo dobili jednačinu za odredjivanje vremena odvajanja, odnosno, jednačinu koja nam daje položaj tačke odvajanja na konturi, u intervalu izmedju tačke u kojoj najpre dodje do odvajanja i tačke gde bi se nalazila pri dostizanju stacionarnog stanja, u sv kom trenutku vremena.

Ova se jednačina može napisati i u razvijenom obliku

$$F_0''(0) + \left[F_1''(0) + U'\Phi_1''(0)\right]\tau + \left[F_2''(0) + U'\Phi_2''(0) + U'^2 H_2''(0) + UU'' H_{a2}''(0)\right]\cdot$$

$$\tau^2 + \left[F_3''(0) + U'\Phi_3''(0) + U'^2 H_K''(0) + UU'' H_{a3}''(0) + U'^3 L_3''(0) + \right.$$

$$\left. + UU'U'' L_{a3}''(0) + U^2U''' L_{b3}'''(0)\right]\tau^3 + \cdots\cdots = 0$$

Te smo dobili po jedan red po τ sa koeficijentima u zagradama[ ] koji su funkcije od x. Gornju jednačinu ćemo rešiti po τ, a zatim iz τ naći vreme t.

Put predjen telom do trenutka odvajanja

$$S = \int_0^{t_s} \Omega \, dt$$

ili pošto je $d\tau = \frac{\Omega^2}{\nu} dt$ to je

$$S = \nu \int_0^{\tau_s} \frac{d\tau}{\Omega(\tau)}$$

§8 NAČIN UPOTREBE METODE

U ovom paragrafu iznećemo uputstvo za upotrebu date metode. Obično su problemi teorije graničnog sloja tako postavljeni da je spoljni raspored brzina zadata funkcija. Zato ćemo i posmatrati ovaj normalan slučaj. Medjutim, isto tako može biti unapred zadata i glavna funkcija $\alpha(\tau)$.

Način kako se upotrebljava metoda pokazaćemo na slučaju A. Kod slučaja A je:

$$\Omega(t) = \Omega_0 + \Omega_1 t + \cdots\cdots$$

sa $\Omega_0 \neq 0$

Radi lakšeg rada preporučljivo je sve veličine učiniti bezdimenzionim, kao što smo već pokazali u § 3

$$\bar{t} = \frac{V}{L} t \qquad \bar{y} = y \frac{\sqrt{Re}}{L} \qquad \text{sa} \quad Re = \frac{V^2 T}{\nu}$$

$$\bar{\Omega}(\bar{t}) = \frac{\Omega(t)}{V} \qquad \bar{\Omega}_k = \frac{\Omega_k}{\Omega_0} \frac{L^k}{V^k} \qquad \bar{\alpha}_k = \alpha_k Re^k \qquad \bar{F}_k = F_k Re^k$$

$$\bar{\Phi}_k = \Phi_k Re^k, \; \ldots\ldots\ldots, \; \bar{\tau} = \frac{\tau}{Re}$$

Za karakterističnu veličinu V zgodno je uzeti Ω_0.

1. Najpre se sračunava prelaz od koordinata $\bar{t}$, $\bar{y}$ na koordinate

$$\bar{\tau} = \int \bar{\Omega}^2(\bar{t}) d\bar{t} \qquad \bar{\eta} = \frac{\bar{\Omega}(\bar{t})}{\sqrt{3\bar{\tau}(\bar{t})}} \bar{y} = \bar{g}(\bar{t}) \cdot \bar{y}$$

Pošto se faktor $\bar{g}(\bar{t})$ i kasnije javlja to ga je korisno posebno proračunati.

2. Iz datih koeficijenata $\bar{\Omega}_i$ lako se sračunavaju koeficijenti $\bar{\alpha}_k$ preko formula 26.

3. Tangencijalni napon se sračunava prema formuli

$$\left[\frac{\partial \bar{u}}{\partial \bar{y}}\right]_{\bar{y}=0} = \frac{N(x,\bar{t})\,Re^{1/2}}{\rho\,\Omega_0^2} = \bar{U}(x)\,\bar{\Omega}(\bar{t})\frac{\bar{\Omega}(\bar{t})}{\sqrt{3\bar{\tau}}}\,\bar{F}_{\eta\eta}(x,0,\bar{\tau}) =$$

$$= \bar{U}(x,\bar{t})\;\bar{g}(\bar{t})\left(\sum_{K=0}\bar{F}_K''(0)\bar{\tau}^K + U'\sum_{K=1}\bar{\Phi}_K''(0)\bar{\tau}^K + \cdots\right)$$

4. Na odgovarajući način lako se dobija i debljina istiskivanja δ^* prema 12.

$$\frac{\delta^* Re^{1/2}}{L} = \frac{1}{\bar{g}(\bar{t})}\left\{\lim_{\eta\to\infty}(\eta - \bar{F}_0) - \left[\sum_K \bar{F}_K(\infty)\bar{\tau}^K + U'\sum_K \bar{\Phi}_K(\infty)\bar{\tau}^K + \cdots\right]\right\}$$

5. Ako želimo da dobijemo pojedine profile brzina na različitim mestima i u različitim trenucima vremena koristimo za izračunavanje

$$\frac{u(x,\bar{y},\bar{t})}{\Omega_0} = \bar{U}(x)\,\bar{\Omega}(\bar{t})\left[\sum_{K=0}\bar{F}_K'(\eta)\bar{\tau}^K + U'\sum_{K=1}\bar{\Phi}_K'(\eta)\bar{\tau}^K + \cdots\right]$$

6. Ako se želi odrediti vreme odvajanja služimo se izrazom dobivenim u § 6

$$\left[\sum_{K=0}\bar{F}_K''(0)\bar{\tau}^K + \sum_{K=1}\bar{\Phi}_K''(0)\bar{\tau}^K U' + \cdots\cdots\right] = 0$$

7. Za dobivanje predjenog puta do početka odvajanja služimo se izrazom takodje dobivenim u § 6

$$\bar{s} = \frac{s}{L} = \int_0^{\tau_s}\frac{d\bar{\tau}}{\bar{\Omega}} = \int_0^{t_s}\bar{\Omega}\,dt$$

Iz izloženog uputstva očigledno je, da je primena metode vrlo prosta, jedino što ostaje dosta proste računice.

Pošto se nemože dati dokaz konvergencije specijalnog i stepenih redova, odnosno ocena ostataka ako se zadržimo na n-om članu svakog stepenog reda i na m-om članu specijalnog reda, to se dobrota konvergencije mora oceniti od oka. U praktičnom radu verovatno je dovoljno zadržati se na trećem članu ili četvrtom članu specijalnog reda i na 6-im članovima stepenih redova.

Kod Blaziusovog reda u slučaju stacionarnih graničnih slojeva postojala je poznata proba tj. ispunjenje prve konturne veze graničnog sloja neophodno je kod egzaktnog rešenja. Ista proba je neophodna i u našem slučaju. Iz prve konturne veze graničnog sloja

$$U_t + UU_x = -\nu \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$$

vodeći računa o

$$u = U(x)\Omega(t)\, F_\eta(x,\eta,\tau)$$

sleduje

$$U(x)\dot{\Omega} + UU'\Omega^2 = -\nu \frac{\Omega^3}{\nu^2 3\tau} U F_{\eta\eta\eta}$$

ili
$$\dot{\Omega} + \Omega^2 U' = -\nu \Omega g^2 F_{\eta\eta\eta}(x,0,\tau)$$

odnosno ako se unese izraz za funkciju $F_{\eta\eta\eta}(x,o,\tau)$

$$\dot{\Omega} + \Omega^2 U' = -\nu\Omega g^2 \Big[\sum_{K=0} F_K'''(0)\tau^K + U'\sum_{K=1} \Phi_K'''(0)\tau^K + U'^2 \sum_{K=2} H_K'''(0)\tau^K + \cdot$$

Odavde je

$$\dot{\Omega} = -\nu\Omega g^2 \sum_{K=0} F_K'''(0)\tau^K$$

$$\Omega^2 = -\nu\Omega g^2 \sum_{K=1} \Phi_K'''(0)\tau^K$$

$$0 = \nu\Omega g^2 \sum_{K=2} H_K'''(0)\tau^K$$

$$0 = \nu\Omega g^2 \sum_{K=2} H_{aK}'''(0)\tau^K$$

ili pošto je

$$\dot{\Omega} = \frac{\Omega_0^2}{\ell}\dot{\bar{\Omega}} \qquad g = \frac{\Omega}{\nu\sqrt{3\tau}} = \frac{\Omega_0^2}{\nu^2 Re}\frac{\bar{\Omega}^2}{3\bar{\tau}} = \frac{\Omega_0^2}{\nu^2 Re}\bar{g}$$

sleduje

$$\dot{\bar{\Omega}} = -\bar{\Omega}\,\bar{g}^2 \sum_{K=0} \bar{F}_K''(0)\bar{\tau}^K$$

$$\ell\bar{\Omega} = -\bar{g}^2 \sum_{K=1} \bar{\Phi}_K'''(0)\bar{\tau}^K$$

$$0 = \sum_{K=2} \bar{H}_K'''(0)\bar{\tau}^K$$

. . . .

. . . .

Izrazi se mogu dalje transformisati ako se vodi računa da je

$$\alpha(\tau) = \frac{3\nu\tau\dot{\Omega}}{\Omega^3} = \frac{\dot{\Omega}}{\nu\Omega}\frac{1}{g^2} \qquad \beta(\tau) = \frac{3\nu\tau}{\Omega} = \frac{1}{\nu}\frac{\Omega}{g^2}$$

ili

$$\bar{\alpha}(\bar{\tau}) = \frac{\dot{\bar{\Omega}}}{\bar{\Omega}\,\bar{q}^2} \qquad \bar{\beta}(\bar{\tau}) = \ell\,\frac{\bar{\Omega}}{\bar{q}^2}$$

Tako konačno transformisani izrazi imaju sledeći oblik

$$\sum_{K=0} \bar{F}_K'''(0)\,\bar{\tau}^K = -\bar{\alpha}(\bar{\tau})$$

$$\sum_{K=1} \bar{\Phi}_K'''(0)\,\bar{\tau}^K = -\bar{\beta}(\bar{\tau})$$

$$\sum_{K=2} \bar{H}_K'''(0)\,\bar{\tau}^K = 0$$

$$\sum_{K=2} \bar{H}_{aK}'''(0)\,\bar{\tau}^K = 0$$

. . . .

. . . .

Ako se u praktičnim računima zaustavimo na nekom stepenu $\bar{\tau}$ onda treba ispitati sa kako dobrom približnošću su zadovoljeni gornji uslovi tj. dali je

$$\sum_{K=0} \bar{F}_K'''(0)\,\bar{\tau}^K \approx -\bar{\alpha}(\bar{\tau})$$

$$\sum_{K=1} \bar{\Phi}_K'''(0)\,\bar{\tau}^K \approx -\bar{\beta}(\bar{\tau})$$

$$\sum_{K=2} \bar{H}_K'''(0)\,\bar{\tau}^K \approx 0$$

§ 9 REŠAVANJE DIFERENCIJALNIH JEDNAČINA

Slučaj A: $\alpha_0 = 0$

Zaustavljajući se na drugom članu specijalnog reda (druga aproksimacija) i na članovima do četvrtog stepena stepenih redova dobićemo dva rekurzivna sistema diferencijalnih jednačina .

Sistem I:

$$F_0''' + \frac{3}{2}\eta F_0'' = 0$$

$$f_1''' + \frac{3}{2}\eta f_1'' - 3f_1' = -(1 - F_0' - \eta F_0'')$$

$$f_{11}''' + \frac{3}{2}\eta f_{11}'' - 6f_{11}' = f_1' + \eta f_1''$$

$$f_2''' + \frac{3}{2}\eta f_2'' - 6f_2' = -(1 - F_0' - \eta F_0'')$$

. . . .

Sistem II:

$$\varphi_1''' + \frac{3}{2}\eta\varphi_1'' - 3\varphi_1' = -(1 - F_0'^2 + F_0F_0'')$$

$$\varphi_{11}''' + \frac{3}{2}\eta\varphi_{11}'' - 6\varphi_{11}' = (\varphi_1' + \eta\varphi_1'') - (-2F_0'f_1' + F_0f_1'' + f_1F_0'')$$

$$\varphi_2''' + \frac{3}{2}\eta\varphi_2'' - 6\varphi_2' = -(1 - F_0'^2 + F_0F_0'')$$

. . . .

Uvodjenjem nove promenljive

$$\eta_1 = \sqrt{\frac{3}{4}}\ \eta$$

gornji sistemi se transformišu na nove oblike

Sistem I:

$$F_0''' + 2\eta_1 F_0'' = 0$$

$$f_1''' + 2\eta_1 f_1'' - 4f_1' = -\frac{4}{3}(1-F_o' - \eta_1 F_o'')$$

$$f_{11}''' + 2\eta_1 f_{11}'' - 4\cdot 2f_{11}' = \frac{4}{3}(f_1' + \eta_1 f_1'')$$

$$f_2''' + 2\eta_1 f_2'' - 4\cdot 2f_2' = -\frac{4}{3}(1-F_o' - \eta F_o'')$$

. . . .

Sistem II:

$$\varphi_1''' + 2\eta_1\varphi_1'' - 4\varphi_1' = -\frac{4}{3}(1 - F_o'^2 + F_o F_o'')$$

$$\varphi_{11}''' + 2\eta_1\varphi_{11}'' - 4\cdot 2\varphi_{11}' = \frac{4}{3}\left[(\varphi_1' + \eta\varphi_1'') - (-2F_o'f_1' + F_o''f_1'' + f_1 F_o'')\right]$$

. . . .

Ako pogledamo homogene delove gornjih diferencijalnih jednačina videćemo da su one sve linearne diferencijalne jednačine drugog reda oblika

$$Y'' + 2zY' - 4\alpha Y = 0$$

Smenom

$$Y = e^{-\frac{z^2}{2}} D(z)$$

gornja diferencijalna jednačina svodi se na oblik

$$D'' + (-4\alpha - 1 - z^2)\cdot D = o$$

ili ako se uvede smena promenljivih relacijom

$$z_1 = \sqrt{2}\, z$$

diferencijalna jednačina dobija sledeći oblik

$$D'' + \left(-2\alpha - 1 + \frac{1}{2} - \frac{z_1^2}{4}\right)\cdot D = o$$

Ako se uvede oznaka $-2\alpha - 1 = p$ gornja jednačina dobija konačan oblik

$$D'' + \left(p + \frac{1}{2} - \frac{z_1^2}{4}\right)\cdot D = o$$

Dobineva diferencijalna jednačina je Veber-ova, a rešenja su Veber-ove funkcije ili funkcije paraboličnog cilindra [5] .

Za p = o, 1, 2 . . . . rešenja gornje diferencijalne jednačine su:

$$D_p(z_1)\ ; \quad D_p(-z_1); \quad D_{-p-1}(i\, z_1); \quad D_{-p-1}(-i\cdot z_1)$$

Iz asimptotskog razvoja za funkciju $D_p(z_1)$ i $D_{-p-1}(i z_1)$ u granicama $-\frac{3}{4}\pi < \arg z_1 < \frac{1}{4}\pi$, očigledno je, da je odnos ova dva rešenja različit od konstante, te su prema tome ova dva rešenja linearno nezavisna. Linearna kombinacija ova dva rešenja daće rešenje date diferencijalne jednačine

$$D = C_1 D_p(z_1) + C_2 D_{-p-1}(i \cdot z_1)$$

Ako $z_1 \to \infty$ tada $D_{-p-1}(i z_1) \to \infty$ te konstanta $C_2 = 0$ u cilju konačnosti rešenja, stoga je

$$D = C_1 D_p(z_1)$$

Funkcija Veber-a $D_p(z_1)$ razlaže se u redove vida

$$D_p(z_1) = \frac{2^{-\frac{p}{2}-1} e^{-z_1^2/4}}{\Gamma(-p)} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(\frac{m-p}{2}\right)}{m!} (z_1\sqrt{2})^m$$

pri $p \neq 0, 1, 2, \ldots\ldots\ldots$ i

$$D_p(z_1) = 2^{-p/2} e^{-z_1^2/4} \sum_{k=0}^{[\frac{p}{2}]} \frac{(-1)^k\, p!}{k!(p-2k)!} (z_1\sqrt{2})^{p-2k}$$

pri $p = 0, 1, 2, \ldots\ldots\ldots$

Ponašanje funkcije pri velikim vrednostima modula z_1 pretstavljeno je asimptotskim redom

$$D_p(z_1) \sim e^{-z_1/4} z_1^p \left\{ 1 - \frac{p(p-1)}{2 \cdot z_1^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4 \cdot z_1^4} - \cdots \right\}$$

$$\text{za } -\frac{3}{4}\pi \leq \arg z \leq \frac{3}{4}\pi$$

Funkcija paraboličnog cilindra vezana je i sa nizom specijalnih funkcija. Evo nekih

$$D_p(z_1) = \frac{\Gamma(1/2)\, 2^{p/2+1/4}}{\Gamma\left(\frac{1-p}{2}\right)} z_1^{-1/2} M_{p/2+1/4;\,-1/4}(z_1^2/2) +$$

$$+ \frac{\Gamma(-1/2)\, 2^{p/2+1/4}}{\Gamma(-p/2)} z_1^{-1/2} M_{p/2+1/4;\,1/4}(z_1^2/2) \qquad |\arg z| < \frac{3}{4}\pi$$

$$D_p(z_1) = 2^{p/2+1/4} z_1^{-1/2} W_{p/2+1/4;\,-1/4}(z_1^2/2)$$

gde su $M_{k,m}(z_1)$ i $W_{k,m}(z_1)$ -rešenja jednačine Vitaker-a.

Pri p=n - celom pozitivnom, funkcija Vebera predstavlja se polinomima Hermita

$$D_n(z_1) = 2^{-n/2}\, e^{-z_1^2/4}\, H_n\left(\frac{z_1}{2}\right) \; ; \; H_n(\xi) = (-1)^n\, e^{\xi^2} \frac{d^n}{d\xi^n}\left(e^{-\xi^2}\right)$$

a pri p=-n - celom negativnom, preko funkcije verovatnoće

$$D_{-n-1}(z_1) = \sqrt{\frac{\pi}{2}}\, \frac{(-1)^n}{n!}\, e^{-z_1^2/4}\, \frac{d^n}{dz_1^n}\left\{ e^{z_1^2/2}\left[1-\Phi\left(x_1/\sqrt{2}\right)\right]\right\}$$

$$\Phi(z_1) = \frac{2}{\sqrt{\pi}} \int_0^{z_1} e^{-x^2}\, dx$$

Integralna reprezentacija za funkciju paraboličnog cilindra

$$D_p(z_1) = \frac{\Gamma(p+1)}{2\pi i}\, e^{-z_1^2/4} \int_{\infty}^{0+} e^{-z_1 t - \frac{t^2}{2}} (-t)^{-p-1}\, dt = \frac{1}{\Gamma(-p)}\, e^{-z_1^2/4} \int_0^{\infty} e^{-z_1 t - \frac{t^2}{2}} \cdot (-t)^{-p-1}\, dt$$

Kod nas je $p=-2\alpha-1$ i $z_1=\sqrt{2}\, z$ te je:

$$D_{-2\alpha-1}(\sqrt{2}\,z) = \frac{1}{\Gamma(2\alpha+1)}\, e^{-z^2/4} \int_0^{\infty} e^{-2zt-t^2} (\sqrt{2}\, t)^{2\alpha} \sqrt{2}\, dt =$$

$$= \frac{2^{\alpha+1/2}}{\Gamma(2\alpha+1)}\, e^{-z^2/4} \int_0^{\infty} e^{-2zt-t^2}\, t^{2\alpha}\, dt$$

Smenom $t=\gamma - z$ izraz se transformiše na novi oblik

$$D_{-2\alpha-1}(\sqrt{2}\,z) = 2^{\alpha-1/2} \sqrt{\pi}\, e^{z^2/2} \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_z^{\infty} (\gamma - z)^{2\alpha} e^{-\gamma^2} d\gamma$$

ili vodeći računa da izraz

$$g_\alpha(z) = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_z^{\infty} (\gamma - z)^{2\alpha} e^{-\gamma^2} d\gamma$$

predstavlja Gaus-ovu funkciju greške, dobija se da je

$$D_{-2\alpha-1}(\sqrt{2}\, z) = 2^{\alpha-1/2} \sqrt{\pi}\, e^{z^2/2} g_\alpha(z)$$

Na taj način smo uspostavili vezu, izmedju funkcije parboličnog cilindra i Gaus-ove funkcije greške, koja će nam kasnije biti potrebna.

Kada je $z=0$ tada je

$$g_\alpha(0) = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_0^{\infty} \gamma^{2\alpha} e^{-\gamma^2} d\gamma = \frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}\,\Gamma(2\alpha+1)}$$

a funkcija paraboličnog cilindra za nultu vrednost modula z imaće vrednost

$$D_{-2\alpha-1}(o) = 2^{\alpha-1/2}\sqrt{\pi}\,\frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}\,\Gamma(2\alpha+1)} = 2^{\alpha-1/2}\frac{2^{1-2\alpha}\sqrt{\pi}\,\Gamma(2\alpha)}{\Gamma(\alpha)\,\Gamma(2\alpha+1)}$$

jer je po Ležandrovoj formuli

$$\Gamma(\alpha+1/2) = 2^{1-2\alpha}\sqrt{\pi}\,\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}$$

Vodeći računa da je

$$\Gamma(2\alpha+1) = 2\alpha\cdot\Gamma(2\alpha) \quad i$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

to je

$$D_{-2\alpha-1}(o) = \frac{2^{-\alpha-1/2}\sqrt{\pi}}{\Gamma(\alpha+1)}$$

ili pošto je $-2\alpha-1 = p$ to je

$$D_{p}(o) = \frac{2^{p/2}\sqrt{\pi}}{\Gamma(\frac{1-p}{2})}$$

Rešenje početne diferencijalne jednačine je sada

$$Y = C\,e^{-z^2/2}D_{-2\alpha-1}(\sqrt{2}\,z)$$

Neka je za $z=o$; $Y=Yo$ tada je

$$Yo = C\,\frac{2^{-\alpha-1/2}\sqrt{\pi}}{\Gamma(\alpha+1)}$$

ili odavde

$$C = Yo\frac{\Gamma(\alpha+1)}{2^{-\alpha-1/2}\sqrt{\pi}}$$

te je

$$Y = Yo\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,2^{-\alpha-1/2}}\,e^{-z^2/2}D_{-2\alpha-1}(\sqrt{2}\,z)$$

ili vodeći računa o vezi funkcije paraboličnog cilindra sa Gausovom funkcijom greške dobija se konačno rešenja

$$Y = Yo\;2^{2\alpha}\,\Gamma(\alpha+1)\;g_{\alpha}(z)$$

Znači da se rešenje diferencijalne jednačine može odmah tražiti u obliku

$$Y = C g_{\alpha}(z)$$

zadovoljavajući navedeni granični uslov sleduje:

$$Yo = Cg_\alpha(o) = C\frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}\,\Gamma(2\alpha+1)} = C\,2^{-2\alpha}\frac{1}{\Gamma(\alpha+1)}$$

a odavde je

$$C = Yo\;2^{2\alpha}\,\Gamma(\alpha+1)$$

te je sada rešenje

$$Y = Yo\;2^{2\alpha}\,\Gamma(\alpha+1)\;g_\alpha(z)$$

Nadjimo sada rekurentnu formulu za Gausovu funkciju greške. Pri tome ćemo poći od izraza

$$z\cdot g_{\alpha-1/2}(z) = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty z\,(\gamma-z)^{2\alpha-1}e^{-\gamma^2}d\gamma$$

ili ako dodamo i oduzmemo γ kod z pod oznakom integrala dobiće se

$$z\cdot g_{\alpha-1/2}(z) = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty (z-\gamma+\gamma)(\gamma-z)^{2\alpha-1}e^{-\gamma^2}d\gamma =$$

$$= -\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty(\gamma-z)^{2\alpha}e^{-\gamma^2}d\gamma + \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty\gamma\,(\gamma-z)^{2\alpha-1}e^{-\gamma^2}d\gamma$$

Kod prvog integrala izvršimo množenje sa $\frac{2\alpha}{2\alpha}$ pa vodeći računa da je

$$\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\,\frac{2\alpha}{2\alpha} = \frac{2^2\alpha}{\sqrt{\pi}\,\Gamma(2\alpha+1)}$$

dobiće se

$$z\cdot g_{\alpha-1/2}(z) = -\frac{2^2\alpha}{\sqrt{\pi}\,\Gamma(2\alpha+1)}\int_z^\infty(\gamma-z)^{2\alpha}e^{-\gamma^2}d\gamma + \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty\gamma(\gamma-z)^{2\alpha-1}e^{-\gamma^2}d\gamma$$

Podjimo sada od izraza

$$\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty\frac{d}{d\gamma}\left[(\gamma-z)^{2\alpha-1}e^{-\gamma^2}\right]d\gamma = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty(2\alpha-1)(\gamma-z)^{2\alpha-2}e^{-\gamma^2}d\gamma -$$

$$-\frac{4}{\sqrt{\pi}\,\Gamma(2\alpha)}\int_z^\infty\gamma\,(\gamma-z)^{2\alpha-1}e^{-\gamma^2}d\gamma$$

Pošto je integral na levoj strani u granicama od z do ∞ ravan nuli $(\gamma-z)^{2\alpha-1}e^{-\gamma^2}\Big|_z^\infty = 0$ to je

$$\int_z^\infty\gamma\,(\gamma-z)^{2\alpha-1}e^{-\gamma^2}d\gamma = \frac{1}{2}\int_z^\infty(2\alpha-1)(\gamma-z)^{2\alpha-2}e^{-\gamma^2}d\gamma$$

Sada je

$$z\cdot g_{\alpha-1/2}(z) = -2\alpha\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)}\int_z^\infty(\gamma-z)^{2\alpha}e^{-\gamma^2}d\gamma + \frac{1}{2}(2\alpha-1)\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha)}\cdot$$

$$\cdot\int_z^\infty(\gamma-z)^{2\alpha-2}e^{-\gamma^2}d\gamma = -2\alpha\,g_\alpha(z) + \frac{1}{2}\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha-1)}\int_z^\infty(\gamma-z)^{2\alpha-2}e^{-\gamma^2}d\gamma$$

$$= -2\alpha\,g_\alpha(z) + \frac{1}{2}\,g_{\alpha-1}(z)$$

odnosno konačno

$$z\, g_{\alpha-1/2}(z) = -2\alpha\, g_{\alpha}(z) + \frac{1}{2} g_{\alpha-1}(z)$$

Navešćemo još neke osnovne operacije sa funkcijom $g_{\alpha}(z)$:

$$\frac{dg_{\alpha}}{dz} = -2\alpha \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_{z}^{\infty} (\gamma - z)^{2\alpha-1} e^{-\gamma^2} d\gamma =$$

$$= 2\alpha \frac{-2}{\sqrt{\pi}\, 2\alpha\, \Gamma(2\alpha)} \int_{z}^{\infty} (\gamma - z)^{2\alpha-1} e^{-\gamma^2} d\gamma = g_{\alpha-1/2}(z)$$

Znači:

$$g'_{\alpha}(z) = -g_{\alpha-1/2}(z)$$

$$\int g_{\alpha}(z)dz = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_{z}^{\infty} e^{-\gamma^2} d\gamma \int (\gamma - z)^{2\alpha} dz =$$

$$= -\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+2)} \int_{z}^{\infty} (\gamma - z)^{2\alpha+1} e^{-\gamma^2} d\gamma =$$

$$= -g_{\alpha+1/2}(z)$$

Znači da je

$$\int g_{\alpha}(z)dz = -g_{\alpha+1/2}(z)$$

Sada možemo konačno pristupiti rešavanju diferencijalnih jednačina oba sistema, budući da smo pripremili sve što nam je potrebno za ta rešavanja.

Sistem I:

1. $$F_o''' + 2\eta_1 F_o'' = 0$$

sa graničnim uslovima

$$F_o = F_o' = 0 \qquad \eta_1 = 0$$
$$F_o' = 1 \qquad \eta_1 = \infty$$

Ako se uvede smena

$$F_o' = 1 - S'$$

dobija se diferencijalna jednačina sa novim graničnim uslovima

$$S''' + 2\eta_1 S'' = 0$$

$$S' = 1 \qquad \eta_1 = 0$$
$$S' = 0 \qquad \eta_1 = \infty$$

Rešenje date diferencijalne jednačine je

$$S' = C\, g_0(\eta_1)$$

Za date granične uslove je

$$C = 1$$

te je resenje

$$S' = Cg_0(\eta_1) = g_0(\eta_1)$$

ili

$$F_0' = 1 - g_0(\eta_1)$$

Vodeći računa o operacijama integriranja i diferenciranja funkcije greške, lako se dobija da je

$$F_0'' = g_{-1/2}(\eta_1)$$
$$F_0 = \eta_1 + g_{1/2} - \frac{1}{2}\frac{1}{\Gamma(3/2)}$$

Ubuduće ćemo umesto $\Gamma(3/2)$ pisati samo Γ, pošto ćemo sve Gama funkcije svoditi na $\Gamma(3/2)$

2. $f_1''' + 2\eta_1 f_1'' - 4f_1' = -\frac{4}{3}(1 - F_0' - \eta_1 F_0'')$

ili ako se unesu vrednosti za F_0' i F_0'' i vodeći računa o rekurentnoj formuli, posle sredjivanja dobija se:

$$f_1''' + 2\eta_1 f_1'' - 4f_1' = -\frac{4}{3} g_0 + \frac{2}{3} g_1$$

sa graničnim uslovima

$$f_1 = f_1' = 0 \qquad \eta_1 = 0$$
$$f_1' = 0 \qquad \eta_1 = \infty$$

Rešenje homogenog dela je

$$f'_{1h} = Cg_1$$

Partikularno rešenje potražićemo u obliku

$$f'_{1p} = C_1 g_o + C_2 g_{-1}$$

Zadovoljavajući diferencijalnu jednačinu dobijaju se uslovi za odredjivanje nepoznatih konstanata.

$$-4\,C_1 = -\frac{4}{3}$$

$$-8\,C_2 = \frac{2}{3}$$

odavde je

$$C_1 = \frac{1}{3}\;;\quad C_2 = -\frac{1}{12}$$

te je rešenje

$$f_1' = Cg_1 + \frac{1}{3}\,g_o - \frac{1}{12}\,g_{-1}$$

Iz prvog graničnog uslova

$$o = C\,\frac{1}{2^2} + \frac{1}{3}\,g_o$$

dobija se $\quad C = -\frac{4}{3}$

i konačno

$$f_1' = -\frac{4}{3}\,g_1 + \frac{1}{3}\,g_o - \frac{1}{12}\,g_{-1}$$

Lako se nalaze

$$f_1'' = \frac{4}{3}\,g_{1/2} - \frac{1}{3}\,g_{-1/2} + \frac{1}{12}\,g_{-3/2}$$

$$f_1 = \frac{4}{3}\,g_{3/2} - \frac{1}{3}\,g_{1/2} + \frac{1}{12}\,g_{-1/2} - \frac{1}{36}\,\frac{1}{r}$$

3. $\quad f_{11}''' + 2\eta_1 f_{11}'' - 4\cdot 2\,f_{11}' = \frac{4}{3}\,(f_1' + \eta_1 f_1'')$

ili ako se unesu vrednosti za f_1' i f_1'' i sredi dobija se da je

$$f_{11}''' + 2\eta_1 f_{11}'' - 4\cdot 2 f_1' = \frac{4}{3}\left[-4g_1 + g_o - \frac{1}{2}\,g_1 + \frac{1}{24}\,g_{-2}\right]$$

sa graničnim uslovima

$f_{11} = f_{11}' = o$ $\qquad \eta_1 = o$

$f_{11}' = o$ $\qquad \eta_1 = \infty$

Rešenje homogenog dela je

$$f_{11h}' = Cg_2$$

Partikularno rešenje potražićemo u obliku

$$f_{11p}' = C_1g_1 + C_2g_o + C_3g_{-1} + C_4g_{-2}$$

Zadovoljavajući diferencijalnu jednačinu dobijaju se uslovi za odredjivanje nepoznatih konstanti

$$-4C_1 = -\frac{16}{3}$$

$$-8C_2 = \frac{4}{3}$$

$$-12C_3 = -\frac{1}{9}$$

$$-16C_4 = \frac{1}{18}$$

odavde su $C_1 = \frac{4}{3}$; $C_2 = -\frac{1}{6}$; $C_3 = \frac{1}{9\cdot 12}$; $C_4 = -\frac{1}{16\cdot 18}$

Sada je rešenje

$$f_{11}' = Cg_2 + \frac{4}{3}g_1 - \frac{1}{6}g_o + \frac{1}{9\cdot 12}g_{-1} - \frac{1}{16\cdot 18}g_{-2}$$

Zadovoljavajući prvi granični uslov dobija se

$$o = C\frac{1}{2^5} + \frac{4}{3}\frac{1}{2^2} - \frac{1}{6}$$

a odavde je

$$C = -\frac{2^4}{3}$$

te je konačno rešenje

$$f_{11}' = -\frac{2^4}{3}g_2 + \frac{4}{3}g_1 - \frac{1}{6}g_o + \frac{1}{9\cdot 12}g_{-1} - \frac{1}{16\cdot 18}g_{-2}$$

Lako se nalaze i:

$$f_{11}'' = \frac{2^4}{3} g_{3/2} - \frac{4}{3} g_{1/2} + \frac{1}{6} g_{-1/2} - \frac{1}{9 \cdot 12} g_{-3/2} + \frac{1}{15 \cdot 18} g_{-5/2}$$

$$f_{11} = \frac{2^4}{3} g_{5/2} - \frac{4}{3} g_{3/2} + \frac{1}{6} g_{1/2} - \frac{1 \cdot g_{-1/2}}{9 \cdot 12} + \frac{1}{16 \cdot 18} g_{-3/2} - \frac{1}{2160} \cdot \frac{1}{\Gamma}$$

4. $f_2''' + 2\eta_1 f_2'' - 4 \cdot 2 f_2' = -\frac{4}{3} (1 - F_o' - \eta_1 F_o'') = -\frac{4}{3} g_o + \frac{2}{3} g_{-1}$

sa graničnim uslovima

$$f_2 = f_2' = 0 \qquad \eta_1 = 0$$

$$f_2' = 0 \qquad \eta_1 = \infty$$

Rešenje homogenog dela

$$f_{2h}' = C g_2$$

Partikularno rešenje treba potražiti u obliku

$$f_{2p}' = C_1 g_o + C_2 g_{-1}$$

Iz uslova

$$-8C_1 = -\frac{4}{3} \qquad C_1 = \frac{1}{6}$$

$$-12C_2 = \frac{2}{3} \qquad \text{je} \qquad C_2 = -\frac{1}{18}$$

Rešenje je sada

$$f_2' = C g_2 + \frac{1}{6} g_o - \frac{1}{18} g_{-1}$$

Zadovoljavajući granične uslove

$$0 = C \frac{1}{2^5} + \frac{1}{6}$$

dobija se da je

$$C = -\frac{2^4}{3}$$

te je konačno rešenje

$$f_2' = -\frac{2^4}{3} g_2 + \frac{1}{6} g_o - \frac{1}{18} g_{-1}$$

Lako se nalazi da je

$$f_2'' = \frac{2^4}{3} g_{3/2} - \frac{1}{6} g_{-1/2} + \frac{1}{18} g_{-3/2}$$

$$f_2 = \frac{2^4}{3} g_{5/2} - \frac{1}{6} g_{1/2} + \frac{1}{18} g_{-1/2} - \frac{1}{60} \frac{1}{\Gamma}$$

5. $$f_{111}''' + 2\eta_1 f_{111}'' - 4\cdot 3 f_{111}' = \frac{4}{3}(f_{11}' + \eta_1 f_{11}'') = \frac{4}{3}\Big[-2^4 \frac{5}{3} g_2 + \frac{20}{3} g_1 - \frac{5}{6} g_0 + \frac{2}{27} g_{-1} + \frac{5}{18\cdot 48} g_{-2} + \frac{1}{18\cdot 32} g_{-3}\Big]$$

sa graničnim uslovima

$$f_{111} = f_{111}' = 0 \qquad \eta_1 = 0$$

$$f_{111}' = 0 \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$f_{111h}' = Cg_3$$

a partikularno treba tražiti u obliku

$$f_{111p}' = C_1 g_2 + C_2 g_1 + C_3 g_0 + C_4 g_{-1} + C_5 g_{-2} + C_6 g_{-3}$$

Zadovoljavajući datu diferencijalnu jednačinu dobijamo uslove

$$-4C_1 = -\frac{4}{3} 2^4 \frac{5}{3} \qquad \text{iz kojih je} \qquad C_1 = \frac{5}{9} 2^4$$

$$-8C_2 = \frac{4}{3}\frac{20}{3} \qquad C_2 = -\frac{10}{9}$$

$$-12C_3 = -\frac{4}{3}\frac{5}{6} \qquad C_3 = \frac{5}{54}$$

$$-16C_4 = \frac{4}{3}\frac{2}{27} \qquad C_4 = -\frac{1}{6\cdot 27}$$

$$-20C_5 = \frac{4}{3}\frac{5}{18\cdot 48} \qquad C_5 = -\frac{1}{3\cdot 18\cdot 48}$$

$$-24C_6 = \frac{4}{3}\frac{1}{18\cdot 32} \qquad C_6 = -\frac{1}{18\cdot 18\cdot 32}$$

Rešenje je sada

$$f_{111}' = Cg_3 + \frac{5}{9} 2^4 g_2 - \frac{10}{9} g_1 + \frac{5}{54} g_0 - \frac{1}{6\cdot 27} g_{-1} - \frac{1}{48\cdot 54} g_{-2} - \frac{1}{18^2\cdot 32} g_{-3}$$

Zadovoljavajući granične uslove

$$0 = \frac{1}{2^6 \cdot 3 \cdot 2} C + \frac{5}{9} 2^4 \frac{1}{2^4 \cdot 2} - \frac{10}{9} \frac{1}{2^2} + \frac{5}{54}$$

nalazimo konstantu

$$C = -2^7 \frac{5}{18}$$

Konačno rešenje je

$$f'_{111} = -\frac{2^7 \cdot 5}{18} g_3 + \frac{2^4 \cdot 5}{9} g_2 - \frac{10}{9} g_1 + \frac{5}{54} g_0 - \frac{1}{6 \cdot 27} g_{-1} - \frac{1}{18^2 \cdot 32} g_{-3} - \frac{1}{48 \cdot 54} g_{-2}$$

Lako je pronaći i:

$$f''_{111} = \frac{2^7 \cdot 5}{18} g_{5/2} - \frac{2^4 \cdot 5}{9} g_{3/2} + \frac{10}{9} g_{1/2} - \frac{5}{54} g_{-1/2} + \frac{1}{6 \cdot 27} g_{-3/2} + \frac{1}{48 \cdot 54} \cdot g_{-5/2} + \frac{1}{18^2 \cdot 32} g_{-7/2}$$

$$f_{111} = \frac{2^7 \cdot 5}{18} g_{7/2} - \frac{2^4 \cdot 5}{9} g_{5/2} + \frac{10}{9} g_{3/2} - \frac{5}{54} g_{1/2} + \frac{1}{6 \cdot 27} g_{-1/2} + \frac{1}{48 \cdot 54} g_{-3/2} + \frac{1}{18^2 \cdot 32} g_{-5/2} + \frac{1}{27 \cdot 672} \frac{1}{\Gamma}$$

6. $$f'''_{12} + 2\eta_1 f''_{12} - 4 \cdot 3 f'_{12} = \frac{4}{3}\left[f'_1 + \eta_1 f''_1 + f'_2 + \eta_1 f''_2\right] =$$

$$= \frac{4}{3}\left[-\frac{4}{3} g_1 + \frac{7}{6} g_0 - \frac{2}{18} g_{-1} + \frac{5}{72} g_{-2} - \frac{2^4 \cdot 5}{3} g_2\right]$$

sa graničnim uslovima

$$f_{12} = f'_{12} = 0 \qquad \eta_1 = 0$$

$$f'_{12} = 0 \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$f'_{12h} = C g_3$$

a partikularno treba tražiti u obliku

$$f'_{12p} = C_1 g_2 + C_2 g_1 + C_3 g_0 + C_4 g_{-1} + C_5 g_{-2}$$

Smenom u diferencijalnu jednačinu dobija se

$$-4C_1 = -\frac{4}{3}\,\frac{2^4\cdot 5}{3} \quad \text{a odavde je} \quad C_1 = \frac{2^4\cdot 5}{9}$$

$$-8C_2 = -\frac{16}{9} \qquad C_2 = \frac{2}{9}$$

$$-12C_3 = \frac{28}{18} \qquad C_3 = \frac{7}{54}$$

$$-16C_4 = \frac{-8}{18\cdot 3} \qquad C_4 = \frac{1}{108}$$

$$-20C_5 = \frac{20}{72\cdot 3} \qquad C_5 = -\frac{1}{216}$$

Rešenje je sada

$$f'_{12} = Cg_3 - \frac{2^4\cdot 5}{9} g_2 + \frac{2}{9} g_1 - \frac{7}{54} g_o + \frac{1}{108} g_{-1} - \frac{1}{216} g_{-2}$$

Zadovoljavajući granične uslove

$$o = C\,\frac{1}{2^6\cdot 3\cdot 2} - \frac{2^4\cdot 5}{9}\,\frac{1}{2^4\cdot 2} + \frac{2}{9}\,\frac{1}{2^2} - \frac{7}{54}$$

nalazimo konstantu

$$C = \frac{19}{9}\,2^6$$

Konačno rešenje je:

$$f'_{12} = \frac{2^6\cdot 19}{9} g_3 - \frac{2^4\cdot 5}{9} g_2 + \frac{2}{9} g_1 - \frac{7}{54} g_o + \frac{1}{108} g_{-1} - \frac{1}{216} g_{-2}$$

Lako se mogu naći

$$f''_{12} = -\frac{2^6\cdot 19}{9} g_{5/2} + \frac{2^4\cdot 5}{9} g_{3/2} - \frac{2}{9} g_{1/2} + \frac{7}{54} g_{-1/2} - \frac{1}{108} g_{-3/2} + $$

$$+ \frac{1}{216} g_{-5/2}$$

$$f_{12} = -\frac{2^6\cdot 19}{9} g_{1/2} + \frac{2^4\cdot 5}{9} g_{5/2} - \frac{2}{9} g_{3/2} + \frac{7}{54} g_{1/2} - \frac{1}{108} g_{-1/2} + $$

$$+ \frac{1}{216} g_{-3/2} - \frac{3}{140}\,\frac{1}{\Gamma}$$

7. $f_3''' + 2\eta_1 f_3'' - 4\cdot 3 f_3' = -\frac{4}{3}(1-F_0' - \eta_1 F_0'') = -\frac{4}{3} g_0 + \frac{2}{3} g_{-1}$

Sa graničnim uslovima

$$f_3 = f_3' = o \qquad \eta_1 = o$$

$$f_3' = o \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$f_{3h}' = Cg_3$$

a partikularno treba tražiti u obliku

$$f_{3p}' = C\, g_o + C_2 g_{-1}$$

Zadovoljavajući diferencijalnu jednačinu dobijaju se uslovi

$$-12C_1 = -\frac{4}{3} \quad \text{iz kojih je} \quad C_1 = \frac{1}{9}$$

$$-16C_2 = \frac{2}{3} \qquad C_2 = -\frac{1}{24}$$

Rešenje je

$$f_3' = Cg_3 + \frac{1}{9} g_o - \frac{1}{24} g_{-1}$$

Ako se zadovolji prvi od graničnih uslova

$$o = C\frac{1}{2^6\cdot 3\cdot 2} + \frac{1}{9}$$

dobiće se da je $C = -\frac{2^7}{3}$

i konačno rešenje

$$f_3' = -\frac{2^7}{3} g_3 + \frac{1}{9} g_o - \frac{1}{24} g_{-1}$$

Možemo odmah naći f_3'' i f_3

$$f_3'' = \frac{2^7}{3} g_{5/2} - \frac{1}{9} g_{-1/2} + \frac{1}{24} g_{-3/2}$$

$$f_3 = \frac{2^7}{3} g_{7/2} - \frac{1}{9} g_{1/2} + \frac{1}{24} g_{-1/2} - \frac{3}{280} \frac{1}{\Gamma}$$

8. $$f_{1111}''' + 2\eta_1 f_{1111}'' - 4\cdot 4\, f_{1111}' = \frac{4}{3} (f_{111}' + \eta_1 f_{111}'') =$$

$$= \frac{4}{3} \Big[- \frac{35}{18}\, 2^7 g_3 + \frac{35}{9}\, 2^4 g_2 - \frac{70}{9} g_1 + \frac{35}{54} g_o - \frac{13}{12\cdot 27} g_{-1} +$$

$$+ \frac{11}{48\cdot 54} g_{-2} + \frac{7}{18^2\cdot 32} g_{-3} + \frac{1}{18^2\cdot 64} g_{-4} \Big]$$

Sa graničnim uslovima

$$f_{1111} = f_{1111}' = o \qquad \eta_1 = o$$

$$f_{111}' = o \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$f_{111h}' = C g_4$$

a partikulatno ćemo tražiti u obliku

$$f_{1111p}' = C g_3 + C_2 g_2 + C_3 g_1 + C_4 g_o + C_5 g_{-1} + C_6 g_{-2} + C_7 g_{-3} +$$

$$+ C_8 g_{-4}$$

Smenom u diferencijalnu jednačinu dobijaju se uslovi

$- 4C_1 = - \frac{4}{3} \frac{35}{18} 2^7$ iz kojih je $C_1 = \frac{35}{54} 2^7$

$- 8C_2 = \frac{4}{3} \frac{35}{9} 2^4$ $\qquad C_2 = -\frac{35}{54} 2^4$

$-12C_3 = - \frac{4}{3} \frac{35}{9} 2$ $\qquad C_3 = \frac{35}{81} 2$

$-16C_4 = \frac{4}{3} \frac{35}{54}$ $\qquad C_4 = - \frac{35}{12\cdot 54}$

$-20C_5 = - \frac{4}{3} \frac{13}{12\cdot 27}$ $\qquad C_5 = \frac{13}{12\cdot 15\cdot 27}$

$-24C_6 = \frac{4}{3}\frac{11}{48\cdot 54}$ $\qquad$ $C_6 = -\frac{11}{18\cdot 48\cdot 54}$

$-28C_7 = \frac{4}{3}\frac{7}{18^2\cdot 32}$ $\qquad$ $C_7 = -\frac{1}{3\cdot 18^2\cdot 32}$

$-32C_8 = \frac{4}{3}\frac{1}{18^2\cdot 64}$ $\qquad$ $C_8 = -\frac{2}{3\cdot 18^2\cdot 32^2}$

Rešenje je sada

$$f'_{1111} = C g_4 + \frac{35}{54}2^7 g_3 - \frac{35}{54}2^4 g_2 + \frac{70}{81}g_1 - \frac{35}{12\cdot 54}g_0 + \frac{13}{12\cdot 15\cdot 27}g_{-1} -$$
$$- \frac{11}{18\cdot 48\cdot 54}g_{-2} - \frac{1}{3\cdot 18^2\cdot 32}g_{-3} - \frac{2}{3\cdot 18^2\cdot 32^2}g_{-4}$$

Zadovoljavajući granični uslov dobija se

$$0 = C\frac{1}{2^8\cdot 4!} + \frac{35}{54}2^7\frac{1}{2^6\cdot 3!} - \frac{35}{54}2^4\frac{1}{2^4 2!} + \frac{70}{81}\frac{1}{2^2} - \frac{35}{12\cdot 54}$$

odnosno

$$C = -\frac{35}{54}2^9$$

te je konačno rešenje sada

$$f'_{1111} = -\frac{35}{54}2^9 g_4 + \frac{35}{54}2^7 g_3 - \frac{35}{54}2^4 g_2 + \frac{70}{81}g_1 - \frac{35}{12\cdot 54}g_0 + \frac{13}{12\cdot 15\cdot 27}g_{-1} -$$
$$- \frac{11}{18\cdot 48\cdot 54}g_{-2} - \frac{1}{3\cdot 18^2\cdot 32}g_{-3} - \frac{2}{3\cdot 18^2\cdot 32^2}g_{-4}$$

Vrlo lako se mogu naći f''_{1111} i f_{1111} znajući elementarna operisanja sa Gaus-ovom funkcijom greške, te zbog toga nema smisla ni pisati izraze za njih, a i ubuduće, kod ostalih funkcija to nećemo raditi.

9. $$f'''_{112} + 2\eta_1 f''_{112} - 4\cdot 4 f'_{112} = \frac{4}{3}\left[f'_{11} + \eta_1 f''_{11} + (f'_{12} + \eta_1 f''_{12})\right] = \frac{4}{3}\left[7\ \frac{19}{9}2^6 g_3 -\right.$$
$$\left. - \frac{78}{9}2^4 g_2 + \frac{106}{9}g_1 - \frac{58}{54}g_0 + \frac{1}{6}g_{-1} - \frac{23}{864}g_{-2} + \frac{7}{8\cdot 216}g_{-3}\right]$$

sa graničnim uslovima

$f_{112} = f'_{112} = 0$ $\qquad$ $\eta_1 = 0$

$f'_{112} = 0$ $\qquad$ $\eta_1 = \infty$

Homogeno rešenje je

$$f'_{112h} = C g_4$$

a partikularno tražimo u obliku

$$f'_{112p} = C_1 g_3 + C_2 g_2 + C_3 g_1 + C_4 g_0 + C_5 g_{-1} + C_6 g_{-2} + C_7 g_{-3}$$

Smenom u diferencijalnu jednačinu dobijamo uslove

$$-4C_1 = \frac{4}{3} 7 \frac{19}{9} 2^6 \qquad C_1 = -\frac{4 \cdot 19}{27 \cdot 4} 2^6 \cdot 7$$

a odavde

$$-8C_2 = -\frac{4}{3}\frac{78}{9} 2^4 \qquad C_2 = \frac{78}{54} 2^4$$

$$-12 C_3 = \frac{4}{3}\frac{106}{9} \qquad C_3 = -\frac{106}{81}$$

$$-16 C_4 = -\frac{4}{3}\frac{58}{54} \qquad C_4 = \frac{1}{12}\frac{58}{54}$$

$$-20 C_5 = \frac{4}{3}\frac{1}{6} \qquad C_5 = -\frac{1}{90}$$

$$-24 C_6 = -\frac{4}{3}\frac{23}{864} \qquad C_6 = \frac{1}{6}\frac{23}{864}\frac{1}{3}$$

$$-28 C_7 = \frac{4}{3}\frac{7}{8}\frac{1}{216} \qquad C_7 = -\frac{1}{24}\frac{1}{216}$$

Rešenje je sada

$$f'_{112} = C g_4 - \frac{7 \cdot 19}{27} 2^6 g_3 + \frac{78}{54} 2^4 g_2 - \frac{106}{81} g_1 + \frac{58}{12 \cdot 54} g_0 -$$

$$- \frac{1}{90} g_{-1} + \frac{23}{18 \cdot 864} g_{-2} - \frac{1}{24 \cdot 216} g_{-3}$$

Zadovoljavajući granične uslove dobija se da je

$$0 = C \frac{1}{2^8 \cdot 4!} - \frac{7 \cdot 19}{27}\frac{2^6}{2^6 \cdot 3!} + \frac{78}{54} 2^4 \frac{1}{2^4\, 2!} - \frac{106}{81}\frac{1}{2^2\, 1!} + \frac{58}{12 \cdot 54}$$

a odavde je

$$C = \frac{218}{54} 2^9$$

te je konačno rešenje

$$f'_{112} = \frac{218}{54} 2^9 g_4 - \frac{133}{27} 2^6 g_3 + \frac{78}{54} 2^4 g_2 - \frac{106}{81} g_1 + \frac{58}{12 \cdot 54} g_0 -$$

$$- \frac{1}{90} g_{-1} + \frac{23}{18 \cdot 864} g_{-2} - \frac{1}{24 \cdot 216} g_{-3}$$

10. $$f_{13}''' + 2\eta_1 f_{13}'' - 4 \cdot 4 f_{13}' = \frac{4}{3}\left[f_1' + \eta_1 f_1'' + f_3' + \eta_1 f_3''\right] = \frac{4}{3}\Big[-\frac{7}{3} 2^7 g_3 +$$

$$+ \frac{2^6}{3} g_2 - 4 g_1 + \frac{10}{9} g_0 - \frac{7}{3 \cdot 24} g_{-1} + \frac{3}{48} g_{-2}\Big]$$

sa graničnim uslovima

$f_{13} = f'_{13} = 0 \qquad \eta_1 = 0$

$f'_{13} = 0 \qquad \eta_1 = \infty$

Homogeno rešenje je

$$f'_{13h} = C\,g_4$$

a partikularno rešenje je oblika

$$f'_{13p} = C_1 g_3 + C_2 g_2 + C_3 g_1 + C_4 g_0 + C_5 g_{-1} + C_6 g_{-2}$$

Smenom u diferencijalnu jednačinu dobićemo uslove

$$-4C_1 = -\frac{28}{9}2^7 \qquad C_1 = \frac{7}{9}2^7$$

$$-8C_2 = \frac{4}{3}\,\frac{1}{3}2^6 \qquad C_2 = -\frac{1}{18}2^6$$

$$-12C_3 = -\frac{4}{3}4 \qquad C_3 = \frac{4}{9}$$

$$-16C_4 = \frac{4}{3}\,\frac{10}{9} \qquad C_4 = -\frac{10}{12\cdot 9}$$

$$-20C_5 = -\frac{4}{3}\,\frac{7}{3}\,\frac{1}{24} \qquad C_5 = \frac{7}{24\cdot 45}$$

$$-24C_6 = \frac{4}{3}\,\frac{3}{48} \qquad C_6 = -\frac{1}{96\cdot 3}$$

Rešenje je sada

$$f'_{13} = Cg_4 + \frac{7}{9}2^7 g_3 - \frac{1}{18}2^6 g_2 + \frac{4}{9}g_1 - \frac{5}{54}g_0 + \frac{7}{24\cdot 45}g_{-1} - \\ - \frac{1}{3\cdot 96}g_{-2}$$

Zadovoljavajući prvi granični uslov

$$0 = C\frac{1}{2^8 4!} + \frac{7}{9}2^7\frac{1}{2^6\,3!} - \frac{1}{18}2^6\frac{1}{2^4\cdot 2!} + \frac{4}{9}\,\frac{1}{2^2 1!} - \frac{5}{54}$$

možemo odrediti konstantu

$$C = -2^{10}$$

Konačno rešenje je

$$f'_{13} = -2^{10}g_4 + \frac{7}{9}2^7 g_3 - \frac{1}{18}2^6 g_2 + \frac{4}{9}g_1 - \frac{5}{54}g_0 + \frac{7}{24\cdot 45}g_{-1} - \\ - \frac{1}{3\cdot 96}g_{-2}$$

11. $f'''_{22} + 2\eta_1 f''_{22} - 4\cdot 4\, f'_{22} = \frac{4}{3}\left(f'_2 + \eta_1 f''_2\right) = \frac{4}{3}\Big[-\frac{5}{3}2^4 g_2 + \frac{2^3}{3}g_1 + \frac{1}{6}g_0 -$

$- \frac{1}{36} g_{-1} + \frac{1}{36} g_{-2} \Big]$

sa graničnim uslovima

$$f_{22} = f_{22}' = 0 \qquad \eta_1 = 0$$
$$f_{22}' = 0 \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$f_{22h}' = C g_4$$

a partikularno tražimo u obliku

$$f_{22p}' = C_1 g_2 + C_2 g_1 + C_3 g_0 + C_4 g_{-1} + C_5 g_{-2}$$

Zadovoljavajući diferencijalnu jednačinu dobijamo uslove

$$-8C_1 = -\frac{4}{3}\frac{5}{3}2^4 \qquad C_1 = \frac{5}{18}2^4$$
$$-12C_2 = \frac{4}{3}\frac{2^3}{3} \qquad \text{iz kojih je} \qquad C_2 = -\frac{1}{27}2^3$$
$$-16C_3 = \frac{4}{3}\frac{1}{6} \qquad C_3 = -\frac{1}{3\cdot 24}$$
$$-20C_4 = -\frac{4}{3}\frac{1}{36} \qquad C_4 = \frac{1}{15\cdot 36}$$
$$-24C_5 = \frac{4}{3}\frac{1}{36} \qquad C_5 = -\frac{1}{18\cdot 36}.$$

Rešenje je sada

$$f_{22}' = C g_4 + \frac{5}{18}2^4 g_2 - \frac{1}{27}2^3 g_1 - \frac{1}{3\cdot 24} g_0 + \frac{1}{15\cdot 36} g_{-1} -$$
$$- \frac{1}{18\cdot 36} g_{-2}$$

Zadovoljavajući prvi granični uslov

$$0 = C\frac{1}{2^8\cdot 4!} + \frac{5}{18}2^4\frac{1}{2^4 2!} - \frac{1}{27}2^3\frac{1}{2^2 1!} - \frac{1}{3\cdot 24}$$

određujemo konstantu

$$C = -\frac{11}{9}2^8$$

Konačno rešenje je

$$f_{22}' = -\frac{11}{9}2^8 g_4 + \frac{5}{18}2^4 g_2 - \frac{1}{27}2^3 g_1 - \frac{1}{3\cdot 24} g_0 + \frac{1}{15\cdot 36} g_{-1} -$$
$$- \frac{1}{18\cdot 36} g_{-2}$$

12. $f_4''' + 2\eta_1 f_4'' - 4\cdot 4 f_4' = -\frac{4}{3}(1 - F_0' - \eta_1 F_0'') = -\frac{4}{3} g_0 + \frac{2}{3} g_{-1}$

sa graničnim uslovima

$$f_4 = f_4' = 0 \qquad \eta_1 = 0$$

$$f_4' = 0 \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$f_{4h}' = Cg_4$$

a partikularno tražimo u obliku

$$f_{4p}' = C_1 g_0 + C_2 g_{-1}$$

Smenom u jednačinu dobijaju se uslovi

$$-16C_1 = -\frac{4}{3} \qquad \text{iz kojih je} \qquad C_1 = \frac{1}{12}$$

$$-20C_2 = \frac{2}{3} \qquad\qquad C_2 = -\frac{1}{30}$$

Rešenje je

$$f_4' = Cg_4 + \frac{1}{12} g_0 - \frac{1}{30} g_{-1}$$

Zadovoljavajući granične uslove

$$0 = C\frac{1}{2^8 4!} + \frac{1}{12}$$

odredjujemo konstantu

$$C = -2^9$$

Konacno rešenje je

$$f_4' = -2^9 g_4 + \frac{1}{12} g_0 - \frac{1}{30} g_{-1}$$

Sistem II:

$$1. \quad \varphi_1''' + 2\eta_1\varphi_1'' - 4\varphi_1' = -\frac{4}{3}(1 - F_0'^2 + F_0F_0'') = -\frac{4}{3}\Big[2g_0 - \frac{1}{2}\frac{1}{\Gamma} g_{-1/2} + \frac{1}{2} g_{-1} - g_0^2 + g_{1/2} g_{-1/2}\Big]$$

Sa graničnim uslovima

$$\varphi_1 = \varphi_1' = 0 \qquad \eta_1 = 0$$
$$\varphi_1' = 0 \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$\varphi_{1h}' = Cg_1$$

a partikularno treba potražiti u obliku

$$\varphi_{1p}' = C_1 g_o + C_2 g_{-1/2} + C_3 g_{-1} + C_4 g_{1/2}^2 + C_5 g_o g_1$$

Smenom u diferencijalnu jednačinu dobijamo uslove

$$-4C_1 = -2\frac{4}{3} \quad \text{iz kojih su} \quad C_1 = \frac{2}{3}$$
$$-6C_2 = \frac{4}{3}\frac{1}{2}\frac{1}{\Gamma} \qquad C_2 = -\frac{1}{9}\frac{1}{\Gamma}$$
$$-8C_3 = -\frac{4}{3}\ \frac{1}{2} \qquad C_3 = \frac{1}{12}$$
$$2C_4 = \frac{4}{3} \qquad C_4 = \frac{2}{3}$$
$$2C_5 = -\frac{4}{3} \qquad C_5 = -\frac{2}{3}$$

Rešenje je sada

$$\varphi' = Cg_1 + \frac{2}{3} g_o - \frac{1}{9}\frac{1}{\Gamma} g_{-1/2} + \frac{1}{12} g_{-1} + \frac{2}{3} g_{1/2}^2 - \frac{2}{3} g_o g_1$$

Zadovoljavajući prvi granični uslov

$$o = C\frac{1}{2^2} + \frac{2}{3} - \frac{1}{9}\frac{1}{\Gamma}\frac{1}{\Gamma} + \frac{2}{3}\left(\frac{1}{2\Gamma}\right)^2 - \frac{2}{3}\ \frac{1}{2^2}$$

odredjujemo konstantu

$$C = -\frac{2}{3}\left[3 + \frac{1}{3}\ \frac{1}{\Gamma^2}\right]$$

Sada je konačno rešenje

$$\varphi_1' = -\frac{2}{3}\left[3 + \frac{1}{3}\ \frac{1}{\Gamma^2}\right] g_1 + \frac{2}{3} g_o - \frac{1}{9}\frac{1}{\Gamma} g_{-1/2} + \frac{1}{12} g_{-1} +$$
$$+ \frac{2}{3} g_{1/2}^2 - \frac{2}{3} g_o g_1$$

$$2.\quad \varphi_{11}''' + 2\eta_1\varphi_{11}'' - 4\cdot 2\,\varphi_{11}' = \frac{4}{3}\Big[(\varphi_1' + \eta\varphi_1'') - (-2F_0'f_1' + F_0 f_1'' + f_1 F_0'')\Big] =$$

$$= \frac{4}{3}\Big[-2\Big(3 + \frac{1}{3}\frac{1}{\Gamma^2}\Big)g_1 + \frac{1}{3}\Big(5 + \frac{1}{3}\frac{1}{\Gamma^2}\Big)g_0 + \frac{2}{3}\frac{1}{\Gamma}g_{1/2} -$$

$$- \frac{5}{36}\frac{1}{\Gamma}g_{-1/2} - \frac{7}{12}g_{-1} + \frac{7}{72}\frac{1}{\Gamma}g_{-3/2} - \frac{1}{12}g_{-2} -$$

$$- \frac{2}{3}g_0^2 - \frac{1}{12}g_{-1/2}^2 - 2g_0 g_1 + \frac{1}{3}g_{1/2}g_{-1/2} +$$

$$+ \frac{1}{6}g_0 g_{-1} - \frac{4}{3}g_{-1/2}g_{3/2} + \frac{1}{3}g_{-1}g_1 - \frac{1}{12}g_{-3/2}g_{1/2}\Big]$$

Sa graničnim uslovima

$$\varphi_{11} = \varphi_{11} = 0 \qquad \eta_1 = 0$$

$$\varphi_{11}' = 0 \qquad \eta_1 = \infty$$

Homogeno rešenje je

$$\varphi_{11h}' = Cg_2$$

a partikularno treba tražiti u obliku

$$\varphi_{11p_1}' = C_1 g_1 + C_2 g_{1/2} + C_3 g_0 + C_4 g_{-1/2} + C_5 g_{-1} + C_6 g_{-3/2} + C_7 g_{-2}$$

$$\varphi_{11p_2}' = C_8 g_1^2 + C_9 g_{1/2}^2 + C_{10} g_0^2 + C_{11} g_{3/2} g_{1/2} + C_{12} g_1 g_0 + C_{13} g_{1/2} g_{-1/2} +$$

$$+ C_{14} g_0 g_2 + C_{15} g_{-1/2} g_{3/2} + C_{16} g_{-1} g_1$$

Smenom u datu diferencijalnu jednačinu dobijamo uslove

$$-4C_1 = -\frac{4}{3}\,2\,\Big(3 + \frac{1}{3}\,\frac{1}{\Gamma^2}\Big) \quad \text{iz kojih je} \quad C_1 = \frac{2}{3}\Big(3 + \frac{1}{3}\,\frac{1}{\Gamma^2}\Big)$$

$$-6C_2 = \frac{4}{3}\,\frac{2}{3}\,\frac{1}{\Gamma} \qquad C_2 = -\frac{4}{27}\,\frac{1}{\Gamma}$$

$$-8C_3 = \frac{1}{3}\,\frac{4}{3}\,\Big(5 + \frac{1}{3}\,\frac{1}{\Gamma^2}\Big) \qquad C_3 = -\frac{1}{18}\Big(5 + \frac{1}{3}\,\frac{1}{\Gamma^2}\Big)$$

$$-10C_4 = -\frac{4}{3}\,\frac{5}{36}\,\frac{1}{\Gamma} \qquad C_4 = \frac{1}{54}\,\frac{1}{\Gamma}$$

$$-12C_5 = -\frac{4}{3}\,\frac{7}{12} \qquad C_5 = \frac{7}{108}$$

$$-14C_6 = \frac{4}{3}\,\frac{7}{72}\,\frac{1}{\Gamma} \qquad C_6 = -\frac{1}{108}\,\frac{1}{\Gamma}$$

$$-16C_7 = -\frac{4}{3}\,\frac{1}{12} \qquad C_7 = \frac{1}{144}$$

i za drugi deo

$2C_8 - 4C_9 = 0$	$C_8 = -\frac{4}{3}$
$2C_9 - 8C_{10} = -\frac{4}{3}\,\frac{2}{3}$	$C_9 = -\frac{2}{3}$
$2C_{10} = -\frac{4}{3}\,\frac{1}{12}$	$C_{10} = -\frac{1}{18}$
$2C_{11} - 4C_{12} = -2\,\frac{4}{3}$	$C_{11} = -\frac{4}{3}$
$2C_{12} - 8C_{13} = \frac{4}{3}\,\frac{1}{3}$	$C_{12} = \frac{2}{3}$
$2C_{13} = \frac{4}{3}\,\frac{1}{6}$	$C_{13} = \frac{1}{9}$
$2C_{14} - 4C_{15} = -\frac{4}{3}\,\frac{4}{3}$	$C_{14} = -\frac{8}{9}$
$2C_{15} - 8C_{16} = \frac{4}{3}\,\frac{1}{3}$	$C_{15} = 0$
$2C_{16} = -\frac{4}{3}\,\frac{1}{12}$	$C_{16} = -\frac{1}{18}$

Rešenje je sada

$$\varphi'_{11} = C g_2 + \frac{2}{3}\left(3 + \frac{1}{3}\,\frac{1}{\Gamma^2}\right) g_1 - \frac{4}{27}\,\frac{1}{\Gamma}\, g_{1/2} - \frac{1}{18}\left(5 + \frac{1}{3}\,\frac{1}{\Gamma^2}\right) g_0 +$$

$$+ \frac{1}{54}\,\frac{1}{\Gamma}\, g_{-1/2} + \frac{7}{108}\, g_{-1} - \frac{1}{108}\,\frac{1}{\Gamma}\, g_{-3/2} + \frac{1}{144}\, g_{-2} - \frac{4}{3}\, g_1^2 -$$

$$- \frac{2}{3}\, g_{1/2}^2 - \frac{1}{18}\, g_0^2 - \frac{4}{3}\, g_{1/2} g_{3/2} + \frac{2}{3}\, g_0 g_1 + \frac{1}{9}\, g_{-1/2} g_{1/2} - \frac{8}{9}\, g_0 g_2 -$$

$$- \frac{1}{18}\, g_{-1} g_1$$

Smenom u diferencijalnu jednačinu pretpostavljenog rešenja odredili smo konstante od C_1 do C_{16}, a zadovoljavajući prvi granični uslov možemo odrediti i konstantu C, sledeće:

$$0 = C\,\frac{1}{2^2} + \frac{2}{3}\left(3 + \frac{1}{3}\,\frac{1}{\Gamma^2}\right)\frac{1}{2^2} - \frac{4}{27}\,\frac{1}{\Gamma}\cdot\frac{1}{2\Gamma} - \frac{1}{18}\left(5 + \frac{1}{3}\,\frac{1}{\Gamma^2}\right) +$$

$$+\frac{1}{54}\frac{1}{\Gamma}\frac{1}{\Gamma}-\frac{1}{108}\frac{1}{\Gamma}\left(-\frac{2}{\Gamma}\right)-\frac{4}{3}\left(\frac{1}{2^2}\right)^2-\frac{2}{3}\left(\frac{1}{2\Gamma}\right)^2-\frac{1}{18}-\frac{4}{3}\frac{1}{2\Gamma}\frac{1}{2^2\cdot 3}\frac{1}{\Gamma}+$$

$$+\frac{2}{3}\frac{1}{2^2}+\frac{1}{9}\frac{1}{\Gamma}\frac{1}{2\Gamma}-\frac{8}{9}\frac{1}{2^5}$$

Odavde je

$$C=-\frac{2^6}{3}\left(\frac{1}{3}-\frac{1}{4}\frac{1}{\Gamma^2}\right)$$

Rešenje je sada

$$\varphi_{11}'=-\frac{2^6}{3}\left(\frac{1}{3}-\frac{1}{4}\frac{1}{\Gamma^2}\right)g_2+\frac{2}{3}\left(3+\frac{1}{3}\frac{1}{\Gamma^2}\right)g_1-\frac{4}{27}\frac{1}{\Gamma}g_{1/2}-\frac{1}{18}\left(5+\frac{1}{3}\frac{1}{\Gamma^2}\right)g_0+$$

$$+\frac{1}{54}\frac{1}{\Gamma}g_{-1/2}+\frac{7}{108}g_{-1}-\frac{1}{108}\frac{1}{\Gamma}g_{-3/2}+\frac{1}{144}g_{-2}-\frac{4}{3}g_1^2-\frac{2}{3}g_{1/2}^2-$$

$$-\frac{1}{18}g_0^2-\frac{4}{3}g_{1/2}g_{3/2}+\frac{2}{3}g_0g_1+\frac{1}{9}g_{-1/2}g_{1/2}-\frac{8}{9}g_0g_2-\frac{1}{18}g_{-1}g_1$$

3. $\varphi_2'''+2\eta_1\varphi_2''-4\cdot 2\cdot\varphi_2'=-\frac{4}{3}\left(1-F_0'^2+F_0F_0''\right)=-\frac{4}{3}\Big[2g_0-\frac{1}{2}\frac{1}{\Gamma}g_{-1/2}+$

$$+\frac{1}{2}g_{-1}-g_0^2+g_{1/2}g_{-1/2}\Big]$$

sa graničnim uslovima

$$\varphi_2=\varphi_2'=0 \qquad \eta_1=0$$

$$\varphi_2'=0 \qquad \eta_1=\infty$$

Homogeno rešenje je

$$\varphi_{2h}'=C\,g_2$$

a partikularno potražimo u obliku

$$\varphi_{2p_1}'=C_1g_0+C_2g_{-1/2}+C_3g_{-1}$$

$$\varphi_{2p_2}'=C_4g_{3/2}^2+C_5g_1^2+C_6g_{1/2}^2+C_7g_1g_2+C_8g_{1/2}g_{3/2}+C_9g_0g_1$$

Smenom u datu diferencijalnu jednačinu dobićemo uslove

$-8C_1=-\frac{8}{3}$	iz kojih je	$C_1=\frac{1}{3}$
$-10C_2=\frac{2}{3}\frac{1}{\Gamma}$		$C_2=-\frac{1}{15}\frac{1}{\Gamma}$
$-12C_3=-\frac{2}{3}$		$C_3=\frac{1}{18}$

i za drugi deo

$$C_4 = 0$$

$$2C_5 - 4C_6 = 0 \qquad C_5 = \frac{4}{3}$$

$$2C_6 = \frac{4}{3} \qquad C_6 = \frac{2}{3}$$

$$C_7 = 0$$

$$2C_8 - 4C_9 = 0 \qquad C_8 = -\frac{4}{3}$$

$$2C_9 = -\frac{4}{3} \qquad C_9 = -\frac{2}{3}$$

Da je $C_4 = C_7 = 0$ to je očigledno jer smo u partikularnom rešenju pretpostavili veći red Gaus-ove funkcije nego što odgovara samoj diferencijalnoj jednačini.

Rešenje je

$$\varphi_2' = C g_2 + \frac{1}{3} g_0 - \frac{1}{15}\frac{1}{\Gamma} g_{-1/2} + \frac{1}{18} g_{-1} + \frac{4}{3} g_1^2 + \frac{2}{3} g_{1/2}^2 - \\ - \frac{4}{3} g_{1/2} g_{3/2} - \frac{2}{3} g_0 g_1$$

Zadovoljavajući prvi granični uslov

$$0 = C\frac{1}{2^5} + \frac{1}{3} - \frac{1}{15}\frac{1}{\Gamma}\frac{1}{\Gamma} + \frac{4}{3}\frac{1}{2^4} + \frac{2}{3}\left(\frac{1}{2\Gamma}\right)^2 - \frac{4}{3}\frac{1}{2\Gamma}\cdot\frac{1}{2^2\cdot 3\cdot\Gamma} - \frac{2}{3}\frac{1}{2^2}$$

dobija se

$$C = -2^5\left(\frac{1}{4} + \frac{2}{45}\frac{1}{\Gamma^2}\right)$$

Sada je rešenje

$$\varphi_2' = -2^5\left(\frac{1}{4} + \frac{2}{45}\frac{1}{\Gamma^2}\right) g_2 + \frac{1}{3} g_0 - \frac{1}{15}\frac{1}{\Gamma} g_{-1/2} + \frac{1}{18} g_{-1} + \frac{4}{3} g_1^2 + \\ + \frac{2}{3} g_{1/2}^2 - \frac{4}{3} g_{1/2} g_{3/2} - \frac{2}{3} g_0 g_1$$

Nadalje ćemo davati samo početne jednačine i njihova rešenja,obzirom da smo na dosta jednačina do sada iscrpno pokazali sam postupak rešavanja istih.

$$4.\quad \varphi_{111}''' + 2\eta_1 \varphi_{111}'' - 4\cdot 3\varphi_{111}' = \frac{4}{3}\Big[(\varphi_{11}' + \eta\varphi_{11}'') - (-2F_0' f_{11}' - f_1'^2 + F_0 f_{11}'' +$$

$$+ f_1 f_1'' + f_{11}F_0''\Big] = \frac{4}{3}\Big[-\frac{2}{3}^{6}\Big(\frac{7}{6} - \frac{5}{4}\frac{1}{\Gamma^2}\Big)g_2 + \frac{8}{3}\frac{1}{\Gamma}g_{3/2} + \frac{2}{3}\Big(\frac{31}{3} - \frac{3}{\Gamma^2}\Big)g_1$$

$$- \frac{25}{27}\frac{1}{\Gamma}g_{1/2} - \frac{1}{18}\Big(17 + \frac{7}{3}\frac{1}{\Gamma^2}\Big)g_0 + \frac{321}{2160}\frac{1}{\Gamma}g_{-1/2} + \frac{1}{36}\Big(1 + \frac{1}{3}\frac{1}{\Gamma^2}\Big)g_{-1}$$

$$+ \frac{1}{144}\frac{1}{\Gamma}g_{-3/2} - \frac{5}{72}g_{-2} + \frac{11}{108\cdot 16}\frac{1}{\Gamma}g_{-5/2} - \frac{17}{144\cdot 32}g_{-3} - \frac{68}{9}g_1^2 +$$

$$+ \frac{4}{9}g_{1/2}^2 + \frac{7}{18}g_0^2 + \frac{1}{27}g_{-1/2}^2 + \frac{1}{144}g_{-1}^2 - \frac{8}{9}g_0 g_1 - \frac{32}{3}g_{1/2}g_{3/2} -$$

$$- \frac{3}{18}g_{-1/2}g_{1/2} - \frac{2}{27}g_{-1}g_0 - \frac{1}{96}g_{-3/2}g_{-1/2} + \frac{88}{9}g_0 g_2 + \frac{26}{9}g_{-1/2}g_{3/2}$$

$$- \frac{1}{18}g_{-1}g_1 + \frac{1}{108}g_{-3/2}g_{1/2} + \frac{1}{144}g_{-2}g_0 - \frac{16}{3}g_{-1/2}g_{5/2} + \frac{4}{9}g_{-1}g_2 -$$

$$- \frac{1}{9}g_{-3/2}g_{3/2} + \frac{1}{36}g_{-2}g_1 - \frac{1}{288}g_{-5/2}g_{1/2}$$

sa graničnim uslovima

$$\varphi_{111} = \varphi_{111}' = 0 \qquad \eta_1 = 0$$
$$\varphi_{111}' = 0 \qquad \eta_1 = \infty$$

Rešenje:

$$\varphi_{111}' = -384\Big(\frac{1219}{648} + \frac{1423}{1890}\frac{1}{3}\frac{1}{\Gamma^2}\Big)g_3 + \frac{64}{9}\Big(\frac{7}{6} - \frac{5}{4}\frac{1}{\Gamma^2}\Big)g_2 -$$

$$- \frac{16}{27}\frac{1}{\Gamma}g_{3/2} - \frac{1}{9}\Big(\frac{31}{3} - \frac{3}{\Gamma^2}\Big)g_1 + \frac{10}{81}\frac{1}{\Gamma}g_{1/2} + \frac{1}{9}\Big(17 + \frac{7}{3}\frac{1}{\Gamma^2}\Big)g_0 -$$

$$- \frac{107}{7\cdot 1080}\frac{1}{\Gamma}g_{-1/2} - \frac{1}{12\cdot 36}\Big(1 + \frac{1}{3}\frac{1}{\Gamma^2}\Big)g_{-1} - \frac{1}{27\cdot 72}\frac{1}{\Gamma}g_{-3/2} +$$

$$+ \frac{1}{216}g_{-2} - \frac{1}{24\cdot 108}\frac{1}{\Gamma}g_{-5/2} + \frac{17}{18\cdot 32\cdot 144}g_{-3} + \frac{16}{27}g_{3/2}^2 +$$

$$+ \frac{228}{81}g_1^2 + \frac{51}{81}g_{1/2}^2 + \frac{5}{81}g_0^2 + \frac{1}{216}g_{-1/2}^2 - \frac{384}{27}g_1 g_2 - \frac{96}{27}g_{1/2}\cdot$$

$$g_{3/2} - \frac{20}{27}g_0 g_1 - \frac{17}{162}g_{-1/2}g_{1/2} - \frac{1}{144}g_{-1}g_0 + \frac{328}{27}g_{1/2}g_{5/2} +$$

$$+ \frac{76}{27}g_0 g_2 + \frac{2}{9}g_{-1/2}g_{3/2} + \frac{7}{162}g_{-1}g_1 + \frac{1}{216}g_{-3/2}g_{1/2} -$$

$$- \frac{32}{9}g_0 g_3 - \frac{2}{27}g_{-1}g_2 - \frac{1}{432}g_{-2}g_1$$

$$\cdot\ \varphi_{12}''' + 2\eta_1\varphi_{12}'' - 4\ 3\varphi_{12}' = \frac{4}{3}\left[(\varphi_2' + \eta_1\varphi_2'') - (-2F_0'f_1' + F_0 f_1'' + f_1 F_0'')\right] =$$

$$= \frac{4}{3}\Big[-5\cdot 2^5\left(\frac{1}{4} + \frac{2}{25}\frac{1}{\Gamma^2}\right)g_2 + 2^4\left(\frac{1}{4} + \frac{2}{45}\frac{1}{\Gamma^2}\right)g_1 + \frac{2}{3}\frac{1}{\Gamma}g_{1/2} + \frac{1}{3}g_0 +$$

$$+ \frac{5}{36}\frac{1}{\Gamma}g_{-1/2} - \frac{7}{18}g_{-1} + \frac{3}{40}\frac{1}{\Gamma}g_{-3/2} - \frac{5}{72}g_{-2} + 4g_1^2 - \frac{2}{3}g_0^2 - \frac{1}{12}g_{-1/2}^2$$

$$- \frac{8}{3}g_{1/2}g_{3/2} + \frac{2}{3}g_0g_1 + \frac{1}{3}g_{-1/2}g_{1/2} + \frac{1}{6}g_{-1}g_0 - \frac{2}{3}g_{3/2}g_{-1/2} +$$

$$+ \frac{1}{3}g_1g_{-1} - \frac{1}{12}g_{1/2}g_{-3/2}$$

sa graničnim uslovima

$$\varphi_{12} = \varphi_{12}' = 0 \qquad \eta_1 = 0$$

$$\varphi_{12}' = 0 \qquad \eta_1 = \infty$$

Rešenje:

$$\varphi_{12}' = -2^7\left(\frac{29}{144} - \frac{601}{1260}\frac{1}{\Gamma^2}\right)g_3 + \frac{5}{3}2^5\left(\frac{1}{4} + \frac{2}{45}\frac{1}{\Gamma^2}\right)g_2 - \frac{2^3}{3}\left(\frac{1}{4} + \frac{2}{45}\frac{1}{\Gamma^2}\right)$$

$$g_1 - \frac{4}{45}\frac{1}{\Gamma}g_{1/2} - \frac{1}{27}g_0 + \frac{5}{21\cdot 18}\frac{1}{\Gamma}g_{-1/2} + \frac{7}{12\cdot 18}g_{-1} - \frac{1}{180}\frac{1}{\Gamma}g_{-3/2}$$

$$+ \frac{1}{216}g_{-2} - 8g_{3/2}^2 - \frac{16}{3}g_1^2 - \frac{4}{3}g_{1/2}^2 - \frac{1}{18}g_0^2 + \frac{56}{9}g_1g_2 + 4g_{1/2}g_{3/2} +$$

$$+ \frac{8}{9}g_0g_1 + \frac{1}{9}g_{-1/2}g_{1/2} - \frac{16}{9}g_{1/2}g_{5/2} - \frac{8}{9}g_0g_2 - \frac{1}{9}g_{-1/2}g_{3/2} -$$

$$- \frac{1}{18}g_{-1}g_1$$

6\. $$\varphi_{21}''' + 2\eta_1\varphi_{21}'' - 4\cdot 3\varphi_{21}' = \frac{4}{3}\Big[\frac{2^5}{3}g_2 + \frac{8}{3}\frac{1}{\Gamma}g_{3/2} - \frac{2}{3}\left(13 + \frac{1}{3}\frac{1}{\Gamma^2}\right)$$

$$g_1 + \frac{1}{3}\left(6 + \frac{1}{3}\frac{1}{\Gamma^2}\right)g_0 - \frac{1}{15}\frac{1}{\Gamma}g_{-1/2} - \frac{5}{9}g_{-1} + \frac{1}{12}\frac{1}{\Gamma}g_{-3/2} -$$

$$- \frac{1}{72}g_{-2} + \frac{8}{3}g_{1/2}^2 - \frac{1}{3}g_0^2 - \frac{16}{3}g_{1/2}g_{3/2} - \frac{2}{3}g_0g_1 - \frac{2}{3}g_{1/2}$$

$$g_{-1/2} + \frac{1}{9}g_0g_{-1} + \frac{32}{3}g_0g_2 - \frac{1}{3}g_1g_{-1} - \frac{1}{18}g_{1/2}g_{-3/2}$$

$$- \frac{16}{3}g_{5/2}g_{-1/2}$$

sa istim graničnim uslovima kao i kod pret[illegible]

Rešenje:

$$\varphi_{21}' = \frac{2^7}{9}\left(\frac{7}{16} + \frac{61}{28}\frac{1}{\Gamma^2}\right) g_3 - \frac{2^5}{9} g_2 - \frac{16}{27}\frac{1}{\Gamma} g_{3/2} + \frac{1}{9}\left(13 + \frac{1}{3}\frac{1}{\Gamma^2}\right) g_1 -$$

$$- \frac{1}{27}\left(6 + \frac{1}{3}\frac{1}{\Gamma^2}\right) g_0 + \frac{2}{315}\frac{1}{\Gamma} g_{-1/2} + \frac{5}{108} g_{-1} - \frac{1}{162}\frac{1}{\Gamma} g_{-3/2} +$$

$$+ \frac{1}{1080} g_{-2} + \frac{16}{9} g_{3/2}^2 + \frac{8}{9} g_1^2 - \frac{2}{9} g_{1/2}^2 - \frac{40}{9} g_1 g_2 - \frac{4}{9} g_{1/2} g_{3/2}$$

$$+ \frac{2}{27} g_{-1/2} g_{1/2} + \frac{32}{9} g_{1/2} g_{5/2} - \frac{48}{27} g_0 g_2 - \frac{12}{27} g_{-1/2} g_{3/2} -$$

$$- \frac{1}{27} g_{-1} g_1 - \frac{32}{9} g_3 g_0$$

7. $$\varphi_3''' + 2\eta_1 \varphi_3'' - 4\cdot 3\,\varphi_3' = -\frac{4}{3}\left[1 - F_0'^2 + F_0 F_0''\right] = -\frac{4}{3}\left[2 g_0 - \frac{1}{2}\frac{1}{\Gamma} g_{-1/2}\right.$$

$$\left. + \frac{1}{2} g_{-1} - g_0^2 + g_{-1/2} g_{1/2}\right]$$

Rešenje:

$$\varphi_3' = -2^7\left(\frac{39}{72} + \frac{51}{378}\frac{1}{\Gamma^2}\right) g_3 + \frac{2}{9} g_0 - \frac{1}{21}\frac{1}{\Gamma} g_{-1/2} + \frac{1}{24} g_{-1} +$$

$$+ \frac{16}{3} g_{3/2}^2 + \frac{8}{3} g_1^2 + \frac{2}{3} g_{1/2}^2 - \frac{16}{3} g_1 g_2 - \frac{8}{3} g_{1/2} g_{3/2} - \frac{2}{3} g_0 g_1$$

8.

$$\varphi_{1111}''' + 2\eta_1 \varphi_{1111}'' - 4\cdot 4\,\varphi_{1111}' = \frac{4}{3}\left[-384\left(\frac{8293}{648} + \frac{1423}{1890}\frac{7}{3}\frac{1}{\Gamma^2}\right) g_3 +\right.$$

$$+ \frac{160}{9}\frac{1}{\Gamma} g_{5/2} + 64\left(\frac{3717}{648} + \frac{221}{3780}\frac{1}{\Gamma^2}\right) g_2 - \frac{180}{27}\frac{1}{\Gamma} g_{3/2} - \frac{1}{9}\left(\frac{85}{3} -\right.$$

$$\left. - 49\frac{1}{\Gamma^2}\right) g_1 + \frac{1721}{1620}\frac{1}{\Gamma} g_{1/2} + \frac{1}{54}\left(113 + 5\frac{1}{\Gamma^2}\right) g_0 - \frac{761319}{7348320}\frac{1}{\Gamma} g_{-1/2}$$

$$- \frac{1}{1296}\left(1193 + 167\frac{1}{\Gamma^2}\right) g_{-1} + \frac{1803}{3780\cdot 36}\frac{1}{\Gamma} g_{-3/2} - \frac{1}{12\cdot 216}\left(46 - \frac{1}{\Gamma^2}\right)\cdot$$

$$g_{-2} + \frac{65}{6\cdot 48\cdot 108}\frac{1}{\Gamma} g_{-5/2} - \frac{261}{18\cdot 32\cdot 144} g_{-3} + \frac{5}{96\cdot 216}\frac{1}{\Gamma} g_{-7/2} -$$

$$- \frac{9}{32\cdot 36\cdot 144} g_{-4} - \frac{80}{27} g_{3/2}^2 + \frac{852}{81} g_1^2 + \frac{1}{9} g_{1/2}^2 + \frac{11}{81} g_0^2 -$$

$$- \frac{1}{48} g_{-1/2}^2 - \frac{1}{12\cdot 54} g_{-1}^2 - \frac{1}{12\cdot 16\cdot 18} g_{-3/2}^2 - \frac{1648}{27} g_1 g_2 +$$

$$+\frac{28}{3} g_{1/2} g_{3/2} + \frac{8}{27} g_0 g_1 - \frac{41}{162} g_{-1/2} g_{1/2} + \frac{5}{144} g_{-1} g_0 - \frac{1}{288} g_{-1/2} g_{-3/2} +$$

$$+ \frac{1}{32 \cdot 54} g_{-2} g_{-1} - \frac{592}{27} g_{1/2} g_{5/2} - \frac{472}{27} g_0 g_2 - \frac{16}{9} g_{-1/2} g_{3/2} + \frac{2}{81} g_{-1} g_1$$

$$+ \frac{1}{4 \cdot 324} g_{-3/2} g_{1/2} - \frac{1}{24 \cdot 108} g_{-2} g_0 - \frac{1}{18 \cdot 18 \cdot 32} g_{-5/2} g_{-1/2} +$$

$$+ \frac{608}{9} g_0 g_3 + \frac{22015}{27 \cdot 128} g_{-1/2} g_{5/2} - \frac{4}{9} g_{-1} g_2 + \frac{4}{81} g_{-3/2} g_{3/2} -$$

$$- \frac{7}{4 \cdot 324} g_{-2} g_1 - \frac{1}{6 \cdot 432} g_{-5/2} g_{1/2} + \frac{1}{16 \cdot 18^2} g_{-3} g_0 - \frac{320}{9} g_{-1/2} g_{7/2}$$

$$+ \frac{16}{9} g_{-1} g_3 - \frac{4}{9} g_{-3/2} g_{5/2} + \frac{1}{27} g_{-2} g_2 - \frac{1}{12 \cdot 18} g_{-5/2} g_{3/2} + \frac{1}{2 \cdot 432}$$

$$g_{-3} g_1 - \frac{1}{18 \cdot 18 \cdot 32} g_{-7/2} g_{1/2} \Big]$$

Rešenje:

$$\varphi'_{1111} = - \frac{2^{11}}{27} \left(\frac{12872183}{128 \cdot 576} + \frac{5960700835821}{1837080 \cdot 11648 \cdot 11} \frac{1}{\Gamma^2} \right) g_4 + \frac{384}{3} \Big(\frac{8293}{648} +$$

$$+ \frac{1423}{1890} \frac{7}{3} \frac{1}{\Gamma^2} \Big) g_3 - \frac{320}{81} \frac{1}{\Gamma} g_{5/2} - \frac{32}{3} \left(\frac{3717}{648} + \frac{221}{3780} \frac{1}{\Gamma^2} \right) g_2 +$$

$$+ \frac{72}{81} \frac{1}{\Gamma} g_{3/2} + \frac{1}{81} \left(\frac{85}{3} - 49 \frac{1}{\Gamma^2} \right) g_1 - \frac{3442}{21 \cdot 1620} \frac{1}{\Gamma} g_{1/2} - \frac{1}{12 \cdot 54} \Big(113 +$$

$$+ 5 \frac{1}{\Gamma^2} \Big) g_0 + \frac{761319}{54 \cdot 1837080} \frac{1}{\Gamma} g_{-1/2} + \frac{1}{15 \cdot 1296} \left(1193 + 167 \frac{1}{\Gamma^2} \right) g_{-1} -$$

$$- \frac{1803}{22 \cdot 27 \cdot 3780} \frac{1}{\Gamma} g_{-3/2} + \frac{1}{12 \cdot 18 \cdot 216} \left(46 - \frac{1}{\Gamma^2} \right) g_{-2} - \frac{65}{6 \cdot 36 \cdot 26 \cdot 108} \frac{1}{\Gamma} g_{-5/2}$$

$$+ \frac{261}{18 \cdot 21 \cdot 32 \cdot 144} g_{-3} - \frac{1}{9 \cdot 48 \cdot 216} \frac{1}{\Gamma} g_{-7/2} + \frac{1}{3 \cdot 32 \cdot 32 \cdot 144} g_{-4} -$$

$$- \frac{800}{81} g_2^2 - \frac{320}{81} g_{3/2}^2 - \frac{74}{27} g_1^2 - \frac{38}{81} g_{1/2}^2 - \frac{23}{9 \cdot 54} g_0^2 - \frac{13}{18 \cdot 216} g_{-1/2}^2$$

$$- \frac{1}{36 \cdot 144} g_{-1}^2 - \frac{1376}{81} g_{3/2} g_{5/2} + \frac{320}{27} g_1 g_2 + \frac{38}{27} g_{1/2} g_{3/2} + \frac{49}{9 \cdot 27} g_0 g_1$$

$$+ \frac{5}{108} g_{-1/2} g_{1/2} + \frac{1}{16 \cdot 27} g_{-1} g_0 + \frac{1}{48 \cdot 54} g_{-3/2} g_{-1/2} - \frac{11440}{9 \cdot 27} g_1 g_3 -$$

$$- \frac{3944}{9 \cdot 27} g_{1/2} g_{5/2} - \frac{278}{9 \cdot 27} g_0 g_2 - \frac{15}{9 \cdot 27} g_{-1/2} g_{3/2} - \frac{19}{18 \cdot 108} g_{-1} g_1$$

$$- \frac{1}{27\cdot 36} g_{-3/2} g_{1/2} - \frac{1}{18\cdot 27\cdot 32} g_{-2} g_0 + \frac{145919}{32\cdot 81} g_{1/2} g_{7/2} + \frac{29183}{64\cdot 81}$$

$$g_0 g_3 + \frac{28}{81} g_{-1/2} g_{5/2} + \frac{26}{9\cdot 27} g_{-1} g_2 + \frac{1}{108} g_{-3/2} g_{3/2} + \frac{5}{9\cdot 432} g_{-2} g_1 +$$

$$+ \frac{1}{16\cdot 8\cdot 27} g_{-5/2} g_{1/2} - \frac{640}{27} g_0 g_4 - \frac{8}{27} g_{-1} g_3 - \frac{1}{12\cdot 27} g_{-2} g_2 -$$

$$- \frac{1}{18\cdot 27\cdot 32} g_{-3} g_1$$

9. $$\varphi_{121}''' + 2\eta_1 \varphi_{121}'' - 4\cdot 4 \varphi_{121}' = \frac{4}{3}\Big[- \frac{2^7}{9}\Big(\frac{559}{16} - \frac{427}{28}\frac{1}{\Gamma^2}\Big) g_3 - \frac{19}{9}\cdot$$

$$\cdot 2^5 \frac{1}{\Gamma} g_{5/2} + \frac{2^5}{9}\Big(\frac{65}{8} + \frac{22}{7}\frac{1}{\Gamma^2}\Big) g_2 + \frac{20}{9}\frac{1}{\Gamma} g_{3/2} + \frac{1}{9}\Big(101 - 17\frac{1}{\Gamma^2}\Big) g_1 -$$

$$- \frac{4}{45}\frac{1}{\Gamma} g_{1/2} - \frac{1}{27}\Big(64 + \frac{13}{3}\frac{1}{\Gamma^2}\Big) g_0 + \frac{227}{7\cdot 216}\frac{1}{\Gamma} g_{-1/2} + \frac{1}{27}\Big(3 + \frac{5}{12}\frac{1}{\Gamma^2}\Big)\cdot$$

$$g_{-1} + \frac{253}{54\cdot 40\cdot 7}\frac{1}{\Gamma} g_{-3/2} - \frac{221}{1080\cdot 2} g_{-2} + \frac{13}{162\cdot 8}\frac{1}{\Gamma} g_{-5/2} - \frac{1}{160} g_{-3} -$$

$$- 8 g_{3/2}^2 - \frac{4}{9} g_1^2 - \frac{14}{9} g_{1/2}^2 + \frac{17}{54} g_0^2 + \frac{1}{24} g_{-1/2}^2 + \frac{1}{108} g_{-1}^2 + 8 g_1 g_2 -$$

$$- \frac{116}{9} g_{1/2} g_{3/2} + \frac{10}{9} g_0 g_1 + \frac{11}{54} g_{-1/2} g_{1/2} - \frac{13}{108} g_{-1} g_0 - \frac{1}{72} g_{-3/2}\cdot$$

$$\cdot g_{-1/2} + \frac{1184}{9} g_{1/2} g_{5/2} + 12 g_0 g_2 + \frac{20}{9} g_{-1/2} g_{3/2} + \frac{11}{54} g_{-1} g_1 -$$

$$- \frac{1}{216} g_{-3/2} g_{1/2} + \frac{1}{108} g_{-2} g_0 - \frac{2464}{9} g_0 g_3 - \frac{64}{9} g_{-1/2} g_{5/2} + \frac{20}{9}\cdot$$

$$g_{-1} g_2 + \frac{4}{27} g_{-3/2} g_{3/2} + \frac{5}{108} g_{-2} g_1 - \frac{1}{216} g_{-5/2} g_{1/2} + \frac{64\cdot 19}{9} g_{-1/2} g_{7/2} +$$

$$+ \frac{16}{9} g_{-1} g_3 - \frac{4}{9} g_{-3/2} g_{5/2}$$

Rešenje:

$$\varphi_{121}' = - \frac{2^{11}}{27}\Big[\frac{1329}{64} + \frac{3243691}{60\cdot 77\cdot 117}\frac{1}{\Gamma^2}\Big] g_4 + \frac{2^7}{27}\Big(\frac{559}{16} - \frac{427}{28}\frac{1}{\Gamma^2}\Big) g_3 +$$

$$+ \frac{19}{81} 2^6 \frac{1}{\Gamma} g_{5/2} - \frac{2^4}{27}\Big(\frac{65}{8} + \frac{22}{7}\frac{1}{\Gamma^2}\Big) g_2 - \frac{8}{27}\frac{1}{\Gamma} g_{3/2} - \frac{1}{81}\Big(101 - 17\frac{1}{\Gamma^2}\Big) g_1 +$$

$$+ \frac{8}{21\cdot 45}\frac{1}{\Gamma} g_{1/2} + \frac{1}{12\cdot 27}\Big(64 + \frac{13}{3}\frac{1}{\Gamma^2}\Big) g_0 - \frac{227}{9\cdot 21\cdot 108}\frac{1}{\Gamma} g_{-1/2} -$$

$$-\frac{1}{15\cdot 27}\left(3+\frac{5}{12}\frac{1}{r^2}\right)g_{-1} - \frac{253}{11\cdot 20\cdot 21\cdot 54}\frac{1}{r}g_{-3/2} + \frac{221}{36\cdot 1080}g_{-2} -$$

$$-\frac{1}{12\cdot 162}\frac{1}{r}g_{-5/2} + \frac{1}{21\cdot 160}g_{-3} + \frac{272}{9}g_2^2 + \frac{160}{9}g_{3/2}^2 + \frac{122}{27}g_1^2 +$$

$$+\frac{25}{27}g_{1/2}^2 + \frac{29}{4\cdot 81}g_0^2 + \frac{1}{6\cdot 27}g_{-1/2}^2 - \frac{592}{9}g_{3/2}g_{5/2} - \frac{960}{27}g_1g_2 -$$

$$-\frac{182}{27}g_{1/2}g_{3/2} - \frac{101}{81}g_0g_1 - \frac{14}{81}g_{-1/2}g_{1/2} - \frac{1}{108}g_{-1}g_0 + \frac{3904}{27}g_1g_3$$

$$+\frac{256}{9}g_{1/2}g_{5/2} + \frac{138}{27}g_0g_2 + \frac{49}{81}g_{-1/2}g_{3/2} + \frac{49}{3\cdot 108}g_{-1}g_1 + \frac{1}{3\cdot 54}\cdot$$

$$g_{-3/2}g_{1/2} - \frac{4736}{26}g_{1/2}g_{7/2} + \frac{32}{9}g_0g_3 + \frac{56}{27}g_{-1/2}g_{5/2} + \frac{8}{81}g_{-1}g_2 -$$

$$-\frac{1}{3\cdot 108}g_{-2}g_1 + \frac{64\cdot 38}{27}g_0g_4 - \frac{8}{27}g_{-1}g_3$$

10. $$\varphi_{112}''' + 2\eta_1\varphi_{112}'' - 4\cdot 4\,\varphi_{112}' = \frac{4}{3}\Big[-7\cdot 2^7\left(\frac{29}{144}-\frac{601}{1260}\frac{1}{r^2}\right)g_3 +$$

$$+2^4\left(\frac{203}{36}-\frac{1103}{945}\frac{1}{r^2}\right)g_2 + \frac{8}{3}\frac{1}{r}g_{3/2} - \frac{2^3}{3}\left(\frac{17}{4}+\frac{26}{45}\frac{1}{r^2}\right)g_1 -$$

$$-\frac{109}{3\cdot 45}\frac{1}{r}g_{1/2} + \frac{1}{3}\left(\frac{17}{9}+\frac{8}{45}\frac{1}{r^2}\right)g_0 + \frac{257}{2160}\frac{1}{r}g_{-1/2} + \frac{1}{72}g_{-1} +$$

$$+\frac{11}{56\cdot 90}\frac{1}{r}g_{-3/2} - \frac{5}{108}g_{-2} + \frac{13}{16\cdot 180}\frac{1}{r}g_{-5/2} - \frac{7}{8\cdot 216}g_{-3} -$$

$$-\frac{112}{3}g_{3/2}^2 - \frac{152}{9}g_1^2 - \frac{4}{3}g_{1/2}^2 + \frac{7}{18}g_0^2 + \frac{1}{27}g_{-1/2}^2 + \frac{1}{144}g_{-1}^2 + \frac{136}{9}\cdot$$

$$g_1g_2 + \frac{44}{9}g_{1/2}g_{3/2} + \frac{2}{3}g_0g_1 + \frac{7}{18}g_{-1/2}g_{1/2} - \frac{2}{27}g_{-1}g_0 - \frac{1}{96}\cdot$$

$$g_{-3/2}g_{-1/2} - \frac{32}{9}g_{1/2}g_{5/2} + \frac{68}{9}g_0g_2 + \frac{2}{9}g_{-1/2}g_{3/2} - \frac{1}{9}g_{-1}g_1 +$$

$$+\frac{1}{6\cdot 18}g_{-3/2}g_{1/2} + \frac{1}{16\cdot 9}g_{-2}g_0 - \frac{40}{9}g_{-1/2}g_{5/2} + \frac{4}{9}g_{-1}g_2 - \frac{1}{18}\cdot$$

$$g_{-3/2}g_{3/2} + \frac{1}{36}g_{-2}g_1 - \frac{1}{16\cdot 18}g_{-5/2}g_{1/2}\Big]$$

Rešenje:

$$\varphi_{112}' = \frac{2^{11}}{81}\left(\frac{7809}{32}+\frac{44612}{7\cdot 105}\frac{1}{r^2}\right)g_4 + \frac{7}{3}2^7\left(\frac{29}{144}-\frac{601}{1260}\frac{1}{r^2}\right)g_3 -$$

$$-\frac{8}{3}\left(\frac{203}{36}-\frac{1103}{945}\frac{1}{r^2}\right)g_2-\frac{16}{45}\frac{1}{r}g_{3/2}+\frac{8}{27}\left(\frac{17}{4}+\frac{26}{45}\frac{1}{r^2}\right)g_1+$$

$$+\frac{218}{45\cdot 63}\frac{1}{r}g_{1/2}-\frac{1}{36}\left(\frac{17}{9}+\frac{8}{45}\frac{1}{r^2}\right)g_0-\frac{257}{27\cdot 1080}\frac{1}{r}g_{-1/2}-\frac{1}{18\cdot 60}g_{-1}-$$

$$-\frac{1}{3\cdot 28\cdot 90}\frac{1}{r}g_{-3/2}+\frac{5}{18\cdot 108}g_{-2}-\frac{1}{48\cdot 90}\frac{1}{r}g_{-5/2}-\frac{1}{24\cdot 216}g_{-3}-$$

$$-\frac{1200}{81}g_2^2+\frac{408}{81}g_{3/2}^2+\frac{330}{81}g_1^2+\frac{67}{81}g_{1/2}^2+\frac{23}{324}g_0^2+\frac{1}{216}g_{-1/2}^2-$$

$$-\frac{1056}{81}g_{3/2}g_{5/2}-\frac{936}{81}g_1g_2-\frac{300}{81}g_{1/2}g_{3/2}-\frac{56}{81}g_0g_1-\frac{77}{9\cdot 72}$$

$$g_{-1/2}g_{1/2}-\frac{1}{144}g_{-1}g_0+\frac{2064}{81}g_1g_3+\frac{1128}{81}g_{1/2}g_{5/2}+\frac{180}{81}g_0g_2+$$

$$+\frac{28}{81}g_{-1/2}g_{3/2}+\frac{17}{12\cdot 27}g_{-1}g_1+\frac{1}{8\cdot 27}g_{-3/2}g_{1/2}-\frac{192}{27}g_{1/2}g_{7/2}-$$

$$-\frac{96}{27}g_0g_3-\frac{4}{27}g_{-1/2}g_{5/2}-\frac{2}{27}g_{-1}g_2-\frac{1}{8\cdot 27}g_{-3/2}g_{3/2}-\frac{1}{16\cdot 27}g_{-2}g_1$$

11. $$\varphi_{31}'''+2\eta_1\varphi_{31}''-4\cdot 4\varphi_{31}'=\frac{4}{3}\left[\frac{2^9}{3}g_3+\frac{2^6}{3}\frac{1}{r}g_{5/2}-\frac{2^6}{3}g_2-2\left(3+\frac{1}{3}\frac{1}{r^2}\right)\right.$$

$$g_1+\frac{1}{9}\left(11+\frac{1}{r^2}\right)g_0-\frac{113}{9\cdot 280}\frac{1}{r}g_{-1/2}-\frac{19}{36}g_{-1}+\frac{11}{6\cdot 24}\frac{1}{r}g_{-3/2}-$$

$$-\frac{1}{16}g_{-2}+\frac{4}{3}g_{1/2}^2-\frac{2}{9}g_0^2+\frac{1}{24}g_{-1/2}^2-\frac{2}{3}g_0g_1-\frac{1}{9}g_{-1/2}g_{1/2}+$$

$$+\frac{1}{12}g_{-1}g_0+\frac{2^8}{3}g_0g_3+\frac{1}{3}g_1g_{-1}-\frac{1}{24}g_{1/2}g_{-3/2}-\frac{2^7}{3}g_{1/2}g_{5/2}-$$

$$\left.-\frac{2^7}{3}g_{-1/2}g_{7/2}\right]$$

Rešenje:

$$\varphi_{31}'=-\frac{2^{11}}{9}\left(\frac{237}{32}+\frac{2833}{630}\frac{1}{r^2}\right)g_4-\frac{2^9}{9}g_3-\frac{2^7}{27}\frac{1}{r}g_{5/2}+\frac{2^5}{9}g_2+$$

$$+\frac{2}{9}\left(3+\frac{1}{3}\frac{1}{r^2}\right)g_1-\frac{1}{4\cdot 27}\left(11+\frac{1}{r^2}\right)g_0+\frac{113}{81\cdot 420}\frac{1}{r}g_{-1/2}+\frac{19}{15\cdot 36}g_{-1}-$$

$$-\frac{1}{12\cdot 18}\frac{1}{r}g_{-3/2}+\frac{1}{16\cdot 48}g_{-2}+\frac{32}{3}g_2^2+\frac{16}{3}g_{3/2}^2+\frac{4}{3}g_1^2+\frac{2}{27}g_{1/2}^2+$$

$$+\frac{1}{36}g_0^2+\frac{384}{27}g_{3/2}g_{5/2}+\frac{192}{27}g_1g_2+\frac{48}{27}g_{1/2}g_{3/2}+\frac{10}{27}g_0g_1+$$

$+ \frac{1}{18} g_{-1/2} g_{1/2} - \frac{2^8}{9} g_1 g_3 - \frac{1}{36} g_{-1} g_1 + \frac{2^9}{9} g_{1/2} g_{7/2} - \frac{2^8}{9} g_0 g_4$

12. $\varphi_{22}''' + 2\eta_1 \varphi_{22}'' - 4\cdot 4 \varphi_{22}' = \frac{4}{3}\Big[-2^5\Big(\frac{11}{12} + \frac{10}{45}\frac{1}{r^2}\Big) g_2 + \frac{8}{3}\frac{1}{r} g_{3/2} +$

$+ 2^5\Big(\frac{1}{24} + \frac{1}{45}\frac{1}{r^2}\Big) g_1 + \frac{2}{3} g_0 - \frac{1}{15}\frac{1}{r} g_{-1/2} - \frac{13}{36} g_{-1} + \frac{11}{5\cdot 36}\frac{1}{r} g_{-3/2} -$

$- \frac{1}{18} g_{-2} + 4 g_1^2 + \frac{2}{3} g_{1/2}^2 - \frac{2}{3} g_0^2 - \frac{1}{18} g_{-1/2}^2 - \frac{20}{3} g_{1/2} g_{3/2} + \frac{2}{3} g_0 g_1 +$

$+ \frac{1}{3} g_{-1/2} g_{1/2} + \frac{1}{9} g_{-1} g_0 + \frac{16}{3} g_0 g_2 + \frac{1}{3} g_{-1} g_1 - \frac{16}{3} g_{-1/2} g_{5/2} -$

$- \frac{1}{18} g_{1/2} g_{-3/2} \Big]$

Rešenje:

$\varphi_{22}' = -\frac{2^{11}}{9}\Big(\frac{7}{4} + \frac{23}{210}\frac{1}{r^2}\Big) g_4 + \frac{2^4}{3}\Big(\frac{11}{12} + \frac{10}{45}\frac{1}{r^2}\Big) g_2 - \frac{16}{45}\frac{1}{r} g_{3/2} -$

$- \frac{2^5}{9}\Big(\frac{1}{24} + \frac{1}{45}\frac{1}{r^2}\Big) g_1 - \frac{1}{18} g_0 + \frac{2}{15\cdot 27}\frac{1}{r} g_{-1/2} + \frac{13}{15\cdot 36} g_{-1} - \frac{1}{15\cdot 18}\frac{1}{r} g_{-3/2}$

$+ \frac{1}{18^2} g_{-2} - \frac{80}{3} g_2^2 - \frac{40}{3} g_{3/2}^2 - 4 g_1^2 - \frac{20}{27} g_{1/2}^2 - \frac{1}{27} g_0^2 + \frac{304}{9} g_{3/2} g_{5/2} +$

$+ \frac{152}{9} g_1 g_2 + \frac{16}{3} g_{1/2} g_{3/2} + \frac{22}{27} g_0 g_1 + \frac{2}{27} g_{-1/2} g_{1/2} + \frac{32}{9} g_1 g_3 +$

$+ \frac{16}{9} g_{1/2} g_{5/2} - \frac{12}{27} g_0 g_2 - \frac{2}{27} g_{-1/2} g_{3/2} - \frac{1}{27} g_{-1} g_1 - \frac{64}{9} g_{1/2} g_{7/2} -$

$- \frac{32}{9} g_0 g_3$

13. $\varphi_{13}''' + 2\eta_1 \varphi_{13}'' - 4\cdot 4 \varphi_{13}' = \frac{4}{3}\Big[-7\cdot 2^7\Big(\frac{39}{72} + \frac{51}{378}\Big)\frac{1}{r^2}\Big) g_3 + 2^6\Big(\frac{39}{72} +$

$+ \frac{51}{378}\frac{1}{r^2}\Big) g_2 + \frac{2}{3}\frac{1}{r} g_{1/2} + \frac{2}{9} g_0 - \frac{5}{36}\frac{1}{r} g_{-1/2} - \frac{23}{3\cdot 24} g_{-1} + \frac{11}{7\cdot 24}\frac{1}{r}$

$g_{-3/2} - \frac{1}{16} g_{-2} + \frac{16}{3} g_{3/2}^2 + \frac{16}{3} g_1^2 - \frac{2}{3} g_{1/2}^2 - g_0^2 - \frac{1}{12} g_{-1/2}^2 + \frac{16}{3} g_1 g_2 -$

$- \frac{8}{3} g_{1/2} g_{3/2} + \frac{10}{3} g_0 g_1 + \frac{2}{3} g_{1/2} g_{-1/2} + \frac{1}{6} g_0 g_{-1} - 8 g_0 g_2 - \frac{4}{3} g_{-1/2} g_{3/2}$

$$+\frac{1}{3}g_{-1}g_1-\frac{1}{12}g_{-3/2}g_{1/2}$$

Rešenje:

$$\varphi_{13}' = -\frac{2^{11}}{9}\left(\frac{239}{32}+\frac{2\,987}{14\cdot 90}\frac{1}{\Gamma^2}\right)g_4+\frac{7}{3}2^7\left(\frac{39}{72}+\frac{51}{378}\frac{1}{\Gamma^2}\right)g_3-\frac{2^5}{3}\left(\frac{39}{72}+\right.$$

$$\left.+\frac{51}{378}\frac{1}{\Gamma^2}\right)g_2-\frac{4}{63}\frac{1}{\Gamma}g_{1/2}-\frac{1}{2\cdot 27}g_0+\frac{5}{18\cdot 27}\frac{1}{\Gamma}g_{-1/2}+\frac{23}{40\cdot 27}g_{-1}-$$

$$-\frac{1}{9\cdot 28}\frac{1}{\Gamma}g_{-3/2}+\frac{1}{9\cdot 32}g_{-2}-\frac{416}{9}g_2^2-\frac{224}{9}g_{3/2}^2-\frac{64}{9}g_1^2-\frac{10}{9}g_{1/2}^2-$$

$$-\frac{1}{18}g_0^2+\frac{784}{9}g_{3/2}g_{5/2}+\frac{376}{9}g_1g_2+\frac{98}{9}g_{1/2}g_{3/2}+\frac{13}{9}g_0g_1+$$

$$+\frac{1}{9}g_{-1/2}g_{1/2}-\frac{256}{9}g_1g_3-\frac{128}{9}g_{1/2}g_{5/2}-\frac{20}{9}g_0g_2-\frac{2}{9}g_{-1/2}g_{3/2}-$$

$$-\frac{1}{18}g_{-1}g_1$$

14. $\varphi_4'''+2\eta_1\varphi_4''-4\cdot 4\,\varphi_4'=-\frac{4}{3}\left[2g_0-\frac{1}{2}\frac{1}{\Gamma}g_{-1/2}+\frac{1}{2}g_{-1}-g_0^2+g_{-1/2}g_{1/2}\right]$

Rešenje:

$$\varphi_4' = -2^{11}\left(\frac{15}{32}+\frac{7}{45}\frac{1}{\Gamma^2}\right)g_4+\frac{1}{6}g_0-\frac{1}{27}\frac{1}{\Gamma}g_{-1/2}+\frac{1}{30}g_{-1}+32g_2^2+$$

$$+16g_{3/2}^2+4g_1^2+\frac{2}{3}g_{1/2}^2-32g_{3/2}g_{5/2}-16g_1g_2-4g_{1/2}g_{3/2}-$$

$$-\frac{2}{3}g_0g_1$$

§ 10 REZIME

Uvodjenjem novih nezavisnih specijalnih promenljivih, umesto starih promenljivih y i t, u obliku

$$\tau = \frac{1}{\nu}\int_0^t \Omega^2\, dt \qquad \eta = \frac{\Omega \cdot y}{\nu\sqrt{3\tau}}$$

gde je:

$\Omega(t)$ – funkcija koja pokazuje ostvarenu nestacionarnost spoljnog potencijalnog strujanja

i pretpostavljanjem oblika strujne funkcije

$$\Psi(x,y,t) = \nu\sqrt{3\tau}\; U(x)\; F(x,\eta,\tau)$$

diferencijalne jednačine nestacionarnih graničnih slojeva svode se na parcijalnu jednačinu 7 u kojoj centralnu ulogu igraju funkcije

$$\alpha(\tau) = \frac{\Omega' \nu\, 3\tau}{\Omega^3} \qquad \beta(\tau) = \frac{\nu\, 3\tau}{\Omega}$$

Podaci svakog posebnog problema još jedino preko njih dolaze eksplicitno do izražaja.

Rešenje ove parcijalne jednačine daje se u obliku specijalnog reda 8 razvijenog po stacionarnoj brzini i njenim izvodima. Za odredjivanje koeficijenata-funkcija ovog reda dobijen je sistem parcijalnih jednačina 9. Rešenja ovog sistema parcijalnih jednačina daju se u obliku stepenih redova po promenljivoj τ sa koeficijentima koji su funkcije redukovanog odstojanja η, a za odredjivanje nepoznatih koeficijenata-funkcija $F_0, F_1, \ldots, \Phi_0, \Phi_1, \ldots$, dobijeni su sistemi običnih diferencijalnih jednačina.

Preko linearnih kombinacija iz dobijenih sistema mogu se eliminisati svi koeficijenti stepenih redova funkcija $\alpha(\tau)$ i $\beta(\tau)$, sem vodećeg koeficijenta α_0, koji je odlučujući za formiranje početnog profila i za dalje ponašanje pri strujanju.

Na osnovi glavne funkcije $\alpha(\tau)$ u § 4 izvršena je klasifikacija problema koji se mogu rešavati ovom metodom. Dalje su u radu iscrpno proučene dve široke klase problema, kojima odgovaraju sledeći oblici brzina spoljnog potencijalnog strujanja

$$U(x,t) = U(x)(\Omega_0 + \Omega_1 t + \Omega_2 t^2 + \Omega_3 t^3 + \cdots)$$

i

$$U(x,t) = U(x)(\Omega_1 t + \Omega_2 t^2 + \Omega_3 t^3 + \cdots)$$

Gornjim oblicima spoljne brzine odgovaraju sledeće vrednosti vodećeg koeficijenta stepenog reda glavne funkcije: $\alpha_0 = 0$ i $\alpha_0 = 1$.

Sem toga u radu je pokazano da, kada se glavna funkcija svede na koeficijenat α_0 tj. bude nezavisna od promenljive τ, da se tada dobijaju poznata slična rešenja koja su iscrpno proučena u radu [4].

Treba naglasiti da se mogu proučiti još mnogi interesantni slučajevi, kojima odgovaraju druge vrednosti koeficijenta α_0. Zatim, metoda se sa uspehom prenosi na trodimenzione, temperaturske i granične slojeve stišljivih fluida. Svi ovi, kao i drugi interesantni problemi, kao što je, prelaz iz nestacionarnog u stacionarno stanje, pitanje stabilnosti nestacionarnih graničnih slojeva, predmet su daljeg autorovog rada u ovoj oblasti.

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P R I L O G

PRIMENA METODE

U cilju ilustracije primene nove metode, rešićemo jedan konkretan primer. Naime, ako je pri stacionarnom opticanju cilindričnog tela potencijalnim ravanskim tokom brzina ravna $U(x)=U_0 \sin \frac{x}{R}$, gde je R poluprečnik cilindra, tada pri promeni brzine sa vremenom po zakonu $\Omega(t) = 1 - 1/2\, t^2 + 1/24\, t^4$, zakon promene brzine spoljnog potencijalnog strujanja ima oblik

$$U(x,t) = U_0 \sin \frac{x}{R} \cdot \left(1 - \frac{1}{2} t^2 + \frac{1}{24} t^4\right)$$

Proračun ćemo provesti dosledno uputstvu iznešenom u § 7. Sleduje:

Iz izraza za $\Omega(t)$ očigledno je

$$\Omega_0 = 1, \quad \Omega_1 = 0, \quad \Omega_2 = -\frac{1}{2}, \quad \Omega_3 = 0, \quad \Omega_4 = \frac{1}{24}$$

gde je Ω_0 u ovom slučaju bezdimenziona veličina, dok U_0 ima dimenziju brzine.

Brzine spoljnjeg potencijalnog strujanja, ako se uvedu bezdimenzione veličine izrazima

$$\frac{U}{U_0} = \bar{U}, \quad \frac{x}{R} = \bar{x}, \quad \frac{\Omega_k}{\Omega_0} \frac{R^k}{U_0^k} = \bar{\Omega}_k, \quad \frac{U_0}{R} t = \bar{t}$$

možemo napisati i u bezdimenzionom obliku

$$\bar{U} = \sin \bar{x} \cdot \left(1 - \frac{1}{2} \bar{t}^2 + \frac{1}{24} \bar{t}^4\right)$$

Nadalje ćemo pripremiti sve formule potrebne za brojni deo proračuna.

1. Najpre ćemo izvršiti prelaz na nove koordinate

$$\bar{\tau} = \int_0^{\bar{t}} \bar{\Omega}^2 d\bar{t} = \bar{t} - \frac{\bar{t}^3}{3} + \frac{\bar{t}^5}{15} - \frac{\bar{t}^7}{336} + \frac{\bar{t}^9}{5184}$$

$$\bar{\eta} = \bar{g}(\bar{t}) \cdot \bar{Y}$$

gde je
$$\bar{g}(\bar{t}) = \frac{1 - \frac{1}{2}\bar{t}^2 + \frac{1}{24}\bar{t}^4}{\sqrt{3\left(\bar{t} - \frac{\bar{t}^3}{3} + \frac{\bar{t}^5}{15} - \frac{\bar{t}^7}{336} + \frac{\bar{t}^9}{5184}\right)}}$$

$$\bar{\Omega} = \frac{\Omega}{U_o}, \quad \bar{\tau} = \frac{\tau}{R_e}, \quad R_e = \frac{U_o R}{\nu}$$

2. Preko formula 26. i 52. lako se pronalaze koeficijenti funkcija $\bar{\alpha}(\bar{\tau})$ i $\bar{\beta}(\bar{\tau})$.

3. Tangencijalni napon sračunava se prema formuli

$$\frac{\Xi(x,t)}{\rho U_o^2} R_e^{1/2} = \bar{U}(\bar{x},\bar{t}) \cdot \bar{g} \left(\sum_{k=0} \bar{F}_k''(0) \bar{\tau}^k + \bar{U}' \sum_{k=1} \bar{\Phi}_k''(0) \bar{\tau}^k + \cdots \right)$$

odnosno

$$\frac{\Xi(x,t)}{\rho U_o^2} R_e^{1/2} = \sin\bar{x} \cdot \bar{\Omega} \cdot \bar{g} \left\{ F_o'' + \bar{\alpha}_1 f_1'' \bar{\tau} + (\bar{\alpha}_1^2 \bar{f}_{11}'' + \bar{\alpha}_2 \bar{f}_2'') \bar{\tau}^2 + \cdots + \right.$$
$$\left. + \cos\bar{x} \left[\bar{\beta}_1 \bar{\varphi}_1 \bar{\tau} + \cdots \right] + \cdots \right\}$$

ovde su:

$$\bar{F}_k = R_e^k F_k, \quad \bar{\Phi}_k = R_e^k \frac{U_o}{R} \Phi_k$$

4. Debljinu istiskivanja sračunavamo po formuli

$$\frac{\delta^* R_e}{R} = \frac{1}{\bar{g}} \left\{ \lim_{\eta_1 \to \infty} (\eta_1 - \bar{F}_o) - \left[\sum_{k=0} \bar{F}_{k(\infty)} \bar{\tau}^k + \bar{U}' \sum_{k=1} \bar{\Phi}_{k(\infty)} \bar{\tau}^k + \cdots \right] \right\}$$

odnosno

$$\frac{\delta^* R_e}{R} = \frac{1}{\bar{g}} \left\{ \lim_{\eta_1 \to \infty} (\eta_1 - \bar{F}_o) - \left[\bar{\alpha}_1 \bar{f}_1 \bar{\tau} + (\bar{\alpha}_1^2 \bar{f}_{11} + \bar{\alpha}_2 \bar{f}_2) \bar{\tau}^2 + \cdots + \bar{U}' (\bar{\beta}_1 \bar{\varphi}_1 \bar{\tau} + \cdots) \right.\right.$$
$$\left.\left. + \cdots \right] \right\}$$

5. Za dobijanje profila brzina na raznim mestima duž konture i u raznim trenucima vremena koristimo formulu

$$\frac{u}{U_o} = \bar{u} = \sin\bar{x} \cdot \bar{\Omega} \left[\sum_{k=0} \bar{F}_k'(\eta) \bar{\tau}^k + \bar{U}' \sum_{k=1} \bar{\Phi}_k'(\eta) \bar{\tau}^k + \cdots \right]$$

odnosno

$$\frac{u}{U_0} = \sin\bar{x} \cdot \bar\Omega \Big\{ \bar F_0' + \alpha_1 \bar f_1' \bar\tau + (\alpha_1^2 \bar f_{11}' + \alpha_2 \bar f_2')\bar\tau^2 + \dots + \cos\bar x \cdot \left[\bar\beta_1 \bar\varphi_1' \bar\tau + (\bar\alpha_1 \bar\beta_1 \bar\varphi_{11}' + \bar\beta_2\bar\varphi_2')\bar\tau^2 + \dots\right] + \dots\Big\}$$

6. Vreme odvajanja nalazimo iz uslova

$$\sum_{k=0} \bar F_k''(0)\bar\tau^k + \bar U' \sum_{k=1} \bar\Phi_k''(0)\bar\tau^k + \dots = 0$$

odnosno

$$\bar F_0'' + \bar\alpha_2 \bar f_2'' \bar\tau^2 + (\bar\alpha_2^2 \bar f_{22}'' + \bar\alpha_4 \bar f_4'')\bar\tau^4 + \dots + \bar U'(\bar x)\left[\bar\beta_1\bar\varphi_1''\bar\tau + (\bar\alpha_2\bar\beta_1\bar\varphi_{21}'' + \bar\beta_3\bar\varphi_3'')\bar\tau^2 + \dots\right] + \dots = 0$$

7. Put predjen telom do momenta odvajanja nalazimo iz izraza

$$\bar S = \int_0^{\bar t_s} \bar\Omega \, d\bar t$$

odnosno

$$\bar S = \bar t_s - \frac{1}{6}\bar t_s^3 + \frac{1}{120}\bar t_s^5$$

gde je $\bar S = \frac{S}{R}$

8. Na kraju možemo izvršiti probu, zadovoljenje prve ~~konrutne~~ konturne veze tj. dali je

$$\sum_{k=0} \bar F_k'''(0)\bar\tau^k = -\bar\alpha(\bar\tau)$$

$$\sum_{k=1} \bar\Phi_k'''(0)\bar\tau^k = -\bar\beta(\bar\tau)$$

odnosno

$$\bar F_0''' + \bar\alpha_2\bar f_2'''\bar\tau^2 + (\bar\alpha_2^2 \bar f_{22}''' + \bar\alpha_4 \bar f_4''')\bar\tau^4 + \dots = -(\bar\alpha_2\bar\tau^2 + \bar\alpha_4\bar\tau^4 + \dots)$$

$$\bar\beta_1\bar\varphi_1'''\bar\tau + (\bar\alpha_2\bar\beta_1\bar\varphi_{21}''' + \bar\beta_3\bar\varphi_3''')\bar\tau^2 + \dots = -(\bar\beta_1\bar\tau + \bar\beta_3\bar\tau^3 + \dots)$$

Na dalje ćemo ići od tačke do tačke i nalaziti brojne vrednosti traženih veličina zaustavljajući se na drugom članu specifičnog reda i 4-tim članovima stepenih redova. Resultati će biti svrstani u tablice i dijagrame. Pošto će proračun biti vršen na običnoj ručnoj mašini, to razumljivo i tačnost neće biti velika, ali će ipak moći da se stekne jasna slika o primeni metode. Kasnije, ako to dozvole tehnička sredstva izvršiće se tabulisanje svih universalnih funkcija i ponovo rešiti isti primer, kao i neke druge, ali sa daleko većom tačnošću.

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1. Tabela funkcija $\bar{\tau}(\bar{t})$, $\bar{q}(\bar{t})$ i $\bar{\Omega}(\bar{t})$

$\bar{t}$	$\bar{\tau}$	$\bar{q}$	$\bar{\Omega}$
0,5	0,4603945	0,7446800	0,8750000
1,0	0,7305500	0,39403321	0,5833333
1,2	0,7802177	0,23949260	0,3664000

2. $\bar{\alpha}_0 = 0$ $\qquad$ $\bar{\beta}_0 = 0$

$\bar{\alpha}_1 = 0$ $\qquad$ $\bar{\beta}_1 = 3$

$\bar{\alpha}_2 = -3$ $\qquad$ $\bar{\beta}_2 = 0$

$\bar{\alpha}_3 = 0$ $\qquad$ $\bar{\beta}_3 = \frac{3}{2}$

$\bar{\alpha}_4 = -5$ $\qquad$ $\bar{\beta}_4 = 0$

gde je

$$\bar{\alpha}_k = \alpha_k R_e^k \qquad \bar{\beta}_k = \beta_k R_e^k$$

3. Tangencijalni napon

Pošto je

$$\bar{F}_0''(o) = \frac{1}{\Gamma} = 1,128379167095512$$

$$\bar{f}_2''(o) = \frac{1}{6\Gamma} = 0{,}188063194515919$$

$$\bar{f}_{22}''(o) = \frac{1}{9\cdot 24\cdot 35}\ \frac{1}{\Gamma} = 0{,}000149256903583$$

$$\bar{f}_4''(o) = \frac{13}{84}\ \frac{1}{\Gamma} = 0{,}1746301038053373$$

$$\bar{\varphi}_1''(o) = \frac{1}{3}\ \frac{1}{\Gamma}\left(1 + \frac{1}{3}\ \frac{1}{\Gamma^2}\right) = 0{,}53584093756107$$

$$\bar{\varphi}_{21}''(o) = \frac{1}{9}\ \frac{1}{\Gamma}\left(\frac{119}{40} - \frac{1429}{630}\ \frac{1}{\Gamma^2}\right) = 0{,}01072084333209$$

$$\bar{\varphi}_3''(o) = \frac{1}{15}\ \frac{1}{\Gamma}\left(2 + \frac{408}{189}\ \frac{1}{\Gamma^2}\right) = 0{,}357319197430$$

to ceme tangencijalni napon dobija sledeći izraz

$$\frac{N Re^{1/2}}{\varrho U_o^2} = \bar{U}(\bar{x})\cdot\bar{\Omega}(\bar{t})\bar{g}(\bar{t})\Big[1{,}128379167095512 - 0{,}56418958354774\cdot\bar{\tau}^2$$

$$-0{,}871807210494438\cdot\bar{\tau}^4 + \bar{U}'(\bar{x})\,(1{,}60752281268321\cdot\bar{\tau} +$$

$$+ 0{,}439481206156\cdot\bar{\tau}^3)$$

Tabela promene tangencijalnog napona sa vremenom

$\bar{t}$	$N Re^{1/2}/\varrho U_o^2$
0,5	$\sin\bar{x}\,(0{,}6318015234938 + \cos\bar{x}\cdot 0{,}51018931210477)$
1,0	$\sin\bar{x}\,(0{,}13307323365856 + \cos\bar{x}\cdot 0{,}3093176643500)$
1,2	$\sin\bar{x}\,(0{,}0405297523801 + \cos\bar{x}\cdot 0{,}128374264395)$

Tabela promene tangencijalnog napona sa vremenom i duž konture

$\bar{t}$	$\frac{N Re^{1/2}}{\varrho U_o^2}$		
	$\bar{x} = 30°$	$\bar{x} = 60°$	$\bar{x} = 90°$
0,5	0,5368203868	0,76807869842	0,631801523497
1,0	0,2004758053	0,24918460112	0,133073233699
1,2	0,0758528583	0,0906879636	0,0405297524

$HRe^{1/2}/\rho U_0^2$		
$\bar{x}=120°$	$\bar{x}=150°$	$\bar{x}=180°$
0,3263394484	0,0949811 367	0,00000
- 0,01869377589	- 0,06740 85716	0,00000
- 0,020488001	- 0,035323106	0,00000

Sve ove rezultate možemo predstaviti i na dijagramu (vidi uprilogu sl. 1).

Sa dijagrama je očigledno da u trenutku $\bar{t}$ =0,5 na konturi se još nije javilo odvajanje graničnog sloja, ali se oseća tendencija pojave početka odvajanja. U trenutku $\bar{t}$=1,0 tačka odvajanja se već nalazi na oko 115° od prednje zaustavne tačke, dok se u trenutku $\bar{t}$ = 1,2 nalazi na oko 110° od prednje zaustavne tačke i teži vrednosti 108,5°. Sem toga sa porastom vremena tangencionalni napon opada i postaje ravan nuli kada je $\Omega = 0$.

4. Debljina istiskivanja

Pošto je

$$\eta_1 - \bar{F}_0 = \eta_1 - (\eta_1 + g_{1/2} - \frac{1}{2}\frac{1}{\Gamma}) = \frac{1}{2}\frac{1}{\Gamma} - g_{1/2}$$

to je

$$\lim_{\eta_1 \to \infty} (\eta_1 - \bar{F}_0) = \frac{1}{2}\frac{1}{\Gamma} = 0,56418958355$$

jer $g_{1/2} \to 0$ kada $\eta_1 \to \infty$

Lako se dobija da je

$$\bar{f}_2(\infty) = -\frac{1}{60}\frac{1}{\Gamma} = -0,01880631951$$

$$\bar{f}_{22}(\infty) = -\frac{2369}{34020}\frac{1}{\Gamma} = 0,079238621596$$

$$\bar{f}_4(\infty) = -\frac{65}{7560}\frac{1}{\Gamma} = -0,0097016686724$$

$$\bar{\varphi}_1(\infty) = 0,15483207311203$$

$$\bar{\varphi}_{21}(\infty) = 0,026402372 0112$$

$$\bar{\varphi}_3(\infty) = 0,080408 0239850$$

Sada možemo napisati i konačan izraz za debljinu istiskivanja

$$\frac{\delta^* Re^{1/2}}{R} = \frac{1}{s}\Big\{0{,}56418958355 - \big[0{,}05641895853\cdot\bar{\tau}^2 - 0{,}664639251002\cdot$$

$$\bar{\tau}^4 + \bar{U}'(0{,}4644403927\cdot\bar{\tau} - 0{,}1175433102893\cdot\bar{\tau}^3)\big]\Big\}$$

ili u obliku tabele, za razne trenutke i razna mesta na konturi

$\bar{t}$	$\frac{\delta^* Re^{1/2}}{R}$			
	$\bar{x}=0°$	$\bar{x}=30°$	$\bar{x}=60°$	$\bar{x}=90°$
0,5	0,50984690827	0,5462597270	0,64574605932	0,7816452104
1,0	1,08883199421	1,18879947352	1,46192851071	1,835029027
1,2	1,95949807690	2,13111391993	2,5999991046	3,2405001123

$\frac{\delta^* Re^{1/2}}{R}$		
$\bar{x}=120°$	$\bar{x}=150°$	$\bar{x}=180°$
0,9175443614	1,017030694	1,0534435125
2,2081215437	2,4812505809	2,5812180602
3,8810011300	4,349886303	4,521908148

Rezultati su prikazani i na dijagramu (vidi u prilogu sl.2) Sa dijagrama se vidi; kada $\bar{\Omega}\to 0$ da $\frac{\delta^* Re^{1/2}}{R}\to\infty$, odnosno sa porastom vremena debljina istiskivanja se uvećava pošto se $\bar{\Omega}$ smanjuje. U intervalu od $\bar{x} = 0°$ do $\bar{x} = 108{,}5°$ karakter krive je isti kao i u slučaju stacionarnog kretanja.

4. Za profil brzina u raznim trenucima i na raznim mestima duž konture dobija se izraz

$$\frac{u}{U(x,t)} = \bar{F}_0' - 3\bar{F}_2'\bar{\tau}^2 + (9\bar{F}_{22}' - 5\bar{F}_4')\bar{\tau}^4 + \bar{U}'\big[3\bar{\varphi}_1'\bar{\tau} + (-9\bar{\varphi}_{21}' +$$

$$+ \frac{3}{2}\bar{\varphi}_3')\bar{\tau}^3 = K(\bar{\tau}) + \bar{U}'K_1(\bar{\tau})$$

ili u obliku tabele

η	$\frac{u}{U(x,t)}$
0,0	0,0000000000
0,4	$0{,}42839235505 - 0{,}01619961 \cdot \bar{\tau}^2 - 0{,}0123105 \cdot \bar{\tau}^4 +$
0,8	$0{,}742100964708 + 0{,}06187260 \cdot \bar{\tau}^2 + 0{,}08525639 \cdot \bar{\tau}^4 +$
1,2	$0{,}910313978229 + 0{,}06559362 \cdot \bar{\tau}^2 + 0{,}06191647 \cdot \bar{\tau}^4 +$
1,6	$0{,}976348383345 + 0{,}0354054 \cdot \bar{\tau}^2 - 0{,}00517588 \cdot \bar{\tau}^4 +$
2,0	$0{,}995322265019 + 0{,}01137867 \cdot \bar{\tau}^2 + 0{,}00081817 \cdot \bar{\tau}^4 +$

$\frac{u}{U(x,t)}$
0,0000000000
$+ \bar{U}' (0{,}35972664 \; \bar{\tau} - 0{,}07372633167 \; \bar{\tau}^3)$
$+ \bar{U}' (0{,}3396024 \; \bar{\tau} - 0{,}0760787886 \; \bar{\tau}^3)$
$+ \bar{U}' (0{,}1932891 \; \bar{\tau} - 0{,}037716666459 \; \bar{\tau}^3)$
$+ \bar{U}' (0{,}0760290 \; \bar{\tau} - 0{,}0135472746 \; \bar{\tau}^3)$
$+ \bar{U}' (0{,}01743387 \; \bar{\tau} - 0{,}0039826523 \; \bar{\tau}^3)$

$\bar{t} = 0{,}5$

$\frac{u}{U_0}$

η	$\bar{x} = 30°$	$\bar{x} = 60°$	$\bar{x} = 90°$	$\bar{x} = 120°$	$\bar{x} = 150°$
0,0	0,0000000	0,0000000	0,0000000	0,0000000	0,0000000
0,4	0,2456977	0,3816247	0,3713549	0,2615842	0,1256571
0,8	0,3889074	0,6316726	0,6641616	0,5186951	0,2756541
1,2	0,4362224	0,7331196	0,8111241	0,6717960	0,3749000
1,6	0,4375791	0,7569390	0,8588366	0,7306174	0,4162575
2,0	0,4399329	0,7595096	0,8730888	0,7527327	0,4331559

$\frac{u}{U(x,t)}$

η	$\bar{x} = 30°$	$\bar{x} = 60°$	$\bar{x} = 90°$	$\bar{x} = 120°$	$\bar{x} = 150°$
0,0	0,0000000	0,0000000	0,0000000	0,0000000	0,0000000
0,4	0,5615948	0,5036114	0,4244056	0,3451997	0,2872163
0,8	0,8880170	0,8335053	0,7590418	0,6845783	0,6300666
1,2	0,9970830	0,9674618	0,9269990	0,8865362	0,8569149
1,6	1,0116093	0,9988951	0,9815275	0,9641599	0,9514457
2,0	1,0155609	1,0022874	0,9978158	0,9933441	0,9900706

$\bar{t} = 1,0$

$\frac{u}{U(x,t)}$

η	$\bar{x} = 30°$	$\bar{x} = 60°$	$\bar{x} = 90°$	$\bar{x} = 120°$	$\bar{x} = 150°$
0,0	0,0000000	0,0000000	0,0000000	0,0000000	0,0000000
0,4	0,6189369	0,5332666	0,4168402	0,2992138	0,2135435
0,8	0,9885477	0,9085944	0,7993775	0,6901606	0,6102072
1,2	1,0274465	1,0174529	0,9629580	0,9084630	0,8689694
1,6	1,0309724	1,0230341	0,9937982	0,9669823	0,9466240
2,0	1,0305283	1,0259350	0,9993956	0,9928663	0,9880691

$\frac{u}{U_0}$

η	$\bar{x} = 30°$	$\bar{x} = 60°$	$\bar{x} = 90°$	$\bar{x} = 120°$	$\bar{x} = 150°$
0,0	0,0000000	0,0000000	0,0000000	0,0000000	0,0000000
0,4	0,1805232	0,2693978	0,2428054	0,1511980	0,0622834
0,8	0,2883267	0,4590219	0,4663009	0,3486691	0,1779773
1,2	0,3084212	0,5140009	0,5617223	0,4589409	0,2533322
1,6	0,3094372	0,5158101	0,5797123	0,4849480	0,2760980
2,0	0,3152244	0,5182860	0,5829774	0,5015892	0,2881862

$\bar{t} = 1{,}2$

$$\frac{u}{U(x,t)}$$

η	$\bar{x}=30^\circ$	$\bar{x}=60^\circ$	$\bar{x}=90^\circ$	$\bar{x}=120^\circ$	$\bar{x}=150^\circ$
0,0	0,0000000	0,0000000	0,0000000	0,0000000	0,0000000
0,4	0,6267084	0,5367936	0,4139692	0,2911448	0,2012299
0,8	1,0094951	0,9257364	0,8113212	0,6969061	0,6131473
1,2	1,0298039	1,0289687	0,9731875	0,9174062	0,8765710
1,6	1,0300614	1,0288263	0,9998190	0,9708118	0,9495767
2,0	1,0311437	1,0292445	1,0025521	0,9958597	0,9909605

$$\frac{u}{U_0}$$

η	$\bar{x}=30^\circ$	$\bar{x}=60^\circ$	$\bar{x}=90^\circ$	$\bar{x}=120^\circ$	$\bar{x}=150^\circ$
0,0	0,0000000	0,0000000	0,0000000	0,0000000	0,0000000
0,4	0,1148129	0,1703300	0,1480143	0,0923823	0,0368650
0,8	0,1849395	0,2937361	0,2972681	0,2211283	0,1123286
1,2	0,1909881	0,3260020	0,3565759	0,2911022	0,1605878
1,6	0,1923712	0,3264569	0,3663337	0,3080483	0,1739625
2,0	0,1957911	0,3302434	0,3672434	0,3159962	0,1815440

Zaustavljajući se na drugom stepenu $\bar{\tau}$, za profile brzina dobija se izraz

$$\frac{u}{U(x,t)} = \bar{F}_0' - 3\bar{f}_2'\bar{\tau}^2 + \bar{U}'\,3\,\varphi_1'\bar{\tau}$$

odnosno

η	$\frac{u}{U(x,t)}$
0,0	0,00000000000
0,4	$0{,}42839235505 - 0{,}01619961\,\bar{\tau}^2 + \bar{U}'\,0{,}35972664\,\bar{\tau}$
0,8	$0{,}74210096471 + 0{,}06187260\,\bar{\tau}^2 + \bar{U}'\,0{,}33960240\,\bar{\tau}$
1,2	$0{,}91031397823 + 0{,}06559362\,\bar{\tau}^2 + \bar{U}'\,0{,}1932891\,\bar{\tau}$
1,6	$0{,}97634838334 + 0{,}03540541\,\bar{\tau}^2 + \bar{U}'\,0{,}07602934\,\bar{\tau}$
2,0	$0{,}99532226502 + 0{,}01137870\,\bar{\tau}^2 + \bar{U}'\,0{,}01743387\,\bar{\tau}$

$\overset{M}{U}(x,\bar{t})$

η	$\bar{x}= 30^\circ$	$\bar{x}= 60^\circ$	$\bar{x}= 90^\circ$	$\bar{x}= 120^\circ$	$\bar{x}= 150^\circ$
0,0	0,0000000	0,0000000	0,0000000	0,0000000	0,0000000
0,4	0,6473378	0,5511457	0,4197466	0,2883474	0,1921554
0,8	0,98998141	0,8991706	0,7751224	0,6510741	0,5602633
1,2	1,0676174	1,0159286	0,9453213	0,8747139	0,8230252
1,6	1,0433461	1,0230157	0,9952442	0,8674728	0,94714423
2,0	1,0106051	1,0159432	0,9995771	0,9932069	0,9885451

Na dijagramima sl. 3, 4, 5, 6 i 7 rezultati su prikazani grafički. Očigledno, sa porastom vremena oseća se tendencija sužavanja profila u donjem delu (blizu konture). Pri $\bar{t}$ = 0,5 idući nizvodno prema zadnjoj zaustavnoj tački napadni ugao izmedju normale i tangente na profil brzina u tački na samom zidu konture teži vrednosti 0°, mada ga ne dostiže.

Pri $\bar{t}$=1, 0 i $\bar{t}$ =1,2 ova pojava se već javila na $116,5^\circ$ odnosno $110,5^\circ$, a idući prema zadnjoj zaustavnoj tački ugao postaje negativan, tj. na samu konturu javlja se povratno kretanje. Ovo kretanje kod nas nije registrovano zbog isuviše velikog koraka kojim se išlo η = 0,4. Medjutim, iz izraza za profil brzina, očigledno je da će pri $\eta \to 0$ vrednost K $(\bar{\tau})$ biti negativna, što je očigledno iz ponašanja funkcija $\bar{F}_0'$, $\bar{f}_2'$, $\bar{f}_{22}'$ i $\bar{f}_4'$. K $(\bar{\tau})$ će imati znak funkcija $\bar{f}_2'$ i $\bar{f}_4'$.

Sem toga, ako se u izrazu za profil brzina zaustavimo na drugom stepenu $\bar{\tau}$ tada se vidi da profil vrlo malo odstupa od profila dobijenog zaustavljanjem na 4-om stepenu $\bar{\tau}$, ali samo do tačke odvajanja odnosno pojave singulariteta, kada se konvergencija naglo kvari, što govori da je konvergencija idući nizvodno od prednje zaustavne tačke prema tački odvajanja lošija i tada treba ići na veći broj članova po $\bar{\tau}$ u specijalnom redu.

5. Vreme odvajanja

Vreme odvajanja nalazimo iz uslova da je tangencijalni napon na konturi raven nuli, sleduje

$$1,12837916709512 - 0,564189583547 76\ \bar{\tau}^2 - 0,87180721049444\ \bar{\tau}^4 + \bar{U}'(x)\ (1,60752281268321\ \bar{\tau} + 0,439481206156\ \bar{\tau}^3) = 0$$

Tabela položaja tačke odvajanja u raznim trenucima vremena

$\bar{x}_s$	$\bar{t}_s$
180°	0,5870
160°	0,6134
150°	0,6383
130°	0,7421
116°30	1,0000
110°30	1,2000

Sa dijagrama (sl. 8) je očigledno da se tačka odvajanja od prvog trenutka odvajanja pomera usvodno najpre brže, a zatim sve sporije, što se više udaljava od zadnje zaustavne tačke, a što je fizički i potpuno opravdano.

6. Predjeni put

Put koji telo predje od početnog do nekog posmatranog trenutka nalazimo iz izraza

$$\bar{s} = \int_0^{\bar{t}} \bar{\Omega}\, d\bar{t}$$

odnosno

$$\bar{s} = \bar{t} - \frac{\bar{t}^3}{6} + \frac{\bar{t}^5}{120}$$

Tabela i dijagram predjenog puta za razne trenutke vremena

$\bar{t}$	$\bar{s}$
0,5	0,47942708
1,0	0,84166667
1,2	0,93227361
1,5	1,00078125

Očigledno da će u početku telo u manjim vremenskim intervalima prelaziti veći put, dok će kasnije za isti vremenski interval prelaziti sve manji i manji put, sve dok se ne zaustavi tj. dok $\bar{\Omega}(\bar{t})$ ne postane = 0 (tada je $\bar{s}$ = = const.).

7. Provera prve konturne veze

Proverimo da li je ispunjen uslov

$$UU_t = -\nu\,(u_{yy})_{y=0}$$

odnosno

$$\sum_{k=0} \bar{F}'''_{k(o)}\,\bar{\tau}^{k} = -\alpha(\bar{\tau})$$

$$\sum_{k=1} \bar{\Phi}'''_{k(o)}\,\bar{\tau}^{k} = -\bar{\beta}(\bar{\tau})$$

Ako se vodi računa da je

$$\frac{d^3 f}{d\eta^3} = \frac{3}{4}\,\frac{d^3 f}{d\eta_1^3}\ , \quad \text{jer je} \quad \eta_1 = \sqrt{\frac{3}{4}}\,\eta$$

gornji uslovi mogu se napisati i u obliku

$$\frac{3}{4}\left[\bar{F}_0''' - 3\bar{f}_2'''\,\bar{\tau}^2 + (9\bar{f}_{22}''' - 5\bar{f}_4''')\,\bar{\tau}^4\right] = -(-3\bar{\tau}^2 - 5\bar{\tau}^4)$$

$$\frac{3}{4}\left[3\bar{\varphi}_1'''\,\bar{\tau} + (-9\bar{\varphi}_{21}''' + \frac{3}{2}\bar{\varphi}_3''')\,\bar{\tau}^3\right] = -(3\bar{\tau} + \frac{3}{2}\bar{\tau}^3)$$

Lako se pronalazi da je

$\bar{F}'''_{0(o)} = 0$ $\qquad\qquad$ $\bar{\varphi}_1''' = -\frac{4}{3}$

$\bar{f}'''_{2(o)} = -\frac{4}{3}$

$\bar{f}'''_{22(o)} = 0$ $\qquad\qquad$ $\bar{\varphi}_{21}''' = 0$

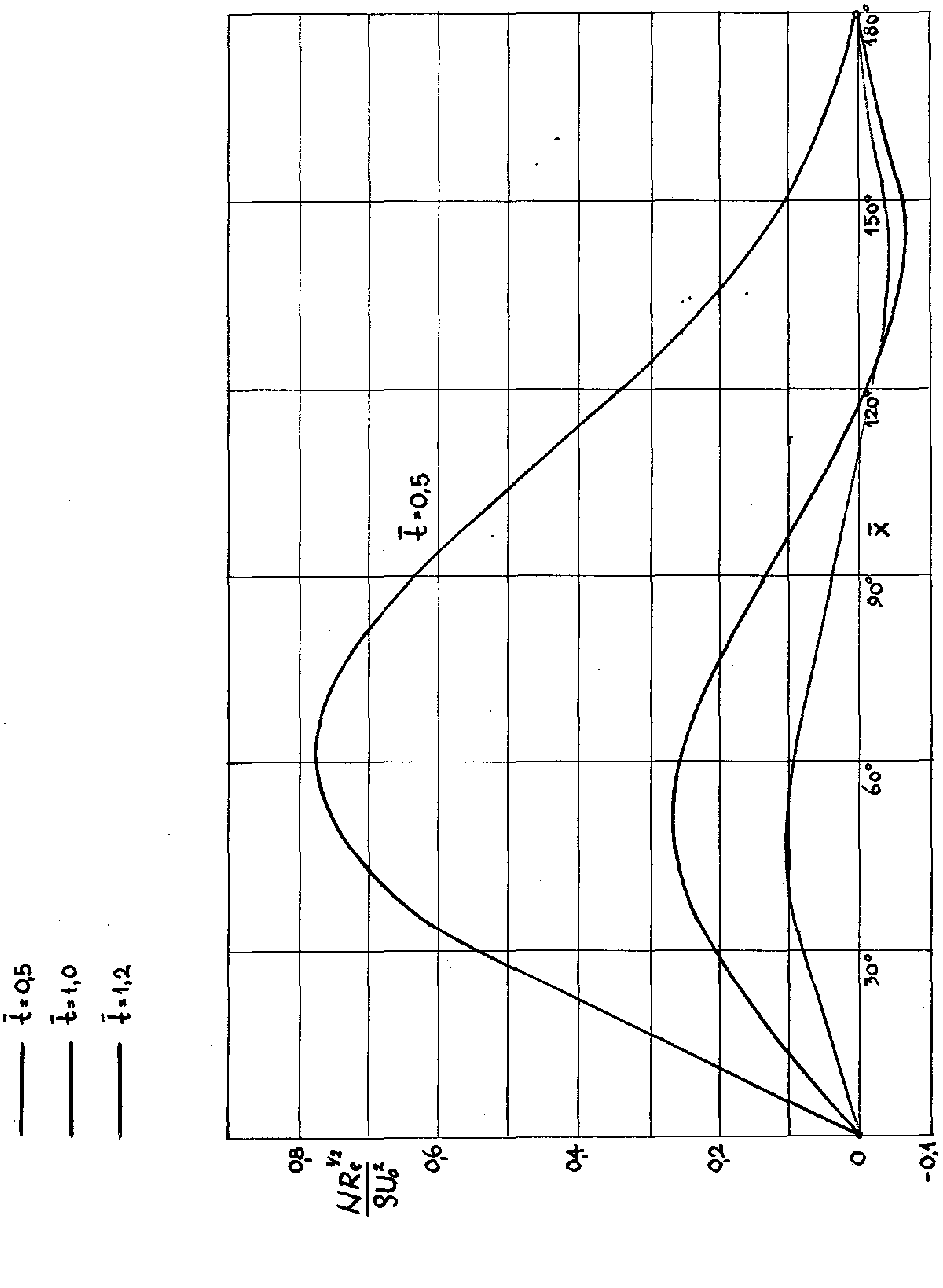
$\bar{f}'''_{4(o)} = -\frac{4}{3}$ $\qquad\qquad$ $\bar{\varphi}_3''' = -\frac{4}{3}$

i gornji uslovi svode na konačan oblik

$$\frac{3}{4}(4\bar{\tau}^2 + 5\frac{4}{3}\bar{\tau}^4) = 3\bar{\tau}^2 + 5\bar{\tau}^4$$

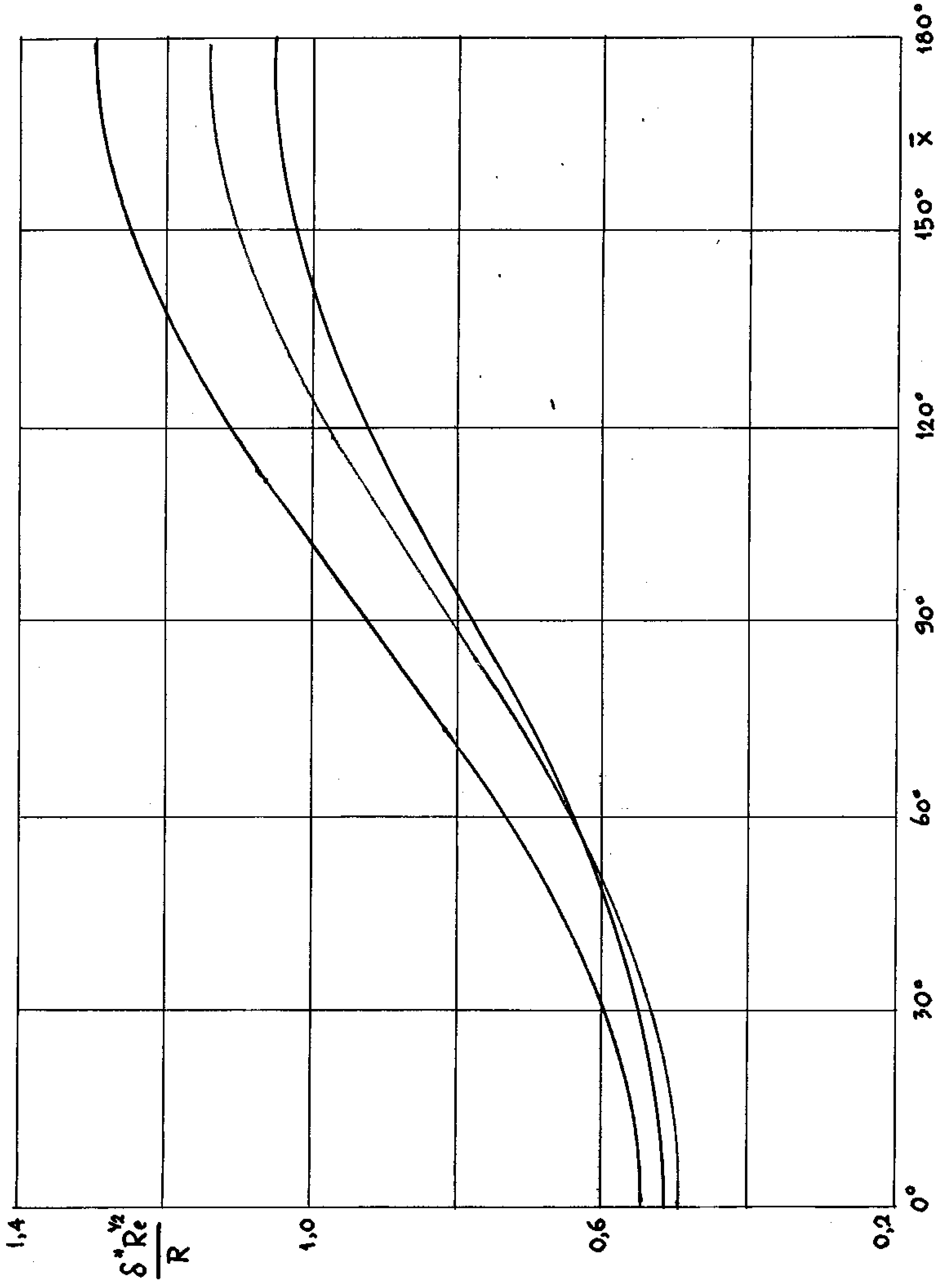
$$\frac{3}{4}(4\bar{\tau} + 2\bar{\tau}^3) = 3\bar{\tau} + \frac{3}{2}\bar{\tau}^3$$

odakle se vidi da je prva konturna veza zaista zadovoljena.

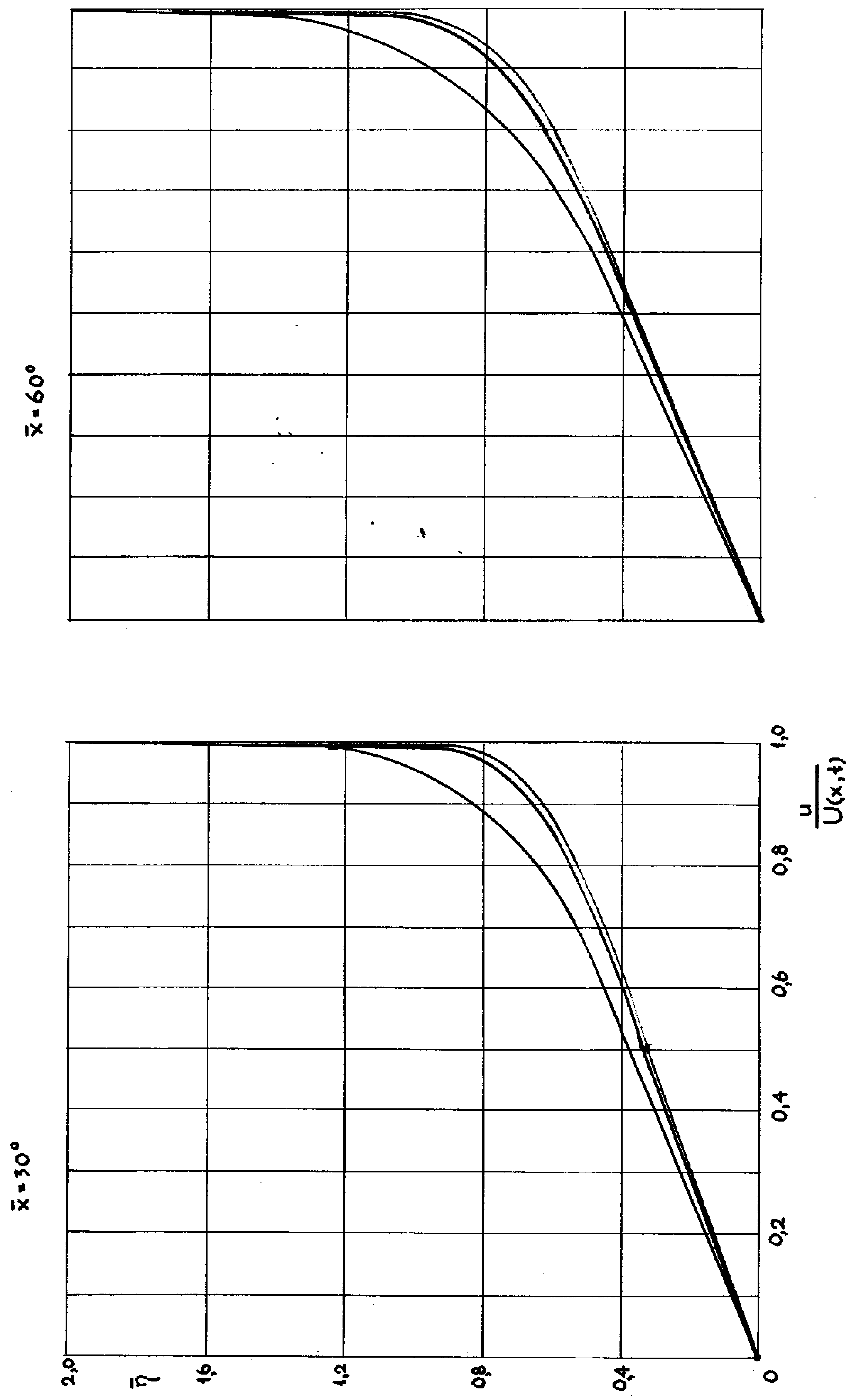


t̄=0,5
t̄=1,0
t̄=1,2
t̄=0,5
0,8
0,6
0,4
0,2
0
-0,1
$\frac{N Re^{1/2}}{\rho U_0^2}$
30°
60°
90°
120°
150°
180°
x̄

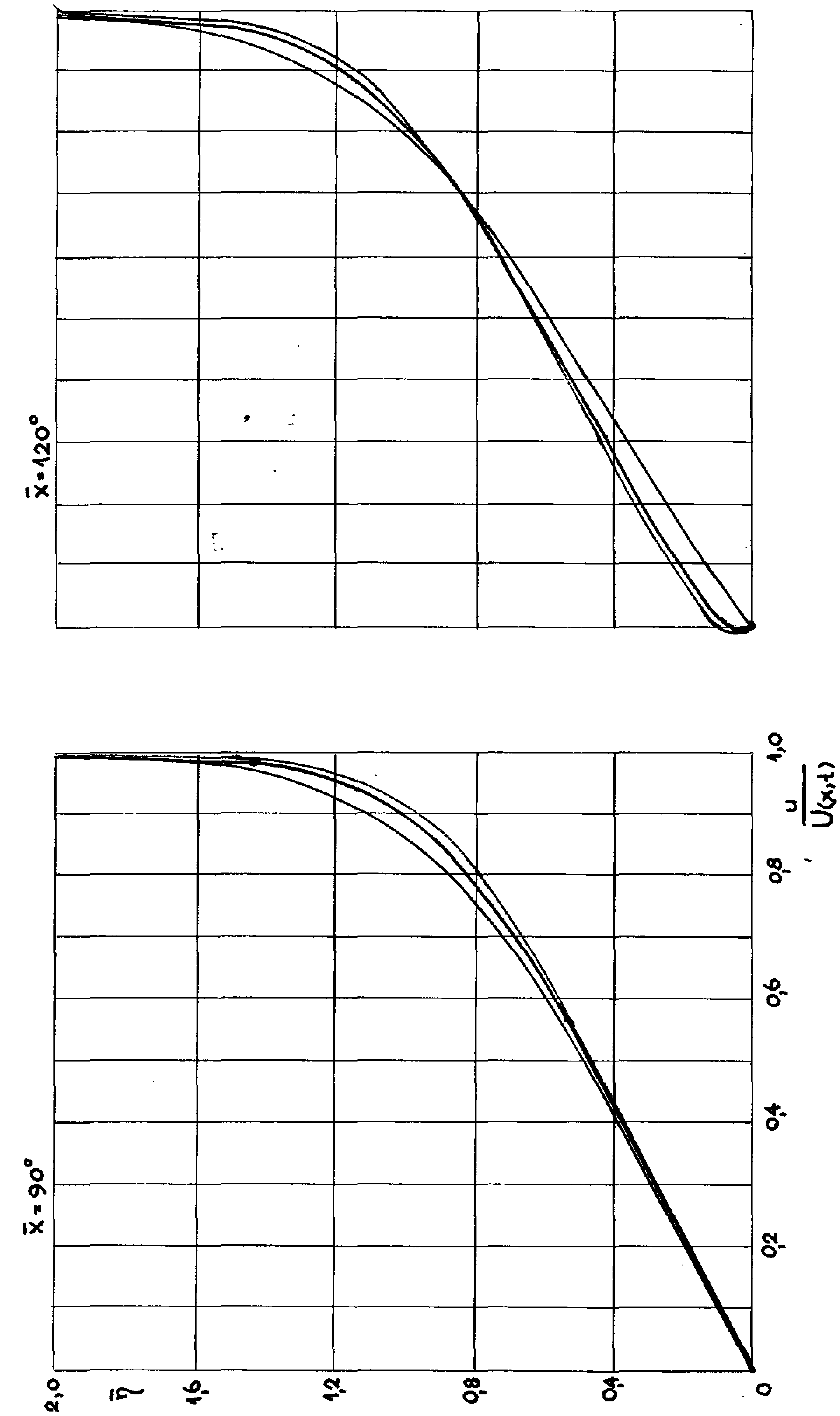
sl.1



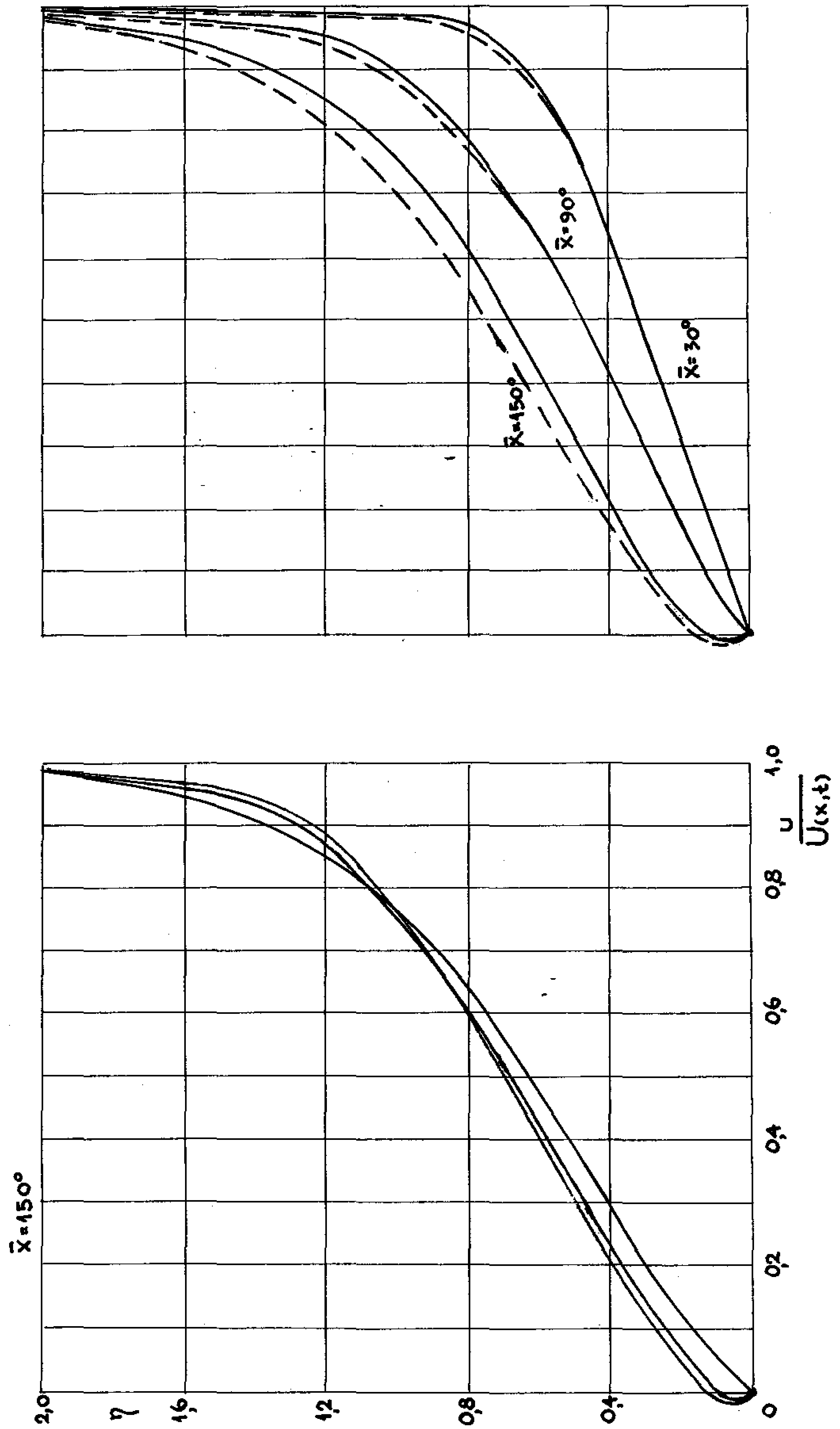
sl.2



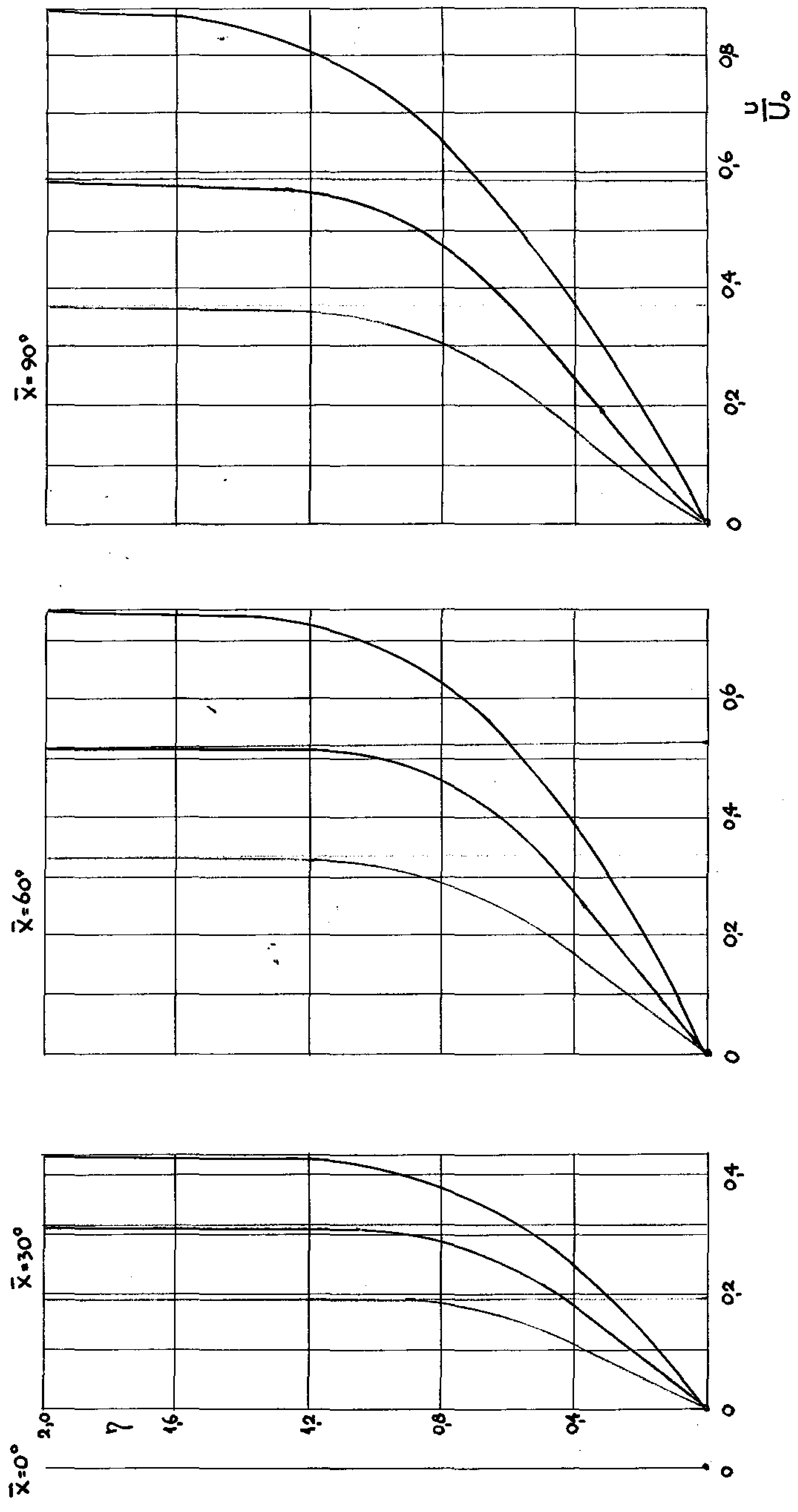
sl.3



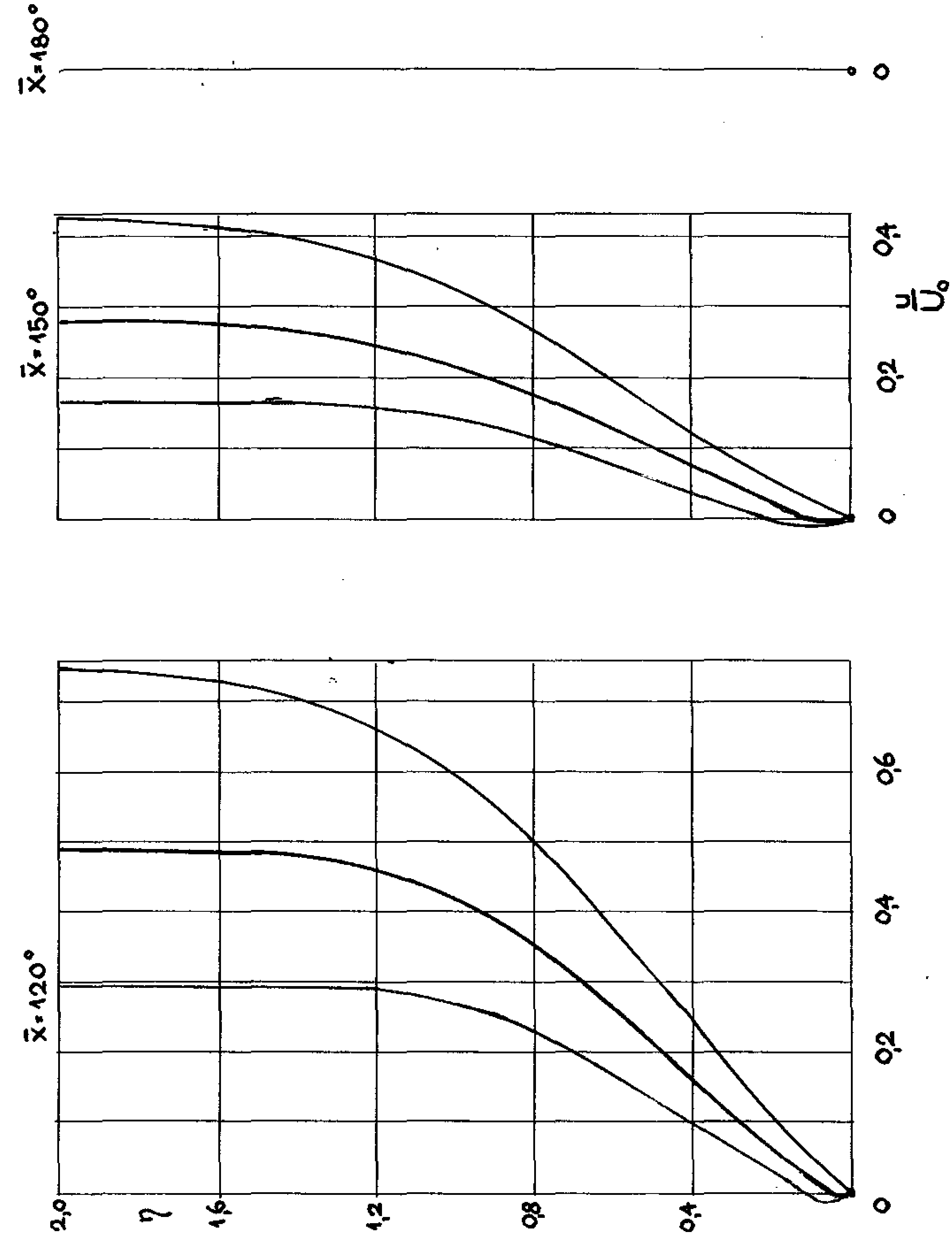
sl.4



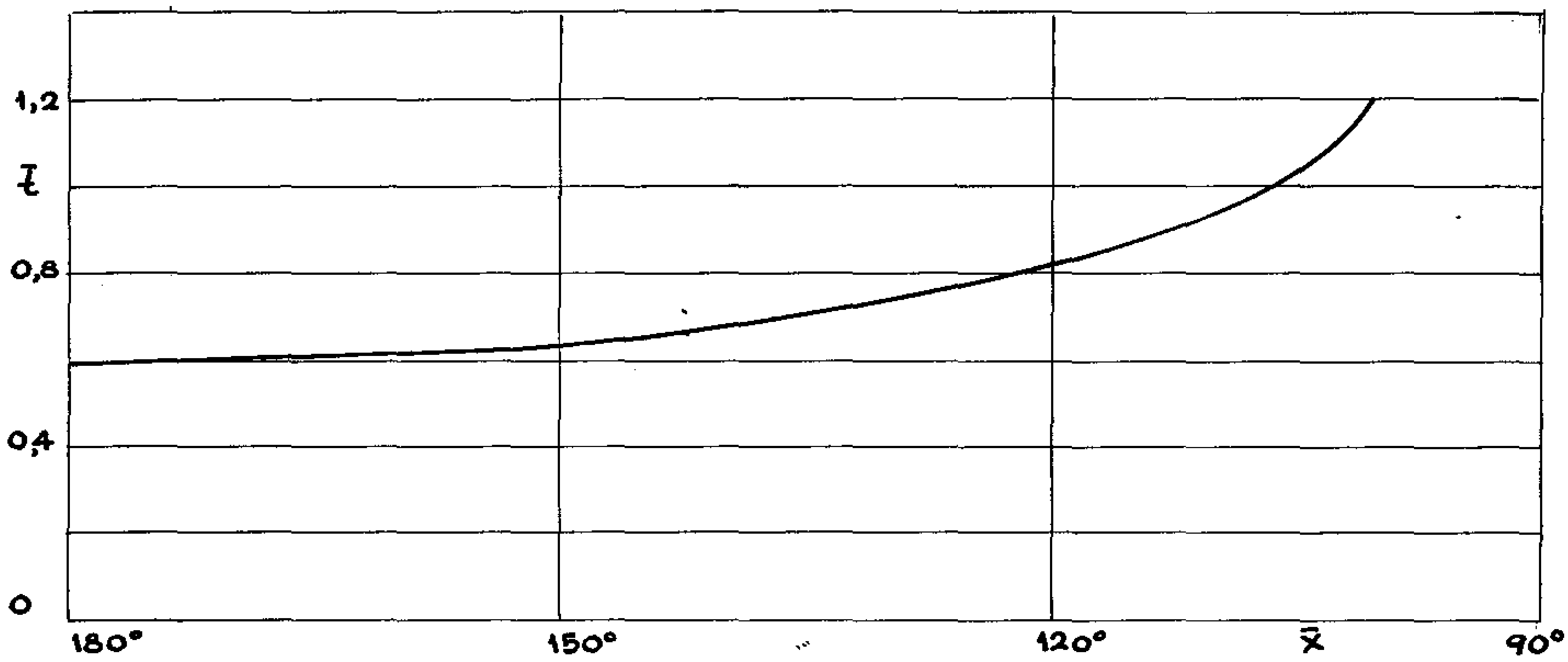
sl.5



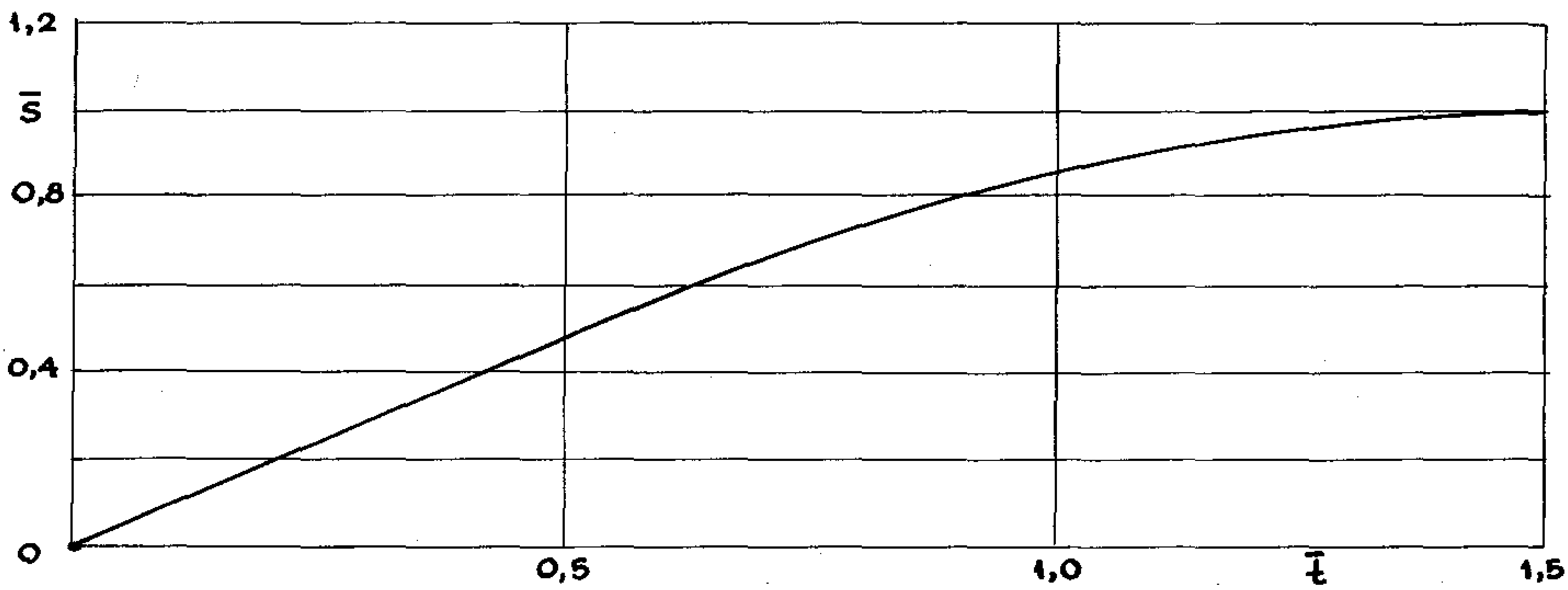
sl.6



sl. 7



sl.8



sl.9

T A B L I C E

funkcije

$$g_{\alpha}(\eta) = \frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)} \int_{\eta}^{\infty} (\gamma-\eta)^{2\alpha} e^{-\gamma^2} d\gamma$$

η	g_3	$g_{5/2}$	g_2	$g_{3/2}$
,0	0,00299725939	0,0100793469	0,0321917193	0,0950136980
,2	0,00142534209	0,00507603751	0,0172600051	0,0546216664
,4	0,000648841055	0,00244596298	0,00884931219	0,0300011764
,6	0,000282134800	0,00112515602	0,00432780123	0,0157004428
,8	0,000116942896	0,000493002044	0,00201407326	0,00780824070
,0	0,0000461129096	0,00020532505	0,00088991264	0,00368129755
,2	0,000017265207	0,00008111508	0,00022850956	0,00164160472
,4	0,000006126715	0,00003033965	0,00014744354	0,00069094966
,6	0,000002057040	0,00001072474	0,00005520316	0,00027396894
,8	0,000000652393	0,00000357685	0,00001938179	0,00010215822
,0	0,000000190547	0,00000112376	0,00000623066	0,00003576111
,2	0,000000050401	0,00000033211	0,00000199482	0,00001174012
,4	0,000000014570	0,00000006147	0,00000044041	0,00000360836
,6	0,000000003630	0,00000002401	0,00000015896	0,00000103723
,8	0,000000000847	0,00000000586	0,00000004065	0,00000027855
,0	0,000000000185	0,00000000612	0,00000000972	0,0000006982

η	g_1	$g_{1/2}$	$1-g_0$	$g_{-1/2}$
0,0	0,250694324	0,564471507	0,00000000000	1,12837916710
0,2	0,156095870	0,386801070	0,22270258920	1,08413478710
0,4	0,092733217	0,252253551	0,42839235505	0,96154129884
0,6	0,0524005012	0,156013507	0,60385609085	0,78724531714
0,8	0,0280830697	0,0912192238	0,74210096471	0,59498578626
1,0	0,0142369601	0,0502796510	0,84270079295	0,41510749721
1,2	0,00681100038	0,0260619615	0,91031397823	0,26734434700
1,4	0,00306820229	0,0126763531	0,95228511976	0,15894170768
1,6	0,00129894761	0,0057748272	0,97634838334	0,08722905865
1,8	0,00051591553	0,00245999407	0,98909050164	0,04419172333
2,0	0,00019194258	0,00097851119	0,99532226502	0,02066698535
2,2	0,00006679935	0,00036299704	0,99813715370	0,00892215506
2,4	0,00002171943	0,00012545357	0,99931148610	0,00355564868
2,6	0,00000659065	0,00004035569	0,99976396558	0,00130805005
2,8	0,00000186462	0,00001207306	0,99992498681	0,00044420794
3,0	0,00000049143	0,00000335670	0,99997790950	0,00013925305

	g_{-1}	$g_{-3/2}$	g_{-2}	$g_{-5/2}$
,0	0,00000000000	-2,25675833419	0,00000000000	13,54065000400
,2	0,43365391484	-1,99480800827	-2,52602305595	10,95583250551
,4	0,76923302307	-1,30769617922	-4,12308908942	4,45770572680
,6	0,94469211805	-0,44085632176	-4,30779605834	-2,52421733940
,8	0,95197725801	+0,33319204030	-3,27480176780	-7,23883507044
,0	0,83021499484	0,83021499484	-1,66042998960	-8,30214994800
,2	0,64162643281	1,00521474474	-0,16040660820	-6,40166732665
,4	0,44503678149	0,92821957284	0,81886767807	-3,27648814536

1,6	0,27913298763	0,718767443143	1,18352386749	-0,52532828269
1,8	0,15909020399	0,48614108772	1,10726781976	1,08008990991
2,0	0,08266794142	0,28933779497	0,82667941400	1,57069088660
2,2	0,03925748229	0,15488861193	0,52447996334	1,37838017613
2,4	0,01706711367	0,07481084825	0,29082361692	0,94708969940
2,6	0,00680186026	0,03275357325	0,14311113987	0,55916196525
2,8	0,00248756348	0,01304193964	0,06308463540	0,27502246462
3,0	0,00083552831	0,00473466376	0,02506554940	0,12198567346

TABLICA UNIVERZALNIH FUNKCIJA

η	f_2'	f_{22}'	f_4'
0,0	0,0000000000	0,0000000000	0,0000000000
0,4	0,0053998714	0,0024632734	0,0068960324
0,8	-0,0206242036	0,0022577135	-0,0129674342
1,2	-0,2186454723	-0,0011106783	-0,0143825991
1,6	-0,0118018032	-0,0046990987	-0,0074232114
2,0	-0,0037928919	-0,0012313700	-0,0023801092
2,4	-0,0009090214	-0,0004631921	-0,0005135097
2,8	-0,0001347592	-0,0000944546	-0,0000772031
3,2			

η	φ_1'	φ_{21}'	φ_3'
0,0	0,0000000000	0,0000000000	0,0000000000
0,4	0,1199088825	0,01773984463	0,0574081913
0,8	0,1132008047	0,0176723304	0,0553147957
1,2	0,0644297045	0,0093322755	0,0308492157
1,6	0,0253430023	0,0003520292	0,0120902473
2,0	0,0038112936	0,0010137205	0,0034272216
2,4	0,0013955524	0,0002127441	0,0006744942
2,			