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KNJIGA 12

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J. BULATOVIĆ, J. VUKMIROVIĆ, S. ŽIVANOVIĆ

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ПОСЕБНА ИЗДАЊА МАТЕМАТИЧКОГ ИНСТИТУТА У БЕОГРАДУ

Математички институт у Београду у својим *Псебним издањима* објављиваће: монографије актуелних питања из математике и механике, оригиналне чланке већег обима, оригиналне нумеричке таблице итд. Ова публикација није периодична.

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1. (1963) *D. S. Mitrinović* et *R. S. Mitrinović:*
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2. (1963) *K. Milošević-Rakočević:*
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11. (1974) *Zlatko P. Mamuzić:*
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PREFACE

This paper is an outcome of the seminar „Theory of spectral multiplicity in Hilbert space with application to stochastic process" that was held in the Mathematical Institute in Belgrade, during 1971—1973.

Chapter I contains the material necessary for the understanding of Chapter II. According to Plesner's theory of spectral types ([15]) and „regularizing transposition" of Stone ([18]), by „geometrical" reasoning (Lemma 2, Ch. I) the well known theorem on the complete system of unitary invariants of a self-adjoint operator in Hilbert space is proved. The preliminary knowledge for this chapter the reader can find, for example, in the standard book by N. I. Ahiezer and I. M. Glazman.

Applications of the results presented in Chapter I to stochastic processes considered as curves in Hilbert space are given in Chapter II. The knowledge required for this chapter the reader can find in Doob ([3]) or, for example, in the book by Cramér and Leadbetter ([7]).

Appendices I and II consider examples of Markov's processes and random fields.

Appenix III conteats one part of Cramér's results shown in the work [6] which we have seen after this work had been in print.

The essential progress in Cramér's theory has been made by Yu. A. Rozanov: Theory of Innovation Processes (in Russian), Moscow, 1974. Rozanov's book became available to us in the course of printing of this work; this is a reason why a survey of Rozanov's results is here missing.

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INTRODUCTION

Let $\{x(t), a \leqslant t \leqslant b\}$ be a complex-valued, second ordered stochastic process, i. e. $E|x(t)|^2 < +\infty$ for each $t \in [a, b]$ $(E(x(t)) = 0$, for each $t \in [a, b])$. In the correlation theory of stochastic processes, all properties of the process $\{x(t)\}$ are defined and determined in terms of its correlation function $r(s, t) = Ex(s)\overline{x(t)}$, $s, t \in [a, b]$. The connection of two second ordered processes $\{x(t)\}$ and $\{y(t)\}$ is defined by their cross-correlation function $\rho(s, t) = Ex(s)\overline{y(t)}$, $s, t \in [a, b]$. One of the main problems of the correlation theory is the problem of linear prediction: to find the random variable $\hat{x}(s; t)$, $s < t$, as a quadratic mean limit of the sequence $\sum_{k: t_{kn} \leq s} c_{kn} x(t_{kn})$, $n = 1, 2, \ldots$ such that

$$E|x(t - \hat{x}(s; t)|^2$$

is minimal. A more general problem is the problem of linear filtration: to obtain the random variable $\tilde{y}(s; t)$, $s < t$, as a quadratic mean limit of a sequence $\sum_{k: t_{kn} \leq s} c_{kn} y(t_{kn})$, $n = 1, 2, \ldots$ such that

$$E|x(t) - \tilde{y}(s; t)|^2$$

is minimal.

Studying the stationary sequence $\{x_k, k = \ldots, -1, 0, +1, \ldots\}$, A. N. Kolmogorov ([13]) introduced the Hilbert space method in the correlation theory of stochastic processes for the first time. Random variables $x, y, \ldots$ of finite dispersion are considered as elements of Hilbert space $\mathcal{H}$ with the scalar product defined by $(x, y) = Ex\bar{y}$, $x, y \in \mathcal{H}$. Hence the stochastic process $\{x(t)\}$ is a curve in the space $\mathcal{H}$. The problem of linear prediction is so reduced to the projection problem. Now, $\hat{x}(s; t)$ is a projection of $x(t)$ on the subspace $\mathcal{H}(x; s)$, where $\mathcal{H}(x; s)$ is the smallest subspace spanned by the variables $x(u)$, where $u \leqslant s$. Wold's representation of stationary sequence is

$$x_n = \sum_{k=-\infty}^{n} c_{n-k} z_k, \quad n = \ldots, -1, 0, +1, \ldots, \tag{1}$$

where $\{z_n, \ n = \ldots, -1, 0, +1, \ \ldots\}$ is a sequence of the mutually orthogonal random variables such that

$$\mathcal{H}(x; n) = \mathcal{H}(z; n) \text{ for each } n = \ldots, -1, 0, +1, \ldots \tag{2}$$

Applying Stone's representation of the group of unitary operators, Kolmogorov gave the effective expression for the coefficients c_i, $i = 0, 1, \ldots$ in Wold's representation.

The equality (2) plays the fundamental role in the correlation theory. It shows that the stationary sequence $\{x_n\}$ can be substituted by the sequence $\{z_n\}$ and therefore that all the information about $\{x_n\}$ is contained in $\{z_n\}$. Also, $\{z_n\}$ can be determined by means of $\{x_n\}$. For example, from (1) and (2) it follows that a linear prediction can be expressed by

$$\hat{x}_{m,n} = \sum_{k=-\infty}^{m} c_{n-k} z_k, \quad m < n.$$

Krein, Hanner and Kahrhunen (see, for instance [3]) extended Kolmogorov's result to the case of a stationary process $\{x(t), -\infty < t < +\infty\}$ with a continuous parameter. In this case Wold's representation of the process $\{x(t)\}$ is a stochastic integral (as quadratic mean integral) of a process with orthogonal increments $\{z(t), -\infty < t < +\infty\}$, i. e.

$$x(t) = \int_{-\infty}^{t} g(t-u) \, z(du), \quad t \in (-\infty, +\infty) \tag{3}$$

where

$$\mathcal{H}(x; t) = \mathcal{H}(z; t), \quad t \in (-\infty, +\infty), \tag{4}$$
$$E|z(dt)|^2 = dt.$$

Let us notice once more that (4) shows that, in the framework of the correlation theory, the processes $\{x(t)\}$ and $\{z(t)\}$ carry the same information, and that $\{z(t)\}$, being the process with orthogonal increments, is easier to apply.

It is now natural to study whether Wold's representation (3) can be extended to the second ordered process $\{x(t), a \leqslant t \leqslant b\}$ in a general case, i. e. the possibility of the representation

$$x(t)=\int_a^t g(t,u)\,z(du), \quad t\in[a,b], \tag{5}$$

where $\{z(t),\ a\leqslant t\leqslant b\}$ is a process with orthogonal increments and

$$\mathcal{H}(x;t)=\mathcal{H}(z;t), \quad t\in[a,b). \tag{6}$$

The first example of the second ordered process for which the representation (5) is impossible was given by Hida in 1960. However that process had rather pathological properties (for example, the discontinuity in quadratic mean at each point).

H. Cramér ([4]) solved the problem of Wold's representation in general form. It follows, by simple geometrical reasoning, that every second ordered process $\{x(t),\ a\leqslant t\leqslant b\}$ can be represented in the form

$$x(t)=\int_a^t \sum_{n=1}^{M} g_n(t,u)\,z_n(du), \tag{7}$$

where $\{z_n(t),\ a\leqslant t\leqslant b\}$, $n=\overline{1,M}$ are mutually orthogonal processes with orthogonal increments and

$$\mathcal{H}(x;\mathrm{t})=\sum_{n=1}^{M}\oplus\,\mathcal{H}(z_n;t), \quad t\in[a,b].$$

It is evident that the represantion (7) is not uniquely determined. The question is which properties of the representation (7) are determined in terms of the correlation function $r(s,t)$ of the process $\{x(t)\}$. Applying the theorem of the complete system of unitary invariants of a self-adjoint operator in a separable Hilbert space, Cramér pointed out that among the representations (7) exists one for which M is minimal (min $\{M\}=N$, N may be infinite) and that the measures induced by distribution functions $F_n(t)=E\,|\,z_n(t)\,|^2$, $a\leqslant t\leqslant b$, $n=\overline{1,N}$ can be ordered by absolute continuity:

$$F_1>F_2>\ldots>F_N.$$

The equivalence classes ρ_n, $n=\overline{1,N}$, of the measures induced by F_n, respectively, are uniquely determined by the correlation function $r(s,t)$. The sequence

$$\rho_1>\rho_2>\ldots>\rho_N \tag{8}$$

is called the spectral type of the process $\{x(t)\}$.

Cramér's main result is that for any given sequence (8) there exists a stochastic process $\{x(t)\}$, continuous in quadratic mean (and harmonizable), whose spectral type is that sequence.

The representation

$$x(t)=\int\limits_a^t \sum_{n=1}^{N} g_n(t,u)\, z_n(du), \quad t\in[a,b], \tag{9}$$

of the process $\{x(t),\ a\leqslant t\leqslant b\}$ satisfying (8) and

$$\mathcal{H}(x;t)=\sum_{n=1}^{N}\oplus\,\mathcal{H}(z_n;t), \quad t\in[a,b], \tag{10}$$

will be called Cramér's representation. Equality (10) shows that any process $\{z_n(\mathrm{t})\}$, $n=\overline{1,N}$ is determined by $\{x(t)\}$ and, conversely, that the process $\{x(t)\}$ is determined by the processes $\{z_n(t)\}$, $n=\overline{1,N}$.

Now, the main problem of the whole theory is to determine explicitly the pectral type of the given process in terms of its correlation function (Cramér, [5]).

Kallianpur and Mandrekar ([12]) extended Cramér's theory to an n-dimensional process and, more generally, to the process $\{x(t,\varphi),\ a\leqslant t\leqslant b,\ \varphi\in\Phi\}$, where Φ is a Hausdorff space with a denumerable base. The other generalizations of that theory and some special classes of processes are considered in Rozanov ([17]), Mandrekar ([14]), Rozanov and Ivković ([11]).

From the continuity of the process $\{x(t)\}$ it follows that the corresponding space $\mathcal{H}(x)$ $(=\mathcal{H}(x;b))$ is separable. The analog theory in the case of non-separable space $\mathcal{H}(x)$ can be developed using Plesner's generalized spectral types (Плеснер [15], Halmos [8]).

Chapter I

THE COMPLETE SYSTEM OF UNITARY INVARIANTS OF A SELF-ADJOINT OPERATOR IN SEPARABLE HILBERT SPACE

I.1. **The concept of spectral theory of self-adjoint operators**

The main aim of this chapter is to desribe the set of all self-adjoint operators unitary equivalent to a given self-adjoint operator A, defined on a separable Hilbert space $\mathcal{H}$.

Two operators*) A_1 and A_2 defined on Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively ($\mathcal{H}_1$ and $\mathcal{H}_2$ may coincide) are said to be *unitary equivalent* if there exists an isomorphism U between $\mathcal{H}_1$ and $\mathcal{H}_2$ such that

$$A_2 = U A_1 U^{-1}. \tag{I.1}$$

If $\mathcal{H}_1 \equiv \mathcal{H}_2$ the operator U is called a unitary operator.

The problem of unitary equivalence is to find the necessary and sufficient conditions for A_1 and A_2 (in our case they are self-adjoint operators) under which exists the isomorphism U so that (I.1.) holds.

From the geometrical point of view, there is no difference between unitary equivalent operators. So, the „description" of A_1 is, at the same time, the „description" of the whole class of operators unitary equivalent to A_1.

To find out if two operators A_1 and A_2 are unitary equivalent we need, according to the definition, to prove the existence of the isomorphism U satisfying (I.1.). In general, this is rather comlicated. Therefore, we shall solve the equivalent problem called: the finding of a complete system of unitary invariants of a self-adjoint operator. This means that we shall correspond an „object" $\mathbf{F}_A$ to a self-adjoint operator A such that:

(1) If A_1 and A_2 are unitary equivalent, then $\mathbf{F}_{A_1} = \mathbf{F}_{A_2}$;

(2) If $\mathbf{F}_{A_1} = \mathbf{F}_{A_2}$, then self-adjoint operators A_1 and A_2 are unitary equivalent;

(3) For each „object" $\mathbf{F}$ there exists a self-adjoint operator A such that $\mathbf{F}_A = \mathbf{F}$.

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*) The term „operator" means a transformation of the space $\mathcal{H}$ inso itself.

It should be noted that the spectrum of a self-adjoint operator satisfies only (1) and (3); so, the spectrum is not the complete system of unitary invariants.

The conditions (1), (2) and (3) describe a biunique correspondence between the set of all equivalence classes of self-adjoint operators and the set of all „objects" **F**. We shall see that the „objects" **F** are simpler than the corresponding operators and that their effective construction will be possible. Furthermore, the definition of an „object" **F** does not depend on the theory of operators.

Further on, when nothing else is explicitly mentioned, all operators will be considered as self-adjoint operators defined on the same separable Hilbert space $\mathcal{H}$.

Let

$$\{E(t),\ a \leqslant t \leqslant b\} \tag{I.2}$$

be a resolution of the identity in $\mathcal{H}$, defined on some finite or infinite interval $[a, b]$. If M is a Borel set, then $E(M)$ means

$$E(M) = \int_M E(dt).$$

For any fixed $x \in \mathcal{H}$

$$\rho_x(M) = \| E(M)x \|^2 \tag{I.3}$$

is a measure over $[a, b]$. Let $\mathcal{M}$ be the set of all measures $\rho_x(\cdot)$, $x \in \mathcal{H}$: $\mathcal{M} = \{\rho_x(\cdot),\ x \in \mathcal{H}\}$. In $\mathcal{M}$ we introduce the ordering relation $<$ in the following way: $\rho_1(M) < \rho_2(M)$ if the measure $\rho_1(M) = \| E(M)x_1 \|^2$ is absolutely continuous with respect to the measure $\rho_2(M) = \| E(M)x_2 \|^2$. We shall say that $\rho_1(M)$ is *subordinated* to $\rho_2(M)$.

We shall say that $\rho_1(M)$ and $\rho_2(M)$ are *equivalent* $(\rho_1(M) \sim \rho_2(M))$ if $\rho_1(M) < \rho_2(M)$ and $\rho_2(M) < \rho_1(M)$ hold. As „$\sim$" is the equivalence relation, we can consider the set of all equivalence classes $\mathcal{M}/\sim$. The *spectral type* is an equivalence class, i. e. an element of $\mathcal{M}/\sim$. We shall denote by ρ the spectral type determined by the measure $\rho(M)$, and for the measure $\rho(M)$ we shall say that it belongs to the type ρ. The notation $\rho_1 < \rho_2$ has the usual meaning.

Different measures belonging to the same type ρ have the same family of null sets $\mathcal{N}_\rho$, but the families of null sets of different spectral types are different. This means that the spectral type ρ uniquely determines its family $\mathcal{N}_\rho$ and conversely, the family of the null sets $\mathcal{N}_\rho$ uniquely determines the spectral type ρ. It is evident, from the definition of the ordering relation $<$, that the spectral type ρ_1 is subordinated to the spectral type ρ_2 if and only if $\mathcal{N}_{\rho_2} \subset \mathcal{N}_{\rho_1}$. This simple fact enables us to point out that for two arbitrary spectral types there exists a uniquely determined supremum $\rho = \rho_1 + \rho_2 = \sup\{\rho_1, \rho_2\}$, defined by $\rho(M) = \rho_1(M) + \rho_2(M)$. It means that $\mathcal{N}_\rho = \mathcal{N}_{\rho_1} \cap \mathcal{N}_{\rho_2}$ and therefore, ρ is uniquely determined.

The more general statement is true: any at most countable set $\{\rho_1, \rho_2, \ldots\}$ of spectral types from $\mathcal{M}/\sim$ has the supremum. Namely, without the restriction of generality, we can assume that $\sum_i \rho_i(M) < +\infty$ for any Borel set M (for, the measures $\rho_i(M)$ can be substituted by equivalent measures multiplying each $\rho_i(M)$ by a sufficiently small positive number). Let the measure $\rho_i(M)$ belong to the type ρ_i. Then for $\rho(M) = \sum_i \rho_i(M)$ we have: $\mathcal{N}_\rho = \bigcap_i \mathcal{N}_{\rho_i}$ and spectral type $\rho = \sup\{\rho_1, \rho_2, \ldots\}$ is uniquely determined. It follows that any at most countable set of spectral types is bounded.

We shall denote with $\inf\{\rho_1, \rho_2, \ldots\}$ the maximal spectral type subordinated to each ρ_i, $i = 1, 2, \ldots$.

The smallest element of the set $\mathcal{M}/\sim$ is the spectral type 0 identically equal zero on the whole interval $[a, b]$. In this and the next chapter we shall operate with sets of spectral types having the maximal element (we shall see that the separability of $\mathcal{H}$ provides us with that).

Two spectral types ρ_1 and ρ_2 are said to be *orthogonal* if and only if $\inf\{\rho_1, \rho_2\} = 0$.

Let $\mathcal{H}$ be a separable Hilbert space and A a self-adjoint operator defined on it. We shall first consider a spectral type of a subspace of $\mathcal{H}$ related to the operator A.

We shall say that the subspace $\mathfrak{M}$ of $\mathcal{H}$ is *invariant with respect to the operator* A if $Ax \in \mathfrak{M}$ for all $x \in \mathfrak{M}$. The subspace $\mathfrak{M}$ *reduces* A if both $\mathfrak{M}$ and $\mathcal{H} \ominus \mathfrak{M}$ are invariant with respect to A. The operator A_1 induced by A on any subspace $\mathfrak{M}$ which reduces A will be called *the part of the operator* A.

If A is a self-adjoint operator, any subspace invariant with respect to A reduces A ([1], § 46).

It is well known ([1], § 75) that there is a one-to-one corespondence between the class of all self-adjoint operators and the class of all resolutions of the identity on the real axes. Let $\{E(t), a \leqslant t \leqslant b\}$ be a resolution of the identity corresponding to a self-adjoint operatot A and let x be an arbitrary element of $\mathcal{H}$. The subspace $\mathfrak{M}(x) = \mathcal{L}\{E(t)x, a \leqslant t \leqslant b\}$ [1)] of $\mathcal{H}$ will be called *the cyclic subspace of the operator A with the generating element x*. The cyclic subspace reduces A. We shall denote by ρ_x a spectral type determined by the measure $\rho_x(M) = \|E(M)x\|^2$. It can be shown ([1], § 83) that the cyclic subspace $\mathfrak{M}(x)$ of the operator A, generated by $x \in \mathcal{H}$, coincides with the set of all elements of the form

$$y = \int_a^b f(t)\, E(dt)\, x, \tag{I.4}$$

where $f(t)$ is a square integrable function with respect to $\rho_x(M)$, i. e. $f(t) \in \mathcal{L}_2(\rho_x)$. The correspondence $f(t) \leftrightarrow y$ is an isomorphism between the spaces $\mathcal{L}_2(\rho_x)$ and $\mathfrak{M}(x)$.

---

1) $\mathcal{L}\{\cdot\}$ denotes the smallest subspace spanned by the elements in the parantheses

The spectral type ρ_x defined by the measure $\rho_x(M)$ with respect to the resolution of the identity of the operator A is called *the spectral type of the element* x. We say that measures and types of elements of $\mathscr{H}$ *belong* to the operator A. The element $x=0$ is the only element with zero type. The operator A is *the operator with the maximal spectral type* if and only if there exists the maximal spectral type among the types belonging to A. Any element generating the maximal spectral type is called *the element with the maximal spectral type.*

The operator A is *cyclic* if there exists an element $x \in \mathscr{H}$ such that $\mathscr{H}=\mathfrak{M}(x)$, i. e. if the whole space $\mathscr{H}$ is cyclic. It is easy to find the set of spectral types belonging to a cyclic operator A. If x is a generating element of $\mathscr{H}$ and ρ_x its spectral type, then the spectral type σ belongs to the cyclic operator A if and only if $\sigma < \rho_x$. Indeed, if $y \in \mathscr{H}$, then there exists the function $f(t) \in \mathscr{L}_2(\rho_x)$ such that (I.4) holds. Let $\sigma(M)=\|E(M)y\|^2$. Then:

$$\sigma(M)=\int_M |f(t)|^2 \rho_x(dt) < \rho_x(M).$$

Conversely, if $\sigma < \rho_x$, then, according to the Radon-Nicodym theorem, there exists a non-negative ρ_x-integrable function $\varphi(t)$ such that

$$\sigma(M)=\int_M \varphi(t)\, \rho_x(dt).$$

Since $f(t)=\sqrt{\varphi(t)} \in \mathscr{L}_2(\rho_x)$, the element y, corresponding to the function $f(t)$, belongs to $\mathscr{H}$.

It follows that any element with the maximal spectral type in a cyclic space is the generating element. Hence the cyclic operator has the element with the maximal spectral type.

The spectral type of the cyclic operator A is the maximal spectral type belonging to A.

THEOREM 1. Let $\mathfrak{M}(x_1)$ anp $\mathfrak{M}(x_2)$ be cyclic subspaces of the operator A and suppose that the generating elements x_1 and x_2 have mutually orthogonal spectral types. Then the space $\mathfrak{M}(x_1) \oplus \mathfrak{M}(x_2)$ is cyclic and its generating element $x=x_1+x_2$ has a spectral type $\rho_x=\rho_{x_1}+\rho_{x_2}$.

Proof. We shall first show that the subspaces $\mathfrak{M}(x_1)$ and $\mathfrak{M}(x_2)$ are mutually orthogonal. Let y be an element of $\mathfrak{M}(x_2)$, y_1 its projection on $\mathfrak{M}(x_1)$ and $z=y-y_1$. As $\mathfrak{M}(x_1)$ reduces A and z is orthogonal to $\mathfrak{M}(x_1)$, we have

$$(E(M)z, y_1)=(E(M)y_1, z)=0$$

and therefore

$$\| E(M)y \|^2 = \| E(M)z \|^2 + \| E(M)y_1 \|^2.$$

That means that the spectral type ρ_{y_1} is subordinated to the type ρ_y and, because of $\rho_y < \rho_{x_2}$, it holds that $\rho_{y_1} < \rho_{x_2}$. As $\rho_{y_1} < \rho_{x_1}$ and as the spectral types ρ_{x_1} and ρ_{x_2} are mutually orthogonal, it follows that $\rho_{y_1} = 0$, i. e. $y = 0$. Therefore y is orthogonal to $\mathfrak{M}(x_1)$ for any $y \in \mathfrak{M}(x_2)$, i. e. $\mathfrak{M}(x_1)$ is orthogonal to $\mathfrak{M}(x_2)$.

From

$$\| E(M)x \|^2 = \| E(M) \sum_{i=1}^{2} x_i \|^2 = \sum_{i=1}^{2} \| E(M)x_i \|^2 = \rho_{x_1}(M) + \rho_{x_2}(M),$$

we see that $\rho_x = \rho_{x_1} + \rho_{x_2}$ is the spectral type of the element $x = x_1 + x_2$.

Hence the spectral type ρ_x belongs to x and is the maximal spectral type of the orthogonal sum $\mathfrak{M}(x_1) \oplus \mathfrak{M}(x_2)$. Because of $E(M)x = E(M)x_1 + E(M)x_2$, x is the generating element of $\mathfrak{M}(x_1) \oplus \mathfrak{M}(x_2)$, i. e. that space is a cyclic subspace. ▲

Let K_ρ be the operator of multiplying by independent variable in the space $\mathscr{L}_2(\rho)$. One can show ([1], § 83) that the cyclic operator A with the spectral type ρ and the operator K_ρ are isomorphic. The operator K_ρ is called *the canonical representation of the cyclic operator A*.

It follows that any cyclic operator is defined at a separable space since the space on which K_ρ is defined is separable.

The next theorem is a simple generalisation of Theorem 1.

THEOREM 2. The orthogonal sum of at most countable many cycli operators with mutually orthogonal spectral types ρ_i is a cyclic operator. It spectral type is $\rho = \sup \{\rho_i\}$.

We omit the proof.

It is easy now to prove the following:

THEOREM 3. Two cyclic operators are unitary equivalent if and only if they have the same spectral type.

Proof. It is clear (from the definition of the unitary operator) that unitary equivalent cyclic operators have the same spectral type. Conversely, if two cyclic operators have the same spectral type ρ, then both of them are unitary equivalent to K_ρ and therefore they are unitary equivalent. ▲

The problem of unitary equivalence is so solved for cyclic operators. In a general case, for self-adjoint operators in separable Hilbert space, the same problem will be solved by reducing it to the preceding problem. The first step is the following:

THEOREM 4. If A is a self-adjoint operator defined on any fixed (not necessarily separable) Hilbert space $\mathcal{H}$, then $\mathcal{H}$ can be represented as an orthogonal sum of subspaces, cyclic with respect to A.

Proof. Let $\mathcal{P}$ be the partitive set of the family of all mutually orthogonal subspaces of $\mathcal{H}$, cyclic with respect to A. We can introduce the partial ordering in $\mathcal{P}$ by inclusion. According to the axiom of choice, there exists the maximal totally ordered chain of mutually orthogonal subspaces. Let $\mathcal{H}_1$ be their orthogonal sum. We shall show that $\mathcal{H}_1 = \mathcal{H}$. If this equality does not hold, there exists $x \neq 0$ in $\mathcal{H} \ominus \mathcal{H}_1$. Since $\mathcal{H}_1$ reduces A, $\mathcal{H} \ominus \mathcal{H}_1$ reduces A too. Therefore $\mathfrak{M}(x) \subset \mathcal{H} \ominus \mathcal{H}_1$, which contradicts the proposition that the chain is maximal. ▲

Later on we will need the following:

THEOREM 5. Let $\mathcal{H} = \mathfrak{M}(x_0)$ and $\mathfrak{M}(y_0) \subset \mathcal{H}$. Let $f_0 \in \mathcal{L}_2(\rho_{x_0})$ be a function corresponding to the element y_0, and $M_0 = \{t : f_0(t) \neq 0\}$. Denote by $\mathfrak{M}_{M_0}$ the set of all elements of $\mathcal{H}$ such that their corresponding functions in $\mathcal{L}_2(\rho_{x_0})$ vanish (almost everywhere with respect to ρ_{x_0}) outside of the set M_0. Then $\mathfrak{M}(y_0) = \mathfrak{M}_{M_0}$.

Proof. The subspace $\mathfrak{M}(y_0)$ consists of all elements

$$y = \int_a^b g(t)\, E(dt)\, y_0 = \int_a^b g(t) f_0(t)\, E(dt)\, x_0,$$

where the function $g(t)$ satisfies the condition

$$\int_a^b |g(t)|^2 \rho_{y_0}(dt) = \int_a^b |g(t) f_0(t)|\, \rho_{x_0}(dt) < +\infty.$$

Therefore the functions $g(t) f_0(t) \in \mathcal{L}_2(\rho_{x_0})$ correspond to the elements of $\mathfrak{M}(y_0)$. If $t \notin M_0$ then $g(t) f_0(t) = 0$ and therefore all elements of $\mathfrak{M}(y_0)$ belong to $\mathfrak{M}_{M_0}$. Conversely, let $y_1 \in \mathfrak{M}_{M_0}$ and $g_1(t) \in \mathcal{L}_2(\rho_{x_0})$ be the function correspondin to y_1. We set

$$g(t) = \begin{cases} \dfrac{g_1(t)}{f_0(t)}, & t \in M_0, \\ 0, & t \notin M_0. \end{cases}$$

Then $g_1(t) = g(t) f_0(t)$ and therefore $y_1 \in \mathfrak{M}(y_0)$. Hence $\mathfrak{M}(y_0) = \mathfrak{M}_{M_0}$. ▲

Suppose that $\mathcal{H}=\mathfrak{M}(x)$ and y_1 and y_2 are arbitrary elements of $\mathcal{H}$ with the corresponding functions $f_1, f_2 \in \mathcal{L}_2(\rho_x)$. Let $M_1=\{t: f_1(t)\neq 0\}$ and $M_2= =\{t: f_2(t)\neq 0\}$. From Theorem 5 it follows that the elements y_1 and y_2 generate the same cyclic subspace if and only if $M_1=M_2$ almost everywhere with respect to ρ_x. This means that the cyclic subspace of a cyclic space $\mathcal{H}$ is uniquely determined by its spectral type.

COROLLARY 1. In a cyclic space different cyclic subspaces have different spectral types.

I.2. The canonical representation of self-adjoint operators. Unitary invariants

Theorem 4 shows that Hilbert space $\mathcal{H}$ can be represented as an orthogonal sum of subspaces cyclic with respect to a self-adjoint operator A, i. e.

$$\mathcal{H}=\sum_{k\geqq 1}\oplus\,\mathfrak{M}(x_k). \tag{I.5}$$

In a general case, spectral types ρ_{x_k} are not comparable and the cardinality of the set $\{\rho_{x_k}\}$ is not uniquely determined. In Theorem 7 we shall prove that $\mathcal{H}$ can be represented as an orthogonal sum of cyclis subspaces $\mathfrak{M}(z_k)$ such that

$$\rho_{z_1}>\rho_{z_2}>\cdots. \tag{I.6}$$

In order to show that this representation has some invariant properties we shall prove some preliminary facts.

Let Σ_ρ be the set of all spectral types ρ and let $\mathcal{B}$ be Borel σ-algebra over the segment $[a, b]$ (on which ρ is defined). To each Borel set $N\in\mathcal{B}$ we correspond the measure

$$\rho_N(M)=\rho(M\cap N).$$

The spectral type of the measure $\rho_N(M)$ is subordinated to ρ. We define the mapping Γ_ρ from $\mathcal{B}$ to Σ_ρ by

$$\Gamma_\rho: N\rightarrow\rho_N.$$

LEMMA 1. The mapping Γ_ρ is a homomorphism of σ-algebra $\mathcal{B}$ onto Σ_ρ.

Proof. Since $\Gamma_\rho([a, b])=\rho_{[a, b]}$ and if $N_1\subset N_2$, then $\rho_{N_1}<\rho_{N_2}$, so that Γ_ρ is a homomorphism of $\mathcal{B}$ into Σ_ρ. It remains to be shown that Γ_ρ is a homomorphism of $\mathscr{B}$ onto Σ_ρ. For each $\sigma \in \Sigma_\rho$ there exists a non-negative ρ-measurable function $f(t)$ such that

$$\sigma(M) = \int_M f(t)\,\rho(dt),$$

for any $M \in \mathscr{B}$. The set $N = \{t : f(t) \neq 0\}$ belongs to $\mathscr{B}$. Since the measure $\sigma(M)$ has the same family of null sets as the measure

$$\rho_N(M) = \rho(M \cap N) = \int_M \chi_N(t)\,dt$$

we get $\Gamma_\rho(N) = \sigma$. ▲

COROLLARY 2. For each spectral type α subordinated to a given spectral type ρ there exists the uniquely determined spectral type τ such that $\rho = \alpha + \tau$.

This follows directly from the preceding lemma and the fact that $N_\tau = [a, b] \setminus N_\alpha$.

THEOREM 6. Let ρ_1 and ρ_2 be given spectral types and

$$\rho_2 = \inf\{\rho_1, \rho_2\} + \tau.$$

Then the spectral types ρ_1 and τ are orthogonal.

Proof. Suppose that $\inf\{\rho_1, \tau\} \neq 0$. That means that there exists a spectral type $\tau_1 \neq 0$ such that

$$\tau = \inf\{\rho_1, \tau\} + \tau_1.$$

Then

$$\rho_2 = \inf\{\rho_1, \rho_2\} + \inf\{\rho_1, \tau\} + \tau_1.$$

From $\rho_2 > \tau$ it follows that

$$\inf\{\rho_1, \rho_2\} > \inf\{\rho_1, \tau\}$$

and $\rho_2 = \inf\{\rho_1, \rho_2\} + \tau_1$ which contradicts to Lemma 1. ▲

REMARK 1. Theorem 6 is in fact the well-known Lebesgue theorem on the additive decomposition of a given measure $\rho_2(M)$ into two parts: one absolutely continuous with respect to a given measure $\rho_1(M)$ and the second singular with respect to $\rho_1(M)$.

LEMMA 2. Let $\mathfrak{M}(x)$ and $\mathfrak{M}(y)$ be two mutually orthogonal cyclic subspaces. Then there exist elements z_1 and z_2 in $\mathfrak{M}(x) \oplus \mathfrak{M}(y)$ such that $\mathfrak{M}(x) \oplus \mathfrak{M}(y) = \mathfrak{M}(z_1) \oplus \mathfrak{M}(z_2)$ and for spectral types ρ_{z_1} and ρ_{z_2} holds $\rho_{z_1} > \rho_{z_2}$.

Proof. According to Theorem 6 there exists the uniquely determined spectral type τ such that

$$\rho_y = \inf\{\rho_x, \rho_y\} + \tau$$

and

$$\inf\{\rho_x, \tau\} = 0.$$

Since $\tau < \rho_y$ there exists the element u in $\mathfrak{M}(y)$ with the spectral type τ. We set $z_1 = x + u$. Because of the orthogonality of spectral types ρ_x and $\rho_y = \tau$ we get

$$\rho_{z_1} = \rho_x + \tau$$

and

$$\mathfrak{M}(z_1) = \mathfrak{M}(x) \oplus \mathfrak{M}(u) \tag{I.7}$$

(Theorem 1). In $\mathfrak{M}(y)$ there exists the element z_2 with the spectral type

$$\rho_{z_2} = \inf\{\rho_x, \rho_y\}.$$

As $\rho_{z_1} > \rho_x$ and $\rho_x > \rho_{z_2}$ we have $\rho_{z_1} > \rho_{z_2}$. Since $\inf\{\rho_{z_2}, \tau\} = 0$ we get

$$\mathfrak{M}(y) = \mathfrak{M}(z_2) \oplus \mathfrak{M}(u),$$

or

$$\mathfrak{M}(z_2) = \mathfrak{M}(y) \ominus \mathfrak{M}(u). \tag{I.8}$$

From (I.7) and (I.8) it follows that

$$\mathfrak{M}(z_1) \oplus \mathfrak{M}(z_2) = \mathfrak{M}(x) \oplus \mathfrak{M}(y).$$ ▲

Now we shall prove the following:

THEOREM 7. Let A be a self-adjoint operator defined on a separable Hilbert space $\mathcal{H}$. Then there exists a represantation

$$\mathcal{H} = \sum_{k=1}^{N} \oplus \mathfrak{M}(z_k), \tag{I.9}$$

such that

$$\rho_{z_1} > \rho_{z_2} > \ldots > \rho_{z_N}. \tag{I.6}$$

(The number N may be finite or infinite.)

Proof. Let (I.5) be one representation of $\mathcal{H}$. According to Lemma 2, there exist the elements u_{1k} and u_{2k} in $\mathfrak{M}(x_k)$, $k=\overline{2, N}$, such that $x_k = u_{1k} + u_{2k}$ and the types $\rho_{u_{1k}}$ and $\rho_{u_{2k}} = \inf \{\rho_{x_1 + \sum_{i=2}^{k-1} u_{1i}}, \rho_{x_k}\}$ are mutually orthogonal, i. e. $\rho_{u_{2k}} < \rho_{x_1 + \sum_{i=2}^{k} u_{1i}}$, $k=\overline{2, N}$; it is obvious that the subspace $\mathfrak{M}(x_1) \oplus \sum_{k=2}^{N} \oplus \mathfrak{M}(u_{1k})$ is a cyclic subspace with a generating element $z_1 = x_1 + \sum_{k=2}^{N} u_{1k}$ whose spectral type is the maximal spectral type in $\mathcal{H}$. Hence $\sum_{k=1}^{N} \oplus \mathfrak{M}(x_k) = \mathfrak{M}(z_1) \oplus \sum_{k=2}^{N} \oplus \mathfrak{M}(u_{2k})$ and $\rho_{u_{2k}} < \rho_{z_1}$ $k=\overline{2, N}$. Applying the same procedure to $\sum_{k=2}^{N} \oplus \mathfrak{M}(u_{2k})$ we can choose the element z_2 with the maximal spectral type in the space $\mathcal{H} \ominus \mathfrak{M}(z_1) = \sum_{k=2}^{N} \oplus \mathfrak{M}(u_{2k})$. Evidently: $\rho_{z_1} > \rho_{z_2}$. We continue this procedure until we get the sequence $\{z_k\}$ such that

$$\mathcal{H} = \sum_{k=1}^{N} \oplus \mathfrak{M}(z_k) \qquad \blacktriangle$$

and the relation (I.6) holds.

REMARK 2. From the construction of the sequence $\{z_k\}$ it follows that the cardinality of its non-zero elements is not greater than the cardinality of the sequence $\{x_k\}$. If $\inf \{\rho_{x_k}\} \neq 0$ these two cardinalities are equal.

The representation (I.9) with the condition (I.6) is called *the canonical representation of the space $\mathcal{H}$ with respect to the operator A*. The corresponding representation of the operator A as a sum of its parts, defined on these subspaces, is called *the canonical representation of the operator A*. The sequence (I.6) is called *the spectral type of the operator A*.

The uniqueness of the sequence (I.6) follows from the proof of Theorem 7, i. e. (I.6) is independent of the choice of $\{x_k\}$. However the elements $z_1, z_2, \ldots, z_N$ themselves depend on the choice of $\{x_k\}$.

LEMMA 3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, A a self-adjoint operator defined on $\mathcal{H}_1$ and U an isomorphism hetween $\mathcal{H}_1$ and $\mathcal{H}_2$. If x is an arbitrary element in $\mathcal{H}_1$, then

$$U\mathfrak{M}(x)=\mathfrak{M}(Ux),$$

where the left side denotes the set $\{y: y=Uz,\ z\in\mathfrak{M}(x)\}$.

Proof. Let V be an isomorphism taking $\mathfrak{M}(x)$ to $\mathcal{L}_2(\rho_x)$. Then UV^{-1} is an isomorphism taking $\mathcal{L}_2(\rho_x)$ to $U\mathfrak{M}(x)$, i. e. $U\mathfrak{M}(x)$ is a cyclic subspace of $\mathcal{H}_2$. Since U is an isomorphism, the spectral type ρ_x is the maximal spectral type in $U\mathfrak{M}(x)$ and the spectral type of the element Ux is ρ_x, or $U\mathfrak{M}(x)=$ $=\mathfrak{M}(Ux)$. ▲

The necessary and sufficient conditions for the unitary equivalence of two self-adjoint operators are given by the following:

THEOREM 8. Two self-adjoint operators are unitary equivalent if and only if they have the same spectral type.

Proof. Let A_1 and A_2 be defined on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Suppose that $A_2=UA_1U^{-1}$, where U is an isomorphism taking $\mathcal{H}_1$ to $\mathcal{H}_2$. Let the sequence $\{z_k^{(1)}\}$ determine the spectral type of A_1. We define a new sequence $\{z_k^{(2)}\}$ by $z_k^{(2)}=Uz_k^{(1)}$, $k=\overline{1,N}$. From Lemma 3 it follows that $\mathfrak{M}(z_k^{(2)})=$ $=U\mathfrak{M}(z_k^{(1)})$, $k=\overline{1,N}$, so $\rho_{z_k^{(2)}}=\rho_{z_k^{(1)}}$, $k=\overline{1,N}$. That shows that unitary equivalent operators have the same spectral type.

Conversely, let $\{z_k^{(1)}\}$ and $\{z_k^{(2)}\}$ be sequences defining canonical representations of $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to A_1 and A_2 respectively and let $\rho_{z_k^{(1)}}\sim\rho_{z_k^{(2)}}$, $k=\overline{1,N}$. According to Theorem 3, there exists the isomorphism V_k taking $\mathfrak{M}(z_k^{(1)})$ to $\mathfrak{M}(z_k^{(2)})$, $k=\overline{1,N}$. Any element $x^{(1)}\in\mathcal{H}_1$ is an orthogonal sum: $x^{(1)}=\sum_{k=1}^{N}x_k^{(1)}$, $x_k^{(1)}\in\mathfrak{M}(z_k^{(1)})$, $k=\overline{1,N}$. We define the operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ by

$$Ux^{(1)}=\sum_{k=1}^{N}V_kx_k^{(1)}.$$

Evidently, U is an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $A_2=UA_1U^{-1}$. ▲

This theorem solves the problem of the complete system of unitary invariants, formulated in I.1. Let us notice that any set $\{\rho_k\}$ of spectral types ρ_k belonging to A is an unitary invariant, but only the sequence (I.6), for which $\mathcal{H}=\sum_{k=1}^{N}\mathfrak{M}(z_k)$ is the *complete* system of unitary invariants of A.

I.3. **The reducibility of self-adjoint operators**

In this section we shall give conditions under which some subspace $\mathfrak{M}$ of a separable Hilbert space $\mathscr{H}$ reduces a self-adjoint operator A. It is well known ([1], § 74) that $\mathfrak{M}$ reduces A if and only if $\mathfrak{M}$ reduces the corresponding resolution of the identity $\{E(t),\ a \leqslant t \leqslant b\}$. Besides, $\mathfrak{M}$ reduces A if and only if $\mathfrak{M}$ is an orthogonal sum of subspaces of $\mathscr{H}$, cyclic with respect to A. Indeed, if $\mathfrak{M}$ reduces A and if $x_1 \in \mathfrak{M}$, then $\mathfrak{M}(x_1) \subset \mathfrak{M}$. If $x_2 \in \mathfrak{M} \ominus \mathfrak{M}(x_1)$, since $\mathfrak{M} \ominus \mathfrak{M}(x_1)$ reduces A, then $\mathfrak{M}(x_2) \subset \mathfrak{M} \ominus \mathfrak{M}(x_1)$, etc. Continuing the procedure, we get the representation $\mathfrak{M} = \sum_k \oplus \mathfrak{M}(x_k)$. On the other hand, if $\mathfrak{M}$ is an orthogonal sum of subspaces, cyclic with respect to A, then $\mathfrak{M}$ reduces A, since any of those cyclic subspaces reduces A.

If we consider $\mathscr{H}$ and $\mathfrak{M}$ in canonical representation, it holds

THEOREM 9. Let $\mathscr{H} = \sum_{n=1}^{N} \oplus \mathfrak{M}(x_n)$ and $\mathfrak{M} = \sum_{n=1}^{N} \oplus \mathfrak{M}(u_n)$ be canonical representation of the space $\mathscr{H}$ and the subspace $\mathfrak{M}$ wich reduces A. Then the spectral type of the part of the operator A in $\mathfrak{M}$ is subordinated to the spectral type of the operator A in $\mathscr{H}$, in the following sense: $M \leqslant N$, $\rho_{u_n} < \rho_{z_n}$, $n = \overline{1, M}$, $\rho_{u_n} = 0$, $n = \overline{M+1, N}$.

Proof. Since ρ_{z_1} is the maximal spectral type in $\mathscr{H}$, it follows that $\rho_{u_1} < \rho_{z_1}$. We can assume that $\rho_{u_1} < \rho_{z_2}$ does not hold. Let us show that the assumption that $\rho_{u_2} < \rho_{z_2}$ is not true yelds to the contradiction. If $\rho_{u_2} < \rho_{z_2}$ does not hold, then there exists a spectral type τ, not identicly equel to zero, orthogonal to ρ_{z_2} and such that $\rho_{u_2} = \inf\{\rho_{u_2}, \rho_{z_2}\} + \tau$, i. e. $\mathfrak{M}(u_2) = \mathfrak{M}(u') \oplus \mathfrak{M}(u'')$, where $\rho_{u'} = \inf\{\rho_{u_2}, \rho_{z_2}\}$, $\rho_{u''} = \tau$. Let U be a unitary operator defined on $\mathfrak{M}$ such that $U\mathfrak{M}(u_1) \subset \mathfrak{M}(z_1)$ and $U\mathfrak{M}(u') \subset \mathfrak{M}(z_2)$. Since $\rho_{u''} \perp \rho_{z_n}$, $n = \overline{2, N}$, the subspace $U\mathfrak{M}(u'')$ can not belong to $\sum_{n=2}^{N} \oplus \mathfrak{M}(z_n)$. Hence, $U\mathfrak{M}(u'') \subset \mathfrak{M}(z_1)$. Since the subspaces $\mathfrak{M}(u_1)$ and $\mathfrak{M}(u'')$ are mutually orthogonal, the subspaces $U\mathfrak{M}(u_1)$ and $U\mathfrak{M}(u'')$ are mutually orthogonal cyclic subspaces of the cyclic subspace $\mathfrak{M}(z_1)$. Therefore the spectral types ρ_{u_1} and $\rho_{u''} = \tau$ are mutually orthogonal, which is in the contradiction with a fact that $\tau < \rho_{u_2} < \rho_{u_1} < \rho_{z_1}$. Hence $\rho_{u_2} < \rho_{z_2}$. The assumption that $\rho_{u_3} < \rho_{z_3}$ does not hold is, by the same reasoning, reduced to the contradiction e. t. c. ▲

In a special case, when A is a cyclic operator, we have:

THEOREM 10. Let $\mathscr{H}$ be a Hilbert space cyclic with respect to A. Let $\mathfrak{M}$ be a subspace of $\mathscr{H}$, and $P_{\mathfrak{M}}$ a projection operator of $\mathscr{H}$ onto $\mathfrak{M}$. The subspace $\mathfrak{M}$ reduces A if and only if $P_{\mathfrak{M}} = \chi(A)$[2], where the function $\chi(A)$ is mea-

[2] If $h \in \mathscr{L}_2(\rho)$, where ρ is the maximal spectral type of the cyclic operator A, then $h(A)$ is the operator defined by

$$h(A)\,x = \int^{b} h(t)E(dt)\,x, \quad x \in \mathscr{H}.$$

surable with respect to the spectral type of A and assumes only the values O and 1.

Proof. Since A is a cyclic operator, there exists an element $x_0 \in \mathcal{H}$ such that $\mathfrak{M}(x_0) = \mathcal{S}\{E(t)x_0, a \leqslant t \leqslant b\} = \mathcal{H}$. As the element $P_{\mathfrak{M}}x_0$ belongs to $\mathcal{H}$, it can be represented in the form $\int_a^b f(s)E(ds)x_0$, $f(s) \in \mathcal{L}_2(\rho_{x_0})$. Then $E(t)P_{\mathfrak{M}}x_0 = \int_a^b f(s)E(t)E(ds)x_0 = \int_a^t f(s)E(ds)x_0$. Since $P_{\mathfrak{M}}E(t)x_0 \in \mathcal{H}$, then $P_{\mathfrak{M}}E(t)x_0 = \int_a^b g(s)E(ds)E(t)x_0 = \int_a^t g(s)E(ds)x_0$, $g(s) \in \mathcal{L}_2(\rho_{x_0})$.

The assumption that $\mathfrak{M}$ reduces A implies $P_{\mathfrak{M}}E(t) = E(t)P_{\mathfrak{M}}$ for all $t \in [a, b]$. Therefore

$$\int_a^t g(s)E(ds)x_0 = \int_a^t f(s)E(ds)x_0 \qquad \text{(I.10)}$$

for all $t \in [a, b]$, or $g(s) = f(s)$ almost everywhere with respect to ρ_{x_0}. The equality (I.10) shows that the operators $P_{\mathfrak{M}}$ and $f(A)$ coincide on the dense set $\{E(t)x_0, a \leqslant t \leqslant b\}$, i. e. they coincide on $\mathcal{H}$. From $P_{\mathfrak{M}}^2 = P_{\mathfrak{M}}$ we get $(f(t))^2 = f(t)$ for all $t \in [a, b]$. That means that the function $f(t)$ can assume only the values 0 and 1.

Conversely, let $P_{\mathfrak{M}} = \chi(A)$. For any $x \in \mathcal{H}$, we have $P_{\mathfrak{M}}x = \int_a^b \chi(s)E(ds)x$ and $E(t)P_{\mathfrak{M}}x = \int_a^t \chi(s)E(ds)x$. Since $P_{\mathfrak{M}}E(t)x = \int_a^b \chi(s)E(ds)E(t)x = \int_a^t \chi(s)E(ds)x$ we conclude that $P_{\mathfrak{M}}E(t)x = E(t)P_{\mathfrak{M}}x$ for all $x \in \mathcal{H}$ and all $t \in [a, b]$.

COROLLARY 3. Any subspace $\mathfrak{M}$ of a cyclic subspace $\mathcal{H} = \mathfrak{M}(x_0)$ reducing A is a cyclic subspace with the generating element $P_{\mathfrak{M}}x_0$.

Chapter II

STOCHASTIC PROCESSES AS CURVES IN HILBERT SPACE

II.1. **Cramér representation**

Further on we assume that all random variables and stochastic processes under consideration are defined on a fixed probability space.

Let $\mathscr{H}$ be a set of complex-valued variables $x, y, \ldots$ with the finite second ordered moment: $E|x|^2 < +\infty$. Without loss of generality we shall assume that $Ex=0$. The set $\mathscr{H}$ becomes a Hilbert space if the scalar product is defined by $(x, y)=Ex\bar{y}$, $x, y \in \mathscr{H}$. The convergence in $\mathscr{H}$ is the convergence in norm: $x_n \to x$ as $n \to \infty$ means $\|x_n - x\| \to 0$ as $n \to \infty$. In terms of probability theory this is the convergence in quadratic mean: $E|x_n - x|^2 \to 0$ when $n \to \infty$.

Let $\{x(t),\ a \leqslant t \leqslant b\}$ be a second ordered process, i. e. $E|x(t)|^2 < +\infty$ for each $t \in [a, b]$ ($Ex(t)=0$ for each $t \in [a, b]$). The parameter t runs through the segment $[a, b]$ which can be finite or infinite. The process $\{x(t),\ a \leqslant t \leqslant b\}$ will be considered as a curve in Hilbert space $\mathscr{H}$.

Let $\mathscr{H}(x; t)$ be the smallest subspace spanned by the variables $x(s)$ for all $s \leqslant t$, i. e. $\mathscr{H}(x; t)$ is a Hilbert space consisting of limits in quadratic mean of all possible sequences

$$\left\{\sum_{k=1}^{n} c_{nk}\, x(t_{nk}),\quad n=1, 2, \ldots\right\},$$

where c_{nk} are complex numbers and $t_{nk} \leqslant t$. We set $\mathscr{H}(x)=\mathscr{H}(x; b)$.

In the sequel we assume that the following two condition are satisfied:

(A) The process $\{x(t)\}$ is continuous in quadratic mean. This condition can be replaced by the weaker one that $\{x(t)\}$ is left-side (or right-side) continuous in quadratic mean for each $t \in [a, b]$.

(B) $\bigcap_{t>a} \mathcal{H}(x; t) = 0$. This condition means that the process is regular or purely nondeterministic.

From (A) it immediately follows that $\mathcal{H}(x)$ is separable. For a base in $\mathcal{H}(x)$ we can choose a countable set $\{x(t_k)\}$, t_k is a rational number in $[a, b]$.

Denote by $E(t)$ (or, $E_x(t)$) projection operator from $\mathcal{H}(x)$ onto $\mathcal{H}(x; t)$. It is easy to see that $\{E(t),\ a \leqslant t \leqslant b\}$ is a resolution of the identity of a self-adjoint operator in $\mathcal{H}(x)$. Indeed, $E(s)E(t) = E(\min\{s, t\})$ for each $s, t \in [a, b]$, $E(t-0) = E(t)$ for each $t \in [a, b]$, $(a) = 0$ (because of the condition (B)) and $E(b) = \mathrm{I}$.

According to the theory of self-adjoint operators in a separable Hilbert space (Theorem 7, Ch. I), there are elements $z_1, z_2, \ldots, z_N$ in $\mathcal{H}(x)$, such that

$$\rho_{z_1} > \rho_{z_2} > \ldots > \rho_{z_N} \tag{II.1}$$

and

$$\mathcal{H}(x) = \sum_{n=1}^{N} \oplus \mathfrak{M}(z_n), \tag{II.2}$$

where N may be infinite.

The number N is minimal in the sense that for any set of elements $y_1, y_2, \ldots, y_M$ in $\mathcal{H}(x)$, satisfying $\mathcal{H}(x) = \sum_{n=1}^{M} \oplus \mathfrak{M}(y_n)$, holds $N \leqslant M$.

For an arbitrary element $z \in \mathcal{H}(x)$ we set $z(t) = E(t)z$. From the properties of the resolution of the identity it follows immediately that $\{z(t);\ a \leqslant t \leqslant b\}$ is the process with the orthogonal increments. The distribution function $F_z(t) = \| z(t) \|^2 = E|z(t)|^2$, $a \leqslant t \leqslant b$, induces a measure belonging to the spectral type ρ_z of the element z. In the sequel, without ambiguity, F_z will be used instead of ρ_z.

The stochastic integral

$$\int_a^b f(t)\, z(dt)\ ,$$

where $\{z(t),\ a \leqslant t \leqslant b\}$ is the process with orthogonal increments and $f \in \mathcal{L}_2(F_z)$, will be considered in the sense of Doob ([3]), Ch. IX). Hence $\mathcal{H}(z)$ is exactly the set of elements of the form $\int_a^b f(t)\, z(dt)$, $f \in \mathcal{L}_2(F_z)$. Since $z(dt) = E(dt)\, z$, it follows that $\mathcal{H}(z)$ coincides with a cyclic subspace $\mathfrak{M}(z)$ generated by the element z. The converse statement is true in the following sense:

LEMMA 1. Let $\{z(t),\ a \leqslant t \leqslant b\}$ be a process with orthogonal increments. Then there exists an element z_0 in $\mathcal{H}(z)$ such that $\mathcal{H}(z) = \mathfrak{M}(z_0)$.

Proof. Let $g \in \mathscr{L}_2(F_z)$ be a positive function almost everywhere with respect to F_z. We set

$$z_0 = \int_a^b g(s)\, z(ds).$$

Since

$$E(t)\, z_0 = \int_a^t g(s)\, z(ds),$$

it follows that measures induced by the distribution functions $\| E(t)\, z_0 \|^2$, $a \leqslant t \leqslant b$, and $F_z(t)$, $a \leqslant t \leqslant b$, are equivalent. Therefore each element $\int_a^b f(t)\, z(dt)$ of $\mathscr{H}(z)$, can be represented in the form

$$\int_a^b f(t) \frac{1}{g(t)} E(dt)\, z_0,$$

i. e. z_0 is the generating element of the cyclic subspace $\mathscr{H}(z)$. ▲

The equality (II.2) can be written in the form

$$\mathscr{H}(x) = \sum_{n=1}^{N} \oplus\, \mathscr{H}(z_n), \tag{II.3}$$

where $z_n(t)$, $a \leqslant t \leqslant b$, $n = \overline{1, N}$ are mutually orthogonal processes with orthogonal increments. Applying $E(t)$ on (II.3) we get

$$\mathscr{H}(x; t) = \sum_{n=1}^{N} \oplus\, \mathscr{H}(z_n; t) \qquad \text{for each } t \in [a, b]. \tag{II.4}$$

From (II.4) it follows

$$x(t) = \sum_{n=1}^{N} \int_a^t g_n(t, u)\, z_n(du) \qquad \text{for each } t \in [a, b]. \tag{II.5}$$

where

$$\sum_{n=1}^{N} \int_{a}^{t} |g_n(t, u)|^2 F_{z_n}(du) < +\infty \quad \text{for each } t \in [a, b].$$

DEFINITION 1. The equality (II.5) is called the *Cramér representation fo the process* $\{x(t), a \leqslant t \leqslant b\}$. The sequence (II.1) is called *the spectral type of the process* $\{x(t)\}$. The number N is called *the multiplicity of the process* $\{x(t)\}$. The spectral type of the process $\{x(t)\}$ will be denoted by ϱ_x or $\mathbf{F}_x$.

EXAMPLE 1. For a wide-sense stationary process $\{x(t), -\infty < t < +\infty\}$ there exists the well know Wold-Kolmogorov representation (see [3])

$$x(t) = \int_{-\infty}^{t} g(t-u)\, z(du), \qquad t \in (-\infty, +\infty),$$

where $\mathscr{H}(x; t) = \mathscr{H}(z; t)$ for each $t \in (-\infty, +\infty)$, $F_z(dt) = dt$ and $g(t) \in \mathscr{L}_2(F_z)$ at the interval $[-\infty, +\infty)$. Hence, the multiplicity of a wide-sense stationary process is $N = 1$ and the spectral type is equivalent to an ordinary Lebesgue measure over $(-\infty, +\infty)$.

In the proof of Lemma 1 we have shown that the multiplicity of a processs with orthogonal increments is $N = 1$ and its spectral type is $\mathbf{F}_z(t) = \| E(t) z_0 \|^2$, $a \leqslant t \leqslant b$.

The fundamental result of the aplication of the theory of spectral multiplicaty in Hilbert space to the theory of stochastic processes is the following:

THEOREM 1. (see [4]) For any given sequence of spectral types

$$\varrho : \rho_1 > \rho_2 > \ldots > \rho_N \tag{II.6}$$

(N may be infinite), there exists a stochastic process $\{x(t)\}$, continuous in quadratic mean, such that (II.6) is its spectral type.

In [4] it is shown that there exists even a harmonizable process $\{x(t)\}$ for which $\varrho_x = \varrho$.

Before proceeding to the proof of Theorem 1, let us make the following notice.

Let $s = \varphi(t)$, $a \leqslant t \leqslant b$, be a diferentiable, strictly increasing function. If we set $y(s) = x(t)$ for $s = \varphi(t)$, then the processes $\{x(t), a \leqslant t \leqslant b\}$ and $\{y(s), \varphi(a) \leqslant s \leqslant \varphi(b)\}$ have equal spectral types in the following sense. Let

$$\mathbf{F}_x : F_{x_1} > F_{x_2} > \ldots > F_{xN_1}$$

and

$$\mathbf{F}_y : F_{y1} > F_{y2} > \ldots > F_{yN_2}$$

be spectral types of $\{x(t)\}$ and $\{y(s)\}$ respectively. Since $\mathcal{H}(y; s) = \mathcal{H}(x; t)$ for $s = \varphi(t)$, $t \in [a, b]$, we have $N_1 = N_2$ and $F_{yn}(s) = F_{xn}(t)$ for $s = \varphi(t)$, $t \in [a, b]$, $n = \overline{1, N_1}$. Therefore we suppose, without loss of generality, that the distribution functions, inducing the spectral type in (II.6), are defined on the segment [0, 1].

Proof of Theorem 1. The proof essentially depends on the existence of disjoint subsets A_1 $A_2, \ldots, A_N$ of [0, 1] $(\bigcup_{n=1}^{N} A_n = [0, 1])$, such that for each n, $n = \overline{1, N}$, and α and β, $0 \leqslant \alpha < \beta \leqslant 1$, the ordinary Lebesgue measure of $A_n \cap [\alpha, \beta]$ is positive. One construction of these sets is given in [5].

According to the Daniell-Kolmogorov theorem, there exist mutually orthogonal processes $\{z_n(t),\ 0 \leqslant t \leqslant 1\}$, $n = \overline{1, N}$ with orthogonal increments for which $F_{z_n}(t) = E\,|\,z_n(t)\,|^2 = F_n(t)$, $0 \leqslant t \leqslant 1$, $n = \overline{1, N}$, where F_n is the distribution function inducing the spectral type ϱ_n in (II.6).

Let the function $\chi_n(t)$, $0 \leqslant t \leqslant 1$, be the indicator-function of the set A_n. We shall first show that the process $\{y_n(t),\ 0 \leqslant t \leqslant 1\}$, defined by

$$y_n(t) = \int_0^t \chi_n(u)\, z_n(u)\, du, \qquad t \in [0, 1],$$

has the spectral type F_n. Obviously, $\mathcal{H}(y_n; t) \subset \mathcal{H}(z_n; t)$ for each $t \in [0, 1]$. On the other hand, for each $t \in A_n$ we have

$$y_n'(t) = \chi_n(t)\, z_n(t) = z_n(t).$$

Since A_n is everywhere dense in [0, 1], we conclude that $\mathcal{H}(y_n'; t) = \mathcal{H}(z_n; t)$ for each $t \in [0, 1]$. As $\mathcal{H}(y_n'; t)$ is always in $\mathcal{H}(y_n; t)$ it follows that $\mathcal{H}(y_n; t) = \mathcal{H}(z_n; t)$ for each $t \in [0, 1]$, i. e. $F_{y_n} = F_n$.

The processes $\{y_n(t)\}$, $n = \overline{1, N}$ are, obviously, mutually orthogonal.

We now define the process $\{x(t),\ 0 \leqslant t \leqslant 1\}$ by

$$x(t) = \sum_{n=1}^{N} \frac{1}{n} y_n(t) = \sum_{n=1}^{N} \frac{1}{n} \int_a^t \chi_n(u)\, z_n(u)\, du, \quad t \in [0, 1] \tag{II.7}$$

(factor $\frac{1}{n}$ insures the convergence of the series in the case $N = \infty$). Obviousiy

$$\mathcal{H}(x;t)\subset\sum_{n=1}^{N}\oplus\mathcal{H}(y_n;t)=\sum_{n=1}^{N}\oplus\mathcal{H}(z_n;t)$$

for each $t\in[0, 1]$.

Any fixed $t\in[0, 1]$ belongs to one and only one set A_n $n=\overline{1, N}$. Let $t\in A_k$. From (II.7) we have

$$x'(t)=\frac{1}{k}z_k(t). \tag{II.8}$$

Since $\mathcal{H}(x';t)\subset\mathcal{H}(x;t)$ for each $t\in[0, 1]$ and as A_k is everywhere dense in [0, 1], from (II.8) we get

$$\mathcal{H}(y_k;t)=\mathcal{H}(z_k;t)\subset\mathcal{H}(x;t)$$

for each $t\in[0, 1]$ and each $k=\overline{1, N}$. Hence

$$\mathcal{H}(x;t)=\sum_{n=1}^{N}\oplus\,\mathcal{H}(z_n;t)$$

for each $t\in[0, 1]$.

The last equality shows that the process $\{x(t)\}$ has the given spectral type (II.6).

The correlation function of the process $\{x(t)\}$ is

$$r(s, t)=Ex(s)\overline{x(t)}=\sum_{n=1}^{N}\frac{1}{n^2}\int_0^s\int_0^t \chi_n(u)\chi_n(v)F_n(min\{u, v\})\,du\,dv.$$

As $r(s, t)$, $0\leqslant s, t\leqslant 1$ is continuous, the process $\{x(t)\}$ is continuous in quadratic mean. ▲

REMARK 1. The following simple construction can be applied for obtaining the process $\{x(t), 0\leqslant t\leqslant 1\}$ with a given spectral type (II.6) (see [9])

$$x(t)=\sum_{n=1}^{N}\frac{1}{n}\int_0^t g_n(t, u)\,z_n(du), \qquad t\in[0, 1], \tag{II.9}$$

where

$$g_n(t,u)=\begin{cases}1, \text{ if } t\in A_n,\\ 0, \text{ otherwise.}\end{cases}$$

The same reasoning as in the proof of Theorem 1, Ch. II, gives

$$\mathcal{H}(x;t)=\sum_{n=1}^{N}\oplus\,\mathcal{H}(z_n;t),$$

where, instead of the metric density of the sets A_n, $n=\overline{1,N}$, it is enough to assume that they are everywhere dense in [0,1]. However the process $\{x(t)\}$, defined by (II.9), is not continuous in quadratic mean because its correlation function

$$r(s,t)=\sum_{n=1}^{N}\frac{1}{n^2}\int_{o}^{\min\{s,t\}} g_n(s,u)g_n(t,u)\,F_n(du)=$$

$$=\begin{cases}F_n(\min\{s,t\}), \text{ if } s \text{ and } t \text{ are in the same set } A_n\ n=\overline{1,N},\\ 0, \text{ otherwise,}\end{cases}$$

is not continuous.

REMARK 2. The process $\{x(t),\ 0\leqslant t\leqslant 1\}$ defined by

$$x(t)=\sum_{n=1}^{N}\frac{1}{n}\int_{o}^{t}\left[\int_{u}^{t}(t-v)\,\chi_n(v)\,dv\right]z_n(du),$$

is continuous and has the spectral type (II.6). This construction is very similar to that in [4]. After showing

$$\mathcal{H}(z_n;t)\subset\mathcal{H}(x'';t)$$

for each $t\in[0,1]$ and each $n=\overline{1,N}$, the proof is analogous to the proof of Theorem 1, Ch. II.

THEOREM 2. ([4]) The spectral type ϱ_x of the process $\{x(t)\}$ is uniquely determined by its correlation function $r(s,t)$.

Proof. We shall prove that two arbitrary processes

$$\{x(t),\ a \leqslant t \leqslant b\} \text{ and } \{y(t),\ a \leqslant t \leqslant b\}$$

with the same correlation function $r(s, t)$ have the same spectral type. We define the operator U by $y(t) = Ux(t)$ for each $t \in [a, b]$ and extend it by linearity to $\mathscr{H}(x)$. Since

$$r(s, t) = (y(s),\ y(t)) = (Ux(s),\ Ux(t)) = (x(s),\ x(t)),$$

$(a \leqslant s,\ t \leqslant b)$ it follows that U is an isomorphism of $\mathscr{H}(x)$ onto $\mathscr{H}(y)$. From the definition of the operator U it follows $\mathscr{H}(y; t) = U\mathscr{H}(x; t)$, i. e. $E_y(t)\,U = UE_x(t)$ for each $t \in [a, b]$. According to the theorem of unitary invariants of self-adjoint operators (Theorem 8, Ch. I) we conclude that $\varrho_y = \varrho_x$. ▲

The converse does not hold, i. e. if two processes have the same spectral type, their correlation functions need not coincide. For example, for a given process $\{x(t)\}$ the process $\{y(t)\}$ is defined by $y(t) = f(t)\,x(t)$, where $f(t)$ is a non-random function such that $0 \leqslant m \leqslant |f(t)| \leqslant M$ for all $t \in [a, b]$. Then $\mathscr{H}(y; t) = \mathscr{H}(x; t)$ for each $t \in [a, b]$, i.e. $E_y(t) = E_x(t)$ for each $t \in [a, b]$ and therefore $\varrho_y = \varrho_x$. On the other hand

$$r_y(s, t) = f(s)\overline{f(t)}\,r_x(s, t) \neq r_x(s, t),\quad a \leqslant s, t \leqslant b.$$

Theorem 2 introduces the problem of expressing the spectral type ϱ_x of the process $\{x(t)\}$ in the terms of its correlation function $r(s, t)$. Before considering this problem we shall give some shorter notations and one definition.

Let $\{\mathbf{z}(t) = (z_n(t))_{n=\overline{1,N}},\ a \leqslant t \leqslant b\}$ be a stochastic process considered as a vector-column, where $\{z_n(t),\ a \leqslant t \leqslant b\},\ n = \overline{1, N}$, are mutually orthogonal processes with orthogonal increments. Set $\mathbf{F}(t) = E\,\mathbf{z}(t)\,\mathbf{z}^\star(t) = (F_{jk}(t))_{k=\overline{1,N}}^{j=\overline{1,N}}$ where $\mathbf{z}^\star(t)$ denotes the transposed matrix of $\mathbf{z}(t)$. The matrix function $\mathbf{F}(t)$, $a \leqslant t \leqslant b$, has non-zero elements only on the principal diagonal and we denote them by $F_{z_n}(t) = F_{nn}(t) = E\,|z_n(t)|^2,\ n = \overline{1, N}$.

Let $\mathscr{L}_2(\mathbf{F})$ be the Hilbert space of all complex-valued vector-row functions $\mathbf{f}(t) = (f_n(t))^{n=\overline{1,N}}$, $a \leqslant t \leqslant b$, for which

$$\int_a^b \mathbf{f}(u)\,\mathbf{F}(du)\,\mathbf{f}^\star(u) < +\infty.$$

The scalar product in $\mathscr{L}_2(\mathbf{F})$ is defined by

$$<\mathbf{f}_1, \mathbf{f}_2> = \int_a^b \mathbf{f}_1(t)\,\mathbf{F}(dt)\,\mathbf{f}_2^\star(t),\ \mathbf{f}_1, \mathbf{f}_2 \in \mathscr{L}_2(\mathbf{F}).$$

DEFINITION 2. The family of functions $\{\mathbf{g}(t, u),\ a\leqslant u\leqslant t\}$, where the parameter $t\in[a, b]$ is *complete* in $\mathscr{L}_2(\mathbf{F})$ if, for any fixed t, from

$$\int_u^s \mathbf{g}(s, u)\,\mathbf{F}(du)\,\mathbf{f}^\star(u)=0 \quad \text{for all } s\in[a, t],$$

it follows that $\mathbf{f}(u)=0$, $a\leqslant u\leqslant t$, almost everywhere with respect to $\mathbf{F}$ (i. e. $\int_a^t \mathbf{f}(u)\,\mathbf{F}(du)\,\mathbf{f}^\star(u)=0$).

The spectral type (II.6) of the process $\{x(t),\ a\leqslant t\leqslant b\}$ can be written in terms of matrix function

$$\mathbf{F}(t)=\begin{pmatrix} F_1(t) & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & F_N(t)\end{pmatrix} \tag{II.6'}$$

where the distribution function $F_n(t)$, $a\leqslant t\leqslant b$, induces the measure which belongs to the spectral type ρ_n, $n=\overline{1, N}$ in (II.6). Hence the Cramér representation of the process $\{x(t),\ a\leqslant t\leqslant b\}$ with the spectral type (II.6') can be written in the form

$$x(t)=\int_a^t \mathbf{g}(t, u)\,\mathbf{z}(du), \quad t\in[a, b], \quad \mathbf{g}(t, u)\in\mathscr{L}_2(\mathbf{F}), \tag{II.10}$$

where $\mathbf{F}(t)=E\,\mathbf{z}(t)\,\mathbf{z}^\star(t)$.

THEOREM 3. The stochastic process $\{x(t),\ a\leqslant t\leqslant b\}$ with the correation function $r(s, t)$, $a\leqslant s, t\leqslant b$ has the spectral type (II.6') if and only if

$$r(s, t)=\int_a^{\min\{s, t\}} \mathbf{g}(s, u)\,\mathbf{F}(du)\,\mathbf{g}^\star(t, u), \quad a\leqslant s, t\leqslant b, \tag{II.11}$$

where the family of functions $\{\mathbf{g}(t, u)$, the parameter $t\in[a, b]\}$ is complete in $\mathscr{L}_2(\mathbf{F})$.

Proof. If $\mathbf{F}$ is the spectral type of the process $\{x(t)\}$, from the Cramér representation (II.10) it follows that

$$r(s,t)=E\,x(s)\,\overline{x(t)}=\left(\int_a^s \mathbf{g}(s,u)\,\mathbf{z}(du),\ \int_a^t \mathbf{g}(t,u)\,\mathbf{z}(du)\right)=$$

$$=\int_a^{\min\{s,t\}} \mathbf{g}(s,u)\,E\,\mathbf{z}(du)\,\mathbf{z}^\star(du)\,\mathbf{g}^\star(t,u)=$$

$$=\int_a^{\min\{s,t\}} \mathbf{g}(s,u)\,\mathbf{F}(du)\,\mathbf{g}^\star(t,u).$$

Let us show that the family of functions $\{\mathbf{g}(t,u)$, the parameter $t\in[a,b]\}$ is complete in $\mathscr{L}_2(\mathbf{F})$. Since (II.10) is the Cramér representation, any element y from $\mathscr{H}(x;t)=\sum_{n=1}^{N}\oplus\mathscr{H}(z_n;t)$ (t is any fixed point in $[a,b]$) is of the form

$$y=\int_a^t \mathbf{f}(u)\,\mathbf{z}(du),\quad \mathbf{f}\in\mathscr{L}_2(\mathbf{F}).$$

The fact that, if $(x(s),y)=0$ for all $s\in[a,t]$, then $y=0$, can be written as: if $\int_a^s \mathbf{g}(s,u)\,\mathbf{F}(du)\,\mathbf{f}^\star(u)=0$ for all $s\in[a,t]$, then $\mathbf{f}(u)=0$ almost everywhere with respect to $\mathbf{F}$ on the segment $[a,t]$. That means that the family $\{\mathbf{g}(t,u)\}$ is complete in $\mathscr{L}_2(\mathbf{F})$.

Conversely, let $\{\mathbf{z}(t)=(z_n(t))_{n=\overline{1,N}},\ a\leqslant t\leqslant b\}$ be a stochastic process for which $E\,\mathbf{z}(t)\,\mathbf{z}^\star(t)=\mathbf{F}(t)$, $a\leqslant t\leqslant b$, and $\mathbf{F}(t)$ is from (II.11). We set

$$x(t)=\int_a^t \mathbf{g}(t,u)\,\mathbf{z}(du),\quad t\in[a,b],\tag{II.12}$$

with $\{\mathbf{g}(t,u)\}$ from (II.11). Let us show that (II.12) is the Cramér representation of the process $\{x(t)\}$. It is sufficient to show that $\mathscr{H}(x;t)=\sum_{n=1}^{N}\oplus\mathscr{H}(z_n;t)$ for each $t\in[a,b]$. Suppose that the last equality does not hold. From (II.12) it follows that $\mathscr{H}(x;t)\subset\sum_{n=1}^{N}\oplus\mathscr{H}(z_n;t)$. Therefore there exists a non-zero element $y\in\sum_{n=1}^{N}\oplus\mathscr{H}(z_n;t)$ orthogonal to $x(s)$ for all $s\in[a,t]$. The element y can be written in the form

$$y=\int_a^t \mathbf{f}(u)\,\mathbf{z}(du),\ \ \mathbf{f}\in\mathscr{L}_2(\mathbf{F}).$$

So, we have

$$(x(s),y)=\int_a^s \mathbf{g}(s,u)\,\mathbf{F}(du)\,\mathbf{f}^\star(u)=0 \text{ for all } s\in[a,t]$$

and

$$\|y\|^2=\int_a^t \mathbf{f}(u)\,\mathbf{F}(du)\,\mathbf{f}^\star(u)>0,$$

which contradicts to the assumption that the family of the functions $\{\mathbf{f}(t,u)$, the parameter $t\in[a,b]\}$ is complete in $\mathscr{L}_2(\mathbf{F})$.

Finally, the correlation function $r(s,t)$ of the process $\{x(t)\}$ defined by (II.12) is given with (II.11). ▲

EXAMPLE 2. Let the disjoint sets A_n, $n=\overline{1,N}$ be everywhere dense in [0,1] and $\bigcup_{n=1}^{N} A_n=[0,1]$. We set

$$g_n(t,u)=\begin{cases}1, \text{ if } u\in[0,t], & t\in A_n,\\ 0, \text{ otherwise}\end{cases}$$

(see Remark 1). It is easy to see that the family of the functions $\{\mathbf{g}(t,u)=(g_n(t,u))^{n=\overline{1,N}}$, the parameter $t\in[0,1]\}$ is complete in $\mathscr{L}_2(\mathbf{F})$, where

$$\mathbf{F}(t)=\begin{pmatrix} t & \dots & 0\\ & \ddots & \\ 0 & \dots & t\end{pmatrix},\ \ 0\leqslant t\leqslant 1. \tag{II.13}$$

Indeed, for each $s\in A_k\cap[0,1]$ and for any vector-row function $\mathbf{f}=(f_n)^{n=\overline{1,N}}\in\mathscr{L}_2(\mathbf{F})$ we have

$$\int_0^s \mathbf{g}(s,u)\,\mathbf{F}(du)\,\mathbf{f}^\star(u)=\int_0^s f_k(u)\,du.$$

If $\int\limits_0^s f_k(u)\,du = 0$ for all $s \in A_k \cap [0, t]$ and as the set A_k is everywhere dense in $[0,1]$, it follows that $f_k(u) = 0$ almost everywhere on the segment $[0,t]$. Hence the function

$$r(s, t) = \int\limits_0^{\min\{s, t\}} \mathbf{g}(s, u)\, \mathbf{F}(du)\, \mathbf{g}^\star(t, u) = \begin{cases} \min\{s, t\}, & \text{if } s \text{ and } t \text{ are in the same set } A_n,\ n = \overline{1, N}, \\ 0, & \text{otherwise} \end{cases} \tag{II.13}$$

is the correlation function of a process whose spectral type is (II.13). An example of a process $\{x(t),\ 0 \leqslant t \leqslant 1\}$ with correlation function (II.13') is

$$x(t) = \int\limits_0^t \mathbf{g}(t, u)\, \mathbf{w}(du), \quad t \in [0,1),$$

where $\mathbf{w}(t) = (\mathrm{w}_n(t))_{n=\overline{1,N}}$ and the processes $\{\mathrm{w}_n(t),\ 0 \leqslant t \leqslant 1\}$ are independent Wiener processes.

RFMARK 3. The analyses of Theorem 3, Ch. II, shows that this theorem holds under some more general conditions in the following sense: Let

$$\mathbf{G}(t) = (G_{jk}(t))_{k=\overline{1,M}}^{j=\overline{1,M}}$$

be a matrix function with non-zero elements $G_{nn}(t)$ $n = \overline{1, M}$ only on the principal diagonal and $G_{nn}(t)$ be distribution function on $[a, b]$ (M may be infinite). Suppose that the function $\mathbf{h}(t, u)$, $a \leqslant u \leqslant t$, for each $t \in [a, b]$ belongs to $\mathscr{L}_2(\mathbf{G})$ and that the process $\{x(t),\ a \leqslant t \leqslant b\}$ is defined by

$$x(t) = \int\limits_a^t \mathbf{h}(t, u)\, \mathbf{z}(du), \tag{II.14}$$

where $E\mathbf{z}(t)\mathbf{z}^\star(t) = \mathbf{G}(t)$, i. e. $\mathbf{z}(t) = (z_n(t))_{n=\overline{1,M}}$ with $E|z_n(t)|^2 = G_{nn}(t)$, $n = \overline{1, M}$. Then

$$\mathscr{H}(x; t) = \sum_{n=1}^{M} \oplus\, \mathscr{H}(z_n; t), \qquad \text{for each } t \in [a\, b] \tag{II.15}$$

if and only if the family of the functions $\{\mathbf{h}(t, u)$, the parameter $t \in [a, b]\}$ is complete in $\mathscr{L}_2(\mathbf{G})$. The representation (II.14) with the condition (II.15) is called *proper canonical* (see [9]). We shall consider this in the next section.

II.2. **The fully submitted process**

In the present section we discuss the relations between canonical and proper canonical representation ([9]), fully submitted process ([17]) and reducibilitty of the resolution of the identity in certain subspaces.

DEFINITION 3. ([13], [17]) The process $\{y(t),\ a \leqslant t \leqslant b\}$ is *submitted* to the process $\{x(t),\ a \leqslant t \leqslant b\}$ if $\mathscr{H}(y; t) \subset \mathscr{H}(x; t)$ for each $t \in [a, b]$.

DEFINITION 4. ([17]) The process $\{y(t),\ a \leqslant t \leqslant b\}$ is *fully submitted* to the process $\{x(t),\ a \leqslant t \leqslant b\}$ if $\mathscr{H}(y; t) \subset \mathscr{H}(x; t)$ and $\mathscr{H}(y) \ominus \mathscr{H}(y; t) \subset$ $\subset \mathscr{H}(x) \ominus \mathscr{H}(x; t)$ for each $t \in [a, b]$.

EXAMPLE 3. In the Cramér representation

$$x(t) = \int_a^t \sum_{n=1}^{N} g_n(t, u)\, z_n(du), \quad t \in [a, b],$$

any process $\{z_n(t)\}$, $n = \overline{1, N}$ is fully submitted to the process $\{x(t)\}$.

EXAMPLE 4. We give an example of a process submitted to a given process, but not fully submitted. Let $\{w(t),\ 0 \leqslant t \leqslant 1\}$ be a Wiener process. The process $\{w_1(t),\ 0 \leqslant t \leqslant 1\}$ defined by

$$w_1(t) = \int_o^t \left(2 - 3\frac{u}{t}\right) w(du), \quad t > 0, \quad w_1(o) = 0,$$

is also a Wiener process, submitted to $\{w(t)\}$. If $\{w_1(t)\}$ is fully submitted to $\{w(t)\}$, then for each $v < t < s$

$$w_1(s) - w_1(t) \in \mathscr{H}(w_1) \ominus \mathscr{H}(w_1; t) \subset \mathscr{H}(w) \ominus \mathscr{H}(w; t) \perp w(v),$$

or

$$\big(w_1(s) - w_1(t), w(v)\big) = \frac{3}{2} v^2 \left(\frac{1}{t} - \frac{1}{s}\right)$$

and therefore, $\{w_1(t)\}$ is not fully submitted to $\{w(t)\}$.

DEFINITION 5. ([9]) Let $\{\mathbf{w}(t)=(w_n(t))_{n=\overline{1,M}},\ a\leqslant t\leqslant b\}$ be a vector column stochastic process, where $\{w_n(t)\}$, $n=\overline{1,M}$ are mutually orthogonal processes with orthogonal increments (M may be infinite). Let the process $\{y(t),\ a\leqslant t\leqslant b\}$ be defined by

$$y(t)=\int_a^t \mathbf{h}(t,u)\,\mathbf{w}(du),\qquad t\in[a,b],\tag{II.16}$$

where $\mathbf{h}(t,u)$, $a\leqslant u\leqslant t$ for each $t\in[a,b]$, belongs to $\mathscr{L}_2(\mathbf{F_w})$ ($\mathbf{h}(t,u)=0$ if $u>t$). The representation (II.16) is the *canonical representation* of the process $\{y(t)\}$ if for all $s\leqslant t$, $s, t\in[a,b]$ holds

$$P_{\mathscr{H}(y;\,s)}\,y(t)=\int_a^s \mathbf{h}(t,u)\,\mathbf{w}(du).$$

EXAMPLE. 5 Let $\{z(t),\ a\leqslant t\leqslant b\}$ be the process with orthogonal increments and $f(t)$. $a\leqslant t\leqslant b$, be an arbitrary function in $\mathscr{L}_2(\rho_z)$. Th representation

$$y(t)=\int_a^t f(u)\,z(du),\qquad a\leqslant t\leqslant b,$$

of the process $\{y(t),\ a\leqslant t\leqslant b\}$ is canonical. Indeed, as $y(t)-y(s)$ is orthogonal to $\mathscr{H}(y;s)$ for every $s\leqslant t$, we have

$$P_{\mathscr{H}(y;\,s)}\,y(t)=\int_a^s f(u)\,z(du).$$

DEFINITION 6. ([9]) The representation

$$y(t)=\int_a^t \mathbf{h}(t,u)\,\mathbf{w}(du),\qquad t\in[a,b],\tag{II.17}$$

of the process $\{y(t),\ a\leqslant t\leqslant b\}$ is *a proper canonical representation* if

$$\mathcal{H}(y;t)=\sum_{n=1}^{M}\oplus\mathcal{H}(\mathrm{w}_n;t)$$

for each $t\in[a,b]$.

DEFINITION 7. ([4], [17]) The process $\{\mathbf{w}(t),\ a\leqslant t\leqslant b\}$ in the proper canonical representation (II.17) of the process $\{y(t),\ a\leqslant t\leqslant b\}$ is *the innovation proces of* $\{y(t)\}$.

EXAMPLE 6. Every Cramér representation is a proper canonical one.

It is evident that every proper canonical representation is the canonical one. The converse need not hold. For instance, if the function $f(t),\ a\leqslant t\leqslant b$, in example 5, is equal to zero on the set of positive ρ_z-measure, then $\mathcal{H}(y;t)$ is a proper subspace of $\mathcal{H}(z;t)$ for at least one $t\in[a,b]$.

We wish to underline the fact that Theorem 7, Ch. I, shows how we can get, starting from any innovation process $\{\mathbf{w}(t)\}$ of the process $\{y(t)\}$, the innovation process $\{\mathbf{z}_y(t)\}$ of $\{y(t)\}$ in the Cramér representation of $\{y(t)\}$.

THEOREM 4. Let $\{x(t),\ a\leqslant t\leqslant b\}$ and $\{y(t),\ a\leqslant t\leqslant b\}$ be two processes and let $\{\mathbf{z}(t),\ a\leqslant t\leqslant b\}$ be an innovation process of $\{x(t)\}$. Then the following three statements are equivalent:

(a) The process $\{y(t)\}$ is fully submitted to the process $\{x(t)\}$;

(b) There exists the function $\mathbf{h}(t,u)\in\mathcal{L}_2(\varrho_z),\ a\leqslant u\leqslant t,\ a\leqslant t\leqslant b$, such that the representation

$$y(t)=\int_a^t\mathbf{h}(t,u)\,\mathbf{z}(du),\quad t\in[a,b],\tag{II.18}$$

is a canonical representation of $\{y(t)\}$.

(c) For each $t\in[a,b]$ the subspace $\mathcal{H}(y;t)$ reduces the resolution of the identity $\{E_x(s),\ a\leqslant s\leqslant b\}$, defined by $\{x(t)\}$.

Proof. We shall first show that (a) and (b) are equivalent. From (II.18) it follows that $\mathcal{H}(y;t)\subset\mathcal{H}(\mathbf{z};t)=\mathcal{H}(x;t)$ for each $t\in[a,b]$. The space $\mathcal{H}(y)\ominus\ominus\mathcal{H}(y;t)$ ts the smallest space spanned by the variables $y(t+h)-P_{\mathcal{H}(y;t)}y(t+h)$ for all $h\in[0,b-t]$. From the canonical representation (II.18) it follows that

$$y(t+h)-P_{\mathcal{H}(y;t)}y(t+h)=$$

$$=\int_t^{t+h}\mathbf{h}(t+h,u)\,\mathbf{z}(du)\in\mathcal{H}(x;t+h)\ominus\mathcal{H}(x;t).$$

Hence $\mathcal{H}(y)\ominus\mathcal{H}(y;t)\subset\mathcal{H}(x)\ominus\mathcal{H}(x;t)$ for each $t\in[a;b]$, i. e. the process $\{y(t)\}$ is fully submitted to the process $\{x(t)\}$.

Conversely, let $\{y(t)\}$ be fully submitted to $\{x(t)\}$. In order to show the existence of the canonical representation (II.18), it is sufficient to show that for all $s\leqslant t$, $s, t\in[a,b]$ holds

$$P_{\mathcal{H}(y;\,s)}y(t)=E_x(s)\,y(t).$$

The last equality follows immediately from

$$E_x(s)\,y(t)=P_{\mathcal{H}(x;\,s)}y(t)=$$

$$=P_{\mathcal{H}(x;\,s)}[P_{\mathcal{H}(y;\,s)}y(t)+P_{\mathcal{H}(y)\ominus\mathcal{H}(y;\,s)}y(t)]=$$

$$=P_{\mathcal{H}(y;\,s)}y(t)+0.$$

Now we shall prove the equivalence of (b) and (c). For all $s<t$, $s, t\in[a,b]$, we have from the canonical representation (II.18) that

$$E_x(s)\,y(t)=\int_a^s \mathbf{h}(t,u)\,\mathbf{z}(du)=P_{\mathcal{H}(y;\,s)}y(t)\in\mathcal{H}(y;s),$$

which means that $\mathcal{H}(y;t)$ is invariant with respect to $E_x(s)$, i. e. $\mathcal{H}(y;t)$ reduces $\{E_x(s),\ a\leqslant s\leqslant b\}$.

Conversely, if $\mathcal{H}(y;t)$ reduces $\{E_x(s)\}$ then, according to the section I.3, there is an innovation process $\{\mathbf{z}(t)=(z_n(t))_{n=\overline{1,M}}\}$ of $\{x(t)\}$ such that

$$\mathcal{H}(y;t)=\sum_{n=1}^{\tilde{M}}\oplus\mathcal{H}(z_n;t),\ \text{for each}\ t\in[a,b],\ \tilde{M}\leqslant M.$$

As $\mathcal{H}(x;t)\ominus\mathcal{H}(y;t)$ also reduces $\{E_x(s)\}$, we have

$$\mathcal{H}(x;t)\ominus\mathcal{H}(y;t)=\sum_{n=\tilde{M}+1}^{M}\oplus\mathcal{H}(z_n;t).$$

Hence the canonical representation of the process $\{y(t)\}$ is

$$y(t)=\int_a^t \mathbf{h}(t,u)\,\mathbf{z}(du),\quad t\in[a,b].$$

Actually, the functions $h_n(t,u)$, $n=\overline{\tilde{M}+1, M}$ in $\mathbf{h}(t,u)=(h_n(t,u))^{n=\overline{1,M}}$ are zero for all $a \leqslant u \leqslant t$ and each $t \in [a,b]$. ▲

Concerning the relation between a canonical and a proper canonical representation in the case when $M=1$ we can prove, by use of Th. 10, Ch. I, the following

THEOREM 5. (see [9]) Let

$$y(t)=\int_a^t h(t,u)\,w(du), \quad t \in [a,b],$$

be a canonical representation of the process $\{y(t)\}$. Then there exists a ρ_w-measurable function $\chi(u)$, assuming the values O and 1, and the process $\{\tilde{w}(t), a \leqslant t \leqslant b\}$ with orthogonal increments, defined by

$$\tilde{w}(t)=\int_a^t \chi(u)\,w(du), \quad t \in [a,b],$$

such that the representation

$$y(t)=\int_a^t h(t,u)\,\tilde{w}(du), \quad t \in [a,b], \tag{II.19}$$

is the proper canonical representation of the process $\{y(t)\}$.

Proof. According to Th. 4, the subspace $\mathcal{H}(y)$ reduces the resolution of the identity $\{E(t)\}$ in a cyclic space $\mathcal{H}(w)$. Let w_o be a generating element of $\mathcal{H}(w)$, such that $w(t)=E(t)\,w_o$, $t \in [a,b]$. According to Th. 10 and Corollary 2, Ch. I,

$$\tilde{w}_0=P_{\mathcal{H}(y)}\,w_o=\int_a^b \chi(u)\,E(du)w_o=\int_a^b \chi(u)\,w(du)$$

is a generating element of $\mathcal{H}(y)$. If we set

$$\tilde{w}(t)=E(t)\tilde{w}_o=\int_a^t \chi(u)\,w(du),$$

then (II.19) is the proper canonical representation of the process $\{y(t)\}$. ▲

When $M>1$ the situation is rather complicated. First of all, holds

THEOREM 6. The representation

$$y(t)=\sum_{n=1}^{M}\int_{a}^{t} h_n(t, u)\, w_n(du), \quad t\in[a, b], \tag{II.20}$$

is the canonical one if and only if for each n, $n=\overline{1, M}$, rhe represetation

$$y_n(t)=\int_{a}^{t} h_n(t, u)\, w_n(du), \quad t\in[a, b],$$

is the canonical representation of the process $\{y_n(t)\}$.

Proof. The space $\mathcal{H}(y; s)$ is the smallest subspace spanned by the elements $\sum_{n=1}^{M} y_n(u)$ when $u\leqslant s$:

$$\mathcal{H}(y; s)=\mathcal{S}\left\{\sum_{n=1}^{M} y_n(u), \quad u\leqslant s\right\}=\sum_{n=1}^{M}\dot{\oplus}\,\mathcal{L}\{y_n(u) \quad u\leqslant s\}. \tag{II.21}$$

Let us notice that $\mathcal{S}\{y_n(u),\ u\leqslant s\}=\mathcal{H}(y_n; s)$, but in a general case

$$\sum_{n=1}^{M}\dot{\oplus}\,\mathcal{S}\{y_n(u), \quad u\leqslant s\}\subset\sum_{u=1}^{M}\oplus\,\mathcal{H}(y_n; s)$$

which we have assined by introducing $\dot{\oplus}$. Hence

$$P_{\mathcal{H}(y;s)}\, y(t)=P_{\sum_{n=1}^{M}\dot{\oplus}\,\mathcal{S}\{y_n(u),\, n\leqslant s\}}\sum_{k=1}^{M} y_k(t)=\sum_{n=1}^{M} P_{\mathcal{H}(y_n;s)}\, y_n(t), \quad s<t, \tag{II.22}$$

If (II.20) is the canonical representation then for each n, $n=\overline{1, M}$,

$$P_{\mathcal{H}(y_n;s)}\, y_n(t)=\int_{a}^{s} h_n(t, u)\mathrm{w}_n(du), \quad s<t, \tag{II.23}$$

and therefore (II.21) is the canonical representation of $\{y_u(t)\}$.

Conversely, if (II.21) is the canonical representation of $\{y_n(t)\}$, $n=\overline{1,M}$, then, according to (II.23) and (II.22), it follows that (II.20) is the canonical representation of $\{y(t)\}$. ▲

EXAMPLE 7. Let $\{w_1(t),\ a\leqslant t\leqslant b\}$ and $\{w_2(t),\ a\leqslant t\leqslant b\}$ be two mutually orthogonal processes with orthogonal increments. Then the representation

$$y(t)=w_1(t)+w_2(t)=\int_a^t w_1(du)+\int_a^t w_2(du),\quad t\in[a,b],$$

is the canonical representation of the process $\{y(t)\}$.

REMARK 4. The last example shows that if all representations (II.21) are the proper canonical ones, the representation (II.20) need not be proper canonical. Therefore in a general case, the canonical representation cannot be reduced to the proper canonical one applying the procedure from Theorem 5 to each of the processes $\{w_n(t)\}$ (compare with [12)].

According to Theorem 4, $\mathcal{H}(y)$ reduces the resolution of the identity in $\sum_{n=1}^{M}\oplus\mathcal{H}(z_n)$ and from Theorem 9, Ch. I, if follows that the multiplicity of the process $\{y(t)\}$ is not greater then M' $(M'\leqslant M)$, where M' is the number of cyclic subspaces $\mathfrak{M}_n$ in the canonical representation of the space $\sum_{n=1}^{M}\oplus\mathcal{H}(z_n)=$ $=\sum_{n=1}^{M'}\oplus\mathfrak{M}_n$.

THEOREM 7. [17] Let the process $\{y(t),\ a\leqslant t\leqslant b\}$ be fully submitted to the process $\{x(t),\ a\leqslant t\leqslant b\}$ and let $\varrho_y=\varrho_x$ with the finite multiplicity $N=N_y=N_x$. Then

$$\mathcal{H}(y;t)=\mathcal{H}(x;t)\ \text{for each}\ t\in[a,b].$$

Proof. If we show that $\mathcal{H}(y)=\mathcal{H}(x)$, then the equality $\mathcal{H}(y;t)=$ $=\mathcal{H}(x;t)$ for each $t\in[a,b]$ follows immediately from the assumption of the theorem. Suppose that $\mathcal{H}(x)\ominus\mathcal{H}(y)\neq0$; then there is an element $z_{x,N+1}\neq0$ in $\mathcal{H}(x)\ominus\mathcal{H}(y)$, such that

$$\rho_{z_{y,1}}>\rho_{z_{y,2}}>\cdots>\rho_{z_{y,N}}>\rho_{z_{x,N+1}}.$$

The fact that the elements $z_{x,n}$, $n=\overline{1,N}$ can be choosen so that $z_{x,n}=z_{y,n}$, $n=\overline{1,N}$, is in contradiction to the assumption that the spectral types of the processes $\{x(t)\}$ and $\{y(t)\}$ are equal. ▲

EXAMPLE 8. ([17]) This simple example shows that the preceding theorem need not hold when N is infinite.

Let all spectral types $\rho_{z_{x,n}}$, $n=\overline{1,\infty}$ be equal and

$$x(t)=\sum_{n=1}^{\infty}\int_{a}^{t} g_n(t,u)\, z_{x,n}(du), \quad t\in[a,b],$$

be the Cramér representation of the process $\{x(t)\}$. If we set

$$y(t)=\sum_{n=2}^{\infty}\int_{a}^{t} g_n(t,u)\, z_{x,n}(du), \quad t\in[a,b],$$

then the spectral types ϱ_x and ϱ_y of the processes $\{x(t)\}$ and $\{y(t)\}$ are equal, but the spaces $\mathcal{H}(x)$ and $\mathcal{H}(y)$ are not equal.

We end this section with two theorems on the relation of spectral types of two processes which are in the relation of full submission.

We say that the spectral type ϱ_y is subordinated to the spectral type ϱ_x ($\varrho_y<\varrho_x$) if $\rho_{z_{y,n}}<\rho_{z_{x,n}}$, $n=\overline{1,N}$, where

$$\varrho_y\ :\ \rho_{z_{y,1}}>\rho_{z_{y,2}}>\ .\ .\ .>\rho_{z_{y,M}}$$

$$\varrho_x\ :\ \rho_{z_{x,1}}>\rho_{z_{x2}}>\ .\ .\ .>\rho_{z_{x,N}}$$

(we assume that $\rho_{z_{y,n}}=0$, $n=\overline{M+1,N}$).

THEOREM 8. ([17]) If the process $\{y(t),\ a\leqslant t\leqslant b\}$ in fully submitted to the process $\{x(t),\ a\leqslant t\leqslant b\}$, then ϱ_y is subordinated to ϱ_x.

Proof. Since the subspace $\mathcal{H}(y)$ reduces the resolution of the identity in $\mathcal{H}(x)$, the proof follows immediately from Theorem 9, Ch. I. ▲

The next theorem is somehow the converse to the preceding one.

THEOREM 9. If the spectral type $\varrho:\ \rho_1>\rho_2>\ldots>\rho_M$ is subordinated to the spectral type ϱ_x then there exists a process $\{y(t),\ a\leqslant t\leqslant b\}$ fully submitted to the process $\{x(t),\ a\leqslant t\leqslant b\}$ and for which $\varrho_y=\varrho$.

Proof. From the facts mentioned on the page 14 we conclude: since $\rho_n<\rho_{x,n}$ in each $\mathcal{H}(z_{x,n})$, $n=\overline{1,N}$, there exists a process with orthogonal increments $\{z_{y,n}(t),\ a\leqslant t\leqslant b\}$ whose spectral type is ρ_n and whose space $\mathcal{H}(z_{y,n})$ reduces $\{E_x(t),\ a\leqslant t\leqslant b\}$. Hence the process $\{y(t),\ a\leqslant t\leqslant b\}$ with the inovation process $\{z_{y,n}(t)\}_{n=\overline{1,M}}$ is fully submitted to the process $\{x(t)\}$. ▲

II.3 The spectral type of some transformations of stochastic processes

The general problem of this section is: if the process $\{y(t),\ a\leqslant t\leqslant b\}$ is a given transformation

$$y(t)=T(t,\ \{x(u),\ a\leqslant u\leqslant b\}),\ t\in[a,b],$$

of the process $\{x(t),\ a\leqslant t\leqslant b\}$, what can be said about the spectral types ϱ_y and ϱ_x?

We shall first consider the operator T defined in the following way: for each $t\in[a,b]$, $Tx(t)$ is an element in $\mathcal{H}(x)$. The process $\{y(t),\ a\leqslant t\leqslant b\}$ is defined by

$$y(t)=Tx(t),\quad t\in[a,b]. \tag{II.24}$$

We extend the operator T by linearity and continuity to $\mathcal{H}(x)$. In such a way T is the linear operator of $\mathcal{H}(x)$ onto $\mathcal{H}(y)$.

EXAMPLE 9. The operator T is defined by

$$y(t)=Tx(t)=x'(t),\quad t\in[a,t],$$

and by linearity and continuity extended to $\mathcal{H}(x)$.

In a general case we cannot make any conclusion about the relations of ϱ_y and ϱ_x, connected by (II.24). The following example shows that even in the case of T being the projetion operator of $\mathcal{H}(x)$ onto a given subspace of $\mathcal{H}(x)$, the process $\{y(t)\}$ need not be regular.

EXAMPLE 10. ([10]) Let $\{w(t),\ 0\leqslant t\leqslant 1\}$ be a Wiener process. It is well known (see, for instance, [16]) that such a process has the representation

$$w(t)=\sum_{k=0}^{\infty}\varphi_k(t)\,z_k,\quad t\in[0,1],$$

where $\varphi_k(t)=\sin\left(k+\frac{1}{2}\right)\pi t,\ 0\leqslant t\leqslant 1,\ k=\overline{0,\infty}$ are the eigenfunctions of the integral operator with the kernel

$$r_w(s,t)=\min\{s,t\},\quad 0\leqslant s,t\leqslant 1,$$

and z_k, $k=\overline{0,\infty}$ are mutually orthogonal random variables for which

$$z_k=\int_0^1 \varphi_k(t)\, w(t)\, dt \in \mathscr{H}(w), \quad k=\overline{0,\infty}.$$

We define the process $\{y(t),\ 0\leqslant t\leqslant 1\}$ by

$$y(t)=\sum_{k=0}^{n} \varphi_k(t)\, z_k,$$

where n is a fixed integer. The space $\mathscr{H}(y)$ $(\subset \mathscr{H}(w))$ is generated by the elements z_k, $k=\overline{0,n}$, and

$$y(t)=P_{\mathscr{H}(y)}\, w(t), \quad t\in[0,1].$$

For any $t>0$ there exist numbers $t_0, t_1, \ldots, t_n$ in $(0,t]$ such that the matrix $(\varphi_k(t_j))_{j=\overline{0,n}}^{k=\overline{0,n}}$ is non-singular. Therefore the linear system

$$\sum_{k=0}^{n} \varphi_k(t_j)\, z_k = y(t_j), \quad j=\overline{0,n},$$

has the unique solution

$$z_k=\sum_{j=0}^{n} c_{kj}\, y(t_j), \quad k=\overline{0,n}.$$

Since $z_k\in\mathscr{H}(y;t)$, $k=\overline{0,n}$, for any $t>0$

$$\bigcap_{t>0} \mathscr{H}(y;t) \ (=\mathscr{H}(y))\neq 0$$

i. e. the process $\{y(t)\}$ is not regular.

EXAMPLE 11. Let the correlation function $r(s,t)$ of the process $\{x(t),\ a\leqslant t\leqslant b\}$ have the derivate $\frac{\partial^2 r(s,t)}{\partial t\, \partial s}$, $s, t\in[a,b]$. Suppose further, that in the Cramér representation

$$x(t)=\int_a^t \mathbf{g}(t,u)\,\mathbf{z}(du), \quad t\in[a,b], \tag{II.25}$$

of the process $\{x(t)\}$ the function $\mathbf{g}(t,u)$, $a\leqslant u\leqslant t\leqslant b$, is continuous with $\mathbf{g}(t,t)=0$ for each $t\in[a,b]$, and that the maximal spectral type $F_{z_1}(t)$ of $\{x(t)\}$ is absolutely continuous. (For instance, the stationary process $\{x(t), -\infty<t<+\infty\}$ with Wold representation

$$x(t)=\int_{-\infty}^t g(t-u)\,z(du), \quad t\in(-\infty,\ +\infty),$$

where $g(t)$, $t\in[0,\ +\infty]$ is continuous and $g(0)=0$ satisfies these conditions.) We shall show that in this case the spectral type of the process $\{x'(t),\ a\leqslant t\leqslant b\}$ is $\mathbf{F}_x$. We set

$$\mathbf{F}_x(dt)=\begin{pmatrix} \varphi_1(t)\,dt & \dots & 0 \\ & \ddots & \\ 0 & \dots & \varphi_N(t)\,dt \end{pmatrix}=\varphi(t)\,dt. \tag{II.26}$$

From (II.25) and (II.26) we have for $s\leqslant t$, $s,t\in[a,b]$

$$r(s,t)=\int_a^s \mathbf{g}(s,u)\,\varphi(u)\,du\,\mathbf{g}^\star(t,u)$$

and

$$\frac{\partial^2 r(s,t)}{\partial t\,\partial s}=\mathbf{g}(s,s)\,\varphi(s)\frac{\partial \mathbf{g}^\star(t,s)}{\partial t}+\int_a^s \frac{\partial\,\mathbf{g}(s,u)}{\partial s}\,\varphi(u)\,du\,\frac{\partial \mathbf{g}^\star(t,u)}{\partial t},$$

or

$$r_{x'}(s,t)=\frac{\partial^2 r(s,t)}{\partial t\,\partial s}=$$

$$=\int_a^s \frac{\partial \mathbf{g}(s,u)}{\partial s}\,\varphi(u)\,du\,\frac{\partial \mathbf{g}^\star(t,u)}{\partial t}=$$

$$= \int_a^s \frac{\partial \mathbf{g}(s, u)}{\partial s} \mathbf{F}(du) \frac{\partial \mathbf{g}^*(t, u)}{\partial t}, \quad s < t, \; s, \; t \in [a, b]. \tag{II.27}$$

We shall show that the family of functions $\left\{\frac{\partial \mathbf{g}(t, u)}{\partial t}, \text{ the parameter } t \in [a, b]\right\}$ is complete in $\mathscr{L}_2(\mathbf{F})$. The condition that for $\mathbf{f} \in \mathscr{L}_2(\mathbf{F})$ and fixed $t \in [a, b]$

$$\int_a^s \frac{\partial \mathbf{g}(s, u)}{\partial s} \mathbf{F}(du) \mathbf{f}^*(u) = 0 \text{ for all } s \in [a, t]$$

we write as

$$\frac{\partial}{\partial s} \int_a^s \mathbf{g}(s, u) \mathbf{F}(du) \mathbf{f}^*(u) = 0 \text{ for all } s \in [a, t],$$

or

$$\int_a^s \mathbf{g}(s, u) \mathbf{F}(du) \mathbf{f}^*(u) = 0 \text{ for all } s \in [a, t].$$

As the family $\{\mathbf{g}(t, \mathrm{u})\}$ is complete in $\mathscr{L}_2(\mathbf{F})$, $\mathbf{f} = 0$ almost everywhere with respect to $\mathbf{F}$. From (II.27) and Theorem 3, Ch. II, it follows that

$$x'(t) = \int_a^t \frac{\partial \mathbf{g}(t, u)}{\partial t} z(du), \quad t \in [a, b],$$

is the Cramér representation of $\{x'(t)\}$, i. e. $\mathbf{F}_{x'} = \mathbf{F}$. ▲

Now we shall consider a more general transformation, the so called non-anticipative transformation. Let $\{\mathbf{z}_x(t), a \leqslant t \leqslant b\}$ be an innovation process of the process $\{x(t), a \leqslant t \leqslant b\}$. The process $\{y(t), a \leqslant t \leqslant b\}$ is a *non-anticipative transformation* of $\{x(t)\}$ if

$$y(t) = T\{x(u), a \leqslant u \leqslant b\} = \int_a^t \mathbf{h}(t, u) \mathbf{z}_x(du), \quad \mathrm{t} \in [a, b]. \tag{II.28}$$

The last equality shows that the process $\{y(t)\}$ is a non-anticipative transformation of the process $\{x(t)\}$ if and only if $\{y(t)\}$ is submitted to $\{x(t)\}$.

For instance $\varrho_x = \varrho_y$ if and only if the family $\{\mathbf{h}(t, u)\}$ is complete in $\mathscr{L}_2(\mathbf{F}_x)$.

Or, if non-anticipative transformation $\{y(t)\}$ is fully submitted to the process $\{x(t)\}$ then $\varrho_y < \varrho_x$ (Th. 8).

EXAMPLE 12. ([5]) Let $\{x(t), -\infty < t < +\infty\}$ be a stationary process with Wold representation

$$x(t) = \int_{-\infty}^{t} g(t-u)\, z(du), \quad t \in (-\infty + \infty),$$

and $q(u)$, $-\infty < u < +\infty$, be a bounded, continuous and everywhere positive function. Let the process $\{y(t) \; -\infty < t < +\infty\}$ be the following non-anticipative transformation of $\{x(t)\}$:

$$y(t) = \int_{-\infty}^{t} g(t-u)\, q(u)\, z(du), \quad t \in (-\infty, +\infty).$$

Since the family $\{g(t-u)$, the parameter $t \in (-\infty, +\infty)\}$ is complete in $\mathscr{L}_2$, it is easy to see that the family $\{g(t-u)\, q(u)$, the parameter $t \in (-\infty, +\infty)\}$ is also complete in $\mathscr{L}_2$. Hence the spectral type of $\{y(t)\}$ is the ordinary Lebesgue measure $(N_y = 1)$.

EXAMPLE 13. ([11]) Let the correlation function $r(s, t)$, $s, t \in [a, b]$ of the process $\{x(t), a \leqslant t \leqslant b\}$ be Riemann integrable function and the function $\varphi(t, u)$, $a \leqslant u \leqslant t$ $(\varphi(t, u) = 0, u > t)$ be such that for each $t \in [a, b]$ the quadratic mean integral

$$\int_a^b \varphi(t, u)\, x(u)\, du$$

exists. We define the process $\{y(t),\; a \leqslant t \leqslant b\}$ as a non-anticipative transformation

$$y(t) = \int_a^t \varphi(t, u)\, x(u)\, du, \quad t \in [a, b]. \tag{II.29}$$

Considering the proper canonical representation

$$x(t) = \int_a^t \mathbf{g}(t, u)\, \mathbf{z}_x(du), \quad t \in [a, b],$$

of $\{x(t)\}$, it is easy to transform (II.29) into the form (II.28):

$$y(t)=\int_a^t \varphi(t,u)\left[\int_a^u \mathbf{g}(u,v)\,\mathbf{z}_x(dv)\right]du=$$

$$=\int_a^t\left[\int_v^t \varphi(t,u)\,\mathbf{g}(u,v)\,du\right]\mathbf{z}_x(dv),\quad t\in[a,b].$$

Let us suppose now that $F_{z_1}(t)$ is absolutely continuous. We shall show that, if the family $\{\varphi(t,u)$, the parameter $t\in[a,b]\}$ is complete in $\mathscr{L}_2$, then $\mathbf{F}_y=\mathbf{F}_x$.

To prove that, it is sufficient to show that the family $\{\int_u^t \varphi(t,v)\,\mathbf{g}(v,u)\,dv$, the parameter $t\in[a,b]\}$ is complete in $\mathscr{L}_2(\mathbf{F}_x)$. Let $\mathbf{f}\in\mathscr{L}_2(\mathbf{F}_x)$ and t be any fixed number from $[a,b]$. If

$$\int_a^s\left[\int_u^s \varphi(s,v)\,\mathbf{g}(v,u)\,dv\right]\mathbf{F}_x(du)\,\mathbf{f}^*(u)=$$

$$=\int_a^s \varphi(s,u)\left[\int_a^u \mathbf{g}(u,v)\,\mathbf{F}_x(dv)\,\mathbf{f}^*(v)\right]du=0$$

for all $s\in[a,t]$, then, by the completness of $\{\varphi(t,u)\}$ in $\mathscr{L}_2$, it follows that

$$\int_a^u \mathbf{g}(u,v)\,\mathbf{F}_x(dv)\,\mathbf{f}^*(v)=0$$

almost everywhere on $[a,t]$. However, because of the continuity of $F_{z_1}(t)\}$, the last equality holds everywhere on $[a,t]$. Since $\{\mathbf{g}(t,u)\}$ is complete in $\mathscr{L}_2(\mathbf{F}_x)$, it follows that $\mathbf{f}=0$ almost everywhere with respect to $\mathbf{F}_x$, as we wanted to prove.

II.4. The stochastic processes regular everywhere and processes with discrete innovation

The regularity of the process $\{x(t),\,a\leqslant t\leqslant b\}$ was defined as the condition that $\bigcap\limits_{t>a}\mathscr{H}(x;t)=0$ or, in other notation,

$$\mathcal{H}(x; a+0) = \sum_{n=1}^{N} \oplus \mathcal{H}(z_n; a+0) = 0, \tag{II.30}$$

where $\{\mathbf{z}(t) = (z_n(t))_{n=\overline{1,N}},\ a \leqslant t \leqslant b\}$ is the innovation process of $\{x(t)\}$ in the Cramér representation. The condition (II.30) is equivalent to

$$\mathbf{F}_x(a+0) = \lim_{t \to a+0} \mathbf{F}_x(t) = 0,$$

or to the condition that the maximal spectral type $F_{z_1}(t)$ in $\mathbf{F}_x$ is continuous at the initial point $t = a$.

Let us notice that if a is a finite number, instead of the process $\{x(t)\}$ on the segment $[a, b]$, we can consider the process $\{x(t)\}$ on the larger segment $[c, b]$, $c < a$, defining $x(t) = 0$ for $t \in [c, a)$. In such a way, the new process $\{x(t)\}$ on $[c, b]$ is always regular. However, we cannot do that if $a = -\infty$. For that reason we shall not accept such an extension of the segment $[a, b]$.

REMARK 5. Any (non-regular) stochastic process $\{x(t),\ a \leqslant t \leqslant b\}$ can be uniquely represented as the sum of two mutually orthogonal processes $\{x_r(t),\ a \leqslant t \leqslant b\}$ and $\{x_s(t),\ a \leqslant t \leqslant b\}$;

$$x(t) = x_r(t) + x_s(t), \quad t \in [a, b], \tag{II.31}$$

where $\{x_r(t)\}$ is a regular process and $\{x_s(t)\}$ is a so-called singular (or deterministic) process, such that $\mathcal{H}(x_s; a+0) = \mathcal{H}(x_s)$. To show that (II.31) is true, it is sufficient to notice that $x_s(t) = P_{\mathcal{H}(x;\, a+0)}\, x(t)$, $t \in [a, b]$.

DEFINITION 8. ([10]) The process $\{x(t),\ a \leqslant t \leqslant b\}$ is *regular at the point* $t_0 \in [a, b]$ if the maximal spectral type $F_{z_1}(t)$ in $\mathbf{F}_x(t)$ is continuous in $t = t_0$. The process $\{x(t),\ a \leqslant t \leqslant b\}$ is *regular everywhere* if it is regular at each point of $[a, b]$.

EXAMPLE 14. A stationary (regular) process $\{x(t),\ -\infty < t < +\infty\}$ is regular everywhere.

DEFINITION 9. The process $\{x(t),\ a \leqslant t \leqslant b\}$ is the process *with discrete innovation* if the maximal spectral type $F_{z_1}(t)$ in $\mathbf{F}_x(t)$ induces a discrete mesaure.

We remark that $F_{z_1}(t)$ does not have the discontinuity at $t = a$, since we consider only regular processes.

THEOREM 10. ([10]) Any process $\{x(t),\ a \leqslant t \leqslant b\}$ can be uniquely represented as the sum of two mutually orthogonal processes

$$x(t)=x_1(t)+x_2(t), \tag{II.32}$$

where $\{x_1(t),\ a \leqslant t \leqslant b\}$ is regular everywhere and $\{x_2(t),\ a \leqslant t \leqslant b\}$ is the process with discrete innovation.

Proof. Let $\{\mathbf{z}(t)=(z_n(t))_{n=\overline{1,N}},\ a \leqslant t \leqslant b\}$ be the innovation process in the Cramér representation of the process $\{x(t),\ a \leqslant t \leqslant b\}$. We write the distribution function $F_{z_n}(t),\ a \leqslant t \leqslant b$, as the sum

$$F_{z_n}(t)=F_{z_{n1}}(t)+F_{z_{n2}}(t), \quad t \in [a, b],$$

where $F_{z_{n1}}(t)$ is a continuous distribution function and $F_{z_{n2}}(t)$ induces a discrete measure. In other words, the spectral type F_{z_n} is the sum of two orthogonal spectral types:

$$F_{z_n}=F_{z_{n1}}+F_{z_{n2}}.$$

According to Theorem 1, Ch. I, there exist two mutually orthogonal processes with orthogonal increments $\{z_{n1}(t),\ a \leqslant t \leqslant b\}$ and $\{z_{n2}(t),\ a \leqslant t \leqslant b\}$ with spectral types $E_{z_{n1}}$ and $F_{s_{n2}}$ respectively, such that

$$z_n(t)=z_{n1}(t)+z_{n2}(t), \quad t \in [a, b],$$

$$\mathcal{H}(z_n; t)=\mathcal{H}(z_{n1}; t) \oplus \mathcal{H}(z_{n2}; t), \quad t \in [a, b].$$

Since

$$F_{z_1} > F_{z_2} > \cdots > F_{z_N}$$

we have

$$F_{z_{11}} > F_{z_{21}} > \cdots > F_{z_{N1}}$$

and

$$F_{z_{12}} > F_{z_{22}} > \cdots > F_{s_{N2}}$$

Introducing $\{\mathbf{z}_1(t)=(z_{n1}(t))_{n=\overline{1,N}},\ a \leqslant t \leqslant b\}$ and $\{\mathbf{z}_2(t)=(z_{n2}(t))_{n=\overline{1,N}},\ a \leqslant t \leqslant b\}$ we can write the Cramér representation

$$x(t)=\int_a^t \mathbf{g}(t, u)\,\mathbf{z}(du), \quad t \in [a, b],$$

of $\{x(t)\}$ as

$$x(t)=\int_a^t \mathbf{g}(t, u)\,\mathbf{z}_1(du)+\int_a^t \mathbf{g}(t, v)\,\mathbf{z}_2(du), \quad t \in [a, b],$$

which proves (II.32). Finally, the uniqueness of (II.32) follows by the standard procedure. Let

$$x(t) = \tilde{x}_1(t) + \tilde{x}_2(t), \quad t \in [a, b],$$

be another decomposition of $\{x(t)\}$. Then

$$x_1(t) - \tilde{x}_1(t) = \tilde{x}_2(t) - x_2(t) \quad t \in]a, b],$$

which is a contradiction because the process $\{x_1(t) - \tilde{x}_1(t), a \leqslant t \leqslant b\}$ is everywhere regular and the process $\{\tilde{x}_2(t) - x_2(t), a \leqslant t \leqslant b\}$ is the process with discrete innovation. ▲

REMARK 6. According to the well-known Lebesgue theorem any distribution function $F(t)$, $a \leqslant t \leqslant b$, has the unique decomposition

$$F(t) = F_{ac}(t) + F_d(t) + F_s(t), \quad t \in [a, b],$$

where $F_{ac}(t)$ is the distribution function inducing the measure which is absolutely continuous with respect to the ordinary Lebesgue measure, $F_d(t)$ induces the discrete measure and $F_s(t)$ is continuous distribution function which induces the singular measure (with respect to the ordinary Lebesgue measure). Now, similarly to the preceding theorem, any process $\{x(t), a \leqslant t \leqslant b\}$ can be uniquely represented as a sum of three mutually orthogonal processes

$$x(t) = x_1(t) + x_2(t) + x_3(t), \quad t \in [a, b],$$

where $\{x_1(t), a \leqslant t \leqslant b\}$ has an absolutely continuous maximal spectral type, $\{x_2(t), a \leqslant t \leqslant b\}$ has discrete innovation and $\{x_3(t), a \leqslant t \leqslant b\}$ has a continuous maximal spectral type singular with respect to the ordinary Lebesgue measure.

Let $\{x(t), a \leqslant t \leqslant b\}$ be a process with discrete innovation. The Cramér representation of that process has a simpler form since for the self-adjoint operator A, defined by the resolution of the identity $\{E_x(s), a \leqslant s \leqslant b\}$ the set $\{t_1, t_2, \ldots\}$ of discontinuity points of the maximal spectral type $F_{z_1}(t)$ of $\{x(t)\}$ is the set of all eigenvalues of A. (see [1], §82) The multiplicity N_k of the eigenvalue t_k, $k = 1, 2, \ldots$ is the number of the members of the sequence

$$F_{z_1}(t) > F_{z_2}(t) > \ldots > F_{z_N}(t),$$

which has discontinuity at the point $t = t_k$ and $N = \sup_k N_k$. Let $z_n(t_k)$, $n = \overline{1, N_k}$ be mutually orthogonal eigenvectors corresponding to the eigenvalue t_k and let $\mathcal{H}_x(t_k)$ be the space generated by $z_n(t_k)$, $n = \overline{1, N_k}$. Then

$$\mathcal{H}(x; t) = \sum_{t_k \leq t} \oplus \mathcal{H}_x(t_k), \quad t \in [a, b], \tag{II.33}$$

or

$$x(t)=\sum_{t_k\leq t}\sum_{n=1}^{N_k} g_n(t, t_k)\, z_n(t_k), \quad t\in[a, b]. \tag{II.34}$$

Introducing $\mathbf{z}(t_k)=(z_n(t_k))_{n=\overline{1,N}}$, $k=1, 2, \ldots,$ where $z_n(t_k)=0$ for $n=\overline{N_{k+1}, N}$ from (II.34) we get the Cramér representation

$$x(t)=\sum_{t_k\leq t} \mathbf{g}(t, t_k)\, \mathbf{z}(t_k), \quad t\in[a, b], \tag{II.35}$$

of the process $\{x(t)\}$ with discrete innovation. The form (II.35) (or (II.34)) shows that the study of such a process is more simple then, for instance, the study of everywhere regular one. So, it holds

THEOREM 11. Let $\{x(t), a\leqslant t\leqslant b\}$ be the process with discrete innovation in a finite set of points $\{t_1, t_2, \ldots, t_l\}$ $(t_k\neq a, k=\overline{1, l})$, T be a bounded operator in $\mathcal{H}(x)$ and let the process $\{y(t), a\leqslant t\leqslant b\}$ be defined by:

$$y(t)=Tx(t), \quad t\in[a, b]. \tag{II.36}$$

Then $\mathbf{F}_y<\mathbf{F}_x$.

Proof. Applying T on (II.33) we have

$$\mathcal{H}(y; t)=\sum_{t_k\leq t} \mathcal{H}_y(t_k), \quad t\in[a, b],$$

where

$$\mathcal{H}_y(t_k)=T\mathcal{H}_x(t_k), \quad k=\overline{1, l}. \tag{II.37}$$

From (II.37) it follows that $\dim \mathcal{H}_y(t_k)\leqslant \dim \mathcal{H}_x(t_k)$, $k=\overline{1, l}$, or the multiplicty N_{yk} of the eigenvalue t_k with respect to $\{E_y(s), a\leqslant s\leqslant b\}$ is not greater then the multiplicity N_{xk} of the eigenvalue t_k with respect to $\{E_x(s), a\leqslant s\leqslant b\}$ $(k=\overline{1, l})$. It means that $\mathbf{F}_y<\mathbf{F}_x$. ▲

The next example shows that Theorem 10. need not hold if the set of discontinuity points is not finite.

EXAMPLE 15. Let $\{w(t), 0\leqslant t\leqslant 1\}$ be a given Wiener process and let the process $\{x(t), -1\leqslant t\leqslant 1\}$ be defined in a following way:

$$x(t)=\begin{cases}0, & -1\leqslant t\leqslant 0,\\ w\left(\dfrac{1}{2^n}\right), & \dfrac{1}{2^n}<t<\dfrac{1}{2^{n-1}},\ n=1,2,\ldots,\\ w(1), & t=1.\end{cases}$$

The spectral type $\mathbf{F}_x(t)$ has the discontinuity points $t_k=\dfrac{1}{2^k}$, $k=1, 2, \ldots$ and $t=0$ is its point of continuity.

The operator T and the process $\{y(t),\ -1\leqslant t\leqslant 1\}$ are defined in the following way

$$y(t)=Tx(t)=\begin{cases}0, & -1\leqslant t\leqslant 0,\\ \dfrac{1}{2^n}w\left(\dfrac{1}{2}\right), & \dfrac{1}{2^n}\leqslant t\leqslant \dfrac{1}{2^{n-1}},\ n=1,2,\ldots,\\ w\left(\dfrac{1}{2}\right), & t=1.\end{cases}$$

The only increasing point of the spectral type $\mathbf{F}_y$ is $t=0$ and hence $\mathbf{F}_y$ is not subordinated to $\mathbf{F}_x$.

Appendix I

THE SPECTRAL TYPE OF WIDE-SENSE MARKOV PROCESS

The class of wide-sense Markov processes is one of the simplest classes of second ordered processes. In this section we shall expose one simple procedure ([10]) for effective obtaining the spectral type of Markov process in terms of its correlation function. Multidimensional wide-sense Markov processes were studied in [9] and [14].

The process $\{x(t),\ a \leqslant t \leqslant b\}$ is the (wide-sense) Markov process if for any $s, t \in [a, b]$, $s \leqslant t$, the projection of $x(t)$ on $\mathcal{H}(x; s)$ coincides with the projection of $x(t)$ on the element $x(s)$:

$$P_{\mathcal{H}(x;\, s)}\, x(t) = a(t, s)\, x(s), \quad s \leqslant t.$$

It is easy to show that the scalar function $a(t, s)$, defined for $s \leqslant t$, $s, t \in \in [a, b]$ is

$$a(t, s) = \frac{r(t, s)}{r(s, s)}, \quad s \leqslant t, \tag{1}$$

where $r(t, s)$ is the correlation function of $\{x(t)\}$.

According to the theorem of three perpendiculars, we get following transitive property of $a(t, s)$: for any $t_1 \leqslant t_2 \leqslant t_3$, $t_1, t_2\, t_3 \in [a, b]$ we have

$$a(t_3, t_1) = a(t_3, t_2) \cdot a(t_2, t_1), \tag{2}$$

To avoid some non-essential difficulties, we shall assume in the sequel that $r(t, s) \neq 0$ for each $t, s \in [a, b]$ (see [9] and [11]).

Let s_0 be any fixed point from $[a, b]$. We define ([9]) the function $g(t)$, $a \leqslant t \leqslant b$, by

$$g(t) = \begin{cases} \dfrac{1}{a(s_0, t)}, & t \in [a, s_0], \\ a(t, s_0), & t \in (s_0, b]. \end{cases} \tag{3}$$

From (2) it follows that for any $s \leqslant t$, $s, t \in [a, b]$ we have

$$a(t, s) = \frac{g(t)}{g(s)}.$$

Let the process $\{z(t),\ a \leqslant t \leqslant b\}$ be defined by

$$z(t) = \frac{1}{g(t)} x(t), \quad t \in [a, b]. \tag{4}$$

It is easy to show that $\{z(t)\}$ is a process with orthogonal increments and that

$$P_{\mathcal{H}(z;\, s)}\, z(t) = z(s), \quad s \leqslant t.$$

Indeed, since $\mathcal{H}(z; t) = \mathcal{H}(x; t)$ for each $t \in [a, b]$, we have for $s \leqslant t$

$$P_{\mathcal{H}(z;\, s)}\, z(t) = P_{\mathcal{H}(z;\, s)} \frac{1}{g(t)} x(t) = \frac{1}{g(t)} a(t, s)\, x(s) = z(s).$$

From (4) it follows that the processes $\{x(t)\}$ and $\{z(t)\}$ have the same spectral type. Hence

$$\mathbf{F}_x(t) = F_z(t) = \frac{1}{|g(t)|^2} \cdot r(t, t),$$

or, from (3) and (1), we have

$$\mathbf{F}_x(t) = \begin{cases} \dfrac{|r(s_0, t)|^2}{r(t, t)}, & t \in [a, s_0], \\[2ex] \left|\dfrac{r(s_0, s_0)}{r(t, s_0)}\right|^2 \cdot r(t, t), & t \in (s_0, b]. \end{cases} \tag{5}$$

It remains to be shown that the spectral type $\mathbf{F}_x(t)$ does not depend on the choice of the point s_0. For another s_1 (say $s_0 < s_1$) we have

$$\tilde{\mathbf{F}}_x(t) = \begin{cases} \dfrac{|r(s_1, t)|_2}{r(t, t)}, & t \in [a, s_1], \\[2ex] \left|\dfrac{r(s_1, s_1)}{r(t, s_1)}\right|^2 \cdot r(t, s), & t \in (s_1, b]. \end{cases} \tag{6}$$

From (1) and (2) it follows that

$$\tilde{\mathbf{F}}_x(t) = \left| \frac{r(s_1, s_0)}{r(s_0, s_0)} \right|^2 \mathbf{F}_x(t), \quad t \in [a, b].$$

As the factor $\left| \frac{r(s_1, s_0)}{r(s_0, s_0)} \right|^2$ is positive for all $s_0, s_1 \in [a, b]$, we conclude that the distribution functions (5) and (6) belong to the same spectral type.

We remark that from

$$x(t) = g(t)\, z(t), \quad t \in [a, b],$$

we get

$$r(t, s) = g(t)\, \mathbf{F}_x(\min\{s, t\})\, \overline{g(s)}, \quad s, t \in [a, b]. \tag{7}$$

Setting

$$g(t, u) = \begin{cases} g(t), & u \in [a, t], \\ 0\ , & u \in (t, b], \end{cases} \tag{8}$$

we conclude that the representation (7) is the representation (II.11) in Ch. II. Since $g(t) \neq 0$ for all $t \in [a, b]$, the family $\{g(t, u)$, the parameter $t \in [a, b]\}$ defined by (8), is complete in $\mathscr{L}_2(\mathbf{F}_x)$.

Appendix II

THE CRAMÉR REPRESENTATION OF A RANDOM FIELD OVER THE COMPLEX PLANE

In Ch. I we have established the complete system of unitary invariants of a self-adjoint operator in a separable Hilbert space. However, all the mentioned theorems hold even for normal operators defined in a separable Hilbert space (see [18], [15]). That enables us to give the Cramér representation of a random field $\{x(\zeta), \zeta \in D\}$, where the parameter ζ is a complex number and $D=\{\zeta : a \leqslant \operatorname{Re}\zeta \leqslant b,\ c \leqslant \operatorname{Im}\zeta \leqslant d\}$ is a finite or infinite rectangle in a complex plane. We shall give the procedure concisely (see [2]).

Let us consider a field $\{x(\zeta), \zeta \in D\}$, $Ex(\zeta)=0$, $E|x(\zeta)|^2 < +\infty$, $\zeta \in D$, with a correlation function $r(\zeta_1, \zeta_2)=Ex(\zeta_1)\overline{x(\zeta_2)}$, $\zeta_1, \zeta_2 \in D$. Let $\mathcal{H}(x;\zeta)$ be the smallest linear space spanned by random variables $x(\eta)$, where $\operatorname{Re}\eta \leqslant \operatorname{Re}\zeta$, $\operatorname{Im}\eta \leqslant \operatorname{Im}\zeta$.

We shall assume that

(A) the field $\{x(\zeta), \zeta \in D\}$ is continuous in quadratic mean for each $\zeta \in D$;

(B) the field $\{x(\zeta), \zeta \in D\}$ is regular, i. e.

$$\bigcap_{\zeta : \operatorname{Re}\zeta > a} \mathcal{H}(x;\zeta) = \bigcap_{\zeta : \operatorname{Im}\zeta > c} \mathcal{H}(x;\zeta) = 0.$$

Let $E(\zeta)$ be the projection operator of $\mathcal{H}(x)$ onto $\mathcal{H}(x;\zeta)$. According to the assumptions (A) and (B), it follows that $\{E(\zeta), \zeta \in D\}$ is the resolution of the identity of a normal operator T in a separable Hilbert space $\mathcal{H}(x)$ ([1], § 82).

The element $x \in \mathcal{H}(x)$ produces the measure $\rho_x(\cdot)$ over a Borel field of sets from D, defined by

$$\rho_x(\Delta) = \| E(\Delta)x \|^2,$$

where $\Delta=\{\zeta : a_1\leqslant \operatorname{Re}\zeta \leqslant b_1, c_1 \leqslant \operatorname{Im}\zeta\leqslant d_1\}$, $a\leqslant a_1<b_1\leqslant b, c\leqslant c_1<d_1\leqslant d$, is a rectangle in D, and

$$E(\Delta)=E(b_1+d_1i)-E(b_1+c_1i)-E(a_1+d_1i)+E(a_1+b_1i).$$

A field $\{z(\zeta), \zeta\in D\}$ is a field with orthogonal increments if for every pair of disjoint rectangles Δ_1 and Δ_2, the increments $z(\Delta_1)$ and $z(\Delta_2)$ are mutually orthogonal random variables $\big(z(\Delta)=z(b_1+d_1i)-z(b_1+c_1i)-(a_1+d_1i)+ + z(a_1+b_1i)\big)$.

The following theorems are analogous to the theorems in Ch. II.

THEOREM 1. For each field $\{x(\zeta), \zeta\in D\}$ holds the Cramér representation

$$x(\zeta)=\int\limits_{\Delta\zeta}\sum_{n=1}^{N} g_x(\zeta,\eta)\,z_x(d\eta),\quad \zeta\in D$$

$\Delta\zeta=\{\eta : a\leqslant\operatorname{Re}\eta\leqslant\operatorname{Re}\zeta,\ c\leqslant\operatorname{Im}\eta\leqslant\operatorname{Im}\zeta\}$, where $\{z_n(\eta), \eta\in D\}$, $n=\overline{1, N}$ are mutually orthogonal fields with orthogonal increments,

$$\rho_{z_1}<\rho_{z_2}<\ldots<\rho_{z_N} \tag{1}$$

and

$$\mathcal{H}(x;\zeta)=\sum_{n=1}^{N}\oplus\mathcal{H}(z_n;\zeta),\qquad \text{for each } \zeta\in D.$$

The sequence (1) is called the *spectral type of the field* $\{x(\zeta)\}$.

THEOREM 2. The correlation function $r(\zeta_1,\zeta_2)$ of the field $\{x(\zeta)\}$ uniquely determines its spectral type.

THEOREM. 3. For each sequence (1) there exists a field $\{x(\zeta)\}$ such that (1) is its spectral type.

Appendix III

ONE CLASS OF PROCESSES WITH MULTIPLICITY $N=1$

Let

$$x(t)=\int_a^t \mathbf{g}(t,u)\,\mathbf{z}(du)=\sum_{n=1}^{N}\int_a^t g_n(t,u)\,z_n(du), \quad t\in[a,b], \tag{1}$$

be the Cramér representation of the real-valued process $\{x(t)\}$.

THEOREM 1. [6] If each term $\int_a^t g_n(t,u)\,z_n(du)$, $a\leqslant u\leqslant t\leqslant b, n=\overline{1\ N}$, in (1) satisfies the conditions

1. $g_n(t,u)$ and $\dfrac{\partial g_n(t,u)}{\partial t}$ are bounded and continuous for all $u,t: a\leqslant \leqslant u\leqslant t\leqslant b$;

2. $g_n(t,t)>0$ for all $t\in[a,b]$;

3. $F_n(t)=E z_n^2(t)$ is absolutely continuous and $\varphi_n(t)=F'_n(t)$ has at most a finite number of discontinuity points in any finite subinterval of $[a,b]$, then $\{x(t)\}$ has multiplicity $N=1$.

Proof. We shall show that, if $N>1$, then, for $t\in[a,b]$, the family of functions $\{\mathbf{g}(t,u),\ a\leqslant u\leqslant t\}$ is not complete in $\mathscr{L}_2(\mathbf{F})$, which is the contradiction, because (1) is the Cramér representation.

By hypothesis 3, we can find a finite subinterval $[a_1,b_1]$ of $[a,b]$, such that the derivatives $\varphi_1(t)$ and $\varphi_2(t)$ are continuous and positive for all $t\in[a_1,b_1]$. To prove that the family $\{\mathbf{g}(t,u)\}$ is not complete in $\mathscr{L}_2(\mathbf{F})$, it is sufficient to show that there exists the vector-function

$$\mathbf{f}(u)=(f_1(u),\ f_2(u),\ 0,0,\ldots),\quad (f_1(u)\neq 0,\ f_2(u)\neq 0),$$

such that

$$\int_{a_1}^{t} (\mathbf{f}(u))^2 \mathbf{F}(du) = \int_{a_1}^{t} \mathrm{f}_1^2(u)\,\varphi_1(u)\,du + \int_{a_1}^{t} \mathrm{f}_2^2(u)\,\varphi_2(u)\,du > 0, \quad t \in [a_1 b_1], \tag{2}$$

$$\int_{a_1}^{s} \mathbf{g}(s, u)\,\mathbf{f}(u)\,\mathbf{F}(du) = \int_{a_1}^{s} g_1(s, u)\, f_1(u)\,\varphi_1(u)\,du + \int_{a_1}^{s} g_2(s, u) f_2(u)\,\varphi_2(u)\,du = 0 \tag{3}$$

for all $s \in [a_1, t]$.

We may replace the condition 2 by the condition

$$g_n(t, t) = 1, \quad t \in [a, b], \quad n = \overline{1, N},$$

if we transforme $g_n\,(t, u)$ and $z_n(du)$ into $\bar{g}_n(t, u)$ and $\bar{z}_n(du)$, by writing

$$\bar{g}_n(t, u) = \frac{g_n(t, u)}{g_n(t, t)}, \quad \bar{z}_n(du) = g_n(u, u) \cdot z_n(du), \quad n = \overline{1, N}.$$

Because of that, we may suppose that $g_n(t, t) = 1$, $n = \overline{1, N}$, $t \in [a, b]$. By the conditions 1. and 2., the relation (3) may be differentiated with respect to s, so we obtain

$$f_1(s)\,\varphi_1(s) + \int_{a_1}^{s} \frac{\partial g_1(s, u)}{\partial s} f_1(u)\,\varphi_1(u)\,du + f_2(s)\,\varphi_2(s) + \int_{a_1}^{s} \frac{\partial g_2(s, u)}{\partial s} f_2(u)\,\varphi_2(u) = 0$$

for all $s \in [a_1, t]$. This equation is satisfied if, for example,

$$f_1(s)\,\varphi_1(s) + \int_{a_1}^{s} \frac{\partial g_1(s, u)}{\partial s} f_1(u)\,\varphi_1(u)\,du = -1,$$

$$f_2(s)\,\varphi_2(s) + \int_{a_1}^{s} \frac{\partial g_2(s, u)}{\partial s} f_2(u)\,\varphi_2(u)\,du = +1.$$

The last two equations are the integral equations of Volterra type, where $f_1(s)\,\varphi_1(s)$ and $f_2(s)\,\rho_2(s)$ are the unknown functions. By above hypothesis, each of these equations has the uniquely determined solution, which is bounded and continuous for $s \in [a_1, t]$; these solutions are not almost everywhere equal to zero. Thus, the relation (3) is satisfied. Since $\varphi_1(u)$ and $\varphi_2(u)$ are positive for $u \in [a_1, b_1]$, it follows that (2) is also satisfied. ▲

THEOREM 2. [6] If in the representation

$$x(t) = \int_a^t g(t, u)\, z(du), \quad t \in [a, b], \tag{4}$$

of the process $\{x(t)\}$, the functions $g(t, u)$ and $\mathbf{F}(u)$ satisfy the conditions 1, 2, 3 of the preceding theorem and if a is finite, then (4) is also Cramér representation (i. e., in this case, the proper canonical representation) of the process $\{x(t)\}$.

Proof. This theorem will be proved if we can show that the family $\}g(t, u)\}$ is complete in $\mathscr{L}_2(F)$; we will do that like in the preceding theorem. The condition

$$\int_{a_1}^{s} g(s, u) f(u)\, \varphi(u)\, du = 0, \text{ for all } s \in [a_1, t],$$

may be differentiated with respect to s, and we obtain

$$f(s)\, \varphi(s) + \int_{a_1}^{s} \frac{\partial g(s, u)}{\partial s} f(u)\, \varphi(u)\, du = 0, \text{ for all } s \in [a_1, t].$$

This is a homogeneous integral equation of the Volterra type and, under our conditions, it follows that its the only solution is $f(s)\, \varphi(s) = 0$, $s \in [a_1, t]$. Since $\varphi(u) > 0$ in $[a_1, b_1]$, it follows that $f(s) = 0$ for all $s \in [a_1, b_1]$, i. e. almost everywhere with respect to F. ▲

REMARK. If $a = -\infty$, accord to the theory of the integral equations of the Volterra type, Theorem 2 will hold under the additional assumption

$$\int_{-\infty}^{t} \left| \frac{\partial g(t, u)}{\partial t} \right| du < \infty$$

for all $t \in [-\infty, b]$.

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